Energy injection for mechanical systems through the method of Virtual Nonholonomic Constraints

by

Adan Moran-MacDonald

A thesis submitted in conformity with the requirements
for the degree of Master of Applied Science
Graduate Department of Electrical and Computer Engineering
University of Toronto

Abstract

Energy injection for mechanical systems through the method of Virtual Nonholonomic Constraints

Adan Moran-MacDonald
Master of Applied Science
Graduate Department of Electrical and Computer Engineering
University of Toronto
2020

TODO: Fill in the abstract

For Fry.

Acknowledgements

TODO: Fill in the acknowledgements

Contents

List of Symbols

Symbol	Definition
$\mathbf{n}$	The index set $\{1, \dots, n\}$ of natural numbers up to n.
$\mathbb{R}^n$	Real numbers in n dimensions.
$[\mathbb{R}]_T$	Real numbers modulo $T > 0$, with $[\mathbb{R}]_\infty = \mathbb{R}$.
$\mathbb{S}^1$	The unit circle, equivalent to $[\mathbb{R}]_{2\pi}$.
$\mathbb{R}^{n \times m}$	The space of real-valued matrices with n rows and m columns.
I_n	The $n \times n$ identity matrix.
$\mathbf{0}_{n \times m}$	The $n \times m$ matrix of all zeros.
M_i	If M is a vector, the ith element of M.
	If M is a matrix, the ith column of M.
$M_{i,j}$	The value of row i, column j for the matrix M.
$[v_i]$	If $v_i \in \mathbb{R}$, the column vector with value v_i at position i.
	If $v_i \in \mathbb{R}^n$, the matrix with column i given by v_i.
	If $v_i \in \mathbb{R}^{1 \times n}$, the matrix with row i given by v_i.
$\dot{x}$	Derivative of x with respect to time t.
$\nabla_v F$	Gradient of the $\mathbb{R}$-valued function F with respect to v.
dF_v	Total differential (Jacobian) of the function F with respect to v.
	If F is $\mathbb{R}$-valued, equivalent to $(\nabla_v F)^\top$.
$\nabla_v dF_w$	Derivative matrix of the $\mathbb{R}$-valued function F, with (i, j) element $\partial^2 F / (\partial v_i \partial w_j)$.
$\nabla_v^2 F$	Hessian of $F : \mathbb{R}^n \to R$, equivalent to $\nabla_v dF_v$.
$\delta_{i,j}$	The Kronecker delta: 1 if $i = j$ and 0 otherwise.

TEST CITATIONS:
[1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22]

Chapter 1

Introduction

1.1 Literature Review

1.2 Statement of Contributions

1.3 Outline of the Thesis

Chapter 2

Development of Virtual Nonholonomic Constraints

2.1 Preliminaries on Analytical Mechanics

A mechanical system can be represented by N point masses, where each point represents the center of mass of a physical body, along with r *equations of constraint* (EOC) which model the physical restrictions between these masses. The position of each point mass is described using three cartesian coordinates (one for each spatial axis), so the system as a whole can be described by a vector in $\mathbb{R}^{3N}$ with r EOC. The dynamics of the system are computed by deriving the $3N$ *equations of motion* (EOM) produced by Newton's second law $F = ma$. While this technique works for simple systems, it is impossible to apply to a majority of mechanical systems since most of the forces on the system are not explicitly known.

Rather than modeling a mechanical system with point masses and constraints, it is often feasible to represent the position of the system using n independent scalar-valued variables $q_1, \dots, q_n$ called *generalized coordinates*, where $n = 3N - m$ is the number of *degrees of freedom* (DOF) of the system [23].

Each generalized coordinate q_i takes values in $[\mathbb{R}]_{T_i}$, where $T_i = \infty$ if q_i represents a length or $T_i = 2\pi$ if q_i represents an angle. It is convention to collect the coordinates into a *configuration* $q = (q_1, \dots, q_n) \in \mathcal{Q}$ where the *configuration manifold* $\mathcal{Q}$ of the system has the following structure:

$$\mathcal{Q} = [\mathbb{R}]_{T_1} \times \cdot \times [\mathbb{R}]_{T_n}$$

The derivative $\dot{q} = (\dot{q}_1, \dots, \dot{q}_n)$ of a configuration is called a *generalized velocity*, and the combined vector $(q, \dot{q})$ is called a *state* of the system.

