Energy injection for mechanical systems thorugh the method of Virtual Nonholonomic Constraints

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TEST CITATION [1]

1 Virtual Nonholonomic Constraints

1.1 Motivation

TODO: Why do we bother with Hamiltonian? Why can't we do virtual nonholonomic constraints in Lagrangian?

1.2 A Hamiltonian Approach to Virtual Nonholonomic Constraints

Take an underactuated mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla V(q) = B(q)\tau \tag{1}$$

with Lagrangian

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q) \tag{2}$$

where $q = (q_1, \ldots, q_n) \in \mathcal{Q}$ is the configuration vector, the inertia matrix $M(q) = M^T(q)$ is positive definite for all $q \in \mathcal{Q}$, $V(q)$ is the potential of the system, $B : \mathcal{Q} \to \mathbb{R}^{n\times m}$ is a C^1 full rank control matrix, and $\tau \in \mathbb{R}^m$ is the control input.

One can compute the Hamiltonian of this system by performing a Legendre transform on the Lagrangian. Define the conjugate of momenta for q by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \in \mathbb{R}^n$$

Then, the Legendre transform is performed by taking

$$\mathcal{H}(q,p) = p^T\dot{q} - \mathcal{L}(q,\dot{q})$$

It is easy to verify that the Hamiltonian for system (2) is given by the total mechanical energy (3).

$$\mathcal{H}(q,p) = E(q,p) = \frac{1}{2}p^T M^{-1}(q)p + V(q) \tag{3}$$

The equations of motion for the system in Hamiltonian coordinates is given by

$$\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = M^{-1}(q)p \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} + B\tau \end{aligned} \tag{4}$$

Suppose the mechanical system has degree of underactuation one, so that coordinates of the system can be split into an unactuated component $q_u \in [\mathbb{R}]_T$, $T \in \mathbb{R}_{>0}$ which is not influence by control, along with an actuated component q_a; that is, suppose B is of the form

$B(q) = [0_m, B_1^T(q), \ldots, B_n^T(q)]^T$, $B_i^T(q) \in \mathbb{R}^{n-1}$ and $\tau \in \mathbb{R}^{n-1}$. In this case, $q = (q_u, q_a)^T$ and the equations of motion become

$$\begin{aligned}
\dot{q_u} &= e_1^T M^{-1}(q) p \\
\dot{p_u} &= -p^T \frac{\partial M}{\partial q_u} p - \partial_{q_u} V(q) \\
\dot{q_a} &= \begin{bmatrix} 0 \cdots 0 \\ I_{n-1} \end{bmatrix} M^{-1}(q) p \\
\dot{p_a} &= -p^T \frac{\partial M}{\partial q_a} p - \nabla q_a V(q) + \begin{bmatrix} B_1(q) \\ \vdots \\ B_n(q) \end{bmatrix} \tau
\end{aligned} \tag{5}$$

Now we can begin to talk about Virtual Nonholonomic Constraints. In a similar fashion to what was defined for VHCs, let us first define the goal of these new virtual constraints.

Definition 1.1. A relation $h \in C^2\left(\mathcal{Q} \times \mathbb{R}^n; \mathbb{R}^k\right)$ with $h(q,p) = 0$ is a **virtual nonholonomic constraint (VNHC) of order k** if there exists a feedback control $\tau(q,p)$ which stabilizes the constraint manifold

$$\Gamma = \{(q,p) | h(q,p) = 0, dh_q \dot{q} + dh_p \dot{p} = 0\}$$

Define the output of the system to be $e = h(q,p)$. We would like to find $\tau(q,p)$ which drives e to zero to stabilize our constraint manifold Γ. To accomplish this, we will input-output linearize (5) to find $\ddot{e} = -k_p e - k_d \dot{e}$ with $k_p, k_d \in \mathbb{R}_{>0}$.

To characterize a certain class of VNHCs, let us make the following assumption.

Assumption 1.2. We assume our relation h is of the form $h(q,p) = q_a - f(q_u, p_u)$ for some $f \in C^2\left([\mathbb{R}]_T \times \mathbb{R}; \mathbb{R}^{n-1}\right)$.

