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and
DIEGO PALLARA

Functionas of Bounded Variation and Free Discontinuity Problems

TEX typesetting by Ali Darijani

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Contents

Preface

Functions of bounded variation (**BV** functions in the sequel) have had an important role in several classical problems of the calculus of variations, for instance in the theory of graphs with minimal area. More recently, this class of functions has been the natural tool to study several problems characterized by the appearance of discontinuity hypersurfaces; examples come from image segmentation theory and fracture mechanics. The analysis of these problems require a knowledge of some of the basic concepts of geometric measure theory, such as Hausdorff measures and rectifiable sets.

One of the motivations which led us to write this book is the desire to provide a systematic and self-contained presentation of the theory of functions of bounded variation and, at the same time, an elementary introduction to geometric measure theory. In fact, after the classical treatises of V. G. Mazj'a [Maz85], A. I. Vol'pert and S. I. Hudjaev [VK85] and H. Federer [Fed96] (in the latter **BV** functions are presented in the more general framework of the currents), some aspects of the theory of **BV** functions have been treated in the monographs of E. Giusti [Giu84], U. Massari and M.Miranda [**?**], W. Ziemer [**?**], L.C. Evans and R. F. Gariepy [**?**], M.Giaquinta, G. Modica and J. Soucek [**?**], but the analysis of fine properties of **BV** functions and the development of general variational problems in **BV** is not the central goal of any of these monographs. The first half of our book is, instead, explicitly devoted to the theory of **BV** functions, from classical results up to the developments of the last ten years.

Our starting point is, in Chapter 1, abstract measure theory. We assume the reader has an elementary knowledge of the subject, and we emphasize some aspects perhaps less widely known, but fundamental for the development of the book, such as weak convergence in spaces of measures, outer measures and Caratheodory construction. In the second chapter we introduce all the basic ingredients of geometric measure theory, such as Hausdorff measures $\mathcal{H}^k$, covering theorems, rectifiable sets, area and coarea formulae, Minkowski content. Moreover, the chapter contains a brief treatment of Young measures and of the continuity and semicontinuity properties of functionals defined on measures. The aim is to give a quite general presentation, without restricting e.g. to the case of hypersurfaces, which is the only one relevant

for the development of the **BV** theory. In our treatment of geometric measure theory a fundamental role is played by Lipschitz functions: indeed, these functions are more flexible than C^1 functions with respect, for instance, to truncation and extension and, by the classical Rademacher theorem, they are almost everywhere differentiable. Hence, as shown by H. Federer in [**?**], the canonical liberalization techniques can be adapted to this context. In particular , we develop the whole theory without using the link between Lipschitz and C^1 functions provided by the Whitney extension theorem. Another feature of the chapter and of the subsequent one is the emphasis on the so-called blow-up technique, which is used both for the study of the local properties of rectifiable sets and for the fine theory of **BV** functions. In this respect, a unifying concept is that if tangent measure, introduced (adapting with minor variants the original definition of D. Preiss in [**?**]) in Section **??**

$$J(\Gamma, u) := \int_{R\setminus\Gamma} \|\nabla u\|^2 + \alpha\|u-g\|^2 dx + \beta\mathcal{H}^{N-1}(R\cap\Gamma) \tag{0.1}$$

Pisa L. A.
Florence N. F.
Lecce D. P.
June 1999

We dedicate this book to Ennio De Giorgi, who generously
shared with us his deep insight on this subject
and much more

Part I
(VOLUME I)

Chapter 1
TRIGONOMETRIC SERIES AND FOURIER SERIES. AUXILIARY RESULTS

1.1 Trigonometric series

These are series of the form

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x). \tag{1.1}$$

Here x is a real variable and *coefficients* $a_0, a_1, b_1, \ldots$ are independent of x. We may usually suppose, if we wish , that the coefficients are real; when they are complex.

basic usage

```
brew install
    fgjn
```

1.2 Section Heading

Use the template *chapter.tex* together with the document class SVMono (monograph-type books) or SVMult (edited books) to style the various elements of your chapter content conformable to the Springer Nature layout.

1.3 Section Heading

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Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Use the standard `equation` environment to typeset your equations, e.g.

$$a \times b = c \,, \tag{1.2}$$

however, for multiline equations we recommend to use the `eqnarray` environment[1].

$$\left|\nabla U_\alpha^\mu(y)\right| \le \frac{1}{d-\alpha} \int \left|\nabla \frac{1}{|\xi - y|^{d-\alpha}}\right| d\mu(\xi) = \int \frac{1}{|\xi - y|^{d-\alpha+1}}\, d\mu(\xi) \tag{1.3}$$

$$= (d-\alpha+1) \int_{d(y)}^{\infty} \frac{\mu(B(y,r))}{r^{d-\alpha+2}}\, dr \le (d-\alpha+1) \int_{d(y)}^{\infty} \frac{r^{d-\alpha}}{r^{d-\alpha+2}}\, dr \tag{1.4}$$

1.3.1 Subsection Heading

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1.3.1.1 Subsubsection Heading

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Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Paragraph Heading

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[1] In physics texts please activate the class option `vecphys` to depict your vectors in ***boldface-italic*** type - as is customary for a wide range of physical subjects.

