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Brief paper

Orbital stabilization of nonlinear systems via Mexican sombrero energy shaping and pumping-and-damping injection ☆

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ABSTRACT

In this paper we show that a slight modification to the widely popular interconnection and damping assignment passivity-based control method – originally proposed for stabilization of equilibria of nonlinear systems – allows us to provide a solution to the more challenging orbital stabilization problem. Two different, though related, ways how this procedure can be applied are proposed. First, the assignment of an energy function that has a minimum in a closed curve, *i.e.*, with the shape of a Mexican sombrero. Second, the use of a damping matrix that changes "sign" according to the position of the state trajectory relative to the desired orbit, that is, pumping or dissipating energy. The proposed methodologies are illustrated with the example of the induction motor and prove that it yields the industry standard field oriented control.

1. Introduction

In many practical tasks the system under control is required to operate along periodic motions, *i.e.*, walking and running robots, path following, rotating electromechanical systems, AC or resonant power converters, and oscillation mechanisms in biology. As clearly explained in Khalil (2002, Section 8.4) the stability analysis of these behaviors can be recast as a standard equilibrium stabilization problem, but this leads to very conservative results. It is more convenient, instead, to invoke the notion of stability of an invariant set, where the latter is the closed orbit associated to the periodic solution. This approach leads to the important notion of orbital stability (Khalil, 2002, Definition 8.2).

A large number of papers and books have been devoted to analysis of orbital stability of a given dynamical system, see *e.g.*, Cheban (2004), Fradkov and Pogromsky (1998) and Guckenheimer and Holmes (1983). However, there are only a few *constructive tools* available to solve the task of orbital stabilization of a controlled system. A popular approach to address this question is the virtual holonomic constraints (VHC) method, which has been tailored for mechanical systems of co-dimension one (Maggiore & Consolini, 2013; Mohammadi, Maggiore, & Consolini, 2018; Shiriaev, Freidovich, & Gusev, 2010; Shiriaev, Perram, & Canudas-de Wit, 2005). In the VHC method a certain subspace of the state-space is rendered attractive and invariant, leading to a projected dynamics that behaves as oscillators. This is a particular case of the framework adopted in the immersion and invariance (I&I) technique, first reported for equilibrium stabilization in Astolfi and Ortega (2003), and later extended for observer design and adaptive control in Astolfi et al. (2008). In Ortega, Yi, Romero, and Astolfi (2018) it has recently been shown that I&I can also be adapted for orbital stabilization, leading to a procedure that contains, as particular case, the VHC designs. The only modification done to the standard I&I technique is in the definition of the target dynamics that now should be chosen possessing periodic orbits, instead of an equilibrium at the desired point. A main drawback in both the VHC and I&I methods is that the steady-state behavior cannot be fixed *a priori*, but depends on the initial states, see Ortega et al. (2018, Remark 2) for a discussion on this matter.

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An alternative approach to generate oscillations is reported in Stan and Sepulchre (2007), where it is proposed to construct passive oscillators for Lure dynamical systems using "sign-indefinite" feedback static mappings, which is a mechanism similar to the pumping-and-damping injection discussed below. Unfortunately, since the analysis is carried out applying the center manifold theory – that is a local notion – the obtained oscillators are assumed to have small amplitudes. Orbital stabilization designs, for some particular controlled plants, have also been reported in Aracil, Gordillo, and Ponce (2005), Fradkov and Pogromsky (1998) and Spong (1995).

The aim of this paper is to show that the widely popular interconnection and damping assignment passivity based control (IDA-PBC), originally proposed in Ortega and Garcia-Canseco (2004), Ortega, Spong, Gomez, and Blankenstein (2002) and Ortega, van der Schaft, Maschke, and Escobar (2002) for stabilization of equilibria, can be easily be adapted to address the problem of orbital stabilization of general nonlinear systems. This leads to two new *constructive* solutions for this problem that – as usual in PBC – have a clear interpretations from the energy viewpoint. First, the assignment of an energy function that has a minimum in a closed curve, *i.e.*, with the shape of a Mexican sombrero. Second, the use of a damping matrix that changes "sign" according to the position of the state trajectory relative to the desired orbit, that is, pumping or dissipating energy. As usual in all constructive nonlinear controller designs, the success of the proposed methods hinges upon our ability to solve a partial differential equation (PDE).

The remainder of the paper is organized as follows. Section 2 revisits the standard IDA-PBC. Section 3 introduces the problem formulation of orbital stabilization, followed by the constructive main results in Section 4. The application to the induction motor (IM) is reported in Section 5. Interestingly, we prove that the resulting controller exactly coincides with the industry standard direct field-oriented control (FOC) first proposed in Blaschke (1972). In Section 6 the orbital stabilization of pendula is studied. The paper is wrapped-up with conclusions and future work in Section 7.

Notation. $\mathbb{S}$ denotes the unit circle. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, we denote $\|x\|_{\mathcal{A}} := \inf_{y\in\mathcal{A}}|x-y|$, with $|x|^2 := x^\top x$, and $B_\varepsilon(\mathcal{A}) := \{x \in \mathbb{R}^n | \|x\|_{\mathcal{A}} \le \varepsilon\}$. All mappings are assumed smooth. For a full-rank mapping $g(x) \in \mathbb{R}^{n\times m}$ with $(m < n)$, we denote the generalized inverse as $g^\dagger(x) := [g^\top(x)g(x)]^{-1}g^\top(x)$, and $g^\perp(x) \in \mathbb{R}^{(n-m)\times n}$ a full-rank left annihilator of $g(x)$. We define the gradient transpose operator as $\nabla_x := (\partial/\partial x)^\top$. When clear from the context the arguments of the mappings and the operator ∇ are omitted.

Caveat. An abridged version of this paper will be presented in Yi, Ortega, Wu, and Zhang (2019). The full version is available on arXiv (arXiv:1903.04070).

2. Background on IDA-PBC

We consider in the paper systems written in the form

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

with the state $x \in \mathbb{R}^n$ and the control $u \in \mathbb{R}^m$, $m \le n$ and $g(x)$ full rank. To solve the orbital stabilization problem we propose in the paper a variation of the IDA-PBC method (Ortega, van der Schaft, et al., 2002), normally used for regulation tasks. The objective in IDA-PBC is to find a feedback control law $u = \hat{u}(x)$ such that the closed-loop dynamics takes a port-Hamiltonian (pH) form, that is,

$$f(x) + g(x)\hat{u}(x) = \big[\mathcal{J}(x) - \mathcal{R}(x)\big]\nabla H(x) =: f_{c1}(x) \tag{2}$$

with $H(x) \in \mathbb{R}$ the desired Hamiltonian and

$$\mathcal{J}(x) = -\mathcal{J}^\top(x), \;\; \mathcal{R}(x) = \mathcal{R}^\top(x) \tag{3}$$

Fig. 1. The closed-loop Hamiltonian with the target orbit in MSEA-PBC and the Mexican sombrero.

