Energy injection for mechanical systems through the method of Virtual Nonholonomic Constraints

by

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Abstract

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List of Symbols

Symbol	Definition
$\mathbf{n}$	The index set $\{1, \dots, n\}$ of natural numbers up to n.
$\mathbb{R}^n$	Real numbers in n dimensions.
$[\mathbb{R}]_T$	Real numbers modulo $T > 0$, with $[\mathbb{R}]_\infty = \mathbb{R}$.
$\mathbb{S}^1$	The unit circle, equivalent to $[\mathbb{R}]_{2\pi}$.
$\mathcal{Q}$	The configuration manifold of a system.
$\mathbb{R}^{n\times m}$	The space of real-valued matrices with n rows and m columns.
I_n	The $n \times n$ identity matrix.
$\mathbf{0}_{n\times m}$	The $n \times m$ matrix of all zeros.
M_i	If M is a vector, the ith element of M.
	If M is a matrix, the ith column of M.
$M_{i,j}$	The value of row i, column j for the matrix M.
$\dot{x}$	Derivative of x with respect to time t.
$\nabla_v F$	If F is $\mathbb{R}$-valued, the gradient of F with respect to v.
	If $F : \mathbb{R}^m \to \mathbb{R}^{n\times n}$, the block matrix gradient $(\frac{\partial F}{\partial v_1}, \dots, \frac{\partial F}{\partial v_m}) \in \mathbb{R}^{nm\times n}$.
dF_v	Total differential (Jacobian) of F, equivalent to $(\nabla_v F)^\top$.
$\operatorname{Hess} F$	If $F : \mathbb{R}^n \to \mathbb{R}$, the $n \times n$ Hessian matrix of double derivatives of F.
	If $F : \mathbb{R}^n \to \mathbb{R}^k$, the block matrix $(\operatorname{Hess} F_1, \dots, \operatorname{Hess} F_k) \in \mathbb{R}^{n\times nk}$.
$\partial_v \partial_w F$	Derivative matrix of $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, with (i, j) element $\frac{\partial^2 F}{\partial v_i \partial w_j}$.
$\delta_{i,j}$	The Kronecker delta: 1 if $i = j$ and 0 otherwise.
$\otimes$	The matrix kronecker product (see Appendix ???).

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[1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39] [40] [41]

Chapter 1

Introduction

1.1 Literature Review

1.2 Statement of Contributions

1.3 Outline of the Thesis

Chapter 2

Development of Virtual Nonholonomic Constraints

2.1 Preliminaries on Analytical Mechanics

A mechanical system can be represented by N point masses where each point represents the center of mass of a physical body, along with r *equations of constraint* (EOC) which model the physical restrictions between these masses. The position of each point mass is described using three cartesian coordinates (one for each spatial axis), so the system as a whole can be described by a vector in $\mathbb{R}^{3N}$ with r EOC. The dynamics of the system are computed by deriving the $3N$ *equations of motion* (EOM) produced by Newton's second law $F = ma$. While this technique works for simple systems, it is impossible to apply to a majority of mechanical systems since most of the forces on the system are not explicitly known.

Rather than modeling a mechanical system with point masses and constraints, it is often feasible to represent the position of the system using n independent scalar-valued variables $q_1, \dots, q_n$ called *generalized coordinates*, where $n = 3N - r$ is the number of *degrees of freedom* (DOF) of the system [23].

For the robotic systems of interest in this thesis, we assume that each generalized coordinate q_i represents either the distance or the angle between two parts of the system. Mathematically, each q_i takes values in $[\mathbb{R}]_{T_i}$, where $T_i = \infty$ if q_i represents a length or $T_i = 2\pi$ if q_i represents an angle. It is convention to collect the coordinates into a *configuration* $q = (q_1, \dots, q_n) \in \mathcal{Q}$ where the *configuration manifold* $\mathcal{Q}$ of the system is a so-called *generalized cylinder*:

$$\mathcal{Q} = [\mathbb{R}]_{T_1} \times \cdot \times [\mathbb{R}]_{T_n}$$

The derivative $\dot{q} = (\dot{q}_1, \dots, \dot{q}_n)$ of a configuration is called a *generalized velocity* of the system. For arbitrary systems, the space of allowable velocities depends on the current configuration of the system. However, since $\mathcal{Q}$ is a generalized cylinder, we find that $\dot{q} \in \mathbb{R}^n$. The combined vector $(q, \dot{q}) \in \mathcal{Q} \times \mathbb{R}^n$ is called a *state* of the system.

The field of analytical mechanics provides a computational method for finding the EOM of a system in generalized coordinates. The two most common analytical methods for modelling robotic systems are *Lagrangian* and *Hamiltonian* mechanics.

2.1.1 Lagrangian Mechanics

Lagrangian mechanics uses the kinetic energy $T(q, \dot{q})$ and potential energy $P(q)$ of the system to define the Lagrangian $\mathscr{L} : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}$ defined by (2.1) [23].

$$\mathscr{L}(q, \dot{q}) = T(q, \dot{q}) - P(q) \tag{2.1}$$

When the mechanical system is actuated, the EOM are described by n second-order ordinary differential equations (ODEs) obtained from the *Euler-Lagrange equations* with *generalized input forces* $\tau \in \mathbb{R}^k$ (2.2).

$$\frac{d}{dt}\left\{\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right\} - \frac{\partial \mathscr{L}}{\partial q_i} = B_i^\top(q)\tau \tag{2.2}$$

The vector $B_i^\top : \mathcal{Q} \to \mathbb{R}^{1\times k}$ describes how the input forces shape the dynamics of q_i. The matrix $B : \mathcal{Q} \to \mathbb{R}^{n\times k}$ with

$$B(q) = \begin{bmatrix} - & B_1^\top(q) & - \\ & \vdots & \\ - & B_n^\top(q) & - \end{bmatrix}$$

is called the *input matrix* for the system. If $k < n$, we say the system is *underactuated* with degree of underactuation $n - k$.

