WALTER RUDIN
Professor of Mathematics
University of Wisconsin-Madison

Principles of Mathematical Analysis

THIRD EDITION

TEX typesetting by Ali Darijani

January 30, 2023

Darij's Publishing House

Contents

PREFACE

Preface

This book is intended to serve as a text for the course in analysis that is usually taken by advanced undergraduates or by first-year students who study mathematics.

The present edition covers essentially the same topics as the second one, with some additions, a few minor omissions, and considerable rearrangement. I hope that these changes will make the material more accessible and more attractive to the students who take such a course.

Experience has convinced me that it is pedagogically unsound (though logically correct) to start off with the construction of the real numbers from the rational ones. At the beginning, most students simply fail to appreciate the need for doing this. Accordingly, the real number system is introduced as an ordered field with the least-upper-bound property, and a few interesting applications of this property are quickly made. However, Dedekind's construction is not omitted. It is now in an Appendix to Chapter 1, where it may be studied and enjoyed whenever the time seems ripe.

The material on functions of several variables is almost completely rewritten, with many details filled in, and with more examples and more motivation. The proof of the inverse function theorem–the key item in Chapter 9–is simplified by means of the fixed point theorem about contraction mappings. Differential forms are discussed in much greater detail. Several applications of Stokes' theorem are included.

As regards other changes, the chapter on the Riemann-Stieltjes integral has been trimmed a bit, a short do-it-yourself section on the gamma function has been added to Chapter 8, and there is a large number of new exercises, most of them with fairly detailed hints.

I have also included several references to articles appearing in the *American Mathe1natical Monthly* and in *Mathematics Magazine*, in the hope that students will develop the habit of looking into the journal literature. Most of these references were kindly supplied by R. B. Burckel.

Over the years, many people, students as well as teachers, have sent me corrections, criticism, and other comments concerning the previous editions of this book. I have appreciated these, and I take this opportunity to express my sincere thanks to all who have written me.

WALTER RUDIN

DEDICATED TO THE MEMORIES OF

A. RAJCHMAN AND J. MARCINKIEWIC

MY TEACHER AND MY PUPIL

Chapter 1
THE REAL AND COMPLEX NUMBER SYSTEMS

1.1 INTRODUCTION

A satisfactory discussion of the main concepts of analysis (such as convergence, continuity, differentiation, and integration) must be based on an accurately defined number concept. We shall not, however, enter into any discussion of the axioms that govern the arithmetic of the integers, but assume familiarity with the rational numbers (i.e., the numbers of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$).
The rational number system is inadequate for many purposes, both as a field and as an ordered set. (These terms will be defined in Secs. **??** and **??**.) For instance, there is no rational p such that $p^2 = 2$. (We shall prove this presently.) This leads to the introduction of so-called "irrational numbers" which are often written as infinite decimal expansions and are considered to be "approximated" by the corresponding finite decimals. Thus the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

"tends to $\sqrt{2}$." But unless the irrational number $\sqrt{2}$ has been clearly defined, the question must arise: Just what is it that this sequence "tends to"?

This sort of question can be answered as soon as the so-called "real number system" is constructed.

Example

We now show that the equation

$$p^2 = 2 \tag{1.1}$$

is not satisfied by any rational p. If there were such a p, we could write $p = \frac{m}{n}$ where m and n are integers that are not both even. Let us assume this is done. Then 1.1 implies

$$m^2 = 2n^2, \tag{1.2}$$

This shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of 1.2 is divisible by 4, so that n^2 is even, which implies that n is even.

[1]

Remark

Definition

Definition

1.2 ORDERED SETS

Example

Example

Example

Example

Example

Example

Example

1.3 FIELDS

Example

Example

Example

Example

Example

Example

1.4 THE REAL FIELD

Example

Example

Example

Example

THE EXTENDED REAL NUMBER SYSTEM

```
brew install
    fgjn
```

1.9 Section Heading

Use the template *chapter.tex* together with the document class SVMono (monograph-type books) or SVMult (edited books) to style the various elements of your chapter content conformable to the Springer Nature layout.