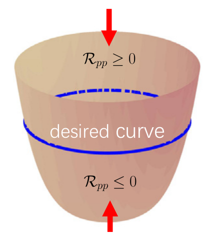

Fig. 2. The desired Hamiltonian and the target orbit in EPD-PBC.

the desired $(n \times n)$ interconnection and damping matrices, respectively. The matching objective (2) is achieved if and only if the following PDE (in $H(x)$) is solved

$$g^\perp(x)f(x) = g^\perp(x)\big[\mathcal{J}(x) - \mathcal{R}(x)\big]\nabla H(x). \tag{4}$$

If this is the case, the control law is given as

$$\hat{u}(x) = g^\dagger(x)\big[\big(\mathcal{J}(x) - \mathcal{R}(x)\big)\nabla H(x) - f(x)\big]. \tag{5}$$

In *regulation* tasks, $H(x)$ has a unique minimum at the desired equilibrium and we choose the matrix $\mathcal{R}(x)$ to be positive semi-definite to inject the damping required to drive the trajectory towards the equilibrium. In this paper we show that, for *orbital stabilization* we select $H(x)$ to have a minimum at the *desired orbit*—see Fig. 1. We will refer to this controller as Mexican sombrero energy assignment (MSEA) PBC.

An alternative option is to select the "sign" of $\mathcal{R}(x)$ to *pump energy* or *inject damping* according to the relative position of the state with respect to the desired orbit—this method is called energy pumping-and-damping (EPD)-PBC. A visual illustration is given in Fig. 2.

3. Problem formulation

We are interested in the paper in the generation, via IDA-PBC, of periodic solutions $X : \mathbb{R}_+ \to \mathbb{R}^n$ which are *asymptotically orbitally stable*. That is,

$$\dot{X}(t) = f_{c1}(X(t)), \quad X(t) = X(t+T), \;\forall t \ge 0,$$

where the closed-loop vector field $f_{c1}(x)$ is given in (2), and the set defined by its associated closed orbit

$$\mathcal{A} := \{x \in \mathbb{R}^n \mid x = X(t), \; 0 \le t \le T\},$$

is attractive.

We consider a particular case of periodic motion, which is defined as follows. First, split the state as $x := \mathrm{col}(x_p, x_\ell)$, with $x_p \in \mathbb{R}^2$, $x_\ell \in \mathbb{R}^{n-2}$. We also partition the matrices $\mathcal{J}(x)$ and $\mathcal{R}(x)$, conformally, as

$$\begin{bmatrix} (\cdot)_{pp} & (\cdot)_{p\ell} \\ (\cdot)_{\ell p} & (\cdot)_{\ell\ell} \end{bmatrix} \sim \begin{bmatrix} \mathbb{R}^{2\times 2} & \mathbb{R}^{2\times(n-2)} \\ \mathbb{R}^{(n-2)\times 2} & \mathbb{R}^{(n-2)\times(n-2)} \end{bmatrix}.$$

Then, define the set $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{C} \times \{x_\ell^*\} \subset \mathbb{R}^n,$$

where $x_\ell^* \in \mathbb{R}^{n-2}$ is a constant vector and $\mathcal{C}$ is a Jordan curve, given in implicit form as

$$\mathcal{C} := \{x_p \in \mathbb{R}^2 \mid \Phi(x_p) = 0\}, \tag{6}$$

on which $\nabla\Phi \neq 0$ with a smooth function $\Phi(x_p) \in \mathbb{R}$.

Remark 1. It is important at this point to clarify the difference between our objective of *orbital* stabilization and the more classical *set* stabilization. The latter is satisfied ensuring $\lim_{t\to\infty}\|x(t)\|_{\mathcal{A}} = 0$, but this does not ensure that the desired periodic motion is generated. Indeed, if the set

$$\mathcal{O} := \{x \in \mathbb{R}^n \mid \|x\|_{\mathcal{A}} = 0\} \tag{7}$$

contains *equilibrium points* of the closed-loop dynamics, the periodic motion is not generated. That is, we want to ensure that the closed-loop vector field (2) satisfies

$$f_{\mathrm{cl}}(x)|_{x\in\mathcal{O}} \neq 0. \tag{8}$$

4. Main results

We propose in this section two methods to solve the orbital stabilization problem posed above, MSEA and EPD-PBC, whose underlying philosophy is described in Section 2. Connections between these two methods are also given.

4.1. Mexican sombrero energy assignment PBC

The successful application of the IDA-PBC procedure described in Section 2 is guaranteed in MSEA with the following.

Assumption 1. There are mappings (3) with $\mathcal{R}(x) \geq 0$ and a function $H_0 : \mathbb{R}^{n-1} \to \mathbb{R}$ verifying

$$\arg\min H_0(x_0, x_\ell) = (0, x_\ell^*) \quad \text{(isolated)}, \tag{9}$$

which are solutions of the PDE (4), where we defined the function $H(x) := H_0(\Phi(x_p), x_\ell)$.

There are two additional requirements to ensure the success of the MSEA design. First, a detectability-like condition to guarantee attractivity of the desired orbit. Second, to avoid the scenario discussed in Remark 1, we impose a constraint on the interconnection matrix, that ensures there are no equilibrium points in the set $\mathcal{O}$ given in (7). These requirements are articulated in the assumptions of the following proposition.

Proposition 1. *Consider the system (1), verifying Assumption 1, in closed-loop with the control law $u = \hat{u}(x)$ with $\hat{u}(x)$ given in (5). Assume the following.*

H1 *$\mathcal{A}$ is the largest invariant set in the set*

$$\mathcal{Q} := \{x \in \mathbb{R}^n | \nabla^\top H(x)\mathcal{R}(x)\nabla H(x) = 0\} \cap B_\varepsilon(\mathcal{A}),$$

for some $\varepsilon > 0$.

H2 *The (1, 2)-element of $\mathcal{J}(x)$ may be parameterized as*

$$\mathcal{J}_{(1,2)}(x) = \frac{c(x)}{\nabla_{x_0} H_0(x_0, x_\ell)}\Bigg|_{x_0=\Phi(x_p)} \tag{10}$$

for some $c : \mathbb{R}^n \to \mathbb{R}$ satisfying $0 < |c(x)| < \infty,\ \forall x \in \mathcal{A}$.

Then, the closed-loop system is asymptotically orbitally stable.

Proof. The closed-loop system takes the form

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H. \tag{11}$$

From the isolated minimum condition of $H_0(x_0, x_\ell)$ stated in (9), we conclude that the function $H(x)$ is minimal in the set $\mathcal{A}$. Consequently,

$$\nabla H(x)\big|_{x\in\mathcal{A}} = 0,\ \ \nabla^2 H_0(x)\big|_{x\in B_\varepsilon(\mathcal{A})} > 0 \tag{12}$$

for some $\varepsilon > 0$. This shows that the set $\mathcal{Q}$—containing the set $\mathcal{A}$—is non-empty.