Now we solve for τ by finding $\ddot{e}$.

$$\begin{aligned}
e &= h(q,p) = qa - f(q_u, p_u) \\
\Rightarrow \dot{e} &= \dot{q_a} - df_{q_u} \dot{q_u} - df_{p_u} \dot{p_u} \\
&= [-df_{q_u} I_{n-1}] \dot{q} - df_{p_u} \dot{p_u} \\
&= dh_q M^{-1}(q) p - df_{p_u} \left(-\frac{1}{2} p^T \frac{\partial M^{-1}(q)}{\partial q_u} p - \partial_{q_u} V(q) \right) \\
&= dh_q M^{-1}(q) p + \frac{1}{2} df_{p_u} p^T \frac{\partial M^{-1}(q)}{\partial q_u} p + df_{p_u} \partial_{q_u} V(q)
\end{aligned}$$

The control input τ only appears in $\dot{p_a}$ (see (5)). To simplify the analysis, terms in $\ddot{e}$ which do not depend on $\dot{p_a}$ explicitly are lumped together under the symbol $(*)$:

$$\begin{aligned}
\ddot{e} &= dh_q M^{-1}(q)\dot{p} + df_{p_u} p^T \frac{\partial M^{-1}(q)}{\partial q_u}\dot{p} + (*) \\
&= (dh_q M^{-1}(q) + df_{p_u} p^T \frac{\partial M^{-1}(q)}{\partial q_u})B\tau + (*) \\
&= (dh_q M^{-1}(q) + dh_{p_u} p^T \frac{\partial M^{-1}(q)}{\partial q_u})B\tau + (*)
\end{aligned}$$

From the derivations above, one can solve for τ iff the matrix on the left of τ is full rank. Thus, for systems with degree of underactuation one we give the following definition.

Definition 1.3. A VNHC $h(q,p) = 0$ of order $n-1$ is **regular** if $dh_{p_a} = 0$, $dh_{q_a} = (1 \dots 1)^T$, and

$$\operatorname{rank}\left\{(dh_q M^{-1}(q) + dh_{p_u} p^T \frac{\partial M^{-1}(q)}{\partial q_u})B\right\} = n-1$$

everywhere on the constraint manifold Γ. Equivalently, a VNHC h of order $n-1$ is regular if it satisfies Assumption 1.2 and system (5) with output $e = h(q,p)$ is of relative degree $\{2, 2, \dots, 2\}$ everywhere on Γ.

In general, $\dot{e}$ is a function of q_u and $p = (p_u, p_a)^T$. Since the purpose of a regular VNHC is to fully parameterize Γ by (q_u, p_u), it is essential that one can solve for $p_a = p_a(q_u, p_u)$. Unfortunately this often cannot be done, since $\dot{e}$ contains the quadratic term

$$\frac{1}{2} df_{p_u} p^T \frac{\partial M^{-1}(q)}{\partial q_u} p$$

We can solve for p_a if this quadratic term does not exist.

Assumption 1.4. Assume $\partial M(q)/\partial q_u = 0 \Leftrightarrow \partial M^{-1}(q)/\partial q_u = 0$

Under Assumption 1.4, we get that the rank condition for $h(q,p)$ to be a regular VNHC reduces to $\operatorname{rank}(dh_q M^{-1} B) = n-1$. This is the same rank condition as required for Virtual Holonomic Constraints.

Now we solve for p_a on the constraint manifold (when $e = \dot{e} = 0$):

$$\dot{e} = dh_q M^{-1}(q)p + df_{p_u}\partial_{q_u}V(q) = 0$$

$$\Leftrightarrow dh_q M^{-1}(q)e_1 p_u + dh_q M^{-1}(q)\begin{bmatrix} 0 & \cdots & 0 \\ & I_{n-1} & \end{bmatrix} p_a = -df_{p_u}\partial_{q_u}V(q)$$

$$\Leftrightarrow dh_q M^{-1}(q)\begin{bmatrix} 0 & \cdots & 0 \\ & I_{n-1} & \end{bmatrix} p_a = -\left(df_{p_u}\partial_{q_u}V(q) + dh_q M^{-1}(q)e_1 p_u\right)$$

One can linearly solve for p_a if and only if the matrix in front of it is invertible.

