Further constructive results on interconnection and damping assignment control of mechanical systems: The Acrobot example

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SUMMARY

Interconnection and damping assignment passivity-based control is a controller design methodology that achieves (asymptotic) stabilization of mechanical systems endowing the closed-loop system with a Hamiltonian structure with a desired energy function—that qualifies as Lyapunov function for the desired equilibrium. The assignable energy functions are characterized by a set of partial differential equations that must be solved to determine the control law. A class of underactuation degree one systems for which the partial differential equations can be *explicitly solved*—making the procedure truly constructive—was recently reported by the authors. In this brief note, largely motivated by the interesting Acrobot example, we pursue this investigation for two degrees-of-freedom systems where a constant inertia matrix can be assigned. We concentrate then our attention on potential energy shaping and give conditions under which an explicit solution of the associated partial differential equation can be obtained. Using these results we show that it is possible to *swing-up* the Acrobot from some configuration positions in the lower half plane, provided some conditions on the robot parameters are satisfied. Copyright © 2006 John Wiley & Sons, Ltd.

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Notation: For functions $f(q)$, $f : \mathbb{R}^n \to \mathbb{R}$, we denote their gradient and Hessian as $\nabla_q f \triangleq \partial f / \partial q$, $\nabla_q^2 f \triangleq \partial f / \partial q^2$, respectively, and $\nabla_i f \triangleq \partial f / \partial q_i$, $\nabla_{ij} f \triangleq \partial^2 f / \partial q_i \, \partial q_j$—when clear from the context the subindex in ∇ will be omitted. For functions of one variable, $(\cdot)'$ will be used to denote its derivative. All vectors are *column* vectors, even the gradient of a scalar function.

1. INTRODUCTION AND BACKGROUND

In Reference [1] we introduced the interconnection and damping assignment passivity-based control (IDA-PBC) approach to regulate the position of underactuated mechanical systems of the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q E \\ \nabla_p E \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u \tag{1}$$

where $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ are the generalized positions and momenta, respectively, $u \in \mathbb{R}^m$ and $G \in \mathbb{R}^{n \times m}$ with rank $G = m < n$

$$E(q,p) = \tfrac{1}{2} p^\top M^{-1}(q) p + V(q) \tag{2}$$

is the total energy with $M = M^\top > 0$ the inertia matrix, and V the potential energy. The main result of Reference [1] is the proof that for all matrices $M_d(q) = M_d^\top(q) \in \mathbb{R}^{n \times n}$ and functions $V_d(q)$ that satisfy the PDEs

$$0 = G^\perp \{ \nabla_q (p^\top M^{-1} p) - M_d M^{-1} \nabla_q (p^\top M_d^{-1} p) + 2 J_2 M_d^{-1} p \} \tag{3}$$

$$0 = G^\perp \{ \nabla V - M_d M^{-1} \nabla V_d \} \tag{4}$$

for some $J_2(q,p) = -J_2^\top(q,p) \in \mathbb{R}^{n \times n}$ and a full rank left annihilator $G^\perp(q) \in \mathbb{R}^{(n-m) \times n}$ of G, i.e. $G^\perp G = 0$ and rank$(G^\perp) = n - m$, system (1) in closed-loop with the IDA-PBC

$$u = (G^\top G)^{-1} G^\top (\nabla_q E - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p) \tag{5}$$

takes the Hamiltonian form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{6}$$

where the new total energy function is

$$H_d(q,p) = \tfrac{1}{2} p^\top M_d^{-1}(q) p + V_d(q) \tag{7}$$

Further, if M_d is positive definite in a neighbourhood of $q^\star \in \mathbb{R}^n$ and

$$q^\star = \arg \min V_d(q) \tag{8}$$

then $(q^\star, 0)$ is a stable equilibrium point of (6) with Lyapunov function H_d.

2. PROBLEM FORMULATION

Clearly, the success of the IDA-PBC method relies on the possibility of solving the PDEs (3), (4) that identify the energy functions that can be assigned to the closed-loop. In Reference [2] the

authors give a series of conditions on the system and the assignable inertia matrices such that the PDEs can be solved.[||] Also, techniques to solve the PDEs have been reported in References [3, 5] and some geometric aspects of the equations are investigated in Reference [6]. In Reference [7] it is shown that the kinetic energy PDE (3) reduces to an ordinary differential equation (ODE) if the system is of underactuation degree one, that is, if the difference between the number of degrees of freedom and the number of control actions is one—see also Reference [8] for a detailed study of this case. More recently, in Reference [9], we furthermore proved that if the inertia matrix and the force induced by the potential energy (on the unactuated co-ordinate) are independent of the unactuated co-ordinate then the PDEs can be *explicitly solved*.

In this brief note we pursue the investigation of Acosta *et al.* [9] with the aim of relaxing the aforementioned assumption on the potential energy, hence we keep the assumptions

A1. The system has underactuation degree one.
A2. The inertia matrix M depends only on the actuated co-ordinates.

Further, motivated by the interesting Acrobot example [10, 11], we make the following assumption:

A3. The system has two degrees-of-freedom and, without loss of generality, take $G = [0\ \ 1]^{\mathrm{T}}$.