1.10 Section Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Use the standard `equation` environment to typeset your equations, e.g.

$$a \times b = c \,, \tag{1.3}$$

however, for multiline equations we recommend to use the `eqnarray` environment[1].

$$\left|\nabla U_\alpha^\mu(y)\right| \le \frac{1}{d-\alpha} \int \left| \nabla \frac{1}{|\xi - y|^{d-\alpha}} \right| d\mu(\xi) = \int \frac{1}{|\xi - y|^{d-\alpha+1}} \, d\mu(\xi) \tag{1.4}$$

$$= (d-\alpha+1) \int\limits_{d(y)}^{\infty} \frac{\mu(B(y,r))}{r^{d-\alpha+2}} \, dr \le (d-\alpha+1) \int\limits_{d(y)}^{\infty} \frac{r^{d-\alpha}}{r^{d-\alpha+2}} \, dr \tag{1.5}$$

1.10.1 Subsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Further on please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

> Please do not use quotation marks when quoting texts! Simply use the `quotation` environment – it will automatically be rendered in the preferred layout.

1.10.1.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.1, see also Fig. 1.1[2]

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Paragraph Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

---

[1] In physics texts please activate the class option `vecphys` to depict your vectors in ***boldface-italic*** type - as is customary for a wide range of physical subjects.

[2] If you copy text passages, figures, or tables from other works, you must obtain *permission* from the copyright holder (usually the original publisher). Please enclose the signed permission with the manucript. The sources must be acknowledged either in the captions, as footnotes or in a separate section of the book.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

For typesetting numbered lists we recommend to use the `enumerate` environment – it will automatically render Springer's preferred layout.

1. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   a. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   b. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
2. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Subparagraph Heading

In order to avoid simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10, see also Fig. 1.2.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

For unnumbered list we recommend to use the `itemize` environment – it will automatically render Springer's preferred layout.

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development, cf. Table 1.1.
  – Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
  – Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

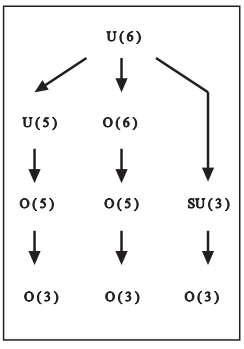

Fig. 1.1 If the width of the figure is less than 7.8 cm use the `sidecapion` command to flush the caption on the left side of the page. If the figure is positioned at the top of the page, align the sidecaption with the top of the figure – to achieve this you simply need to use the optional argument `[t]` with the `sidecaption` command

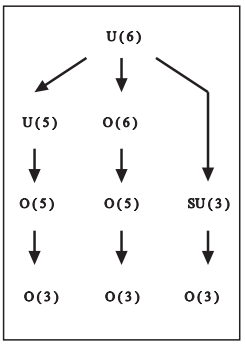

Fig. 1.2 Please write your figure caption here

Table 1.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Run-in Heading Boldface Version Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

Run-in Heading Boldface and Italic Version Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

Run-in Heading Displayed Version
Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

1.11 Section Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

If you want to list definitions or the like we recommend to use the Springer-enhanced `description` environment – it will automatically render Springer's preferred layout.

Type 1 That addresses central themes pertainng to migration, health, and disease. In Sect. 1.9, Wilson discusses the role of human migration in infectious disease distributions and patterns.

Type 2 That addresses central themes pertainng to migration, health, and disease. In Sect. 1.10.1, Wilson discusses the role of human migration in infectious disease distributions and patterns.

1.11.1 Subsection Heading

In order to avoid simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Use the LaTeX automatism for all your cross-references and citations citations as has already been described in Sect. 1.10.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

If you want to emphasize complete paragraphs of texts we recommend to use the newly defined Springer class option `graybox` and the newly defined environment `svgraybox`. This will produce a 15 percent screened box 'behind' your text.

If you want to emphasize complete paragraphs of texts we recommend to use the newly defined Springer class option and environment `svgraybox`. This will produce a 15 percent screened box 'behind' your text.