This leads us to a natural definition.

Definition 1.5. A VNHC $h(q,p)$ is **solvable** (NOTE: actionable? what's a good name?) if

$$\operatorname{rank}\left(dh_q M^{-1}(q)\begin{bmatrix}0 & \cdots & 0\\ & I_{n-1} & \end{bmatrix}\right) = n-1$$

With this analysis and our new definition in hand, we can solve for the dynamics on the constraint manifold.

Theorem 1.6. *Suppose assumptions 1.2 and 1.4 hold. If a regular VNHC $h(q,p) = q_a - f(q_u,p_u)$ is solvable, then the parameterization for p_a on the constraint manifold is given by*

$$\begin{aligned}p_a &= -\left(dh_q M^{-1}(q)\begin{bmatrix}0 & \cdots & 0\\ & I_{n-1} & \end{bmatrix}\right)^{-1}\left(df_{p_u}\partial_{q_u}V(q) + dh_q M^{-1}(q)e_1 p_u\right)\\ &=: g(q_u,p_u)\end{aligned}$$

and the constrained dynamics on Γ are given by

$$\begin{aligned}\dot{q_u} &= e_1^T M^{-1}(q_a)\begin{bmatrix}p_u\\ p_a\end{bmatrix}\Big|_{q_a=f(q_u,p_u),p_a=g(q_u,p_u)}\\ \dot{p_u} &= -\partial_{q_u}V(q_u,q_a)\,\Big|_{q_a=f(q_u,p_u)}\end{aligned}$$

Theorem 1.6 guarantees that, on Γ, q_a is a parameterized completely by (q_u,p_u). Hence, the zero-dynamics on Γ are always two-dimensional regardless of the original dimension n.

2 Energy Injection for the Acrobot

Suppose the acrobot is simple; that is, assume that the acrobot's two links are of equal length l and of equal mass m, with the mass concentrated at the tip of each rod.

Let $q = (q_u, q_a)$ where q_u is the shoulder angle and q_a is the hip angle.

Then the acrobot has hamiltonian given by

$$\mathcal{H}(q,p) = \frac{1}{2}p^T M^{-1}(q)p + V(q)$$

where

$$M^{-1}(q) = \frac{1}{ml^2(2-\cos^2(q_a))}\begin{bmatrix} 1 & -(1+\cos(q_a)) \\ -(1+\cos(q_a)) & (3+2\cos(q_a)) \end{bmatrix}$$

and

$$V(q) = mgl(3 - 2\cos(q_u) - \cos(q_u + q_a))$$

Notice that the acrobot satisfies $\partial M^{-1}(q)/\partial q_u = 0$, which means it may be possible to enforce a regular solvable VNHC to inject energy into the acrobot. One such VNHC is

$$\begin{aligned} h(q,p) &= q_a - f(q_u, p_u) \\ &= q_a - \bar{q_a}\sin(\arctan(p_u, q_u)) \\ &= q_a - \frac{\bar{q_a}p_u}{\sqrt{q_u^2 + p_u^2}} \end{aligned} \tag{6}$$

where $\bar{q_a} \in]0,\pi[$ is a parameter which dictates how much the controlled link can swing back and forth.

To compute the regularity and solvability conditions, first observe that since the acrobot has two coordinates and only p_a is actuatable, the control matrix $B = [0,1]^T$ is constant and full rank everywhere. This also implies that h is both regular and solvable if $dh_q M^{-1}(q)B$ is full rank.