Assumption A3 is critical for our developments. Assumptions A1 and A2, ensures the term $G^\perp \nabla_q (p^\top M^{-1} p)$ in the PDE (3) is zero. In this case (3) can be solved with a constant M_d, taking $J_2 = 0$. This allows us to concentrate our attention on potential energy shaping and the PDE to be solved takes the form

$$\gamma^\top(q_2)\nabla V_d = \nabla_1 V \tag{9}$$

where we have used $G^\perp = [1\ \ 0]$, defined

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \triangleq M^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{10}$$

and taken the constant matrix M_d as

$$M_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \tag{11}$$

The problem that we address in this paper is the derivation of conditions on M and V such that an explicit solution to (9)—satisfying (8)—can be obtained. For simplicity, we will take $q^\star = 0$ and make the observation that, under Assumptions A1 and A3, the origin is an assignable equilibrium for (1) only if

$$\nabla_1 V(0) = 0 \tag{12}$$

that will be assumed in the sequel.

[||] The authors study the PDEs appearing in the Controlled Lagrangian method, but as shown in [3, 4] the PDEs are similar.

3. PRELIMINARY LEMMA

The following lemma, of interest on its own, lies at the core of our subsequent developments. It identifies a class of PDEs of the form (9), whose solution can be transformed into the solution of ordinary differential equations (ODEs).

Lemma 3.1
Consider the PDE in the unknown function $V_d : \mathbb{R}^2 \to \mathbb{R}$

$$a(q_2)\nabla_1 V_d + \nabla_2 V_d = b(q) \tag{13}$$

where $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$. Assume b can be factored as

$$b(q) = c(q_2) + B^\top(q_2)N(q_1) \tag{14}$$

for some $c : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}^p$ and $N : \mathbb{R} \to \mathbb{R}^p$ solution of the ODEs

$$N' = AN \tag{15}$$

for some constant matrix $A \in \mathbb{R}^{p \times p}$ and initial conditions $N(0) \in \mathbb{R}^p$.**

 Then, for all differentiable functions $\Phi : \mathbb{R} \to \mathbb{R}$, a solution of (13) is given by

$$V_d(q) = F(q_2) + H^\top(q_2)N(q_1) + \Phi(q_1 - z(q_2)) \tag{16}$$

with $F : \mathbb{R} \to \mathbb{R}$, $H : \mathbb{R} \to \mathbb{R}^p$ and $z : \mathbb{R} \to \mathbb{R}$ solutions of the ODEs

$$\begin{aligned} F' &= c \\ H' &= -aA^\top H + B \\ z' &= a \end{aligned} \tag{17}$$

Proof
The proof is obtained by direct substitution of (16) into (13). Indeed, differentiating (16)

$$\nabla V_d = \begin{bmatrix} H^\top N' + \Phi' \\ F' + N^\top H' - a\Phi' \end{bmatrix} = \begin{bmatrix} H^\top AN + \Phi' \\ c + B^\top N - a\nabla_1 V_d \end{bmatrix}$$

that, taking into account (14), clearly verifies (13). ∎

 From (16), (17) it is clear that an explicit solution to the PDE (13) is available provided we can compute the integrals

$$\int c(\xi)\,\mathrm{d}\xi, \quad \int a(\xi)\,\mathrm{d}\xi, \quad \int \Psi(\xi, q_2)B(\xi)\,\mathrm{d}\xi$$

where Ψ is the 'state transition matrix' of the H equation in (17)

$$\Psi(\xi, q_2) = \mathrm{e}^{-[\int_\xi^{q_2} a(s)\,\mathrm{d}s]A^\top}$$

 The evaluation of Ψ can sometimes be simplified introducing a change of co-ordinates as follows. Assume the function z is invertible, that is, there exists $\hat{q}_2 : \mathbb{R} \to \mathbb{R}$ such that $z(\hat{q}_2(s)) = s$

**This assumption implies that the dependence of b on q_1 consists of sine, cosine, exponential and polynomial functions generated as solutions of the linear ODEs (15).

for all $s \in \mathbb{R}$. For all functions $x : \mathbb{R} \to \mathbb{R}$ introduce the following notation for the function compositions:

$$\tilde{x}(z) \triangleq x(\hat{q}_2(z))$$

A simple application of the chain rule, i.e. $\mathrm{d}H/\mathrm{d}q_2 = (\mathrm{d}\tilde{H}/\mathrm{d}z)(\mathrm{d}z/\mathrm{d}q_2)$, transforms (17) into

$$\tilde{F}'(z) = \frac{\tilde{c}(z)}{\tilde{a}(z)}$$

$$\tilde{H}'(z) = -A^\top \tilde{H}(z) + \frac{\tilde{B}(z)}{\tilde{a}(z)} \tag{18}$$

where we notice that the function a has been removed from the $\tilde{H}$ equation, yielding a linear forced ODE, whose state transition matrix can be trivially calculated.