From the closed-loop pH dynamics (11) it is clear that

$$\dot{H} = -\nabla^\top H(x)\mathcal{R}(x)\nabla H(x) \leq 0,$$

implying the boundedness of $H(x)$. Together with (12), we conclude the Lyapunov stability of the closed-loop system with respect to $\mathcal{A}$. Thus, given a parameter $\varepsilon > 0$, there always exists an *invariant* set $\mathcal{E}$ such that $\mathcal{A} \subset \mathcal{E} \subset B_\varepsilon(\mathcal{A})$. Now, from the first equation of (12) we get

$$\nabla^\top H(x)\mathcal{R}(x)\nabla H(x)\big|_{x\in\mathcal{A}} = 0.$$

Applying LaSalle's invariance principle, taking into account the trajectory boundedness in $\mathcal{E}$, and the assumption *H1*, we prove the attractivity of $\mathcal{A}$, that is,

$$\lim_{t\to\infty}\|x(t)\|_{\mathcal{A}} = 0, \quad \forall x(t_0) \in \mathcal{E}.$$

The proof is completed establishing the existence of the periodic orbit, that is, verifying (8). Consider the term $\mathcal{J}_{pp}(x)\nabla_{x_p}H(x)$ of the closed-loop dynamics:

$$\begin{aligned}
\mathcal{J}_{pp}(x)\nabla_{x_p}H(x) &= \mathcal{J}_{pp}(x)\nabla_{x_0}H_0(x_0, x_\ell)\nabla\Phi(x_p) \\
&= \begin{bmatrix} 0 & \mathcal{J}_{(1,2)}(x) \\ -\mathcal{J}_{(1,2)}(x) & 0 \end{bmatrix}\nabla_{x_0}H_0(x_0, x_\ell)\nabla\Phi(x_p) \\
&= \begin{bmatrix} 0 & c(x) \\ -c(x) & 0 \end{bmatrix}\nabla\Phi(x_p),
\end{aligned}$$

where we applied in the first identity the chain rule $\nabla_{x_p}H(x) = \nabla_{x_0}H_0(x_0, x_\ell)\nabla\Phi(x_p)$, and used assumption *H2* in the third one. Considering that $\nabla H(x)\big|_{x\in\mathcal{A}} = 0$, the residual dynamics is

$$\dot{x}_p = \begin{bmatrix} 0 & c(x) \\ -c(x) & 0 \end{bmatrix}\nabla\Phi(x_p), \quad \dot{x}_\ell = 0.$$

Now, from $\dot{\Phi} = 0$, we conclude that the set $\mathcal{C}$ is invariant. To prove that the set $\mathcal{A}$ is a periodic orbit, we compute the 1-norm of $\dot{x}_p$ as

$$\|f_{\mathrm{cl}}(x)\|_1 = |c(x)|\|\nabla\Phi(x_p)\|_1 > 0, \quad \forall x \in \mathcal{A}.$$

With the additional Jordan curve assumption, the existence of a periodic orbit is verified, completing the proof. □

Remark 2. The minimum condition (9) implies that $\nabla_{x_0}H_0(x_0, x) = 0$. Consequently, in view of condition (10), the mapping $\mathcal{J}_{p\ell}(x)$ is singular along the orbit. However, the closed-loop dynamics and the feedback law (5) are well-defined everywhere. If the term $\mathcal{J}_{(1,2)}(x)$ is bounded along the orbit the condition (8) is violated. Consequently, the "infinite interconnection" condition *H2* is *necessary* to ensure the orbit exists, otherwise we only achieve set stabilization—see Remark 1.

4.2. Energy pumping-and-damping PBC

In this subsection, we introduce an alternative orbital stabilization methodology: EPD-PBC—where the periodic orbit is enforced by regulating the *energy level* to a constant value. More precisely, we assume the total energy of the closed-loop can be decomposed as

$$H(x) := H_p(x_p) + H_\ell(x_\ell). \tag{13}$$

The function that defines the Jordan curve (6) is given as

$$\Phi(x_p) := H_p(x_p) - H_p^*, \tag{14}$$

with H_p^* the desired energy level for $H_p(x_p)$, which should be larger than the minimal value of $H_p(x_p)$, that is, it should satisfy

$$H_p^* > \min(H_p(x_p)).$$

To enforce the oscillation, the "sign" of (part of) the damping matrix $\mathcal{R}(x)$ changes according to the position of the state x relative to the desired oscillation—whence, to the set $\mathcal{C}$. See Fig. 2.

Similarly to Assumption 1 for MSEA-PBC, in EPD-PBC we require that the PDE (4) is solvable, with an additional constraint on $\mathcal{R}(x)$ to implement the energy pumping-and-damping mechanism.

Assumption 2. There exist functions $H_p(x_p)$ and $H_\ell(x_\ell)$, which have isolated minima in $x_p^* \in \mathbb{R}^2$ and $x_\ell^* \in \mathbb{R}^{n-2}$, respectively, and mappings (3), with

$$\mathcal{R}(x) = \mathrm{diag}\{\mathcal{R}_{pp}(x), \mathcal{R}_{\ell\ell}(x_\ell)\},$$

where $\mathcal{R}_{\ell\ell}(x_\ell) \geq 0$ and the diagonal matrix $\mathcal{R}_{pp}(x)$ satisfies the pumping-and-damping condition

$$\mathcal{R}_{pp}(x)\Phi(x_p) \geq 0 \tag{15}$$

where $\Phi(x_p)$ is given in (14), and

$$\mathcal{R}_{pp}(x) = 0 \quad \Longleftrightarrow \quad \Phi(x_p) = 0. \tag{16}$$

In EPD-PBC besides the detectability-like and the interconnection conditions, we require a technical assumption to complete the proof. That is, $\nabla^\top H_p(x_p)\mathcal{J}_{p\ell}(x) = 0$, in order to "cut off" the energy flow between x_ℓ and x_p partitions.

Proposition 2. *Consider the system (1), verifying Assumption 2, in closed-loop with the control law $u = \hat{u}(x)$ with $\hat{u}(x)$ given in (5). Assume the following.*

H3 *$\{x_\ell^*\}$ is the largest invariant set in the set*

$$\left\{x_\ell \in \mathbb{R}^{n-2} \middle| \nabla^\top H_\ell(x_\ell)\mathcal{R}_{\ell\ell}(x_\ell)\nabla H_\ell(x_\ell) = 0\right\}.$$

H4 *The matrix $\mathcal{J}(x)$ satisfies*

$$\mathcal{J}_{(1,2)}(x) \neq 0, \quad \nabla^\top H_p(x_p)\mathcal{J}_{p\ell}(x) = 0.$$

H5 *For some $\varepsilon_* > 0$*

$$\nabla^2 H_p|_{x_p \in B_{\varepsilon_*}(x_p^*)} > 0,\ max_{B_{\varepsilon_*}(x_p^*)} H_p(x_p) > H_p^*.$$

Then, the closed-loop system is asymptotically orbitally stable with respect to the orbit $\mathcal{A} \cap B_{\varepsilon_*}(x_*)$.

