VNHC and Acrobot Title

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***Abstract*— TODO: Abstract here.**

***Index Terms*— TODO: keywords in alphabetical order, separated by commas.**

I. Introduction

TODO: intro.

A. Notation

We use the following notation in this article. For $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times m}$, define $[A; B] \in \mathbb{R}^{(n+p)\times m}$ as the matrix obtained by stacking A on top of B. For $T > 0$, the set of real numbers modulo T is denoted $[\mathbb{R}]_T$, with $[\mathbb{R}]_\infty := \mathbb{R}$. The gradient of a matrix-valued function $A : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ is the block matrix of stacked partial derivatives, $\nabla_x A := [\frac{\partial A}{\partial x_1}; \ldots ; \frac{\partial A}{\partial x_m}] \in \mathbb{R}^{nm\times n}$. Given two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{r\times s}$, the Kronecker product [1] $A \otimes B \in \mathbb{R}^{nr\times ms}$ is the matrix

$$A \otimes B = \begin{bmatrix} A_{1,1}B & \cdots & A_{1,m}B \\ \vdots & \ddots & \vdots \\ A_{n,1}B & \cdots & A_{n,m}B \end{bmatrix}. \tag{1}$$

The Poisson bracket [2] between the functions $f(q, p)$ and $g(q, p)$ is

$$[f, g] := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}. \tag{2}$$

Finally, the Kronecker delta $\delta_i^j = 1$ if $i = j$ and 0 otherwise.

II. Problem Formulation

TODO: motivate injection/dissipation for energy regulation in an acrobot. Use this to lead into VNHCs. e.g. represent torso as chain of a swing, pivot as seat, and legs as human legs. Replicate fig3.3 from thesis but for acrobot: take (qu,pu)-plane and show how a person moves their legs wrt theta, qu, or pu. Leave the gymnast model until the end, or ignore it entirely.

Alternatively, ff human movement journal says legs move wrt qu, we can use that as a foundation.

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III. Theory of VNHCs

A. Simply Actuated Hamiltonian Systems

Take a mechanical system modelled with generalized coordinates $q = (q_1, \ldots , q_n)$ on a configuration manifold $\mathcal{Q} = [\mathbb{R}]_{T_1} \times \cdot [\mathbb{R}]_{T_n}$, where $T_i = 2\pi$ if q_i is an angle and $T_i = \infty$ if q_i is a displacement. The corresponding generalized velocities are $\dot q = (\dot q_1, \ldots , \dot q_n) \in \mathbb{R}^n$.

Suppose this system has Lagrangian $\mathcal{L}(q, \dot q) = 1/2\ \dot q^T D(q)\dot q - P(q)$, where the potential energy $P : \mathcal{Q} \to \mathbb{R}$ is smooth, and the inertia matrix $D : \mathcal{Q} \to \mathbb{R}^{n\times n}$ is smooth and positive definite for all $q \in \mathcal{Q}$. The *conjugate of momentum* to q is the vector $p := \partial\mathcal{L}/\partial\dot q = D(q)\dot q \in \mathbb{R}^n$. As per [2], the *Hamiltonian* of the system in (q, p) coordinates is

$$\mathcal{H}(q, p) = \frac{1}{2} p^\mathsf{T} D^{-1}(q)p + P(q), \tag{3}$$

with dynamics

$$\begin{cases} \dot q = \nabla_p \mathcal{H}, \\ \dot p = -\nabla_q \mathcal{H} + B(q)\tau, \end{cases} \tag{4}$$

where $\tau \in \mathbb{R}^k$ is a vector of generalized input forces and the input matrix $B : \mathcal{Q} \to \mathbb{R}^{n\times k}$ is full rank for all $q \in \mathcal{Q}$. If $k < n$, we say the system is *underactuated* with degree of underactuation $(n - k)$.

It is easy to show using the matrix Kronecker product that (4) expands to

$$\begin{cases} \dot q = D^{-1}(q)p, \\ \dot p = -\frac{1}{2}(I_n \otimes p^\mathsf{T})\nabla_q D^{-1}(q)p - \nabla_q P(q) + B(q)\tau. \end{cases} \tag{5}$$

Because τ is transformed by $B(q)$, it is not obvious how any particular input force τ_i affects the system. As a first step in addressing this problem, we make the following assumptions.

Assumption 1: The input matrix $B(q) \equiv B \in \mathbb{R}^{n\times k}$ is constant, full rank, and $k < n$.

Assumption 2: There exists a right semi-orthogonal matrix $B^\perp \in \mathbb{R}^{(n-k)\times n}$ which is a left-annihilator for B.