From the function $\tilde{H}$ one can obtain the desired H noting that

$$\tilde{H}(z(s)) = H(\hat{q}_2(z(s))) = H(s) \quad \forall s \in \mathbb{R}$$

4. SOLVING THE POTENTIAL ENERGY PDE

In this section we use Lemma 3.1 and the change of co-ordinates procedure described above to give conditions on M and V such that an explicit solution to (9)—satisfying (8)—can be obtained.

We make the following assumptions:

A4. $\nabla_1 V$ can be factored as

$$\nabla_1 V(q) = c_0(q_2) + B_0^\top(q_2) N(q_1) \tag{19}$$

for some $c_0 : \mathbb{R} \to \mathbb{R}$, $B_0 : \mathbb{R} \to \mathbb{R}^p$ and $N : \mathbb{R} \to \mathbb{R}^p$ solution of the ODE

$$N' = AN \tag{20}$$

for some constant matrix $A \in \mathbb{R}^{p \times p}$ and $N(0) \in \mathbb{R}^p$.

A5. $k_1 > 0$, $k_2 \in \mathbb{R}$, elements of M_d, are such that

$$e_i^\top M^{-1}(0) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0, \quad i = 1, 2$$

with $e_i \in \mathbb{R}^2$ the vectors of the orthonormal Euclidean basis.

A6. The integral

$$z(q_2) \triangleq \int_0^{q_2} \frac{\gamma_1(\sigma)}{\gamma_2(\sigma)} \, \mathrm{d}\sigma \tag{21}$$

where γ is defined in (10), is computable and $z(q_2)$ is a diffeomorphism with an inverse map $\hat{q}_2 : \mathbb{R} \to \mathbb{R}$, i.e. $z(\hat{q}_2(s)) = s, \forall s \in \mathbb{R}$. Furthermore, we can compute the functions

$$\tilde{F}(z) \triangleq \int_0^z \frac{c_0}{\gamma_1}(\hat{q}_2(\sigma)) \, \mathrm{d}\sigma$$

$$\tilde{H}^0(z) \triangleq \int_0^z \mathrm{e}^{(A^\top \sigma)} \frac{B_0}{\gamma_1}(\hat{q}_2(\sigma)) \, \mathrm{d}\sigma \tag{22}$$

Proposition 4.1
Assume $M(q_2)$ and $V(q)$ satisfy conditions A4–A6. Then the function

$$V_d(q) = \tilde{F}(z(q_2)) + N^\top(q_1)\tilde{H}(z(q_2)) + \Phi(q_1 - z(q_2))$$

where

$$\tilde{H}(z) = e^{-A^\top z}[\tilde{H}(0) + \tilde{H}^0(z)] \tag{23}$$

solves the potential energy PDE (9) for all differentiable functions $\Phi : \mathbb{R} \to \mathbb{R}$ and all vectors $\tilde{H}(0) \in \mathbb{R}^2$.

Proof
The proof is a direct application of Lemma 3.1 and the change of co-ordinates described in the previous section. Indeed, under Assumption A5,[††] (9) and (19) can be written in the forms (13) and (14), respectively, with

$$a \stackrel{\triangle}{=} \frac{\gamma_1}{\gamma_2}, \quad b \stackrel{\triangle}{=} \frac{\nabla_1 V}{\gamma_2}, \quad c \stackrel{\triangle}{=} \frac{c_0}{\gamma_2}, \quad B \stackrel{\triangle}{=} \frac{B_0}{\gamma_2}$$

The ODEs (17), in the new co-ordinate z, take the form (18) with

$$\frac{\tilde{c}}{\tilde{a}}(z) = \frac{c_0}{\gamma_1}(\hat{q}_2(z)), \quad \frac{\tilde{B}}{\tilde{a}}(z) = \frac{B_0}{\gamma_1}(\hat{q}_2(z)) \tag{24}$$

The solution of these ODEs, given in the proposition, completes the proof. $\qquad\square$

5. MAIN STABILIZATION RESULT

In the previous section we proposed a parametrization of the assignable potential energy functions in terms of the first column of M_d (the reals $k_1 > 0$, $k_2 \in \mathbb{R}$), the vector $\tilde{H}(0) \in \mathbb{R}^2$ and the (differentiable) function Φ. Here we will impose some additional constraints on these parameters to ensure stability of the closed-loop that, besides positivity of M_d (that is satisfied with a suitable choice of k_3), requires assignment of the desired minimum to V_d, i.e. (8). Towards this end we make the following assumptions:

A7. There exists $k_1 > 0$, $k_2 \in \mathbb{R}$ such that

$$\rho \stackrel{\triangle}{=} [k_1 \ k_2] M^{-1}(0) \begin{bmatrix} \nabla_{11} V(0) \\ \nabla_{12} V(0) \end{bmatrix} > 0$$

A8. $\tilde{H}(0)$ and Φ are such that

$$\tilde{H}^\top(0)AN(0) + \Phi'(0) = 0 \tag{25}$$

$$\tilde{H}^\top(0)A^2N(0) + \Phi''(0) > \frac{[\nabla_{11} V(0)]^2}{\rho} \tag{26}$$

[††]Because of continuity $|\gamma(q)| \neq 0$ for all q in a neighbourhood of zero.