The field of analytical mechanics provides a computational method for finding the EOM of a system in generalized coordinates. The two most common analytical methods for modelling robotic systems are *Lagrangian* and *Hamiltonian* mechanics.

2.1.1 Lagrangian Mechanics

Lagrangian mechanics uses the kinetic energy $T(q, \dot{q})$ and potential energy $P(q)$ of the system to define the Lagrangian $\mathscr{L} : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}$ defined by (2.1) [23].

$$\mathscr{L}(q, \dot{q}) = T(q, \dot{q}) - P(q) \tag{2.1}$$

When the mechanical system is actuated, the EOM are described by n second-order ordinary differential equations (ODEs) obtained from the *Euler-Lagrange equations* with *generalized input forces* $\tau \in \mathbb{R}^k$ (2.2).

$$\frac{d}{dt}\left\{\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right\} - \frac{\partial \mathscr{L}}{\partial q_i} = B_i^\top(q)\tau \tag{2.2}$$

The vector $B_i^\top : \mathcal{Q} \to \mathbb{R}^{1\times k}$ describes how the input forces shape the dynamics of q_i. The matrix $B : \mathcal{Q} \to \mathbb{R}^{n\times k}$ with

$$B(q) = [B_i^\top(q)]$$

is called the *input matrix* for the system. If $k < n$, we say the system is *underactuated* with degree of underactuation $n - k$.

Many actuated mechanical systems have quadratic kinetic energies, so that the Lagrangian can be written explicitly as

$$\mathscr{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} - P(q) \tag{2.3}$$

where the *inertia matrix* $D : \mathcal{Q} \to \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix for all $q \in \mathcal{Q}$ and the potential function $P : \mathcal{Q} \to \mathbb{R}$ is smooth.

2.1.2 Hamiltonian Mechanics

Hamiltonian mechanics provides an equivalent representation of the EOM by converting the n second-order ODEs generated by Lagrangian mechanics into $2n$ first-order ODEs.

To do this, we first define the *conjugate of momentum p_i to q_i* by

$$p_i(q,\dot{q}) = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}(q,\dot{q}) \tag{2.4}$$

To ease notation, we will write $p = (p_1, \dots, p_n)$ and say that p is the *conjugate of momenta to q*. Note that each p_i is a linear function of $\dot{q}$, so one can typically solve for $\dot{q}(q,p)$ by inverting the expressions from (2.4). The combined vector (q,p) is called the *phase* of the system.

Next we define the *Hamiltonian* (2.5) as the Legendre transform of the Lagrangian [24].

$$\mathscr{H}(q,p) = p^\top \dot{q}(q,p) - \mathscr{L}(q,\dot{q}(q,p)) \tag{2.5}$$

The EOM in this framework can be shown to be the $2n$ first-order equations called *Hamilton's equations*

$$\begin{cases} \dot{q} = \nabla_p \mathscr{H} \\ \dot{p} = -\nabla_q \mathscr{H} + B(q)\tau \end{cases} \tag{2.6}$$

where $B(q) \in \mathbb{R}^{n \times k}$ is the same input matrix used by the Lagrangian and $\tau \in \mathbb{R}^k$ is the vector of generalized input forces.