1.11.1.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Theorem 1.1 *Theorem text goes here.*

Definition 1.1 Definition text goes here.

Proof Proof text goes here. □

Paragraph Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.10.

Note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Theorem 1.2 *Theorem text goes here.*

Definition 1.2 Definition text goes here.

Proof Proof text goes here. □

Trailer Head

If you want to emphasize complete paragraphs of texts in an `Trailer Head` we recommend to use

```
\begin{trailer}{Trailer Head}
...
\end{trailer}
```

? Questions

If you want to emphasize complete paragraphs of texts in an `Questions` we recommend to use

```
\begin{question}{Questions}
...
\end{question}
```

> Important

If you want to emphasize complete paragraphs of texts in an `Important` we recommend to use

```
\begin{important}{Important}
...
\end{important}
```

Attention

If you want to emphasize complete paragraphs of texts in an `Attention` we recommend to use

```
\begin{warning}{Attention}
...
\end{warning}
```

Program Code

If you want to emphasize complete paragraphs of texts in an `Program Code` we recommend to use

```
\begin{programcode}{Program Code}
\begin{verbatim}...\end{verbatim}
\end{programcode}
```

Tips

If you want to emphasize complete paragraphs of texts in an `Tips` we recommend to use

```
\begin{tips}{Tips}
...
\end{tips}
```

Overview

If you want to emphasize complete paragraphs of texts in an `Overview` we recommend to use

```
\begin{overview}{Overview}
...
\end{overview}
```

Background Information

If you want to emphasize complete paragraphs of texts in an `Background Information` we recommend to use

```
\begin{backgroundinformation}{Background Information}
...
\end{backgroundinformation}
```

Legal Text

If you want to emphasize complete paragraphs of texts in an `Legal Text` we recommend to use

```
\begin{legaltext}{Legal Text}
...
\end{legaltext}
```

Acknowledgements If you want to include acknowledgments of assistance and the like at the end of an individual chapter please use the `acknowledgement` environment – it will automatically render Springer's preferred layout.

Appendix

When placed at the end of a chapter or contribution (as opposed to at the end of the book), the numbering of tables, figures, and equations in the appendix section continues on from that in the main text. Hence please *do not* use the `appendix` command when writing an appendix at the end of your chapter or contribution. If there is only one the appendix is designated "Appendix", or "Appendix 1", or "Appendix 2", etc. if there is more than one.

$$a \times b = c \tag{1.6}$$

Problems

1.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

1.2 Problem Heading
(a) The first part of the problem is described here.
(b) The second part of the problem is described here.

Appendix A
Chapter Heading

All's well that ends well

Use the template *appendix.tex* together with the Springer document class SVMono (monograph-type books) or SVMult (edited books) to style appendix of your book.

A.1 Section Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations.

A.1.1 Subsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. A.1.

For multiline equations we recommend to use the `eqnarray` environment.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{c} \\ \mathbf{a} \times \mathbf{b} &= \mathbf{c} \end{aligned} \tag{A.1}$$

A.1.1.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. A.1.1.

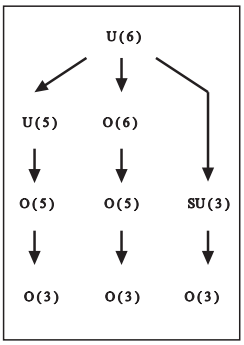

Fig. A.1 Please write your figure caption here

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Table A.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