Assumption A8 is stated in this form for ease of exposition, but there always exist a vector $\tilde{H}(0)$ such that it is satisfied with a quadratic function $\Phi(s) = (k_p/2)s^2$, $k_p > 0$. Indeed, in this case (25) holds taking $\tilde{H}(0)$ orthogonal to $AN(0)$. Furthermore, (26) will be satisfied taking k_p sufficiently large. Assumption A7, on the other hand, imposes a real constraint on the admissible M and V.

We are in position to present our main result, which essentially boils down to proving that, if A7 and A8 hold, V_d has an isolated local minimum at zero.

Proposition 5.1
Consider system (1), (2) satisfying Assumptions A1–A6. Then the IDA-PBC law (5) takes the form

$$u = \tfrac{1}{2}\nabla_2(p^\top M^{-1}p) + \nabla_2 V - [k_2 \ k_3]M^{-1}\nabla V_d$$

and it yields the closed-loop dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & 0 \end{bmatrix}\begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{27}$$

where $H_d(q,p)$ is given by (7) with M_d the constant matrix (11) and $V_d(q)$ as in Proposition 4.1.

Furthermore, if $k_3 > k_2^2/k_1$ and Assumptions A7 and A8 hold, the origin is a stable equilibrium of (27) with Lyapunov function H_d.

Proof
The control expression follows from (5) taking into account Assumptions A1–A3, and the fact that $J_2 = 0$ and M_d is constant.

Proposition 4.1 establishes that, under Assumptions A4–A6, the proposed V_d solves the potential energy PDE (8). Now, the condition on k_3 ensures $M_d > 0$, hence H_d will be a Lyapunov function for the system if V_d has an isolated local minimum at zero. To prove this fact we will find convenient to study the function in the co-ordinates (q_1, z), hence we define

$$\tilde{V}_d(q_1, z) \triangleq \tilde{F}(z) + N^\top(q_1)\tilde{H}(z) + \Phi(q_1 - z)$$

$$\tilde{V}(q_1, z) \triangleq V(q_1, \hat{q}_2(z))$$

The gradient of $\tilde{V}_d$ is

$$\nabla\tilde{V}_d = \begin{bmatrix} \tilde{H}^\top AN + \Phi' \\ \tilde{F}' + N^\top\tilde{H}' - \Phi' \end{bmatrix} = \begin{bmatrix} \tilde{H}^\top AN + \Phi' \\ \dfrac{\tilde{c}}{\tilde{a}} + N^\top\dfrac{\tilde{B}}{\tilde{a}} - \nabla_1\tilde{V}_d \end{bmatrix}$$

where the second identity is obtained using (18). Now, from (19) and (24) we have that

$$\frac{\tilde{c}}{\tilde{a}}(z) + N^\top(q_1)\frac{\tilde{B}}{\tilde{a}}(z) = \frac{1}{\tilde{\gamma}_1(z)}\nabla_1\tilde{V}(q_1, z)$$

which, in view of (12) and the fact that $\hat{q}_2(0) = 0$, is zero at $(q_1, z) = (0, 0)$. Consequently, $\nabla\tilde{V}_d(0) = 0$ if and only if (25) holds.

Some simple calculation show that the Hessian of $\tilde{V}_d$ is given by

$$\nabla^2 \tilde{V}_d(q_1, z) = \begin{bmatrix} \kappa_1(q_1, z) & \kappa_2(q_1, z) - \kappa_1(q_1, z) \\ \kappa_2(q_1, z) - \kappa_1(q_1, z) & \kappa_1(q_1, z) - \kappa_2(q_1, z) + \kappa_3(q_1, z) \end{bmatrix}$$

where the real-valued functions $\kappa_i, i = 1, \ldots, 3$ are defined as

$$\kappa_1 = \tilde{H}^\top A^2 N + \Phi''$$

$$\kappa_2 = N^\top A^\top \frac{\tilde{B}}{\tilde{a}} = \frac{1}{\tilde{\gamma}_1} \nabla_{11} \tilde{V}$$

$$\kappa_3 = \nabla_2 \left(\frac{1}{\tilde{\gamma}_1} \nabla_1 \tilde{V} \right) = \frac{1}{\tilde{\gamma}_1^2} \left(\tilde{\gamma}_2 \nabla_{12} V - \frac{\tilde{\gamma}_2}{\tilde{\gamma}_1} \gamma_1' \nabla_1 V \right)$$