Observe that

$$\begin{aligned} dh_q M^{-1}(q)B &= \frac{1}{ml^2(2-\cos^2(q_a))}\left[\frac{\bar{q_a}p_u q_u}{(q_u^2+p_u^2)^{3/2}}, 1\right]\begin{bmatrix} -(1+\cos(q_a)) \\ (3+2\cos(q_a)) \end{bmatrix} \\ &= \frac{(3+2\cos(q_a))(q_u^2+p_u^2)^{3/2} - \bar{q_a}p_u q_u(1+cos(q_a))}{ml^2(q_u^2+p_u^2)^{3/2}(2-\cos^2(q_a))} \end{aligned} \tag{7}$$

TODO: **Determine if this is full rank everywhere except at** $(q_u, p_u) = (0,0)$

Solving for p_a is straightforward. The computation is shown below.

$$p_a = \left(dh_q M^{-1}(q)B\right)^{-1}$$
$$\left(\frac{mgl\bar{q_a}p_u q_u^2}{(q_u^2+p_u^2)^{3/2}}\left(2\sin(q_u) + sin(q_u+q_a)\right) + \frac{1}{ml^2(2-\cos^2(q_a))}\left[\frac{\bar{q_a}p_u q_u}{(q_u^2+p_u^2)^{3/2}}, 1\right]\begin{bmatrix} 1 \\ -(1+\cos(q_a)) \end{bmatrix}\right)$$

$$
= -\frac{\frac{mgl\bar{q_a}q_u^2(2\sin(q_u)+\sin(q_u+q_a))}{(q_u^2+p_u^2)^{3/2}} + \frac{\bar{q_a}p_u^2q_u}{ml^2(q_u^2+p_u^2)^{3/2}(2-\cos^2(q_a))} - \frac{(1+\cos(q_a))p_u}{ml^2(2-\cos^2(q_a))}}{\frac{(3+2\cos(q_a))(q_u^2+p_u^2)^{3/2}-\bar{q_a}(1+\cos(q_a))q_up_u}{ml^2(q_u^2+p_u^2)^{3/2}(2-\cos^2(q_a))}}
$$

$$
= \frac{\frac{-m^2gl^3\bar{q_a}q_u^2(2\sin(q_u)+\sin(q_u+q_a))(2-\cos^2(q_a))-\bar{q_a}p_u^2q_u+(1+\cos(q_a))(q_u^2+p_u^2)^{3/2}p_u}{ml^2(q_u^2+p_u^2)^{3/2}(2-\cos^2(q_a))}}{\frac{(3+2\cos(q_a))(q_u^2+p_u^2)^{3/2}-\bar{q_a}(1+\cos(q_a))q_up_u}{ml^2(q_u^2+p_u^2)^{3/2}(2-\cos^2(q_a))}}
$$

$$
\Rightarrow p_a = \frac{(1+\cos(q_a))(q_u^2+p_u^2)^{3/2}p_u - m^2gl^3\bar{q_a}q_u^2(2\sin(q_u)+\sin(q_u+q_a))(2-\cos^2(q_a)) - \bar{q_a}q_up_u^2}{(3+2\cos(q_a))(q_u^2+p_u^2)^{3/2} - \bar{q_a}(1+\cos(q_a))q_up_u}
$$

With p_a in hand, it is now possible to solve for the zero dynamics under the constraint. To simplify the notation we define $s_u := \sin(q_u)$, $s_{ua} := \sin(q_u + q_a)$, and $c_a := \cos(q_a)$.

$$
\begin{aligned}
\dot{q_u} &= \frac{1}{ml^2(2-c_a^2)}[1, -(1+c_a)] \begin{bmatrix} p_u \\ p_a \end{bmatrix} \\
\dot{p_u} &= -mgl(2s_u + s_{ua})
\end{aligned}
$$