Many actuated mechanical systems have quadratic kinetic energies, so that the Lagrangian can be written explicitly as

$$\mathscr{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} - P(q) \tag{2.3}$$

where the *inertia matrix* $D : \mathcal{Q} \to \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix for all $q \in \mathcal{Q}$ and the potential function $P : \mathcal{Q} \to \mathbb{R}$ is smooth.

2.1.2 Hamiltonian Mechanics

Hamiltonian mechanics provides an equivalent representation of the EOM by converting the n second-order ODEs generated by Lagrangian mechanics into $2n$ first-order ODEs.

To do this, we first define the *conjugate of momentum p_i to q_i* by

$$p_i(q,\dot{q}) = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}(q,\dot{q}) \tag{2.4}$$

To ease notation, we will write $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ and say that p is the *conjugate of momenta to q*. Note that each p_i is a linear function of $\dot{q}$, so one can typically solve for $\dot{q}(q,p)$ by inverting the expressions from (2.4). The combined vector $(q,p) \in \mathcal{Q} \times \mathbb{R}^n$ is called a *phase* of the system.

The *Hamiltonian* of the system in $\{q,p\}$ coordinates is defined as the Legendre transform (2.5) of the Lagrangian [24].

$$\mathscr{H}(q,p) = p^\top \dot{q}(q,p) - \mathscr{L}(q,\dot{q}(q,p)) \tag{2.5}$$

The EOM in this framework can be shown to be the $2n$ first-order equations called *Hamilton's equations*

$$\begin{cases} \dot{q} = \nabla_p \mathscr{H} \\ \dot{p} = -\nabla_q \mathscr{H} + B(q)\tau \end{cases} \tag{2.6}$$

where $B(q) \in \mathbb{R}^{n\times k}$ is the same input matrix used by the Lagrangian and $\tau \in \mathbb{R}^k$ is the vector of generalized input forces.

If the Lagrangian has quadratic kinetic energy as in (2.3), the conjugate of momenta to q can be computed explicitly by $p = D(q)\dot{q}$.

The resulting Hamiltonian system reduces to (2.7).

$$\mathscr{H}(q,p) = \frac{1}{2}p^\top D^{-1}(q)p + P(q) \tag{2.7}$$

$$\begin{cases} \dot{q} = D^{-1}(q)p \\ \dot{p} = -\frac{1}{2}(I_n \otimes p^\top)\nabla_q D^{-1}(q)p - \nabla_q P(q) + B(q)\tau \end{cases} \tag{2.8}$$

A set of coordinates $\{q,p\}$ which satisfy Hamilton's equations under the Hamiltonian $\mathscr{H}$ are said to be *canonical coordinates* for the system. A change of coordinates $(q,p) \to (Q,P)$ is a *canonical transformation* if $\{Q,P\}$ are canonical coordinates with Hamiltonian $H(Q,P) = \mathscr{H}\left(q(Q,P), p(Q,P)\right)$.

2.2 Simply Actuated Hamiltonian Systems

Given a Hamiltonian mechanical system (2.7), it is not obvious how the input forces τ affect the conjugate of momenta p_i. This is because τ is transformed by the input matrix $B(q)$, which may be quite complicated.

We will define a new class of Hamiltonian systems where the effect of the input forces on the conjugate of momenta is made obvious. This class of systems will form the backbone for the rest of the theory developed in this thesis.

Definition 1. Let $\mathscr{H}$ be an underactuated Hamiltonian system $\mathscr{H}$ with degree of underactuation $(n-k) \geq 0$. A pair of canonical coordinates $\{q, p\}$ for this system are said to be *simply actuated coordinates* if the input matrix $B(q) \in \mathbb{R}^{n \times k}$ is of the form

$$B(q) = \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}$$

The first $(n-k)$ coordinates q_u are called the *unactuated coordinates*, while the remaining k coordinates q_a are called the *actuated coordinates*. We write $q = (q_u, q_a)$ and the corresponding conjugate of momenta $p = (p_u, p_a)$ are the called the unactuated and actuated momenta (respectively).

Definition 2. A Hamiltonian system is said to be *simply actuated* if there exists a canonical transformation from any set of canonical coordinates $\{q, p\}$ into simply actuated coordinates for $\mathscr{H}$.

Under the following assumptions on the input matrix, we will show that the Hamiltonian system (2.7) is simply actuated.

Assumption 1. The input matrix $B(q) \equiv B \in \mathbb{R}^{n \times k}$ is constant, full rank, and $k \leq n$.

Assumption 2. There exists a matrix $B^\perp \in \mathbb{R}^{(n-k)\times n}$ which is right semi-orthogonal (*i.e.*$B^\perp (B^\perp)^\top = I_{(n-k)}$) and which is a left-annihilator for B (*i.e.*$B^\perp B = \mathbf{0}_{(n-k)\times k}$).

Note that if $k = (n-1)$, the existence of any left annihilator A^0 for B implies the left annihilator $B^\perp := A^0/\|A^0\|$ satisfies Assumption 2.

Assumption 3. Assume without loss of generality that the input matrix B is left semi-orthogonal. That is, assume $B^\top B = I_k$.

Proof. Since B is a constant matrix, it has a singular-value decomposition $B = U\Sigma V^\top$ where $U^{-1} = U^\top \in \mathbb{R}^{n\times n}$, $V^{-1} = V^\top \in \mathbb{R}^{k\times k}$, and $\Sigma \in \mathbb{R}^{n\times k}$ is defined by

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \\ - & & \mathbf{0}_{(n-k)\times k} & - \end{bmatrix}$$

where $\sigma_i \neq 0$ because B is full-rank **SOURCE?**. Defining $T \in \mathbb{R}^{k\times k}$ by

$$T = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_k} \end{bmatrix}$$

and assigning the input forces to $\tau = VT\hat{\tau}$, we get a new input matrix for $\hat{\tau} \in \mathbb{R}^k$ given by $\hat{B} = BVT = U\Sigma T$ which is still constant and full-rank. In particular, $\hat{B}^\top\hat{B} = T^\top\Sigma^\top\Sigma^\top T = I_k$. $\square$

Let $\mathbf{B} \in \mathbb{R}^{n\times n}$ be the following matrix:

$$\mathbf{B} = \begin{bmatrix} B^\perp \\ B^\top \end{bmatrix}$$

Since $B^\perp$ is a left annihilator of B and both $B^\perp$ and $B^\top$ are right semi-orthogonal, it is easy to show that $\mathbf{B}$ is orthogonal:

$$\mathbf{B}\mathbf{B}^\top = \begin{bmatrix} B^\perp(B^\perp)^\top & B^\perp B \\ (B^\perp B)^\top & B^\top B \end{bmatrix} = I_n \Rightarrow \mathbf{B}^{-1} = \mathbf{B}^\top$$

The following theorem shows that $\mathbf{B}$ provides a canonical transformation into simply actuated coordinates, so that all unactuated momenta are unaffected by the input forces.