Proof. The closed-loop dynamics takes the form (11) with $\nabla H = \mathrm{col}(\nabla H_p(x_p), \nabla H_\ell(x_\ell))$. From which it is clear that

$$\begin{aligned}\dot{H}_\ell &= -\nabla^\top H_\ell\big(\mathcal{R}_{\ell\ell}(x_\ell)\nabla H_\ell + \mathcal{J}_{p\ell}^\top(x)\nabla H_p\big)\\ &= -\nabla^\top H_\ell \mathcal{R}_{\ell\ell}(x_\ell)\nabla H_\ell \leq 0,\end{aligned}$$

where we have used the assumption *H4*. Applying LaSalle's invariance principle and using the assumption *H3*, we have

$$\lim_{t\to\infty} x_\ell(t) = x_\ell^*.$$

For $H_p(x_p)$ we have

$$\begin{aligned}\dot{H}_p &= -\nabla^\top H_p\big(\mathcal{R}_{pp}(x)\nabla H_p - \mathcal{J}_{p\ell}^\top(x)\nabla H_\ell\big)\\ &= -\nabla^\top H_p \mathcal{R}_{pp}(x)\nabla H_p,\end{aligned}$$

where we used again the assumption *H4*.

Consider the function $V(x_p) := \frac{1}{2}\Phi^2(x_p)$, we have

$$\dot{V} = -\nabla^\top H_p[\Phi(x)\mathcal{R}_{pp}(x)]\nabla H_p \leq 0,$$

where the inequality is the consequence of the pumping-and-damping condition (15). Invoking LaSalle's invariance principle, the state ultimately converges to the largest invariant set of the set

$$\mathcal{Q} = \{x \in \mathbb{R}^n | x_\ell = x_\ell^*,\ [\Phi(x)\mathcal{R}_{pp}(x)]\nabla H_p(x) = 0\}.$$

There are three cases of $[\Phi(x)\mathcal{R}_{pp}(x)]\nabla H_p(x) = 0$, namely,

(i) $[\Phi(x)\mathcal{R}_{pp}(x)] = 0$;
(ii) $\nabla H_p(x) = 0$;
(iii) $\Phi(x)\mathcal{R}_{pp}(x) \neq \mathbf{0}$ and $\nabla H_p(x) \neq \mathbf{0}$ with

$$\nabla H_p(x) \in \mathtt{Ker}(\Phi(x)\mathcal{R}_{pp}(x)) = \mathtt{Ker}(\mathcal{R}_{pp}(x)).$$

First consider Case (iii) with the definition

$$\mathcal{Q}_{iii} := \{x \in \mathbb{R}^n | x_\ell = x_\ell^* \text{ and } x \text{ satisfies Case (iii)}\}.$$

We will prove that $\mathcal{Q}_{iii}$ is not an invariant set by contradiction. Assume that $\mathcal{Q}_{iii}$ is invariant along the closed-loop dynamics. On $\mathcal{Q}_{iii}$ the residual dynamics is

$$\dot{x}_p = [\mathcal{J}_{pp}(x) - \mathcal{R}_{pp}(x)]\nabla H_p(x_p)$$

with $x_\ell = x_\ell^*$. Since $\nabla H_p(x_p) \in \mathtt{Ker}(\mathcal{R}_{pp}(x))$, we have

$$\dot{x}_p = \mathcal{J}_{pp}(x)\nabla H_p. \tag{17}$$

Assumption *H4* ensures $\det(\mathcal{J}_{pp}(x)) \neq 0$, hence from (17) we conclude that there are no equilibrium points in $\mathcal{Q}_{iii}$. From (17) we also conclude that

$$H_p(x_p(t)) \equiv \text{const}, \quad \forall x \in \mathcal{Q}_{iii},\ t \geq 0. \tag{18}$$

Noticing the diagonal condition of $\mathcal{R}_{pp}(x)$, together with $\det(\mathcal{R}_{pp}) = 0$, $\mathcal{R}_{pp} \neq \mathbf{0}$ for Case (iii) and $\nabla H_p(x) \in \mathtt{Ker}(\mathcal{R}_{pp}(x))$, we then have $\nabla_{x_{p1}} H_p \equiv 0$ or $\nabla_{x_{p2}} H_p \equiv 0$, that contradicts the identity (18). Therefore, the set $\mathcal{Q}_{iii}$ is not invariant, excluding the possibility of Case (iii).

For Case (i), from (16) we have

$$\Phi(x)\mathcal{R}_{pp}(x) = 0 \quad \Rightarrow \quad \Phi(x_p) = 0.$$

Together with $\dot{\Phi} = 0$ for all $x \in \mathcal{A}$, it implies the invariance of Case (i). For Case (ii), it yields $x = \mathrm{col}(x_p^*, x_\ell^*) := x_*$. In summary, the largest invariant set in $\mathcal{Q}$ is $\mathcal{A} \cup \{x_*\}$.

We consider the function $W(x) = \Phi(x)$, and, for some small $\varepsilon > 0$, it follows

$$\dot{W} = -\nabla^\top H_p \mathcal{R}_{pp}(x)\nabla H_p \geq 0, \quad \forall x_p \in B_\varepsilon(x_p^*).$$

Therefore, the isolated equilibrium point x_* is unstable. On the other hand, the set $\mathcal{A}$ is attractive.

We proceed now to verify the existence of a periodic orbit. Since x_* is a minimum of $H(x)$ we have that

$$\nabla H(x)\big|_{x=x_*} = 0,\ \nabla^2 H(x)\big|_{x\in B_\varepsilon(x_*)} > 0.$$

If $x \in B_\varepsilon(x_*)$, the function $\mathcal{V}(x) := H(x) - H(x_*)$ qualifies as a Lyapunov function (for the dynamics $\dot{x} = F(x)\nabla H(x)$ with $F(x) > 0$). According to Byrnes (2008, Theorem 4.1), the set $\mathcal{C} \cap B_{\varepsilon_*}(x_p^*)$ defines a Jordan curve. On the set $\mathcal{A} \cap B_{\varepsilon_*}(x_*)$ the residual dynamics is

$$\dot{x}_p = \begin{bmatrix} 0 & \mathcal{J}_{(1,2)}(x) \\ -\mathcal{J}_{(1,2)}(x) & 0 \end{bmatrix}\nabla H_p,\ \dot{x}_\ell = 0.$$

We conclude $|f_{\mathtt{c1}}(x)| \neq 0$, completing the proof. $\square$

Remark 3. The condition *H4* is similar, in nature, to *H2*, but excluding equilibria on the orbit $\mathcal{A} \cap B_{\varepsilon_*}(x_*)$. Noticing that $|\nabla H_p| \neq 0$ on the desired orbit, only "finite interconnection" is adequate in the EPD method for the purpose of orbital stabilization. An example of energy regulation without adequate interconnection – the interconnection matrix becoming zero on the desired orbit – is given in our previous work (Yi, Ortega, & Zhang, 2019), which solves the open problem—using *smooth, time-invariant* state-feedback to achieve almost global asymptotic *regulation* of three-dimensional nonholonomic systems.