If the kinetic energy is quadratic as in (2.3), the conjugate of momenta to q can be computed explicitly:

$$p = D(q)\dot{q}$$

The resulting Hamiltonian system reduces to (2.7).

$$\mathscr{H}(q,p) = \frac{1}{2} p^\top D^{-1}(q) p + P(q) \tag{2.7}$$

$$\begin{cases} \dot{q} = D^{-1}(q)p \\ \dot{p} = -\frac{1}{2}\left[p^\top \frac{\partial D^{-1}}{\partial q_i}(q)p\right] - \nabla_q P(q) + B(q)\tau \end{cases} \tag{2.8}$$

A pair of coordinates (q,p) which satisfy Hamilton's equations under the Hamiltonian $\mathscr{H}$ are said to be *canonical coordinates for $\mathscr{H}$*. A change of coordinates $(q,p) \to (Q,P)$ is said to be a *canonical transformation* if (Q,P) are canonical coordinates for $H(Q,P) = \mathscr{H}\big(q(Q,P), p(Q,P)\big)$.

2.2 Simply Actuated Hamiltonian Systems

Given a Hamiltonian mechanical system (2.7), it is not obvious how the input forces τ affect the conjugate of momenta p_i. This is because τ is transformed by the input matrix $B(q)$, which may be quite complicated.

We will define a new class of Hamiltonian systems where the effect of the input forces on the conjugate of momenta is made obvious. This class of systems will form the backbone for the rest of the theory developed in this thesis.

Definition 1. A pair of canonical coordinates (q, p) for $\mathscr{H}$ are said to be *simply actuated coordinates* if, in these coordinates, the input matrix $B(q) \in \mathbb{R}^{n\times k}$ is of the form

$$B(q) = \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}$$

Definition 2. A Hamiltonian system is said to be *simply actuated* if there exists a canonical transformation into simply actuated coordinates for $\mathscr{H}$.

Under the following assumptions on the input matrix, we will show that the Hamiltonian system (2.7) is simply actuated.

Assumption 1. The input matrix $B(q) \equiv B \in \mathbb{R}^{n\times k}$ is constant, full rank, and $k \leq n$.

Assumption 2. There exists a matrix $B^\perp \in \mathbb{R}^{(n-k)\times n}$ which is right semi-orthogonal (*i.e.*$B^\perp (B^\perp)^\top = I_{(n-k)}$) and which is a left-annihilator for B (*i.e.*$B^\perp B = \mathbf{0}_{(n-k)\times k}$).

Note that if $k = (n-1)$, the existence of a left annihilator A^0 for B implies $B^\perp := A^0/\|A^0\|$ satisfies Assumption 2.

Assumption 3. Assume without loss of generality that the input matrix B is left semi-orthogonal. That is, assume $B^\top B = I_k$.

Proof. Since B is a constant matrix, it has a singular-value decomposition $B = U\Sigma V^\top$ where $U^{-1} = U^\top \in \mathbb{R}^{n\times n}$, $V^{-1} = V^\top \in \mathbb{R}^{k\times k}$, and $\Sigma \in \mathbb{R}^{n\times k}$ is defined by

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_k \\ & - & \mathbf{0}_{(n-k)\times k} & - & \end{bmatrix}$$

where $\sigma_i \neq 0$ because B is full-rank. **SOURCE** Defining $T \in \mathbb{R}^{k \times k}$ by

$$T = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_k} \end{bmatrix}$$

and assigning the input forces to $\tau = VT\hat{\tau}$, we get a new input matrix for $\hat{\tau} \in \mathbb{R}^k$ given by $\hat{B} = BVT = U\Sigma T$ which is still constant and full-rank. In particular, $\hat{B}^\top \hat{B} = T^\top \Sigma^\top \Sigma^\top T = I_k$. □

Let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be the following matrix:

$$\mathbf{B} = \begin{bmatrix} B^\perp \\ B^\top \end{bmatrix}$$

Since $B^\perp$ is a left annihilator of B and both $B^\perp$ and $B^\top$ are right semi-orthogonal, it is easy to show that $\mathbf{B}$ is orthogonal:

$$\mathbf{B}\mathbf{B}^\top = \begin{bmatrix} B^\perp (B^\perp)^\top & B^\perp B \\ (B^\perp B)^\top & B^\top B \end{bmatrix} = I_n \Rightarrow \mathbf{B}^{-1} = \mathbf{B}^\top$$

Definition 3. Define the coordinate transformation $(q_u, q_a) = \mathbf{B}q$, where $q_u = B^\perp q$ and $q_a = B^\top q$. We call q_u the *unactuated coordinates* and q_a the *actuated coordinates* of the system. Their corresponding conjugate of momenta are $(p_u, p_a) = \mathbf{B}p$.