The Hessian is positive-definite if $(\kappa_2 + \kappa_3) > 0$ and $\kappa_1 > \kappa_2^2/(\kappa_2 + \kappa_3)$. Evaluated at the origin, and taking into account that $\nabla_1 V(0) = 0$, we have

$$\kappa_1(0) = \tilde{H}^\top(0) A^2 N(0) + \Phi''(0)$$

$$\frac{\kappa_2^2(0)}{\kappa_2(0) + \kappa_3(0)} = \frac{[\nabla_{11} V(0)]^2}{\rho}$$

with ρ as defined in Assumption A7, completing the proof. $\qquad\square$

6. THE ACROBOT EXAMPLE

In this section we prove that the methodology described above applies to the interesting Acrobot system described in Reference [10]. The equations of motion of the Acrobot (schematic shown in Figure 1) are given by (1), (2) with $n = 2$, $m = 1$,
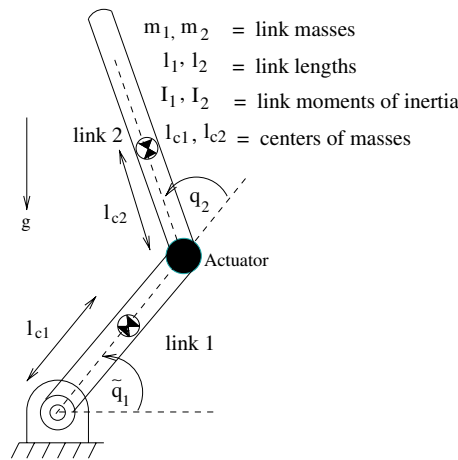
$$M(q_2) = \begin{bmatrix} c_1 + c_2 + 2c_3 \cos q_2 & c_2 + c_3 \cos q_2 \\ c_2 + c_3 \cos q_2 & c_2 \end{bmatrix}$$

$$V(q) = g[c_4 \cos q_1 + c_5 \cos(q_1 + q_2)], \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with g the gravitational constant and the parameters

$$c_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1, \quad c_2 = m_2 l_{c2}^2 + I_2, \quad c_3 = m_2 l_1 l_{c2}$$

$$c_4 = m_1 l_{c1} + m_2 l_1, \quad c_5 = m_2 l_{c2}$$

verifying $c_1 c_2 > c_3^2$.

The control objective is to stabilize the upward equilibrium position, i.e. $q^\star = (0, 0)$.

Figure 1. The Acrobot, with $q_1 = \tilde{q}_1 - \pi/2$.

6.1. Verifying the assumptions

The Acrobot clearly satisfies Assumptions A1–A3. Assumption A4 also holds with

$$c_0(q_2) = 0, \quad B_0(q_2) = -\begin{bmatrix} c_4 g + c_5 g \cos q_2 \\ c_5 g \sin q_2 \end{bmatrix}, \quad N(q_1) = \begin{bmatrix} \sin q_1 \\ \cos q_1 \end{bmatrix}$$

$$A \stackrel{\triangle}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To verify Assumptions A5 and A6 we compute the vector γ given in (10) that, using (11), yields

$$\gamma(q_2) = \frac{1}{\delta(q_2)} \begin{bmatrix} (k_1 - k_2)c_2 - k_2 c_3 \cos q_2 \\ k_2(c_1 + c_2) - k_1 c_2 + c_3(2k_2 - k_1) \cos q_2 \end{bmatrix} \tag{28}$$

where $\delta(q_2) \stackrel{\triangle}{=} c_1 c_2 - c_3^2 \cos^2(q_2) > 0$. The ratio γ_1/γ_2 can be expressed as

$$\frac{\gamma_1}{\gamma_2} = \frac{a_3 + a_4 \cos q_2}{a_1 + a_2 \cos q_2}$$

where

$$a_1 \stackrel{\triangle}{=} k_2(c_1 + c_2) - k_1 c_2, \quad a_2 \stackrel{\triangle}{=} (2k_2 - k_1)c_3, \quad a_3 \stackrel{\triangle}{=} (k_1 - k_2)c_2, \quad a_4 \stackrel{\triangle}{=} -k_2 c_3$$

Even though the function γ_1/γ_2 above can be explicitly integrated, to simplify the derivations we exploit the fact that the ratio takes a constant value $\mu \stackrel{\triangle}{=} a_4/a_2$ when $a_3 a_2 = a_1 a_4$.[‡‡] The latter

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[‡‡]This fact is easily seen noting that

$$\frac{\gamma_1}{\gamma_2} = \frac{a_4}{a_2} + \frac{a_3 a_2 - a_1 a_4}{a_2} \frac{1}{a_1 + a_2 \cos q_2}$$

fixes the following relationship between the free parameters k_1 and k_2:

$$k_1 = \left(1 \pm \sqrt{\frac{c_1}{c_2}}\right) k_2 \tag{29}$$

where we can select either the positive or the negative sign, and the ratio takes the values

$$\frac{\gamma_1}{\gamma_2} = \mu \triangleq \frac{1}{\pm\sqrt{\dfrac{c_1}{c_2}} - 1}$$

From (29) we see that, since $k_1 > 0$, our only choice when $c_1 = c_2$ is the positive sign but, unfortunately, μ is not defined in this case, hence we need the following:

Assumption A6′
$c_1 \neq c_2$.

Under Assumption A6′ (21) becomes $z(q_2) = \mu q_2$, which is obviously a diffeomorphism. Interestingly, for the choice of parameters (29), Assumption A5 is satisfied for all q_2. Indeed, replacing (29) in (28) and doing some simple calculations we get

$$\gamma(q_2) = \frac{k_2\left(\pm\sqrt{c_1 c_2} - c_3 \cos q_2\right)}{\delta(q_2)} \begin{bmatrix} 1 \\ \pm\sqrt{\dfrac{c_1}{c_2}} - 1 \end{bmatrix}$$

which, in view of Assumption A6′, $c_1 c_2 > c_3^2$ and $\delta > 0$, is non-zero for all q_2.