Remark 4. A trivial selection of the mapping $\mathcal{R}_{pp}$ is diag$(0, \Phi(x_p))$, but it is non-unique. This indeed provides an additional degree of freedom to solve the PDE, and the possibility to regulate the convergence speed.

4.3. Comparison of MSEA-PBC and EPD-PBC

In this section we compare the two methods and clarify the parallel between them. To simplify the presentation we relabel the various mappings used in the methods with fonts `mathcal` $(\mathcal{J}, \mathcal{R}, \mathcal{H})$ for MSEA-PBC and `mathbf` $(\mathbf{J}, \mathbf{R}, \mathbf{H})$ for EPD-PBC. We have the following.

Proposition 3. *Consider the system (1), verifying all the assumptions in Proposition 1. Assume the matrix $\mathcal{R}$ is diagonal, $\mathcal{R}_{\ell\ell}$ is a function of x_ℓ, and $\mathcal{R}_{pp}$ is non-zero. If the mapping $\mathcal{H}_0 : \mathbb{R}^{n-1} \to \mathbb{R}$ can be decomposed as*

$$\mathcal{H}_0(x_0, x_\ell) = \mathcal{H}_1(x_0) + \mathcal{H}_\ell(x_\ell), \tag{19}$$

and $\nabla^\top \Phi(x_p)\mathcal{J}_{p\ell}(x) = 0$, then all the assumptions in Proposition 2 are satisfied by selecting the mappings[1]

$$\begin{aligned} &\mathbf{H}_p(x_p) = \Phi(x_p), \qquad \mathbf{H}_\ell(x_\ell) = \mathcal{H}_\ell(x_\ell) \\ &\mathbf{J}(x) = \begin{bmatrix} \mathcal{H}_1'\mathcal{J}_{pp} & \mathcal{J}_{p\ell} \\ -\mathcal{J}_{p\ell}^\top & \mathcal{J}_{\ell\ell} \end{bmatrix}, \ \mathbf{R}(x) = \begin{bmatrix} \mathcal{H}_1'\mathcal{R}_{pp} & \mathbf{0} \\ \mathbf{0} & \mathcal{R}_{\ell\ell} \end{bmatrix}, \end{aligned} \tag{20}$$

and $\mathbf{H}_p^* = 0$. Furthermore, the MSEA and EPD methods yield the same feedback law.

Proof. We first verify the solvability of the matching PDEs—equivalently the coincidence of two closed-loop dynamics. The closed-loop dynamics in Proposition 1 is

$$\begin{aligned} \dot{x} &= \big[\mathcal{J}(x) - \mathcal{R}(x)\big]\nabla\mathcal{H}_0(\Phi(x_p), x_\ell) \\ &= \begin{bmatrix} \mathcal{J}_{pp}(x) - \mathcal{R}_{pp}(x) & \mathcal{J}_{p\ell}(x) \\ -\mathcal{J}_{p\ell}^\top(x) & \mathcal{J}_{\ell\ell}(x) - \mathcal{R}_{\ell\ell}(x) \end{bmatrix} \begin{bmatrix} \mathcal{H}_1'\nabla\Phi \\ \nabla\mathcal{H}_\ell \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{H}_1'(\mathcal{J}_{pp}(x) - \mathcal{R}_{pp}(x)) & \mathcal{J}_{p\ell}(x) \\ -\mathcal{J}_{p\ell}^\top(x) & \mathcal{J}_{\ell\ell}(x) - \mathcal{R}_{\ell\ell}(x) \end{bmatrix} \begin{bmatrix} \nabla\Phi \\ \nabla\mathcal{H}_\ell \end{bmatrix} \\ &= \big[\mathbf{J}(x) - \mathbf{R}(x)\big]\nabla\mathbf{H}(x), \end{aligned}$$

where we have used the assumption $\nabla^\top\Phi(x_p)\mathcal{J}_{p\ell}(x) = 0$ in the third equality. It is obvious that the closed-loop dynamics in Proposition 2 is *exactly* the same with the one in Proposition 1. The matching PDE in Proposition 2 is thus solvable.

Second, we will verify the assumptions in Proposition 2. With the decomposition (19), we have

$$(9) \iff \begin{cases} \arg\min \mathcal{H}_1(x_0) = 0 \\ \arg\min \mathcal{H}_\ell(x_\ell) = x_\ell^*, \end{cases} \tag{21}$$

satisfying the convex properties of $\mathbf{H}_\ell(x_\ell)$ in Proposition 2. We then need to prove that there exists a point x_p^* such that

$$\nabla\mathbf{H}_p(x_p^*) = 0, \quad \nabla^2\mathbf{H}_p(x)\big|_{x\in B_\varepsilon(x_p^*)} > 0. \tag{22}$$

with $\mathbf{H}_p(x_p) = \Phi(x_p)$. To this end, we notice that $\mathcal{C}$ is diffeomorphic to the unit circle, and thus there exists a smooth mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(x_p) = |T(x_p)|^2 - 1$, with $\nabla T \neq 0$ and its inverse mapping $T^{-1}(\cdot)$ is well-defined. By fixing $x_p^* = T^{-1}(\mathbf{0})$, we then have

$$\begin{aligned} \nabla\mathbf{H}_p\Big|_{x=x_p^*} &= 2(\nabla T)^\top T(T^{-1}(\mathbf{0})) = 0 \\ \nabla^2\mathbf{H}_p\Big|_{x\in B_\varepsilon(x_p^*)} &= 2(\nabla T)^\top\nabla T + 2\sum_{i=1}^{2} T_i(x_p)\nabla^2 T_i > 0 \end{aligned}$$

for some small $\varepsilon > 0$, where in the latter inequality we have used the continuity of the mapping and the fact $T_i(x_p^*) = 0$. Thus we have verified the property of the Hamiltonian $\mathbf{H}(x)$ in Proposition 2.

To verify the condition (15), we have

$$\begin{aligned} \mathbf{R}_{pp}(x) = 0 \quad &\iff \quad \mathcal{H}_1'\mathcal{R}_{pp}(x) = 0 \\ &\iff \quad \mathcal{H}_1'(x_0)\Big|_{x_0=\Phi(x_p)} = 0 \\ &\iff \quad \Phi(x_p) = 0, \end{aligned}$$

where we have used the assumption $\mathcal{R}_{pp}(x) \neq \mathbf{0}$ in the second implication, and the fact $\mathcal{H}_1'|_{x_0=0} = 0$ in the last one. Therefore, Eq. (16) is satisfied. According to the property of $\mathcal{H}_1$, for sufficiently small $|x_0|$ we have $\mathcal{H}_1'(x_0) < 0$ for $x_0 < 0$ and $\mathcal{H}_1'(x_0) > 0$ if $x_0 > 0$. It yields

$$\mathbf{R}_{pp}(x)\Phi(x_p) = \mathcal{R}_{pp}(x)\big[\mathcal{H}_1'(\Phi(x_p))\Phi(x_p)\big] \geq 0.$$

Thus, the pumping-and-damping condition – inequality (15) – has been proved. The remaining assumptions *H3* and *H4* can be verified trivially. □

4.4. Remarks

The following remarks are in order specifically emphasizing the connections between the proposed methods and existing methods.