The following theorem shows that these new coordinates are simply actuated coordinates, so that τ_i only affects $p_{a,i}$ and all p_u are unaffected by the input forces.

Theorem 1. *Under Assumptions 1,2, and 3, the change of coordinates $(Q = (q_u, q_a), P = (p_u, p_a))$ is a canonical transformation of the original Hamiltonian system (2.7) into the simply actuated Hamiltonian system (2.9),*

$$\hat{H}(Q, P) = \frac{1}{2} P^\top M^{-1}(Q) P + V(Q) \tag{2.9}$$

$$\begin{cases} \dot{Q} = M^{-1}(Q) P \\ \dot{P} = -\frac{1}{2}\left[P^\top \frac{\partial M^{-1}}{\partial Q_i}(Q) P\right] - \nabla_Q V(Q) + \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix} \tau \end{cases}$$

where

$$
\begin{aligned}
M^{-1}(Q) &= \mathbf{B} D^{-1}(\mathbf{B}^\top Q)\mathbf{B}^\top \\
V(Q) &= P(\mathbf{B}^\top Q)
\end{aligned}
$$

Proof. For any constant matrix A, the transformation $(Q = Aq, P = Ap)$ satisfies $\frac{\partial Q_i}{\partial p_m} = \frac{\partial P_i}{\partial q_m} = 0$ for all $i, m \in \mathbf{n}$. Hence, the Poisson brackets between the new coordinates are zero:

$$
\begin{aligned}
[Q_i, Q_j] &:= \sum_{m=1}^{n} \frac{\partial Q_i}{\partial p_m}\frac{\partial Q_j}{\partial q_m} - \frac{\partial Q_i}{\partial q_m}\frac{\partial Q_j}{\partial p_m} = 0 \\
[P_i, P_j] &:= \sum_{m=1}^{n} \frac{\partial P_i}{\partial p_m}\frac{\partial P_j}{\partial q_m} - \frac{\partial P_i}{\partial q_m}\frac{\partial P_j}{\partial p_m} = 0
\end{aligned}
$$

Since the matrix $A = \mathbf{B}$ is invertible and orthogonal, $(A_i)^\top A_j^\top = (A_i)^\top (A^{-1})_j = \delta_{i,j}$. Using this fact we see that the Poisson brackets between P and Q are given by:

$$
\begin{aligned}
[P_i, Q_j] &= \sum_{m=1}^{n} \frac{\partial P_i}{\partial p_m}\frac{\partial Q_j}{\partial q_m} - \frac{\partial P_i}{\partial q_m}\frac{\partial Q_j}{\partial p_m} \\
&= \sum_{m=1}^{n} A_{i,m}A_{j,m} - 0 \\
&= \sum_{m=1}^{n} A_{i,m}A^\top_{m,j} \\
&= (A_i)^\top A_j^\top \\
&= \delta_{i,j}
\end{aligned}
$$

By (45.10) in [24], the coordinate change $(Q = \mathbf{B}q, P = \mathbf{B}p)$ is a canonical transformation with new Hamiltonian $\hat{H}(Q,P) = \mathscr{H}(\mathbf{B}^\top Q, \mathbf{B}^\top P)$.

Furthermore, since $\dot{P} = \mathbf{B}\dot{p}$, the new input matrix is given by

$$
\mathbf{B}B = \begin{bmatrix} B^\perp B \\ B^\top B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-k)\times k} \\ I_k \end{bmatrix}
$$

so the coordinates (q_u, q_a, p_u, p_a) are simply actuated coordinates for $\hat{H}$ as desired. $\square$

2.3 Virtual Nonholonomic Constraints

Definition 4. A *virtual nonholonomic constraint of order k* is a relation $h(q,p) = 0$ where $h \in C^2\left(?; \mathbb{R}^k\right)$ is smooth, rank $\left(\left[dh_q,\ dh_p\right]\right) = k$ for all $(q,p) \in h^{-1}(0)$, and there exists a feedback controller $\tau(q,p)$ stabilizing the set

$$\Gamma = \left\{(q,p)|h(q,p) = 0, dh_q\dot{q} + dh_p\dot{p} = 0\right\}$$

which is called the *constraint manifold*.