Note that Assumption 2 requires the rows of $B^\perp$ be unit vectors that are mutually orthogonal. In the case that $k = (n - 1)$, the existence of any left annihilator $A^0 \in \mathbb{R}^{1\times n}$ implies the left annihilator $B^\perp := A^0/\|A^0\|$ will be a unit vector satisfying this assumption.

The above assumptions allow us to define a canonical coordinate transformation of (3) which decouples the input forces. To define this transformation we will make use of the following lemma.

Lemma 1: Suppose Assumption 1 holds. Then there exists a nonsingular matrix $\hat T \in \mathbb{R}^{k\times k}$ so that the regular feedback transformation

$$\tau = \hat T \hat\tau$$

has a new input matrix $\hat{B}$ for $\hat{\tau}$ which is left semi-orthogonal.

Proof: Since B is constant and full rank, it has a singular value decomposition $B = U^\mathsf{T}\Sigma V$ where $\Sigma = [\mathrm{diag}\,(\sigma_1, \ldots, \sigma_k)\,; \mathbf{0}_{(n-k)\times k}]$, $\sigma_i > 0$, and $U \in R^{n\times n}$, $V \in \mathbb{R}^{k\times k}$ are unitary matrices [3]. Defining $T = \mathrm{diag}\left(1/\sigma_1^2, \ldots, 1/\sigma_k^2\right)$ and assigning the regular feedback transformation $\tau = VT\hat{\tau}$ yields a new input matrix $\hat{B} = BVT$ for $\hat{\tau}$ such that $\hat{B}^\mathsf{T}\hat{B} = T^\mathsf{T}\Sigma^\mathsf{T}\Sigma T = I_k$. ■

In light of Lemma 1, there is no loss of generality in assuming that the input matrix is left semi-orthogonal. Now, let $\mathbf{B} := [B^\perp; B^\mathsf{T}]$. Since $B^\perp$ is a left annihilator of B and both $B^\perp$ and B^T are right semi-orthogonal, one can easily show that $\mathbf{B}$ is an orthogonal matrix.

Theorem 2: Take the Hamiltonian system (3). Under Assumptions 1 and 2, the coordinate transformation $(\tilde{q} = \mathbf{B}q, \tilde{p} = \mathbf{B}p)$ is a canonical transformation. The resulting dynamics are given by

$$\mathcal{H}(\tilde{q}, \tilde{p}) = \frac{1}{2}\tilde{p}^\mathsf{T} M^{-1}(\tilde{q})\tilde{p} + V(\tilde{q}), \tag{6}$$

$$\begin{cases} \dot{\tilde{q}} = M^{-1}(\tilde{q})\tilde{p}, \\ \dot{\tilde{p}} = -\frac{1}{2}(I_n \otimes \tilde{p}^\mathsf{T})\nabla_{\tilde{q}} M^{-1}(\tilde{q})\tilde{p} \\ \qquad - \nabla_{\tilde{q}} V(\tilde{q}) + \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix} \tau, \end{cases}$$

where $M^{-1}(\tilde{q}) := \mathbf{B}D^{-1}(\mathbf{B}^\mathsf{T}\tilde{q})\mathbf{B}^\mathsf{T}$ and $V(\tilde{q}) := P(\mathbf{B}^\mathsf{T}\tilde{q})$.

Proof: Since $\mathbf{B}$ is constant, this transformation satisfies $\partial\tilde{q}_i/\partial p_j = \partial\tilde{p}_i/\partial q_j = 0$ for all $i, j \in \{1, \ldots, n\}$. This implies the Poisson brackets $[\tilde{q}_i, \tilde{q}_j]$ and $[\tilde{p}_i, \tilde{p}_j]$ are both zero. Then, since $\mathbf{B}$ is orthogonal, $[\tilde{p}_i, \tilde{q}_j] = (\mathbf{B}_i)^\mathsf{T}(\mathbf{B}^\mathsf{T})_j = \delta_i^j$. By (45.10) in [2], this transformation is canonical and the new Hamiltonian is $\mathcal{H}(\mathbf{B}^\mathsf{T}\tilde{q}, \mathbf{B}^\mathsf{T}\tilde{p})$. Finally, since $\dot{\tilde{p}} = \mathbf{B}\dot{p}$, the input matrix for the system in $(\tilde{q}, \tilde{p})$ coordinates is $\mathbf{B}B = [\mathbf{0}_{(n-k)\times k}; I_k]$, which proves the theorem. ■

We call these new coordinates *simply actuated coordinates*. The first $(n-k)$ configuration variables, labelled q_u, are the *unactuated coordinates*; the remaining k configuration variables, labelled q_a, are the *actuated coordinates*. The corresponding (p_u, p_a) are the *unactuated* and *actuated momenta*, respectively.