Remark 5. In the problem formulation, we impose the two-dimensional partition of x_p. It may be argued to be peculiar and stringent. We should underscore that, in many cases, orbital stabilization tasks can be translated into our case. We take the widely studied VHC method, though with a different mechanism from the proposed designs, for instance, and consider an Euler–Lagrange system with states $q \in \mathbb{R}^N$ and $\dot{q} \in \mathbb{R}^N$ the generalized coordinates and velocities. The simplified control task in VHC is to stabilize the invariant manifold

$$\mathcal{M} := \{(q, \dot{q}) \mid \bar{q} = \alpha(q_N)\}, \quad \bar{q} := \mathrm{col}(q_1, \ldots, q_{N-1}),$$

with some mapping $\alpha : \mathbb{R} \to \mathbb{R}^{N-1}$, and guarantee the zero dynamics to have non-trivial periodic solutions. It is clear that on the manifold we have $\dot{\bar{q}} = \eta(q_N, \dot{q}_N) := \nabla\alpha^\top(q_N)\dot{q}_N$. The above-mentioned task appropriately adapts to our problem formulation with $n = 2N$ and the change of coordinate $x_p = \mathrm{col}(q_N, \dot{q}_N)$, $x_\ell = \mathrm{col}(\bar{q} - \alpha(q_N), \dot{\bar{q}} - \eta(q_N, \dot{q}_N))$.

Remark 6. A main drawback of VHC and I&I orbital stabilization technique is that the steady-state behavior cannot be fixed a priori, but depends on the initial states with a notable exception (Mohammadi et al., 2018). The drawback can be circumvented with the IDA methods, but with an additional difficulty in solving PDEs.

Remark 7. In Aracil et al. (2005), Duindam and Stramigioli (2004) and Gomez-Estern, Barreiro, Aracil, and Gordillo (2006) a similar

[1] The notation $\mathcal{H}_1'$ represents the derivative of $\mathcal{H}_1(x_0)$ with respect to x_0. We also have $\mathcal{H}_1'(x_0) = \nabla_{x_0}\mathcal{H}_0(x_0, x_\ell)$ according to (19).

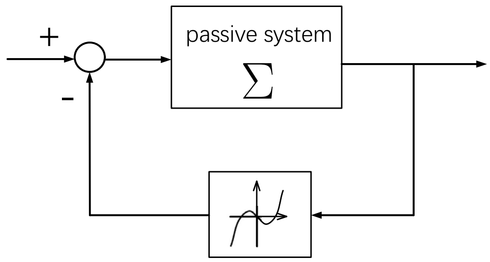

Fig. 3. Lure system with sign-indefinite feedback.

MSEA approach is adopted for some specific dynamical systems. In particular, in Duindam and Stramigioli (2004) the MSEA is imposed to the potential energy in a path following task for *fully actuated* mechanical systems.

Remark 8. In Astrom, Aracil, and Gordillo (2008) the pumping-and-damping injection is applied to *stabilize* pendula at the upright equilibrium almost globally. Some works on energy regulation of nonlinear systems, though not aiming at the generation of oscillation, can be found in Fradkov and Pogromsky (1998), Garofalo and Ott (2017) and Spong (1995).

Remark 9. In Stan and Sepulchre (2007) passive oscillators are constructed for Lure dynamical systems using "sign-indefinite" feedback static mappings, see Fig. 3. After assigning the linearized system with a unique pair of conjugated poles on the imaginary axis, the "sign-indefinite" feedback is adopted to regulate the energy and, whence, achieve the periodic oscillations. Indeed, Stan and Sepulchre (2007, Theorem 2) can be regarded as an EPD controller. It should be underscored that the center manifold theory is applied in the analysis where the center manifold plays the exactly same role as the invariant manifold in VHC. Since the analysis of the latter is carried out applying the center manifold theory – whose nature is intrinsically local – the oscillators resulting from Stan and Sepulchre (2007, Theorem 2) are assumed to have small amplitudes. On the other hand, they circumvent the daunting task of solving PDEs. Whereas, the proposed EPD method has the ability to shape behaviors of the closed-loop dynamics, making them more useful for engineering practice.

Remark 10. The assumptions in Proposition 4 on the equivalence between two proposed methods are relatively mild, namely, the diagonalization of $\mathcal{R}(x)$ and the decomposition of $\mathcal{H}_0(x)$. Despite the equivalence, the realms of applicability of the methods are slightly different. For instance, if a controlled plant endows a pH form, it may be easier to generate oscillations via EPD without solving PDEs; on the other hand, for some systems it is simple to shape the Hamiltonian, *e.g.*, fully-actuated mechanical systems, thus the MSEA method is preferred.

5. Induction motor example

5.1. Dynamic model and control objective

We consider the practical example of speed regulation of current-fed IMs. The normalized dynamics of the IM in the fixed frame is described by

$$\begin{aligned}\dot{\psi}_r &= -R\psi_r + \omega\mathbb{J}\psi_r + Ru \\ \dot{\omega} &= u^\top \mathbb{J}\psi_r - \tau_L, \quad \mathbb{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\end{aligned} \tag{23}$$

where $\psi_r \in \mathbb{R}^2$ is the rotor flux, $\omega \in \mathbb{R}$ is the rotor angular speed, $\tau_L \in \mathbb{R}$ is the constant load torque, $R > 0$ is the rotor resistance, $u \in \mathbb{R}^2$ is the stator current, which is assumed to be the control input and, without loss of generality, we have taken the rotor inertia to be equal to one—see Marino, Tomei, and Verrelli (2010) and Ortega, Loria, Nicklasson, and Sira-Ramirez (1998) for further details. To show the basic idea, we make the assumption that $\tau_L = 0$, which can be removed adding an integral term to the control action (Marino et al., 2010; Ortega et al., 1998).

The control objective is to ensure the asymptotic orbital stabilization of the set

$$\mathcal{A} := \{x \in \mathbb{R}^3 | \ \Phi(x_p) = 0, \ x_\ell = \omega_\star\}, \tag{24}$$

where we introduced the notation $x_p := \psi_r$, $x_\ell := \omega$, and defined the function

$$\Phi(x_p) := |x_p| - \beta_\star, \tag{25}$$

with $\beta_\star > 0$ and $\omega_\star \neq 0$ the desired (constant) references.

