BARTLE

The Elements
of Integration and
Lebesgue Measure

TeX typesetting by Ali Darijani

February 16, 2023

Contents

Preface

This book consists of two separate, but closely related, parts. The first part (Chapters $1-10$) is subtitled *The Elements of Integration;* the second part (Chapters $11-17$) is subtitled *The Elements of Lebesgue measure.* It is possible to read these two parts in either order, with only a bit of repetition.

The Elements of Integration is essentially a corrected reprint of a book with that title, originally published in 1966, designed to present the chief results of the Lebesgue theory of integration to a reader having only a modest mathematical background. This book developed from my lectures at the University of Illinois, Urbana-Champaign, and it was subsequently used there and elsewhere with considerable success. Its only prerequisites are a understanding of elementary real analysis and the ability to comprehend "$\varepsilon - \delta$ arguments". We suppose that the reader has some familiarity with the Riemann integral so that it is not necessary to provide motivation and detailed discussion, but we do not assume that the reader has a mastery of the subtleties of that theory. A solid course in "advanced calculus", an understanding of the first third of my book *The Elements of Real Analysis,* or of most of my book *Introduction to Real Analysis* with D. R. Sherbert provides an adequate background. In preparing this new edition, I have seized the opportunity to correct certain errors, but I have resisted the temptation to insert additional material, since I believe that one of the features of this book that is most appreciated is its brevity.

The Elements of Lebesgue Measure is descended from class notes written to acquaint the reader with the theory of Lebesgue measure in the space $\mathbf{R}^p$. While it is easy to find good treatments of the case $p = 1$, the case $p > 1$ is not quite as simple and is much less frequently discussed. The main ideas of Lebesgue measure are presented in detail in Chapters $10-15$, although some relatively easy remarks are left to the reader as exercises. The final two chapters venture into the topic of nonmeasurable sets and round out the subject.

There are many expositions of the Lebesgue integral from various points of view, but I believe that the abstract measure space approach used here strikes directly towards the most important results: the convergence theorems. Further, this approach is particularly well-suited for students of probability and statistics, as well as students of analysis. Since the book is intended as an introduction, I do not follow all of the

avenues that are encountered. However, I take pains not to attain brevity by leaving out important details, or assigning them to the reader.

Readers who complete this book are certainly not through, but if this book helps to speed them on their way, it has accomplished its purpose. In the References, I give some books that I believe readers can profitably explore, as well as works cited in the body of the text.

I am indebted to a number of colleagues, past and present, for their comments and suggestions; I particularly wish to mention N. T. Hamilton, G. H. Orland, C. W. Mullins, A. L. Peressini, and J. J. Uhl, Jr. I also wish to thank Professor Roy O. Davies of Leicester University for pointing out a number of errors and possible improvements.

ROBERT G. BARTLE

Ypsilanti and Urbana
Novenber 20, 1994

Part I
(The Elements of Integration)

Chapter 1
Introduction

1.1 makhenbakhen

The theory of integration has its ancient and honorable roots in the "method of exhaustion" that was invented by Eudoxos and greatly developed by Archimedes for the purpose of calculating the areas and volumes of geometric figures. The later work of Newton and Leibniz enabled this method to grow into a systematic tool for such calculations.

As this theory developed, it has become less concerned with applications to geometry and elementary mechanics, for which it is entirely adequate, and more concerned with purely analytic questions, for which the classical theory of integration is not always sufficient. Thus a present-day mathematician is apt to be interested in the convergence of orthogonal expansions, or in applications to differential equations or probability. For him the classical theory of integration which culminated in the Riemann integral has been largely replaced by the theory which has grown from the pioneering work of Henri Lebesgue at the beginning of this century. The reason for this is very simple: the powerful convergence theorems associated with the Lebesgue theory of integration lead to more general, more complete, and more elegant results than the Riemann integral admits.

sflk

1.2 Section Heading

Use the template *chapter.tex* together with the document class SVMono (monograph-type books) or SVMult (edited books) to style the various elements of your chapter content conformable to the Springer Nature layout.