Expanding out $\dot{q_u}$ we get

$$
\begin{aligned}
\dot{q_u} = &\frac{p_u}{ml^2(2-c_a^2)} \\
&+ \frac{m^2gl^3\bar{q_a}q_u^2(2s_u+s_{ua})(2-c_a^2)(1+c_a) - (1+c_a)^2(q_u^2+p_u^2)^{3/2}p_u + \bar{q_a}(1+c_a)q_up_u^2}{ml^2(2-c_a^2)((3+2c_a)(q_u^2+p_u^2)^{3/2} - \bar{q_a}(1+c_a)q_up_u)}
\end{aligned}
$$

and, putting everything over one denominator,

$$
\begin{aligned}
\dot{q_a} &= \frac{(3+2c_a)(q_u^2+p_u^2)^{3/2}p_u - (1+c_a)^2(q_u^2+p_u^2)^{3/2}p_u + m^2gl^3\bar{q_a}q_u^2(2s_u+s_{ua})(2-c_a^2)(1+c_a)}{ml^2(2-c_a^2)((3+2c_a)(q_u^2+p_u^2)^{3/2} - \bar{q_a}(1+c_a)q_up_u)} \\
&= \frac{(2-c_a^2)\left((q_u^2+p_u^2)^{3/2}p_u + m^2gl^3\bar{q_a}q_u^2(2s_u+s_{ua})(1+c_a)\right)}{ml^2(2-c_a^2)((3+2c_a)(q_u^2+p_u^2)^{3/2} - \bar{q_a}(1+c_a)q_up_u)}
\end{aligned}
$$

Therefore, the zero-dynamics of the acrobot under the constraint $q_a = \sin(\arctan(p_u, q_u))$ are:

$$
\begin{aligned}
\dot{q_u} &= \frac{(q_u^2+p_u^2)^{3/2}p_u + m^2gl^3\bar{q_a}q_u^2(2s_u+s_{ua})(1+c_a)}{ml^2\left((3+2c_a)(q_u^2+p_u^2)^{3/2} - \bar{q_a}(1+c_a)q_up_u\right)} \\
\dot{p_u} &= -mgl(2s_u + s_{ua})
\end{aligned} \tag{8}
$$

It may be convenient to represent these equations of motion in polar coordinates. Letting $r = \sqrt{q_u^2 + p_u^2}$ be the radius and $\theta = \arctan(p_u, q_u)$ be the angle in the (q_u, p_u) plane and taking derivatives, it is easy to show that

$$
\begin{aligned}
\dot{r} &= \sin(\theta)] \begin{bmatrix} \dot{q_u}(r,\theta) \\ \dot{p_u}(r,\theta) \end{bmatrix} \\
\dot{\theta} &= \frac{1}{r}[-\sin(\theta), \cos(\theta)] \begin{bmatrix} \dot{q_u}(r,\theta) \\ \dot{p_u}(r,\theta) \end{bmatrix}
\end{aligned} \tag{9}
$$

By expanding out (9) then substituting $q_u = r\cos(\theta)$ and $p_u = r\sin(\theta)$ it is straightforward computation to find the equations of motion in cylindrical (r, θ) coordinates. To simplify the notation yet again, define $s(g)_\theta := \sin(g(\theta))$ and $c(g)_\theta := \cos(g(\theta))$ for any function $g : \mathbb{R} \to \mathbb{R}$. The final equations of motion in (r, θ) coordinates are

$$
\begin{aligned}
\dot{r} &= \frac{r^2 c_\theta s_\theta + m^2 g l^3 (2s(rc)_\theta + s(rc + \bar{q}_a s)_\theta)\left((1 + c(\bar{q}_a s)_\theta)\bar{q}_a c_\theta - (3 + 2c(\bar{q}_a s)_\theta) s_\theta r\right)}{ml^2\left((3 + 2c(\bar{q}_a s)_\theta) r - \bar{q}_a (1 + c(\bar{q}_a s)_\theta) c_\theta s_\theta\right)} \\
\dot{\theta} &= -\frac{r s_\theta^2 + m^2 g l^3 (2s(rc)_\theta + s(rc + \bar{q}_a s)_\theta)(3 + 2c(\bar{q}_a s)_\theta) c_\theta}{ml^2\left((3 + 2c(\bar{q}_a s)_\theta) r - \bar{q}_a (1 + c(\bar{q}_a s)_\theta) c_\theta s_\theta\right)}
\end{aligned}
\tag{10}
$$

References

[1] someone, "Something," *Somewhere*, sometime.