5.2. Orbital stabilization of the IM via MSEA-PBC

In the following proposition we show that the aforementioned regulation problem of IMs can be solved via MSEA-PBC.

Proposition 4. *Consider the fixed-frame current-fed IM model (23) and the target set (24), (25).*

P1 *Assumption 1 is satisfied with the choices*

$$\mathcal{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & \frac{k}{\beta_\star}|x_p| \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & -\frac{x_\ell |x_p|}{|x_p| - \beta_\star} & \frac{kR}{\beta_\star}\frac{x_2}{|x_p|} \\ * & 0 & -\frac{kR}{\beta_\star}\frac{x_1}{|x_p|} \\ * & * & 0 \end{bmatrix} \tag{26}$$

and

$$H_0(x_0, x_\ell) = \frac{1}{2}x_0^2 + \frac{1}{2}(x_\ell - \omega_\star)^2. \tag{27}$$

P2 *The controller (5) takes the form*

$$u = \left[\beta_\star I_2 - \frac{k}{\beta_\star}(x_\ell - \omega_\star)\mathbb{J}\right]\frac{x_p}{|x_p|}, \ k > 0. \tag{28}$$

P3 *All the assumptions of Proposition 1 are satisfied.*

Consequently, the closed-loop system is asymptotically orbitally stable with respect to (24). Moreover, the convergence is exponential.

Proof. The fact that Assumption 1 is satisfied with (26)–(28) is easily verified. Assumption *H2* is also satisfied, with $c(x) = -x_\ell |x_p|$, which evaluated in $\mathcal{A}$ yields $-\beta_\star\omega_\star \neq 0$. The largest invariant set in $\{x \in \mathbb{R}^3 | \nabla^\top H(x)\mathcal{R}(x)\nabla H(x) = 0\}$ is $\mathcal{A} \cup \{\mathbf{0}\}$. Some basic Lyapunov analysis shows that the origin is an unstable equilibrium. According to Proposition 1, we conclude that the closed-loop system is almost globally asymptotically orbitally stable.

To establish the exponential orbital stability claim we refer to Hauser and Chung (1994), where it is shown to be equivalent to prove that the transverse coordinate $z := \mathrm{col}(\Phi(x_p), x_\ell - \omega_\star)$ exponentially converges to $(0, 0)$. The proof of Proposition 1 shows that we can always find some invariant compact sets containing $\mathcal{A}$. In these compact sets, $|x_p| \geq c_1$ for some $c_1 > 0$. Thus in the neighborhood of $\mathcal{A}$, we have

$$\dot{z}_2 = -\frac{k}{\beta_\star}|x_p| z_2,$$

then yielding the exponential convergence of z_2 to zero. Now we have

$$\dot{z}_1 = -R|\nabla \Phi|^2 z_1 + \epsilon_t,$$

where ϵ_t is an exponentially decaying term caused by $z_2(0)$. The Jordan curve $\Phi(x) = 0$ implies $\nabla\Phi \neq 0$ in the neighborhood of $\mathcal{A}$, from which we conclude the exponential stability of the transverse coordinate z. □

Remark 11. It can also be shown that the closed-loop system takes the pH form (11) with

$$\mathbf{R}(x) = \begin{bmatrix} R(|x_p| - \beta_\star) & 0 & 0 \\ 0 & R(|x_p| - \beta_\star) & 0 \\ 0 & 0 & \frac{k}{\beta_\star}|x_p| \end{bmatrix}$$

$$\mathbf{J}(x) = \begin{bmatrix} 0 & -\omega|x_p| & \frac{kR}{\beta_\star}\frac{x_2}{|x_p|} \\ * & 0 & -\frac{kR}{\beta_\star}\frac{x_1}{|x_p|} \\ * & * & 0 \end{bmatrix} \tag{29}$$

$$\mathbf{H}(x) = \frac{1}{2}|x_p|^2 + \frac{1}{2}(\omega - \omega_\star)^2, \quad \mathbf{H}_p^\star = \frac{1}{2}\beta_\star^2.$$

and satisfies all the assumptions of Proposition 2. Hence, the IM can be orbitally stabilized with the EPD-PBC also.

5.3. FOC of the IM is an MSEA-PBC

In this subsection we prove that the MSEA controller of Proposition 4 exactly coincides with the industry standard direct FOC first proposed in Blaschke (1972)—see also Marino et al. (2010, Chapter 2.2) and Ortega et al. (1998, Chapter 11.2.1).

Corollary 1. *The MSEA controller (28) of Proposition 4 yields, after a state and input change of coordinates, the classical direct FOC.*

Proof. To prove that (28) coincides – modulo a coordinate change – with the direct FOC we introduce the change of coordinates

$$\psi_r := e^{\mathbb{J}\theta}\lambda, \quad u := e^{\mathbb{J}\theta}v, \quad \dot{\theta} = \omega, \tag{30}$$

that, applied to (23) (with $\tau_L = 0$), yields the well-known current-fed IM dynamics in the rotating frame

$$\dot{\lambda} = -R\lambda + Rv, \quad \dot{\omega} = v^\top \mathbb{J}\lambda. \tag{31}$$

Now, we write (31) in polar coordinates (β, ρ) as

$$\dot{\beta} = -R\beta + Ri_d, \ \dot{\rho} = \frac{R}{\beta}i_q, \ \dot{\omega} = \beta i_q$$

where we have defined

$$\lambda := \beta\begin{bmatrix}\cos\rho \\ \sin\rho\end{bmatrix}, \ \begin{bmatrix} i_d \\ i_q\end{bmatrix} := e^{-\mathbb{J}\rho}v.$$

It is easy to see from the equations above that the objective $\beta(t) \to \beta_\star$, $\omega(t) \to \omega_\star$, which is equivalent to the asymptotic stabilization of the set[2] $\mathcal{A}$, is achieved with the simple control

$$v = e^{\mathbb{J}\rho}\begin{bmatrix} \beta_\star \\ \frac{k}{\beta_\star}(\omega_\star - \omega)\end{bmatrix}, \tag{32}$$

with $k > 0$. This is the famous direct FOC for induction motors. It is a simple exercise to show that (28) is obtained applying to (32) the change of coordinates (30). □

Remark 12. It is interesting to note that, expressed in the rotating coordinates, the direct FOC does not generate a periodic orbit, but only ensures set stability.

[2] Notice that $|\psi_r| = |\lambda| = \beta$.

Remark 13. The application of the main idea of FOC of IMs for smooth regulation of Brockett's nonholonomic integrator was first reported in Escobar, Ortega, and Reyhanoglu (1998), and later adopted in Dixon, Dawson, Zergeroglu, and Zhang (2000) and Morin and Samson (2003) for control of nonholonomic systems.