B. Virtual Nonholonomic Constraints

The notion of a nonholonomic constraint which can be stabilized via state feedback was first described by Griffin and Grizzle in [4]. Horn et. al later extended their results in [5] and derived the dynamics of constrained systems. In this section we reformulate these ideas for the Hamiltonian framework, because the theory is cleaner when using unactuated and actuated momenta. For this reason, we take the system of inquiry to be a Hamiltonian mechanical system in simply actuated coordinates.

Definition 3: A *virtual nonholonomic constraint* (VNHC) *of order* k is a relation $h(q,p) = 0$ where $h : \mathcal{Q}\times\mathbb{R}^n \to \mathbb{R}^k$ is C^2, $\mathrm{rank}\,([dh_q,\, dh_p]) = k$ for all $(q,p) \in h^{-1}(0)$, and there exists a feedback controller $\tau(q,p)$ rendering the *constraint manifold* Γ invariant, where

$$\Gamma = \{(q,p) \mid h(q,p) = 0, dh_q\dot{q} + dh_p\dot{p} = 0\}\,. \tag{7}$$

The constraint manifold is a $2(n-k)$-dimensional closed embedded submanifold of $\mathcal{Q}\times\mathbb{R}^n$. A VNHC thereby allows us to study a reduced-order model of the system: it reduces the original $2n$ differential equations to $2(n-k)$ equations. In particular, if $k = (n-1)$, the constraint manifold is *always* 2-dimensional and its dynamics can be plotted on a plane.

We often want to stabilize a constraint within some neighbourhood of Γ. To see when this is possible, let us define the error output $e = h(q,p)$. If any component of e_i has relative degree 1, we may not be able to stabilize Γ – we can always guarantee $e_i \to 0$, but not necessarily $\dot{e}_i \to 0$. It is for this reason that we define the following special type of VNHC.

Definition 4: A VNHC $h(q,p) = 0$ of order k is *regular* if the output $e = h(q,p)$ is of relative degree $\{2, 2. \ldots, 2\}$ everywhere on the constraint manifold Γ.

The authors of [4], [5] observed that relations which use only the unactuated conjugate of momentum often have vector relative degree $\{2, \ldots, 2\}$. Indeed, we shall now provide a characterization of regularity which shows that regular constraints cannot use the actuated momentum at all.

To ease notation in the rest of this section, we use the following shorthand:

$$\mathcal{A}(q, p_u) := dh_q(q, p_u)M^{-1}(q), \tag{8}$$

$$\mathcal{M}(q,p) := (I_{n-k}\otimes p^\mathsf{T})\nabla_{q_u} M^{-1}(q). \tag{9}$$

Theorem 5: A relation $h(q,p) = 0$ for system (6) is a regular VNHC of order k if and only if $dh_{p_a} = \mathbf{0}_{k\times k}$ and

$$\mathrm{rank}\left((\mathcal{A}(q,p_u) - dh_{p_u}\mathcal{M}(q,p))\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\right) = k,$$

everywhere on the constraint manifold Γ.

Proof: Let $e = h(q,p) \in \mathbb{R}^k$. If $dh_{p_a} \neq \mathbf{0}_{k\times k}$ for some $(q,p) \in \Gamma$, then τ appears in $\dot{e}$ and the VNHC is not of relative degree $\{2, \ldots, 2\}$. Suppose now that $dh_{p_a} = \mathbf{0}_{k\times k}$. Then $\dot{e} = \mathcal{A}(q,p_u)p - dh_{p_u}\left(1/2\ \mathcal{M}(q,p)p + \nabla_{q_u}V(q)\right)$. Taking one further derivative provides $\ddot{e} = (\star) - dh_{p_u}\left(1/2\ d/dt\ \left(\mathcal{M}(q,p)p\right)\right) + \mathcal{A}(q,p_u)[\mathbf{0}_{(n-k)\times k}; I_k]\tau$, where $(\star)$ is a continuous function of q and p. One can further show that $dh_{p_u}\left(1/2\ d/dt\ \left(\mathcal{M}(q,p)p\right)\right) = (\star) + dh_{p_u}\mathcal{M}(q,p)[\mathbf{0}_{(n-k)\times k}; I_k]\tau$. Hence,

$$\ddot{e} = (\star) + (\mathcal{A}(q,p_u) - dh_{p_u}\mathcal{M}(q,p))\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\tau,$$

which we write as $\ddot{e} = E(q,p) + H(q,p)\tau$ for appropriate E and H. From the definition of regularity, the VNHC h is regular when e is of relative degree $\{2, \ldots, 2\}$, which is true if and only if the matrix premultiplying τ is nonsingular, and hence that H is invertible. This proves the theorem. ■