After some lengthy, but straightforward, calculations we see that the remaining integrability conditions of Assumption A6 are easily verifiable with $\tilde{F} = 0$ and $\tilde{H}$ as in (23) with

$$\mathrm{e}^{-A^\top z} = \begin{bmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{bmatrix}$$

$$\tilde{H}^0(z) = \begin{bmatrix} -b_1 \sin z - b_2 \sin\left(\pm\sqrt{\dfrac{c_1}{c_2}}z\right) - b_3 \sin\left(\left(\pm2\sqrt{\dfrac{c_1}{c_2}} - 1\right)z\right) + b_4 \sin\left(\left(\pm\sqrt{\dfrac{c_1}{c_2}} - 2\right)z\right) \\ b_1 \cos z + b_2 \cos\left(\pm\sqrt{\dfrac{c_1}{c_2}}z\right) + b_3 \cos\left(\left(\pm2\sqrt{\dfrac{c_1}{c_2}} - 1\right)z\right) + b_4 \cos\left(\left(\pm\sqrt{\dfrac{c_1}{c_2}} - 2\right)z\right) \end{bmatrix}$$

and the constants $b_i, i = 1 \ldots 4$ defined as

$$b_1 \triangleq \frac{1}{k_2} g\left(\pm\sqrt{c_1 c_2}c_4 + \frac{c_3 c_5}{2}\right), \quad b_2 \triangleq \frac{1}{k_2}\frac{g\mu}{\mu+1}\left(\frac{1}{2}c_3 c_4 \pm \sqrt{c_1 c_2}c_5\right)$$

$$b_3 \triangleq \frac{1}{k_2}\frac{gc_3 c_5 \mu}{2\mu+4}, \quad b_4 \triangleq \frac{1}{k_2}\frac{gc_3 c_4}{2}\frac{\mu}{\mu-1} \tag{30}$$

where we have pulled-out the factor $1/k_2$ to underscore its role as a 'scaling gain' for V_d.

In the co-ordinates (q_1, q_2), the potential energy V_d takes the simple form

$$V_d(q_1, q_2) = N^\top(q_1)\mathrm{e}^{A(\mu q_2)}\tilde{H}(0) + b_1 \cos q_1 + b_2 \cos(q_1 + q_2)$$
$$+ b_3 \cos(q_1 + 2q_2) + b_4 \cos(q_1 - q_2) + \Phi(q_1 - \mu q_2) \tag{31}$$

Once we have established, following Proposition 4.1, the existence of a solution for the potential energy PDE, it remains to verify the stability conditions of Proposition 5.1. For, using (29), we compute

$$\rho = \frac{-gk_2}{c_1 c_2 - c_3^2}\left[\left(-c_3 \pm \sqrt{c_1 c_2}\right)c_4 - c_5\left(-c_1 \pm c_3\sqrt{\frac{c_1}{c_2}}\right)\right] \qquad (32)$$

hence, Assumption A7 is equivalent to

Assumption A7′
The constants k_2 and c_i are such that

$$k_2\left[\left(-c_3 \pm \sqrt{c_1 c_2}\right)c_4 - c_5\left(-c_1 \pm c_3\sqrt{\frac{c_1}{c_2}}\right)\right] < 0$$

Assumption A6′ is generically satisfied and the only 'real' assumption is A7′. However, note that—except for the case when $c_1 = c_2$—the sign of k_2 is not fixed, hence it is also always possible to fulfil this condition.

Finally, Assumption A8 can be shown to hold if and only if the following condition is satisfied:

Assumption A8′
The elements of the vector $\tilde{H}(0)$ and the function Φ satisfy

$$\tilde{H}_1(0) = -\Phi'(0)$$

$$\tilde{H}_2(0) < \Phi''(0) - \frac{g^2(c_4 + c_5)^2}{\rho}$$

6.2. Asymptotic stability

We are in position to state our stabilization result for the Acrobot where, to achieve asymptotic stability, we have added a damping injection term [1].

Proposition 6.1
Consider the Acrobot system satisfying Assumptions A6′ and A7′. Fix

$$k_1 = \left(1 - \sqrt{\frac{c_1}{c_2}}\right)k_2, \quad k_3 > \frac{k_2}{1 - \sqrt{\frac{c_1}{c_2}}}$$