6. Pendulum example

6.1. Local design

We consider a benchmark in nonlinear control—the planar inverted pendulum, which is related to various applications, *e.g.*, the attitude control of space boosters and walking robots. The normalized model given by Astrom et al. (2008)

$$\dot{\theta} = \omega, \quad \dot{\omega} = \sin\theta - u\cos\theta, \tag{33}$$

where $\theta \in \mathbb{S}$ and $\omega \in \mathbb{R}$ denote the angular position and velocity, and the input u is the acceleration of the pivot. In this representation, the angles 0 and π correspond to the upright and downright positions, respectively.

We are interested in asymptotically stabilizing the pendulum oscillating around its upright equilibrium. We define $x = x_p := \mathrm{col}(\theta, \omega)$ in the absence of the x_ℓ partition. The first design is a local result as follows.

Proposition 5. *Consider the model (33) in closed-loop with the control law*

$$u = 2\sin\theta + \omega P(\theta, \omega)\cos\theta \tag{34}$$

with $\frac{1}{\gamma}P(\theta,\omega) = -(\cos\theta - \frac{1}{2})^2 + \frac{1}{2}\omega^2 - H_p^*$, *where* $H_p^* := -(\cos\theta_* - \frac{1}{2})^2$, $\gamma > 0$, *and* $\theta_* \in (-\frac{\pi}{3}, \frac{\pi}{3})$, *the system is* locally *asymptotically orbitally stable. Furthermore, the angle* θ *ultimately oscillates between* $[-\theta_*, \theta_*]$.

Proof of Proposition 5. The proof is available in the full version (arXiv:1903.04070). □

We underscore that the level set $\Phi(x) = 0$ containing two disconnected parts. Hence, the restriction $|x_1| < \frac{1}{3}\pi$ is indispensable. We give the simulation results in Fig. 4 with initial values $(0.1\pi, 0)$ and $(0.3\pi, 0)$, where $\gamma = 5$ and $H_{p*} = -0.0429$. In this figure we show the evaluation of the (2, 2)-element of $\mathcal{R}(x)$, illustrating the pumping-and-damping mechanism.

6.2. Almost global design

The following proposition is an almost global design.

Proposition 6. *Considering Proposition 5, if we select*

$$P(\theta, \omega) = (\frac{3}{2}\cos\theta + \frac{1}{2}\omega^2 - \frac{3}{4})Q(\theta,\omega)$$

and

$$Q(\theta, \omega) := \begin{cases} \gamma_1(H_p(\theta,\omega) - H_p^*), & \theta \in (-\frac{\pi}{3}, \frac{\pi}{3}) \\ \gamma_2, & \theta \in [-\pi, \frac{\pi}{3}] \cup [\frac{\pi}{3}, \pi) \end{cases}$$

then there exist $\gamma_1, \gamma_2 > 0$ *such that the state asymptotically converges to the orbit* $\mathcal{A} \cap \{x \in \mathbb{R}^2 | |x_1| < \frac{\pi}{3}\}$ *or the saddles* $(\frac{\pi}{3}, 0)$, *almost globally on* $\mathbb{S} \times \mathbb{R}$.

Proof. Same in Proof of Proposition 5. □

Fig. 5 gives the simulation results to illustrate Proposition 6 with the $x(0) = (\pi, 0.01)$, $\gamma_1 = 20$, $\gamma_2 = 2, \theta_* = \frac{\pi}{4}$ and $H_{p*} = -0.0429$. The figure illustrates the almost global property with the angle ultimately oscillating between $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

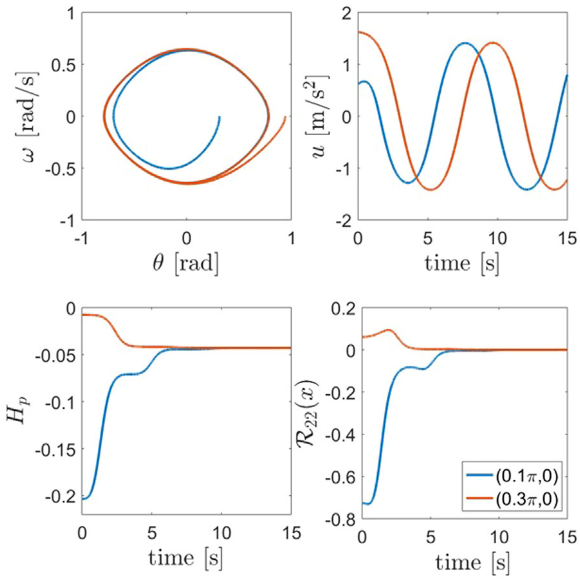

Fig. 4. The dynamics behavior in Proposition 5.

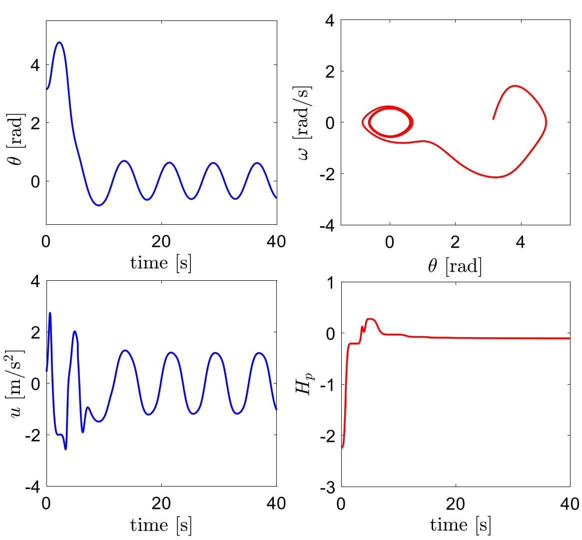

Fig. 5. The dynamics behavior in Proposition 6.

7. Concluding remarks

It has been shown that the IDA-PBC design methodology can be adapted to address the problem of orbital stabilization of nonlinear systems. We propose two different, but related, IDA-PBC designs: MSEA and EPD—whose application, as usual in IDA, requires the solution of a PDE. In the former, the closed-loop Hamiltonian function is shaped to have minima at the desired orbit. For the latter, we regulate the energy to a desired value using a pumping-and-damping dissipation matrix. To ensure asymptotic orbital stability, and not just set attractivity, some constraints are imposed on the interconnection matrix. We then establish connections between the above-mentioned methods.

Currently, research is carried out in the following directions.

- The problem of path following – in a time parameterization-free manner – for mechanical systems. It has been observed that this is closely related to the orbital stabilization problem studied in this paper.
- The connection between the proposed method and the indirect version of FOC is still an open, and interesting, topic.
- Application of the proposed methods to solve some periodic motion control problems in mechanical and power electronic systems, *e.g.* in walking robots and AC (or resonant) power converters.

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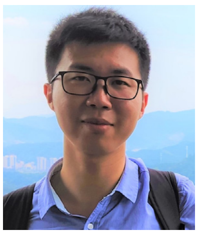

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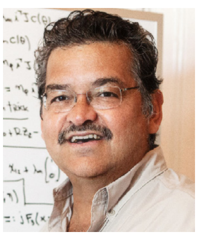

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