Theorem 1. *Under Assumptions 1,2, and 3, the Hamiltonian system (2.7) is simply actuated* *with simply actuated coordinates* $\{Q = \mathbf{B}q, P = \mathbf{B}p\}$. *The resulting dynamics are given by (2.9),*

$$\hat{H}(Q,P) = \frac{1}{2}P^{\top}M^{-1}(Q)P + V(Q) \tag{2.9}$$
$$\begin{cases} \dot{Q} = M^{-1}(Q)P \\ \dot{P} = -\frac{1}{2}(I_n \otimes P^{\top})\nabla_Q M^{-1}(Q)P - \nabla_Q V(Q) + \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\tau \end{cases}$$

where

$$\begin{aligned} M^{-1}(Q) &:= \mathbf{B}D^{-1}(\mathbf{B}^{\top}Q)\mathbf{B}^{\top} \\ V(Q) &:= P(\mathbf{B}^{\top}Q) \end{aligned}$$

Proof. The Poisson bracket between the functions $f(q,p)$ and $g(q,p)$ is defined by [24] as follows:

$$[f,g] := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

For any constant matrix A, the transformation $\{Q = Aq, P = Ap\}$ satisfies $\frac{\partial Q_i}{\partial p_m} = \frac{\partial P_i}{\partial q_m} = 0$ for all $i, m \in \mathbf{n}$. Hence, for our new coordinates $\{Q = \mathbf{B}q, P = \mathbf{B}p\}$,

$$\begin{aligned} [Q_i, Q_j] &:= \sum_{m=1}^{n} \frac{\partial Q_i}{\partial p_m}\frac{\partial Q_j}{\partial q_m} - \frac{\partial Q_i}{\partial q_m}\frac{\partial Q_j}{\partial p_m} = 0 \\ [P_i, P_j] &:= \sum_{m=1}^{n} \frac{\partial P_i}{\partial p_m}\frac{\partial P_j}{\partial q_m} - \frac{\partial P_i}{\partial q_m}\frac{\partial P_j}{\partial p_m} = 0 \end{aligned}$$

Since the matrix $\mathbf{B}$ is orthogonal, $(\mathbf{B}_i)^{\top}\mathbf{B}_j^{\top} = (\mathbf{B}_i)^{\top}(\mathbf{B}^{-1})_j = \delta_{i,j}$. Using this fact we see that the Poisson brackets between P and Q are given by:

$$\begin{aligned}
[P_i, Q_j] &= \sum_{m=1}^{n} \frac{\partial P_i}{\partial p_m}\frac{\partial Q_j}{\partial q_m} - \frac{\partial P_i}{\partial q_m}\frac{\partial Q_j}{\partial p_m} \\
&= \sum_{m=1}^{n} \mathbf{B}_{i,m}\mathbf{B}_{j,m} - 0 \\
&= \sum_{m=1}^{n} \mathbf{B}_{i,m}\mathbf{B}^{\top}_{m,j} \\
&= (\mathbf{B}_i)^{\top}\mathbf{B}^{\top}_j \\
&= \delta_{i,j}
\end{aligned}$$

Therefore, by (45.10) in [24], the coordinate change ($Q = \mathbf{B}q, P = \mathbf{B}p$) is a canonical transformation with new Hamiltonian $\hat{H}(Q,P) = \mathscr{H}(\mathbf{B}^{\top}Q, \mathbf{B}^{\top}P)$.

Furthermore, since $\dot{P} = \mathbf{B}\dot{p}$, the new input matrix is given by

$$\mathbf{B}B = \begin{bmatrix} B^{\perp}B \\ B^{\top}B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}$$

so the coordinates $\{Q = (q_u, q_a), P = (p_u, p_a)\}$ are simply actuated coordinates for $\hat{H}$ as desired. □

2.3 Virtual Nonholonomic Constraints

Let us imagine a child on a swing, who wants to go as high as possible. The child will push off the ground and start swinging with small oscillations, then extend and retract their feet appropriately to gain energy. If a roboticist were designing a machine to do this task, they might design a control mechanism which makes the legs track a trajectory over time. For example, they might tell the robot to extend and retract its legs every two seconds. In ideal situtations, this technique would work perfectly because the leg motion is synchronized with the swing to gain energy as fast as possible.

Most children have an adult pushing the swing to help them go higher; or perhaps the child is swinging on a windy day. In either case, they adjust their leg motion accordingly and do not keep track of time when kicking their legs. Te standard control technique of tracking a function of time, known as *trajectory tracking*, would not work because the disturbance affecting the swing will desynchronize the motion of the legs with the swing and stop the energy-gaining effects.

This is a common issue in the control of many biologically-inspired systems. One method which can provide more realistic, robust motion is the method of of virtual holonomic constraints (VHCs). Instead of a robot's actuators tracking a trajectory over time, VHCs use the actuators to enforce a relation $h(q) = 0$ of the configuration [19]. This method has provided incredible results in the development of walking robots [27], [28], vehicle motion [30], [31], and has even been used to design a snake-like swimming robot [29].

Many authors have attempted to extend these results to enforce a relation $h(q, \dot{q}) = 0$ of the full state. Since these relations use actuators to restrict both the configuration and generalized velocity, they are called virtual *nonholonomic* constraints. This idea has been used for human-robot interaction [32]–[34], error-reduction on time-delayed systems [35], and improving bipedal locomotion [9], [36], [38]. Most interestingly, this nonholonomic approach has been shown to be more robust than standard VHCs when applied to bipedal robotics [37]. This suggests that virtual nonholonomic constraints are more capable of stabilizing specific energy levels while rejecting disturbances, all while producing realistic biological motion. It is for this reason we choose to study nonholonomic constraints in this thesis.