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1. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   a. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   b. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
2. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Subparagraph Heading

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- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development, cf. Table 1.1.
  – Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
  – Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

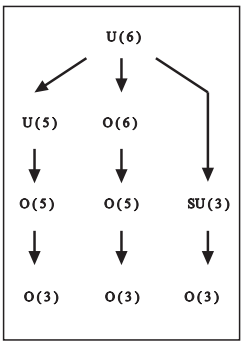

Fig. 1.1 If the width of the figure is less than 7.8 cm use the `sidecapion` command to flush the caption on the left side of the page. If the figure is positioned at the top of the page, align the sidecaption with the top of the figure – to achieve this you simply need to use the optional argument `[t]` with the `sidecaption` command

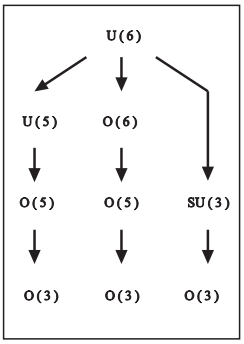

Fig. 1.2 Please write your figure caption here

Table 1.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Run-in Heading Boldface Version Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

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1.4 Section Heading

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Type 1 That addresses central themes pertainng to migration, health, and disease. In Sect. 1.2, Wilson discusses the role of human migration in infectious disease distributions and patterns.

Type 2 That addresses central themes pertainng to migration, health, and disease. In Sect. 1.3.1, Wilson discusses the role of human migration in infectious disease distributions and patterns.

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If you want to emphasize complete paragraphs of texts we recommend to use the newly defined Springer class option and environment `svgraybox`. This will produce a 15 percent screened box 'behind' your text.

1.4.1.1 Subsubsection Heading

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Theorem 1.1 *Theorem text goes here.*

Definition 1.1 Definition text goes here.

Proof Proof text goes here. □

Paragraph Heading

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Theorem 1.2 *Theorem text goes here.*

Definition 1.2 Definition text goes here.

Proof Proof text goes here. □

Trailer Head

If you want to emphasize complete paragraphs of texts in an `Trailer Head` we recommend to use

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\begin{trailer}{Trailer Head}
...
\end{trailer}
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\begin{question}{Questions}
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\end{question}
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\end{important}
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\end{warning}
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\begin{verbatim}...\end{verbatim}
\end{programcode}
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...
\end{legaltext}
```

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Appendix

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$$a \times b = c \tag{1.5}$$

Problems

1.1 A given problem or Excercise is described here. The problem is described here. The problem is described here.

1.2 Problem Heading
(a) The first part of the problem is described here.
(b) The second part of the problem is described here.

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Part II
(VOLUME II)

Appendix A
Chapter Heading

All's well that ends well

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A.1 Section Heading

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For multiline equations we recommend to use the `eqnarray` environment.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{c} \\ \mathbf{a} \times \mathbf{b} &= \mathbf{c} \end{aligned} \tag{A.1}$$

A.1.1.1 Subsubsection Heading

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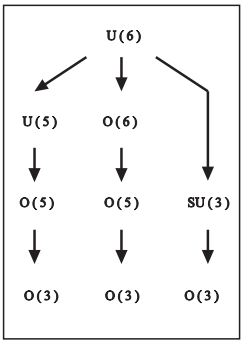

Fig. A.1 Please write your figure caption here

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Table A.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

Acronyms and Abbreviations

Here you can see a list of important acronyms.

ANSI	American National Standards Institute
ASCII	American Standard Code for Information Interchange
CPU	Central Processing Unit
CUDA	Compute Unified Device Architecture
DRAM	Dynamic Random Access Memory
GNU	GNU's Not Unix
GPU	Graphics Processing Unit
grep	g lobal(ly) search r egular e xpression p rint
NVRAM	Non-Volatile Random Access Memory
pip	Pip Installs Packages
RAM	Random Access Memory
SDRAM	Static Random Access Memory
TPU	Tensor Processing Unit

Glossary

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Solutions

Problems of Chapter 1

1.1 The solution is revealed here.

1.2 Problem Heading
(a) The solution of first part is revealed here.
(b) The solution of second part is revealed here.

Index