1.3 Section Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Use the standard `equation` environment to typeset your equations, e.g.

$$a \times b = c \, , \tag{1.1}$$

however, for multiline equations we recommend to use the `eqnarray` environment[1].

$$\left| \nabla U_\alpha^\mu(y) \right| \leq \frac{1}{d-\alpha} \int \left| \nabla \frac{1}{|\xi - y|^{d-\alpha}} \right| d\mu(\xi) = \int \frac{1}{|\xi - y|^{d-\alpha+1}} \, d\mu(\xi) \tag{1.2}$$

$$= (d - \alpha + 1) \int_{d(y)}^{\infty} \frac{\mu(B(y,r))}{r^{d-\alpha+2}} \, dr \leq (d - \alpha + 1) \int_{d(y)}^{\infty} \frac{r^{d-\alpha}}{r^{d-\alpha+2}} \, dr \tag{1.3}$$

1.3.1 Subsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Further on please use the LATEX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

> Please do not use quotation marks when quoting texts! Simply use the `quotation` environment – it will automatically be rendered in the preferred layout.

1.3.1.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LATEX automatism for all your cross-references and citations as has already been described in Sect. 1.3.1, see also Fig. 1.1[2]

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Paragraph Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LATEX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

---

[1] In physics texts please activate the class option `vecphys` to depict your vectors in ***boldface-italic*** type - as is customary for a wide range of physical subjects.

[2] If you copy text passages, figures, or tables from other works, you must obtain *permission* from the copyright holder (usually the original publisher). Please enclose the signed permission with the manucript. The sources must be acknowledged either in the captions, as footnotes or in a separate section of the book.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

For typesetting numbered lists we recommend to use the `enumerate` environment – it will automatically render Springer's preferred layout.

1. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

   a. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   b. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

2. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Subparagraph Heading

In order to avoid simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3, see also Fig. 1.2.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

For unnumbered list we recommend to use the `itemize` environment – it will automatically render Springer's preferred layout.

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development, cf. Table 1.1.

  – Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
  – Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Fig. 1.1 If the width of the figure is less than 7.8 cm use the `sidecapion` command to flush the caption on the left side of the page. If the figure is positioned at the top of the page, align the sidecaption with the top of the figure – to achieve this you simply need to use the optional argument `[t]` with the `sidecaption` command
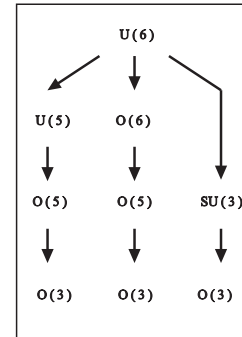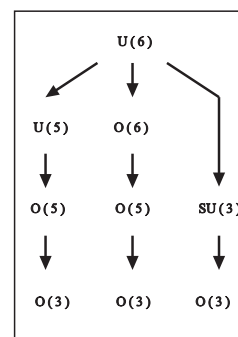
Fig. 1.2 Please write your
figure caption here

Table 1.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA^a	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

^{a} Table foot note (with superscript)

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Run-in Heading Boldface Version Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

Run-in Heading Boldface and Italic Version Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

Run-in Heading Displayed Version
Use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

1.4 Section Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

If you want to list definitions or the like we recommend to use the Springer-enhanced `description` environment – it will automatically render Springer's preferred layout.

Type 1   That addresses central themes pertainng to migration, health, and disease. In Sect. 1.2, Wilson discusses the role of human migration in infectious disease distributions and patterns.

Type 2   That addresses central themes pertainng to migration, health, and disease. In Sect. 1.3.1, Wilson discusses the role of human migration in infectious disease distributions and patterns.

1.4.1 Subsection Heading

In order to avoid simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Use the LaTeX automatism for all your cross-references and citations citations as has already been described in Sect. 1.3.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

> If you want to emphasize complete paragraphs of texts we recommend to use the newly defined Springer class option `graybox` and the newly defined environment `svgraybox`. This will produce a 15 percent screened box 'behind' your text.
>
> If you want to emphasize complete paragraphs of texts we recommend to use the newly defined Springer class option and environment `svgraybox`. This will produce a 15 percent screened box 'behind' your text.