Unlike the theory of VHCs, there does not appear to be a standard definition of virtual nonholonomic constraints. One issue is that Lagrangian dynamics is not particularly suited to finding controllers which stabilize desired constraints. Because of this, all the applications listed above use their own definition of a virtual nonholonomic constraint, which makes it difficult to compare and analyze their work.

This section will provide a new characterization of virtual nonholonomic constraints. The goal is to provide a consistent, rigorous foundation for designing constraints on a general class of systems.

Definition 3. A *virtual nonholonomic constraint* (VNHC) *of order k* is a relation $h(q, p) = 0$ where $h : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^k$ is smooth, rank $\left(\left[dh_q,\ dh_p\right]\right) = k$ for all $(q, p) \in h^{-1}(0)$, and there exists a feedback controller $\tau(q, p)$ stabilizing the set

$$\Gamma = \left\{(q, p) \mid h(q, p) = 0, dh_q\dot{q} + dh_p\dot{p} = 0\right\}$$

which is called the *constraint manifold*.

If we define the error term $e = h(q, p)$, stabilizing Γ is equivalent to solving for $\tau(q, p)$ which drives $e \to 0$ and $\dot{e} \to 0$. Let us imagine that we have no further structure on the VNHC. Then $\dot{e} = dh_q\dot{q} + dh_p\dot{p}$, so τ appears inside $\dot{e}$. To solve for τ explicitly, we must have that dh_p is invertible for all p, which is the same as requiring that $h(q,p)$ is strictly monotonic in p.

This is a very restrictive condition, which many VNHCs will not satisfy. For this reason, it is preferable to have the torque τ appear after two derivatives of e so that there is more freedom in the types of constraints one can use. Mathematically, if τ appears only after two derivatives, one says that e is of *relative degree* $\{2, 2, \ldots, 2\}$. We thus define a special type of VNHC which satisfies this property.

Definition 4. A VNHC of order k $h(q,p) = 0$ is *regular* if the output $e = h(q,p)$ is of relative degree $\{2, 2. \ldots, 2\}$ everywhere on the constraint manifold Γ.

Modifying the results of [9], [36], [37] into the Hamiltonian framework, one observes that relations which use only the unactuated momentum p_u result in τ appearing only after two derivatives of e. To be able to use p_u, we will continue with the following assumption for the rest of the chapter.

Assumption 4. The mechanical system under consideration is a Hamiltonian system with n degrees of freedom and $k < n$ actuators. It is described in simply actuated coordinates $\{q = (q_u, q_a), p = (p_u, p_a)\}$ and has the following dynamics:

$$
\begin{aligned}
&\mathscr{H}(q,p) = p^\top M^{-1}(q)p + V(q)\\
&\qquad\begin{cases}
\dot{q} = M^{-1}p\\
\dot{p} = -\frac{1}{2}(I_n \otimes p^\top)\nabla_q M^{-1}(q)p - \nabla_q V(q) + \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\tau
\end{cases}
\end{aligned}
$$

Notation 1. We will write $q_u \in \mathcal{Q}_u$, $q_a \in \mathcal{Q}_a$ where $\mathcal{Q}_u \times \mathcal{Q}_a = \mathcal{Q}$. We also write $p_u \in \mathscr{P}_u = \mathbb{R}^{n-k}$ and $p_a \in \mathscr{P}_a = \mathbb{R}^k$, so that $p \in \mathscr{P} := \mathscr{P}_u \times \mathscr{P}_a$. In this manner, the phase space of our system can be written as $\mathcal{Q} \times \mathscr{P}$.

Theorem 2. *A VNHC $h(q,p) = 0$ of order k is regular if and only if $dh_{p_a} = 0$ and*

$$
\operatorname{rank}\left(\left(dh_q M^{-1}(q) - dh_{p_u}(I_{n-k} \otimes p^\top)\nabla_{q_u} M^{-1}(q)\right)\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\right) = k
$$

everywhere on the constraint manifold Γ. That is, a regular VNHC is a function with domain $\mathcal{Q} \times \mathscr{P}_u$ which satisfies the rank condition. condition.

Proof. Let $e = h(q,p) \in \mathbb{R}^k$. Then

$$\begin{aligned}\dot e &= dh_q \dot q + dh_p \dot p \\ &= dh_q M^{-1}(q)p + \\ &\begin{bmatrix} dh_{p_u} & dh_{p_a}\end{bmatrix}\left(-\frac{1}{2}\begin{bmatrix}(I_{n-k}\otimes p^\top)\nabla_{q_u}M^{-1}(q)p \\ (I_k \otimes p^\top)\nabla_{q_a}M^{-1}(q)p\end{bmatrix} - \begin{bmatrix}\nabla_{q_u}V(q) \\ \nabla_{q_a}V(q)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{(n-k)\times k} \\ I_k\end{bmatrix}\tau\right)\end{aligned}$$

If $dh_{p_a} \neq \mathbf{0}_{k\times k}$ for some (q,p) on Γ, then τ appears in $\dot e$ and the VNHC is not of relative degree $\{2,2,\dots,2\}$. Hence, we must have that $dh_{p_a} = \mathbf{0}_{k\times k}$ if we want the VNHC to be regular. Proceeding with this assumption, we now find that $h : \mathcal{Q}\times\mathcal{P}_u \to \mathbb{R}^k$, which means

$$\dot e = dh_q M^{-1}(q)p - dh_{p_u}\left(\frac{1}{2}(I_{n-k}\otimes p^\top)\nabla_{q_u}M^{-1}(q)p + \nabla_{q_u}V(q)\right)$$