Let $\Phi(q_1 - \mu q_2) = (k_p/2)(q_1 - \mu q_2)^2$ with

$$\mu = \frac{-1}{\sqrt{\frac{c_1}{c_2}} + 1}, \quad k_p > \frac{g^2(c_4 + c_5)^2}{\rho} + \tilde{H}_2(0)$$

with ρ given by (32) (with negative sign), $\tilde{H}_2(0)$ arbitrary and $\tilde{H}_1(0) = 0$. Then the (globally defined) IDA-PBC law

$$u = -\frac{1}{2}\nabla_{q_2}(p^\top M^{-1}p) - [k_2 \; k_3]M^{-1}\nabla V_d + \nabla_{q_2}V + \frac{k_v}{k_1 k_3 - k_2^2}(k_2 p_1 - k_1 p_2)$$

where $k_v > 0$ is the damping injection gain and

$$\nabla_1 V_d = -\tilde{H}_2(0)\sin(q_1 - \mu q_2) - b_1 \sin q_1 - b_2 \sin(q_1 + q_2) - b_3 \sin(q_1 + 2q_2)$$
$$- b_4 \sin(q_1 - q_2) + k_p(q_1 - \mu q_2)$$

$$\nabla_2 V_d = \tilde{H}_2(0)\mu \sin(q_1 - \mu q_2) - b_2 \sin(q_1 + q_2) - 2b_3 \sin(q_1 + 2q_2)$$
$$+ b_4 \sin(q_1 - q_2) - k_p\mu(q_1 - \mu q_2)$$

achieves (local) asymptotic stabilization of the trivial equilibrium with Lyapunov function H_d.

Proof
It is easy to verify that, with the given selection of Φ and $\tilde{H}(0)$, Assumption A8′ is satisfied. Henceforth, stability follows from Proposition 5.1 and it only remains to prove the asymptotic part.

The derivative of H_d along the trajectories of the closed-loop system is given by

$$\dot{H}_d = -k_v|G^\top M_d^{-1}p|^2 \leqslant 0$$

with $|\cdot|$ the Euclidean norm. Define the residual set as

$$\Pi \triangleq \{(q,p) \in \Omega_c \mid G^\top M_d^{-1}p = 0\}$$

Let $(q,p) \in \Pi$. We have

$$0 = G^\top M_d^{-1}p = G^\top M_d^{-1}M\dot{q} \Leftrightarrow \dot{q}_1 = \frac{\gamma_1}{\gamma_2}\dot{q}_2 = \mu\dot{q}_2 \tag{33}$$

Integrating (33), we obtain $q_1(t) = \mu q_2(t) + C$, where C is the integration constant.

Differentiating the left hand side of (33), we have

$$0 = G^\top M_d^{-1}\dot{p} = -G^\top M^{-1}\nabla V_d \tag{34}$$

where, to obtain the second identity, we have used the fact that the closed–loop dynamics verify

$$\dot{p} = -M_d M^{-1}\nabla_q V_d(q) - k_v G G^\top M_d^{-1}p$$

Now, since $G^\top M^{-1}\nabla V_d$ is an analytic function of q_2, Equation (34) (whose right hand side is not identically zero) has only isolated solutions. Thus, the only possible trajectories lying in Π are of the form $(q_1(t), q_2(t), p_1(t), p_2(t)) = (\bar{q}_2 + C, \bar{q}_2, 0, 0)$, with $\bar{q}_2$ constant.

Since the origin is an *isolated* and stable equilibrium point, we can conclude that trajectories starting sufficiently near to the origin will eventually converge to it, thus proving local asymptotical stability. $\qquad\square$
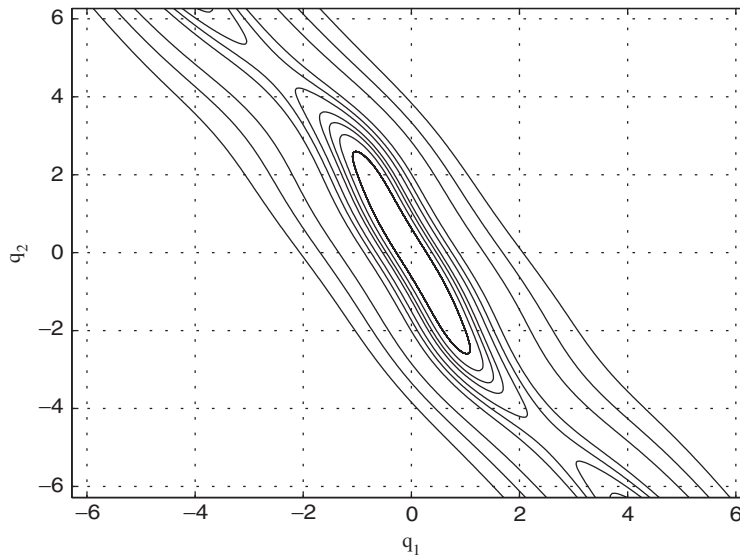
6.3. Simulations

The Acrobot parameters—resulting from $m_1 = 4, m_2 = 1, I_1 = 0.3333, I_2 = 1.3333, l_1 = 1, l_2 = 4, l_{c1} = 0.5000, l_{c2} = 2$—are displayed in Table I.

In (29) we select the negative sign, $k_2 = 1$ and $k_3 = 5.9073$ yielding

$$M_d = \begin{bmatrix} 0.3386 & 1 \\ 1 & 5.9073 \end{bmatrix} > 0, \quad \mu = -0.6019$$

Table I. Acrobot parameters.

c_1	c_2	c_3	c_4	c_5
2.3333	5.3333	2	3	2

Figure 2. Level sets of V_d.