Under additional mild conditions (see [6]), a regular VNHC of order k can be stabilized by the output-linearizing phase-feedback controller

$$\tau(q,p) = -H^{-1}(q,p)\left(E(q,p) + k_p e + k_d\dot{e}\right), \tag{10}$$

where $k_p, k_d > 0$ are control parameters which can be tuned on the resulting error dynamics $\ddot{e} = -k_p e - k_d\dot{e}$.

In Section IV we will enforce a regular constraint on the acrobot of the form $h(q,p) = q_a - f(q_u, p_u)$, where the actuators track a function of the unactuated variables. Intuitively then, the constrained dynamics should be parameterized by (q_u, p_u). Unfortunately, $\dot{q}_u$ depends on p_a, and for general systems one cannot solve explicitly for p_a in terms of (q_u, p_u) because the $\dot{p}$ dynamics contains the coupling term $(I_n \otimes p^\top)\nabla_q M(q)p$.

We now introduce an assumption which allows us to solve for p_a as a function of (q_u, p_u), which in turn allows us to find the constrained dynamics.

Assumption 3: The inertia matrix does not depend on the unactuated coordinates, so that $\nabla_{q_u} M(q) = \mathbf{0}_{n(n-k)\times n}$.

Theorem 6: Let $\mathcal{H}$ be a mechanical system in simply actuated coordinates satisfying Assumption 3. Let $h(q, p_u) = 0$ be a regular VNHC of order k with constraint manifold Γ. Suppose that on Γ one can solve for q_a as a function $q_a = f(q_u, p_u)$. Then the constrained dynamics are given by

$$\left.\begin{aligned} \dot{q}_u &= \left[I_{(n-k)}\ \mathbf{0}_{(n-k)\times k}\right] M^{-1}(q)p \\ \dot{p}_u &= -\nabla_{q_u} V(q) \end{aligned}\right|_{\substack{q_a = f(q_u,p_u) \\ p_a = g(q_u,p_u)}}, \quad (11)$$

where

$$g(q_u, p_u) := \left(\mathcal{A}(q, p_u)[\mathbf{0}_{(n-k)\times k}; I_k]\right)^{-1} (dh_{p_u} \nabla_{q_u} V(q) - \mathcal{A}(q, p_u)[I_{(n-k)}; \mathbf{0}_{k\times(n-k)}]p_u\ )|_{q_a = f(q_u,p_u)}. \quad (12)$$

Proof: Setting $e = h(q, p_u)$ and using Assumption 3, we find that $\dot{e} = \mathcal{A}(q, p_u)p - dh_{p_u}\nabla_{q_u} V(q)$. Notice that $\mathcal{A}(q, p_u)p = \mathcal{A}(q, p_u)[\mathbf{0}_{(n-k)\times k}; I_k]p_a + \mathcal{A}(q, p_u)[I_{n-k}; \mathbf{0}_{k\times(n-k)}]p_u$. Since $h(q, p_u)$ is regular, $\mathcal{A}(q, p_u)$ is invertible. Taking $e = \dot{e} = 0$, solving for p_a, and setting $q_a = f(q_u, p_u)$ completes the proof. ∎

Comparison with existing literature: Horn et.al. provide the constrained dynamics for VNHCs in [7]. Their assumption **H2** is what we call regularity, and our requirement that one can solve for $q_a = f(q_u, p_u)$ on Γ implies their assumption **H3** holds true. The only real distinction between this section and their work is that our constrained dynamics are explicit functions of the Hamiltonian phase coordinates (q_u, p_u). In fact, our constrained dynamics (11) coincide with their system (17) when one chooses the special case $\theta_1 = q_u$ and $\theta_2 = p_u$. This explicit representation will be beneficial when we apply the theory of VNHCs to the acrobot.

IV. The Acrobot VNHC

specialize the theory of VNHCs to the acrobot. consider constraints which depend only on pu, and show arctan vnhc and reduced dynamics. give definition of energy injection/dissipation. conclude with our theorem. Make this section short and sweet.

V. Simulation Results

VI. Experimental Results

VII. Proof of Theorem **TODO: ref theorem**

VIII. Conclusion

References

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