Taking one further derivative provides

$$\begin{aligned}\ddot e = &\frac{d}{dt}\left\{dh_q\right\}M^{-1}(q)p + dh_q\left(\sum_{i=1}^{n}\frac{\partial M^{-1}}{\partial q_i}(q)\dot q_i\right)p + dh_q M^{-1}(q)\dot p - \\ &\frac{d}{dt}\left\{dh_{p_u}\right\}\left(\frac{1}{2}(I_{n-k}\otimes p^\top)\nabla_{q_u}M^{-1}(q)p + \nabla_{q_u}V(q)\right) - \\ &dh_{p_u}\left(\frac{1}{2}\frac{d}{dt}\left\{(I_{n-k}\otimes p^\top)\nabla_{q_u}M^{-1}(q)p\right\} + \frac{d}{dt}\left\{\nabla_{q_u}V(q)\right\}\right)\end{aligned}$$

We will compute the explicit expression of $\ddot e$ in pieces. First we begin with $\frac{d}{dt}\left\{dh_q\right\}$. Since $h = (h^1,\dots,h^k)$ where $h^i : \mathcal{Q}\times\mathcal{P}_u \to \mathbb{R}$, we have that

$$dh_q = \begin{bmatrix} dh_q^1 \\ \vdots \\ dh_q^k\end{bmatrix} = \begin{bmatrix}\frac{\partial h^1}{\partial q_1} & \cdots & \frac{\partial h^1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h^k}{\partial q_1} & \cdots & \frac{\partial h^k}{\partial q_n}\end{bmatrix}$$

The derivative is taken element-wise in each matrix, yielding

$$\frac{d}{dt}\left\{dh_q\right\} = \begin{bmatrix}\sum_{j=1}^n \frac{\partial^2 h^1}{\partial q_1 \partial q_j}\dot q_j & \cdots & \sum_{j=1}^n \frac{\partial^2 h^1}{\partial q_n \partial q_j}\dot q_j \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n \frac{\partial^2 h^k}{\partial q_1 \partial q_j}\dot q_j & \cdots & \sum_{j=1}^n \frac{\partial^2 h^k}{\partial q_n \partial q_j}\dot q_j\end{bmatrix} + \begin{bmatrix}\sum_{l=1}^{n-k} \frac{\partial^2 h^1}{\partial q_1 \partial p_{u_l}}\dot p_{u_l} & \cdots & \sum_{l=1}^{n-k} \frac{\partial^2 h^1}{\partial q_n \partial p_{u_l}}\dot p_{u_l} \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^{n-k} \frac{\partial^2 h^k}{\partial q_1 \partial p_{u_l}}\dot p_{u_l} & \cdots & \sum_{l=1}^{n-k} \frac{\partial^2 h^k}{\partial q_n \partial p_{u_l}}\dot p_{u_l}\end{bmatrix}$$

It is straightforward computation to confirm that each row of this derivative can be written in vector form as follows:

$$\begin{aligned}\frac{d}{dt}\left\{dh_q^i\right\} &= \left[\sum_{j=1}^{n} \frac{\partial^2 h^i}{\partial q_1 \partial q_j}\dot{q}_j \quad \cdots \quad \sum_{j=1}^{n} \frac{\partial^2 h^i}{\partial q_n \partial q_j}\dot{q}_j\right] + \left[\sum_{l=1}^{n-k} \frac{\partial^2 h^i}{\partial q_1 \partial p_{u_l}}\dot{p}_{u_l} \quad \cdots \quad \sum_{l=1}^{n-k} \frac{\partial^2 h^i}{\partial q_n \partial p_{u_l}}\dot{p}_{u_l}\right] \\ &= \dot{q}^\top \operatorname{Hess}_q \left\{h^i\right\}^\top + \dot{p}_u^\top \partial_{p_u}\partial_q h^i \\ &= p^\top M^{-1}(q) \operatorname{Hess}_q \left\{h^i\right\}^\top - \left(\frac{1}{2}p^\top dM_{q_u}^{-1}(q)(I_{n-k}\otimes p) + dV_{q_u}(q)\right)\partial_{p_u}\partial_q h^i\end{aligned}$$

This means that the entire matrix is given by

$$\begin{aligned}\frac{d}{dt}\left\{dh_q\right\} = \left(I_n \otimes \left(p^\top M^{-1}(q)\right)\right) \operatorname{Hess}_q \{h\}^\top - \\ \left(\frac{1}{2}I_n \otimes \left(p^\top dM_{q_u}^{-1}(q)(I_{n-k}\otimes p)\right) + I_n \otimes dV_{q_u}(q)\right)\partial_{p_u}\partial_q h\end{aligned}$$

For the next segment, let $1_j \in \mathbb{R}^n$ be the vector of all zeros, except for a single 1 at position j. We define $C_1 : \mathcal{Q} \times \mathcal{P} \to \mathbb{R}^{n\times n}$ by

$$C_1(q,p) := \sum_{j=1}^{n} \frac{\partial M^{-1}}{\partial q_j}(q)\dot{q}_j = \sum_{j=1}^{n} \frac{\partial M^{-1}}{\partial q_j}(q)\left(1_j^\top M^{-1}(q)p\right)$$

Moving on, one can use a similar approach to $\frac{d}{dt}\left\{dh_q\right\}$ to find the derivative of dh_{p_u} in matrix form. It is given by

$$\begin{aligned}\frac{d}{dt}\left\{dh_{p_u}\right\} = \left(I_n \otimes \left(p^\top M^{-1}(q)\right)\right)\partial_q\partial_{p_u}h - \\ \left(\frac{1}{2}I_n \otimes \left((I_{n-k}\otimes p^\top)\nabla_{q_u}M^{-1}(q)p\right) + I_n \otimes \nabla_{q_u}V(q)\right)\operatorname{Hess}_{p_u}\{h\}^\top\end{aligned}$$

Now observe that the i^{th} row of

$$\frac{1}{2}\frac{d}{dt}\left\{(I_{n-k}\otimes p^\top)\nabla_{q_u}M^{-1}(q)p\right\}$$

is given by

$$\frac{1}{2}\frac{d}{dt}\left\{p^\top \frac{\partial M^{-1}}{\partial q_{u_i}}(q)p\right\} = p^\top \frac{\partial M^{-1}}{\partial q_{u_i}}(q)\dot{p} + \frac{1}{2}p^\top\left(\sum_{j=1}^{n}\frac{\partial^2 M^{-1}}{\partial q_{u_i}\partial q_j}\dot{q}_j\right)p$$