Chapter 3

Application of VNHCS: The Variable Length Pendulum

3.1 Motivation

3.2 The VLP Constraint

Theorem 2. *For the variable-length pendulum, define $\theta := \mathrm{arctan}_2(p, q)$. A VNHC of the form $l = l(\theta)$ injects energy if there exists $l_{avg} \in \mathbb{R}_{>0}$ such that*

$$\left(l(\theta) - l_{avg}\right) sin(2\theta) \leq 0 \ \forall \theta \in \mathbb{S}^1$$

with the property that the inequality is strict for almost every θ.

Proof. Choose, as a candidate anti-Lyapunov function, the energy for the average-length pendulum

$$E_{avg}(q, p) := \frac{1}{2}\frac{p^2}{ml_{avg}^2} + mgl_{avg}(1 - \cos(q))$$

which is non-negative and has derivative

$$\dot{E}_{avg} = \frac{-g\sin(q)p\left(l(\theta)^3 - l_{avg}^3\right)}{l_{avg}^2 l(\theta)^2}$$

We will show that E_{avg} is increasing.

Observe that $\mathrm{sgn}\left(\sin(q)p\right) = \mathrm{sgn}\left(\sin(2\theta)\right)$ and, by Lemma **TODO: REF LEMMA**, $\mathrm{sgn}\left(l(\theta)^3 - l_{avg}^3\right) = \mathrm{sgn}\left(l(\theta) - l_{avg}\right)$.

Then the derivative of E_{avg} is almost always positive, since

$$\begin{aligned} \operatorname{sgn}\left(\dot{E}_{avg}\right) &= \operatorname{sgn}\left(-\sin(q)p\left(l(\theta)^3 - l_{avg}^3\right)\right) \\ &= -\operatorname{sgn}\left(\sin(2\theta)\left(l(\theta) - l_{avg}\right)\right) \\ &\geq 0 \text{ (by assumption)} \end{aligned}$$

Hence, E_{avg} is an anti-Lyapunov function with positive derivative, so the variable-length pendulum is gaining energy. □

3.3 Simulation Results

Chapter 4

Application of VNHCs: The Acrobot

4.1 Motivation

4.2 Previous Approaches

4.3 The Acrobot Constraint

4.3.1 Proving the Acrobot Gains Energy

4.4 Experimental Results

Chapter 5

Conclusion

5.1 Limitations of this Work

5.2 Future Research

Bibliography

[1] F. Udwadia, "Constrained motion of hamiltonian systems," *Nonlinear Dynamics*, vol. 84, no. 3, pp. 1135–1145, Dec. 2015. DOI: 10.1007/s11071-015-2558-3.

[2] K. Ono, K. Yamamoto, and A. Imadu, "Control of giant swing motion of a two-link horizontal bar gymnastic robot," *Advanced Robotics*, vol. 15, no. 4, pp. 449–465, 2001. DOI: 10.1163/156855301750398356.

[3] X. Zhang, H. Cheng, Y. Zhao, and B. Gao, "The dynamical servo control problem for the acrobot based on virtual constraints approach," in *The 2009 IEEE/RSJ International Conference on Intelligent Robots and Systems*, St. Louis, USA: IEEE, Oct. 2009. DOI: 10.1109/IROS.2009.5354570.

[4] A. Mohammadi, M. Maggiore, and L. Consolini, "Dynamic virtual holonomic constraints for stabilization of closed orbits in underactuated mechanical systems," *Automatica*, vol. 94, pp. 112–124, Aug. 2018.

[5] E. Papadopoulos and G. Papadopoulos, "A novel energy pumping strategy for robotic swinging," in *2009 17th Mediterranean Conference on Control and Automation*, Thessaloniki, Greece: IEEE, Jun. 2009. DOI: 10.1109/MED.2009.5164664.