The level sets of V_d, for $k_p = 280$, $\tilde{H}_2(0) = 10$, are shown in Figure 2. We observe from the figure that there are closed sets containing points in the lower half plane of the configuration space, that is, $|q_1| > \pi/2$. This ensures that, starting with zero velocities, the IDA-PBC will *swing-up* the Acrobot from the lower half plane and at the same time asymptotically stabilize its upward equilibrium. To the best of our knowledge this is the first *continuous* controller that achieves these two objectives for the Acrobot. It should, however, be pointed out that our ability to enlarge the domain of attraction is severely stymied as the actual shape of V_d is essentially determined by the robot parameters. Indeed, from (31) it is clear that the effect of the design parameters at our disposal $(\tilde{H}_2(0), k_2, k_p)$[§§] 'disappears' along the line $q_1 = \mu q_2$. On the other hand, V_d may have other extrema at some points arbitrarily close to this line, thus generating spurious equilibria for the closed-loop system.

The time-response of the system starting from the fully stretched downward position, i.e. $(q(0), p(0)) = (-\pi, 0, 0, 0)$ with $k_v = 12$ is depicted in Figure 3. Although this initial condition does not belong to the estimated domain of attraction $\Omega_{\bar{c}}$, the simulation shows that the controller effectively swings-up the Acrobot from the downward position. In order to justify theoretically this simulated evidence, current research is under way to improve our estimate of the basin of attraction.

[§§]Assuming a quadratic Φ.
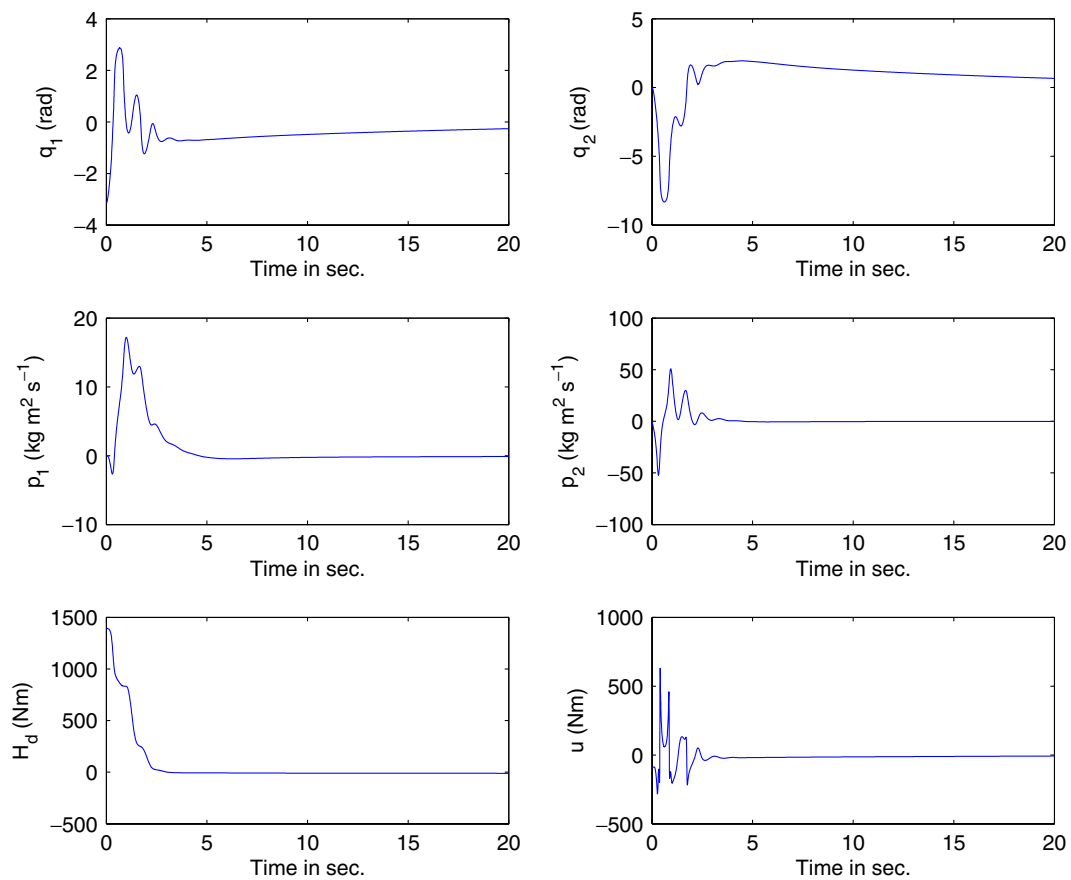
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Figure 3. Transient responses.

7. CONCLUSIONS

Within the framework of IDA-PBC, we have presented a methodology to reduce the potential PDE into an ODE for underactuated mechanical systems with an underactuation degree one, thus overcoming the main stumbling block in its application. The solvability of the ODES's is demonstrated on the Acrobot, wherein the control task of swing-up and balance about the upward equilibrium point is achieved.

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