Define $C_2 : \mathcal{Q} \times \mathcal{P} \to \mathbb{R}^{n(n-k)\times n}$ by

$$C_2(q,p) := \begin{bmatrix} \sum_{j=1}^{n} \frac{\partial^2 M^{-1}}{\partial q_{u_1} \partial q_j} (1_j^\top M^{-1}(q)p) \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial^2 M^{-1}}{\partial q_{u_{n-k}} \partial q_j} (1_j^\top M^{-1}(q)p) \end{bmatrix}$$

This allows us to write the entire derivative in matrix form:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)p\right\} = \\ &(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)\left(-\frac{1}{2}(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)p - \nabla_{q_u} V(q) + \begin{bmatrix}\mathbf{0}_{(n-k)\times k}\\ I_k\end{bmatrix}\tau\right) + \\ &\frac{1}{2}\left(I_{n-k}\otimes p^\top\right)C_2(q,p)p \end{aligned}$$

Finally, we compute the derivative of $\nabla_{q_u} V(q)$:

$$\frac{d}{dt}\left\{\nabla_{q_u} V(q)\right\} = \begin{bmatrix} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial q_{u_1}\partial q_j}\dot{q}_j \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial^2 V}{\partial q_{u_{n-k}}\partial q_j}\dot{q}_j \end{bmatrix} = \partial_{q_u}\partial_q V(q)\dot{q} = \partial_{q_u}\partial_q V(q) M^{-1}(q)p$$

Putting this all together, we can find the explicit form for $\ddot{e}$:

$$\begin{aligned} \ddot{e} = &\frac{d}{dt}\left\{dh_q\right\} M^{-1}(q)p + dh_q C_1(q,p)p - \\ &\frac{1}{2}dh_q M^{-1}(q)((I_n\otimes p^\top)\nabla_q M^{-1}(q)p) + dh_q M^{-1}(q)\nabla_q V(q) - \\ &\frac{d}{dt}\left\{dh_{p_u}\right\}\left(\frac{1}{2}(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)p + \nabla_{q_u} V(q)\right) + \\ &dh_{p_u}(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)\left(\frac{1}{2}(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)p + \nabla_{q_u} V(q)\right) - \\ &\frac{1}{2}dh_{p_u}(I_{n-k}\otimes p^\top)C_2(q,p)p - dh_{p_u}\partial_{q_u}\partial_q V(q)M^{-1}(q)p + \\ &\left(dh_q M^{-1}(q) - dh_{p_u}(I_{n-k}\otimes p^\top)\nabla_{q_u} M^{-1}(q)\right)\begin{bmatrix}\mathbf{0}_{(n-k)\times k}\\ I_k\end{bmatrix}\tau \end{aligned}$$

For shorthand, we'll write $\ddot{e} = E(q,p) + H(q,p)\tau$ where E and H are defined appropriately. From the definition of regularity, the VNHC h is regular when e is of relative degree $\{2,\dots,2\}$, which is true if and only if one can solve for τ when $\ddot{e} = 0$. This is equivalent to requiring that the matrix H be invertible, proving the theorem. $\square$

Using the expression $\ddot{e} = E(q,p) + H(q,p)\tau$ from the proof of Theorem 2, a regular VNHC of order k can be stabilized by the output-linearizing phase-feedback controller (2.10).

$$\tau(q,p) = -H^{-1}(q,p)\left(E(q,p) + k_p e + k_d \dot{e}\right) \tag{2.10}$$

where $k_p, k_d \in \mathbb{R}_{>0}$ are control parameters which can be tuned on the resulting linear system $\ddot{e} = -k_p e - k_d \dot{e}$.

Note that one generally cannot measure conjugate of momenta directly, as sensors on mechanical systems typically only measure the state $(q, \dot{q})$. To implement this controller in practice, one must compute $p = M(q)\dot{q}$ at every iteration. In other words, this controller requires knowledge of the full state of the system.

Now that we have found a controller to enforce a regular VNHC of order k, we would like to solve for the closed-loop dynamics. Intuitively, these dynamics should be parameterized by (q_u, p_u) since q_a is a function of these as specified by $h(q, p_u) = 0$. Unfortunately, $\dot{q}_u$ depends on p_a, and for general systems one cannot solve explicitly for p_a in terms of (q_u, p_u). This is because the $\dot{p}$ dynamics contains the coupling term $(I_n \otimes p^\top)\nabla_{q_u} M(q)p$.

We now introduce a class of systems where explicitly solving for the closed-loop dynamics is feasible.

Definition 5. A mechanical system is —— if $\nabla_{q_u} M(q) = 0$

Theorem 3. *Let $\mathscr{H}$ be a —— mechanical system satisfying Assumption 4. Let $h(q,p_u) = 0$ be a regular VNHC of order k with constraint manifold Γ. Suppose that on Γ one can solve linearly for q_a as a function of q_u, p_u. Then the closed-loop dynamics are given by*

$$\left.\begin{aligned} \dot{q}_u &= M(q)p \\ \dot{p}_u &= -\nabla_{q_u} V(q) \end{aligned}\right| \begin{aligned} h(q,p_u) &= 0 \\ pa &= g(q_u, p_u) \end{aligned} \tag{2.11}$$

where

$$g(q,p_u) := \left.\left(dh_q M^{-1}(q)\begin{bmatrix}\mathbf{0}_{(n-k)\times k} \\ I_k\end{bmatrix}\right)^{-1}\left(dh_{p_u}\nabla_{q_u} V(q) - dh_q M^{-1}(q)\begin{bmatrix} I_{n-k} \\ \mathbf{0}_{k\times(n-k)}\end{bmatrix} p_u\right)\right|_{h(q,p_u)=0}$$
(2.12)