[6] A. J. V. der Schaft, "On the hamiltonian formulation of nonholonomic mechanical systems," *Reports on Mathematical Physics*, vol. 34, no. 2, pp. 225–233, 1994.

[7] O. E. Fernandez, "The hamiltonization of nonholonomic systems and its applications," PhD thesis, University of Michigan, 2009.

[8] S. Wirkus, R. Rand, and A. Ruina, "How to pump a swing," *The College Mathematics Journal*, vol. 29, no. 4, pp. 266–275, 1998. DOI: 10.1080/07468342.1998.11973953.

[9] J. Horn, A. Mohammadi, K. Hamed, and R. Gregg, "Hybrid zero dynamics of bipedal robots under nonholonomic virtual constraints," *IEEE Control Systems Letters*, vol. 3, no. 2, pp. 386–391, Apr. 2019. DOI: 10.1109/lcsys.2018.2888571.

[10] S. C. Anco and G. Bluman, "Integrating factors and first integrals for ordinary differential equations," *European Journal of Applied Mathematics*, vol. 9, pp. 245–259, 1998.

[11] A. Mohammadi, M. Maggiore, and L. Consolini, "On the lagrangian structure of reduced dynamics under virtual holonomic constraints," *ESAIM: Control, Optimization and Calculus of Variations*, vol. 23, no. 3, pp. 913–935, Jun. 2017.

[12] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, Apr. 1998.

[13] J. Lu and L. J. Brown, "A multiple lyapunov functions approach for stability of switched systems," in *2010 American Control Conference*, Baltimore, USA: IEEE, Jul. 2010. DOI: 10.1109/ACC.2010.5530464.

[14] A. M. Bloch, J. E. Marsden, and D. V. Zenkov, "Nonholonomic dynamics," *Notices of the AMS*, vol. 52, no. 3, pp. 320–329, Mar. 2005.

[15] V. Sevrez, E. Berton, G. Rao, and R. J. Bootsma, "Regulation of pendulum length as a control mechanism in performing the backward giant circle in gymnastics," *Human Movement Science*, vol. 28, no. 2, pp. 250–262, Mar. 2009.

[16] B. Piccoli and J. Kulkarni, "Pumping a swing by standing and squatting: Do children pump time-optimally?" *IEEE Control Systems Magazine*, vol. 25, no. 4, pp. 48–56, Aug. 2005.

[17] A. D. Mahindrakar and R. N. Banavar, "A swing-up of the acrobot based on a simple pendulum strategy," *International Journal of Control*, vol. 78, no. 6, pp. 424–429, 2005. DOI: 10.1080/00207170500056835.

[18] T. Henmi, M. Chujo, Y. Ohta, and M. Deng, "Reproduction of swing-up and giant swing motion of acrobot based on a technique of the horizontal bar gymnast," in *Proceedings of the 11th World Congress on Intelligent Control and Automation*, Shenyang, China: IEEE, Jun. 2014. DOI: 10.1109/WCICA.2014.7053137.

[19] M. Maggiore and L. Consolini, "Virtual holonomic constraints for euler-lagrange systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 1001–1008, Apr. 2013.

[20] X. Xin and Y. Liu, "Trajectory tracking control of variable length pendulum by partial energy shaping," *Communications in Nonlinear Science and Numerical Simulations*, vol. 19, no. 5, pp. 1544–1556, May 2014.

[21] X. Wang, "Motion control of a gymnastics robot using virtual holonomic constraints," Master's thesis, University of Toronto, 2016.

[22] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ 07485: Prentice Hall, 2002, ISBN: 0-13-067389-7.

[23] D. T. Greenwood, *Principles of Dynamics*, 2nd ed. Englewood Cliffs, NJ: Prentice Hall, 1987.

[24] E. M. L. L. D. Landau, *Mechanics*, 3rd ed. Butterworth-Heinemann, Jan. 1982, ISBN: 9780750628968.