FURTHER READING

References

1. ARTIN, E.: "The Gamma Function," Holt, Rinehart and Winston, Inc., New York, 1964.
2. BOAS, R. P.: "A Primer of Real Functions," Carus Mathematical Monograph No. 13, John Wiley & Sons, Inc., New York, 1960.
3. BUCK, R. C. (ed.): "Studies in Modern Analysis," Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.
4. BUCK, R. C. (ed.): "Advanced Calculus" 2d ed., McGraw-Hill Book Company, New York, 1965.
5. BURKILL, J. C.: "The Lebesgue Integral," Cambridge University Press, New York, 1951.
6. DIEUDONNE, J.: "Foundations of Modern Analysis," Academic Press, Inc., New York, 1960.
7. FLEMING, w. H.: "Functions of Several Variables," Addison-Wesley Publishing Company, Inc., Reading, Mass., 1965.
8. GRAVES, L. M.: "The Theory of Functions of Real Variables," 2d ed., McGraw-Hill Book Company, New York, 1956.
9. HALMOS, P. R.: "Measure Theory," D. Van Nostrand Company, Inc., Princeton, N.J., 1950.
10. HALMOS, P. R.: "Finite-dimensional Vector Spaces," 2d ed., D. Van Nostrand Company, Inc., Princeton, N.J., 1958.
11. HARDY, G. H.: "Pure Mathematics," 9th ed., Cambridge University Press, New York, 1947.
12. HARDY, G. H. and ROGOSINSKI, W.: "Fourier Series," 2d ed., Cambridge University Press, New York, 1950.
13. HERSTEIN, I. N.: "Topics in Algebra," Blaisdell Publishing Company, New York, 1964.
14. HEWITT, E., and STROMBERG, K. : "Real and Abstract Analysis," Springer Publishing Co., Inc., New York, 1965.
15. KELLOGG, O. D.: "Foundations of Potential Theory," Frederick Ungar Publishing Co., New York, 1940.
16. KNOPP, K.: "Theory and Application of Infinite Series," Blackie & Son, Ltd., Glasgow, 1928.
17. LANDAU, E. G. H.: "Foundations of Analysis," Chelsea Publishing Company, New York, 1951.
18. MCSHANE, E. J.: "Integration," Princeton University Press, Princeton, N.J., 1944.
19. NIVEN, I. M.: "Irrational Numbers," Carus Mathematical Monograph No. 11, John Wiley & Sons, Inc., New York, 1956.
20. ROYDEN, H. L.: "Real Analysis," The Macmillan Company, New York, 1963.
21. RUDIN, W.: "Real and Complex Analysis," 2d ed., McGraw-Hill Book Company, New York, 1974.
22. SIMMONS, G. F.: "Topology and Modern Analysis," McGraw-Hill Book Company, New York, 1963.

23. SINGER, 1. M., and THORPE, J. A.: "Lecture Notes on Elementary Topology and Geometry," Scott, Foresman and Company, Glenview, Ill., 1967.
24. SMITH, K. T.: "Primer of Modern Analysis" Bogden and Quigley, Tarrytown-on-Hudson, N.Y., 1971.
25. SPIVAK, M.: "Calculus on Manifolds," W. A. Benjamin, Inc., New York, 1965.
26. THURSTON, H. A.: "The Number System," Blackie & Son, Ltd., London-Glasgow, 1956.

Acronyms and Abbreviations

Here you can see a list of important acronyms.

ANSI	American National Standards Institute
ASCII	American Standard Code for Information Interchange
CPU	Central Processing Unit
CUDA	Compute Unified Device Architecture
DRAM	Dynamic Random Access Memory
GNU	GNU's Not Unix
GPU	Graphics Processing Unit
grep	g lobal(ly) search r egular e xpression p rint
NVRAM	Non-Volatile Random Access Memory
pip	Pip Installs Packages
RAM	Random Access Memory
SDRAM	Static Random Access Memory
TPU	Tensor Processing Unit

Glossary

Use the template *glossary.tex* together with the Springer document class SVMono (monograph-type books) or SVMult (edited books) to style your glossary in the Springer layout.

GNU GNU is not UNIX

glossary term Write here the description of the glossary term. Write here the description of the glossary term. Write here the description of the glossary term.

glossary term Write here the description of the glossary term. Write here the description of the glossary term. Write here the description of the glossary term.

abs datenwise

glossary term Write here the description of the glossary term. Write here the description of the glossary term. Write here the description of the glossary term.

Solutions

Problems of Chapter 1

1.1 The solution is revealed here.

1.2 Problem Heading
(a) The solution of first part is revealed here.
(b) The solution of second part is revealed here.

Index