Proof. Setting $e = h(q,p_u)$ and using the fact that $\nabla_{q_u} M^{-1}(q) = 0$, we find that

$$\dot{e} = dh_q M^{-1}(q)p - dh_{p_u}\nabla_{q_u} V(q)$$

Observe that

$$\begin{aligned} dh_q M^{-1}(q)p &= dh_q M^{-1}(q)\begin{bmatrix} p_u \\ p_a \end{bmatrix} \\ &= dh_q M^{-1}(q)\begin{bmatrix} I_{n-k} & \mathbf{0}_{(n-k)\times k} \\ \mathbf{0}_{k\times(n-k)} & I_k \end{bmatrix}\begin{bmatrix} p_u \\ p_a \end{bmatrix} \\ &= dh_q M^{-1}(q)\begin{bmatrix} I_{n-k} \\ \mathbf{0}_{k\times(n-k)} \end{bmatrix} p_u + dh_q M^{-1}(q)\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix} p_a \end{aligned}$$

On the constraint manifold, we have $e = \dot{e} = 0$, which means

$$dh_q M^{-1}(q)\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix} p_a = dh_{p_u}\nabla_{q_u}V(q) - dh_q M^{-1}(q)\begin{bmatrix} I_{n-k} \\ \mathbf{0}_{k\times(n-k)} \end{bmatrix} p_u$$

Since h is regular and $\nabla_{q_u}M^{-1}(q) = 0$, we have that

$$\operatorname{rank}\left(dh_q M^{-1}(q)\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\right) = k$$

Solving for p_a gives

$$p_a = \left(dh_q M^{-1}(q)\begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}\right)^{-1}\left(dh_{p_u}\nabla_{q_u}V(q) - dh_q M^{-1}(q)\begin{bmatrix} I_{n-k} \\ \mathbf{0}_{k\times(n-k)} \end{bmatrix} p_u\right)$$

This yields a function $p_a(q, p_u)$. However, on Γ we have $h(q, p_u) = 0$ and can solve for q_a in terms of (q_u, p_u). Hence, on Γ we have $p_a = g(q_u, p_u)$. Since q_a and p_a can be computed directly from (q_u, p_u), the dynamics on Γ are parameterized only by $(\dot{q}_u, \dot{p}_u)$. □

Theorem 3 shows that, for a particular class of systems and constraints, the dynamics on Γ are entirely described by the $2(n-k)$ unactuated coordinates. This is trueregardless of the number of degrees of freedom of the system. This means that Γ is a $2(n-k)$-dimensional surface inside $\mathcal{Q} \times \mathcal{P}$.

The following corollay applies Theorem 3 to systems with only one unactuated coordinate.

Corollary. *Suppose $\mathscr{H}$ is a —- mechanical system satisfying Assumption 4 which has degree of underactuation one. Let $h(q, p_u) = 0$ be a regular VNHC of order $(n-1)$ of the form $h(q, p_u) = q_a - f(q_u, p_u)$ where f is a suitably-defined smooth function. Defining $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^n$,* *the actuated momentum is given by*

$$p_a = -\left(dh_q M^{-1}(q) \begin{bmatrix} \mathbf{0}_{1\times(n-1)} \\ I_{(n-1)} \end{bmatrix} \right)^{-1} \left(\partial_{p_u} f \partial_{q_u} V + dh_q M^{-1}(q) e_1 p_u \right) \Big|_{q_a = f(q_u, p_u)} \tag{2.13}$$

Since $q_u \in [\mathbb{R}]_T$ *for some* $T \in]0, \infty]$ *and* $p_u \in \mathbb{R}$*, the orbit* $(q_u(t), p_u(t))$ *traces out a curve on the 2D-plane* $[\mathbb{R}]_T \times \mathbb{R}$ *which we call the* (q, p)*-plane.*

2.4 Summary of Results

Chapter 3

Application of VNHCS: The Variable Length Pendulum

3.1 Motivation

The variable length pendulum (VLP) is a classical underactuated dynamical system which is often used to model the motion of a person on a swing [8], [16]. The VLP also represents the swinging of a crane, the (simplified) motion of a gymnast on a bar [15], and the tuned-mass-damper systems which stabilize skyscrapers [41], among others.

The motion of the VLP has been well studied (see for instance [39]), and many control mechanisms exist to stabilize trajectories of the system. While many of these control mechanism are time-dependent, [20] offers a time-independent technique to inject energy into the system and stabilize desired energy level sets. The authors designed a controller through a technique called *energy shaping*, and proved that their control mechanism would allow the VLP to achieve any desired energy level set. However, the energy injection mechanism is ad-hoc in the sense that it is not derived from any human-like behaviour. In this chapter, we will use VNHCs to provably stabilize energy level sets as well; the difference is that the control mechanism we derive will stay true to human motion and can be implemented by any person on a swing.

3.2 Dynamics of the Variable Length Pendulum

We will model the VLP as a point mass m connected to a fixed pivot by a massless rod of varying length l with angle $q \in \mathbb{S}^1$ from the vertical, as is seen in Figure 3.1. We will also ignore any damping and frictional forces in this model. In a realistic VLP, the rod

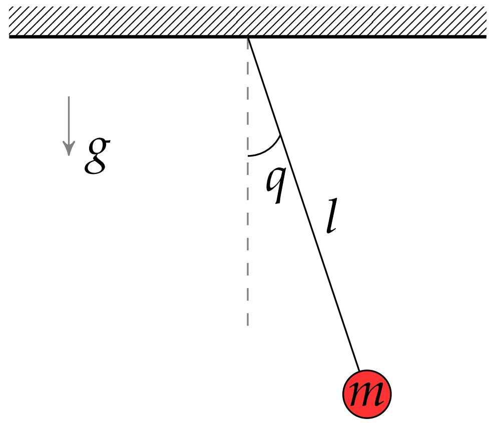

Figure 3.1: The representation of the variable length pendulum as a mass at the tip of a massless rod.

length l varies between some minimum length $\underline{l} \geq 0$ and some maximum length $\bar{l} > \underline{l}$. The configuration of the VLP is the vector $\mathbf{q} := (q, l) \in \mathbb{S}^1 \times [\underline{l}, \bar{l}]$.

Using this configuration, we will compute the Hamiltonian dynamics of the system. The cartesian position of the mass at the tip of the pendulum is given by $x = (l\sin(q), l\cos(q))$, while its velocity is $\dot{x} = (\dot{l}\sin(q) + l\cos q\dot{q}, \dot{l}\cos(q) - l\sin q\dot{q})$. Computing the kinetic energy T yields

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\|\dot{x}\|^2 = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{q}^2\right)$$

The potential energy P with respect to the pivot (under a gravitational acceleration g) is

$$P(\mathbf{q}) = -mgl\cos(q)$$

Collecting the kinetic energy into a quadratic form, we get the Lagrangian

$$\mathscr{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T} D(\mathbf{q})\dot{\mathbf{q}} - P(\mathbf{q}) = \frac{1}{2}\begin{bmatrix}\dot{q} & \dot{l}\end{bmatrix}\begin{bmatrix}ml^2 & 0 \\ 0 & m\end{bmatrix}\begin{bmatrix}\dot{q} \\ \dot{l}\end{bmatrix} + mgl\cos(q)$$

Computing the conjugate of momenta to $\mathbf{q}$, we get

$$\mathbf{p} := \begin{bmatrix}p \\ p_l\end{bmatrix} = \begin{bmatrix}ml^2\dot{q} \\ m\dot{l}\end{bmatrix}$$

Performing the Legendre transform on $\mathscr{L}$ and setting $M(\mathbf{q}) := D(\mathbf{q})$, $V(\mathbf{q}) := P(\mathbf{q})$, we find the Hamiltonian is equal to the total mechanical energy of the system:

$$\mathscr{H} = E = \frac{1}{2}\dot{\mathbf{p}}^\top M^{-1}(\mathbf{q})\dot{\mathbf{p}} + V(\mathbf{q})$$

Taking the appropriate derivatives, the dynamics of the VLP in Hamiltonian form are described in (3.1).

$$\mathscr{H} = \frac{1}{2}\begin{bmatrix} p & p_l \end{bmatrix}\begin{bmatrix} \frac{1}{ml^2} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}\begin{bmatrix} p \\ p_l \end{bmatrix} - mgl\cos(q) \tag{3.1}$$

$$\begin{cases} \dot{q} = \frac{p}{ml^2} \\ \dot{l} = \frac{p_l}{m} \\ \dot{p} = -mgl\sin(q) \\ \dot{p}_l = \frac{p^2}{ml^3} + mg\cos(q) + \tau \end{cases}$$

The control input is a force $\tau \in \mathbb{R}$ affecting the dynamics of p_l, acting colinearly with direction of the rod. We assume the force does not affect the dynamics of p in any way - that is, there is no lateral force such as when a person is pushed on a swing. In this way, the dynamics of (q, p) and (l, p_l) are decoupled.

This decoupling is extremely useful in simplifying the dynamics. Since one can write set $\dot{p}_l$ to any desired function, we can suppose (with some abuse of notation) that l is tracking some function $l(t)$. The closed loop dynamics of the system will be described exclusively by $(\dot{q}, \dot{p})$ with $l = l(t)$; the fact that p_l does not appear in these closed-loop dynamics allows us to ignore the subdynamics $(\dot{l}, \dot{p}_l)$ entirely.

What this means is we can treat l as the control input directly, rather than modelling it as a configuration variable. Re-deriving the Hamiltonian and the dynamics with this in mind, we get the system (3.2) with phase $(q, p) \in \mathbb{S}^1 \times \mathbb{R}$. Note that the control input $l(t)$ and its derivative $\dot{l}(t)$ are both known variables.

$$\mathscr{H}(q, p) = \frac{p^2}{ml^2} - \frac{1}{2}\dot{l}^2 - mgl\cos(q) \tag{3.2}$$

$$\begin{cases} \dot{q} = \frac{p}{ml^2} \\ \dot{p} = -mgl\sin(q) \end{cases}$$

In particular, the Hamiltonian of this simplified model is no longer equal to the total mechanical energy of the system, which is given by (3.3).

$$E(q,p) = \frac{p^2}{ml^2} + \frac{1}{2}\dot{l}^2 - mgl\cos(q) \tag{3.3}$$

3.3 The VLP Constraint

Theorem 4. *For the variable-length pendulum, define $\theta := \text{arctan}_2(p,q)$. A VNHC of the form $l = l(\theta)$ injects energy if there exists $l_{avg} \in \mathbb{R}_{>0}$ such that*

$$\left(l(\theta) - l_{avg}\right) sin(2\theta) \leq 0 \ \forall \theta \in \mathbb{S}^1$$

with the property that the inequality is strict for almost every θ.

Proof. Choose, as a candidate anti-Lyapunov function, the energy for the average-length pendulum

$$E_{avg}(q,p) := \frac{1}{2}\frac{p^2}{ml_{avg}^2} + mgl_{avg}(1 - \cos(q))$$

which is non-negative and has derivative

$$\dot{E}_{avg} = \frac{-g\sin(q)p\left(l(\theta)^3 - l_{avg}^3\right)}{l_{avg}^2 l(\theta)^2}$$

We will show that E_{avg} is increasing.

Observe that $\text{sgn}\left(\sin(q)p\right) = \text{sgn}\left(\sin(2\theta)\right)$ and, by Lemma **TODO: REF LEMMA**, $\text{sgn}\left(l(\theta)^3 - l_{avg}^3\right) = \text{sgn}\left(l(\theta) - l_{avg}\right)$.

Then the derivative of E_{avg} is almost always positive, since

$$\begin{aligned}
\text{sgn}\left(\dot{E}_{avg}\right) &= \text{sgn}\left(-\sin(q)p\left(l(\theta)^3 - l_{avg}^3\right)\right) \\
&= -\text{sgn}\left(\sin(2\theta)\left(l(\theta) - l_{avg}\right)\right) \\
&\geq 0 \text{ (by assumption)}
\end{aligned}$$

Hence, E_{avg} is an anti-Lyapunov function with positive derivative, so the variable-length pendulum is gaining energy. □

3.4 Simulation Results

Chapter 4

Application of VNHCs: The Acrobot

4.1 Motivation

4.2 Previous Approaches

4.3 The Acrobot Constraint

4.3.1 Proving the Acrobot Gains Energy

4.4 Experimental Results

Chapter 5

Conclusion

5.1 Limitations of this Work

5.2 Future Research

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