1.4.1.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Theorem 1.1 *Theorem text goes here.*

Definition 1.1 Definition text goes here.

Proof Proof text goes here.   □

Paragraph Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. 1.3.

Note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

Theorem 1.2 *Theorem text goes here.*

Definition 1.2 Definition text goes here.

Proof Proof text goes here.   □

Trailer Head

If you want to emphasize complete paragraphs of texts in an `Trailer Head` we recommend to use

```
\begin{trailer}{Trailer Head}
...
\end{trailer}
```

? Questions

If you want to emphasize complete paragraphs of texts in an `Questions` we recommend to use

```
\begin{question}{Questions}
...
\end{question}
```

> Important

If you want to emphasize complete paragraphs of texts in an `Important` we recommend to use

```
\begin{important}{Important}
...
\end{important}
```

❗ Attention

If you want to emphasize complete paragraphs of texts in an `Attention` we recommend to use

```
\begin{warning}{Attention}
...
\end{warning}
```

Program Code

If you want to emphasize complete paragraphs of texts in an `Program Code` we recommend to use

```
\begin{programcode}{Program Code}
\begin{verbatim}...\end{verbatim}
\end{programcode}
```

Tips

If you want to emphasize complete paragraphs of texts in an `Tips` we recommend to use

```
\begin{tips}{Tips}
...
\end{tips}
```

Overview

If you want to emphasize complete paragraphs of texts in an `Overview` we recommend to use

```
\begin{overview}{Overview}
...
\end{overview}
```

Background Information

If you want to emphasize complete paragraphs of texts in an `Background Information`
we recommend to use

```
\begin{backgroundinformation}{Background Information}
...
\end{backgroundinformation}
```

Legal Text

If you want to emphasize complete paragraphs of texts in an `Legal Text` we rec-
ommend to use

```
\begin{legaltext}{Legal Text}
...
\end{legaltext}
```

Appendix

When placed at the end of a chapter or contribution (as opposed to at the end of
the book), the numbering of tables, figures, and equations in the appendix section
continues on from that in the main text. Hence please *do not* use the `appendix` com-
mand when writing an appendix at the end of your chapter or contribution. If there
is only one the appendix is designated "Appendix", or "Appendix 1", or "Appendix
2", etc. if there is more than one.

$$a \times b = c \tag{1.4}$$

Problems

1.1 A given problem or Excercise is described here. The problem is described here.
The problem is described here.

1.2 Problem Heading
(a) The first part of the problem is described here.
(b) The second part of the problem is described here.

References

In view of the parallel print and (chapter-wise) online publication of your book at `www.springerlink.com` it has been decided that – as a genreral rule – references should be sorted chapter-wise and placed at the end of the individual chapters. However, upon agreement with your contact at Springer you may list your references in a single seperate chapter at the end of your book. Deactivate the class option `sectrefs` and the `thebibliography` environment will be put out as a chapter of its own.

References may be *cited* in the text either by number (preferred) or by author/year.[3] If the citatiion in the text is numbered, the reference list should be arranged in ascending order. If the citation in the text is author/year, the reference list should be *sorted* alphabetically and if there are several works by the same author, the following order should be used:

1. all works by the author alone, ordered chronologically by year of publication
2. all works by the author with a coauthor, ordered alphabetically by coauthor
3. all works by the author with several coauthors, ordered chronologically by year of publication.

The *styling* of references[4] depends on the subject of your book:

- The *two* recommended styles for references in books on *mathematical, physical, statistical and computer sciences* are depicted in [1, 2, 3, 4, 5] and [6, 7, 8, 9, 10].
- Examples of the most commonly used reference style in books on *Psychology, Social Sciences* are [11, 12, 13, 14, 15].
- Examples for references in books on *Humanities, Linguistics, Philosophy* are [16, 17, 18, 20, 21].
- Examples of the basic Springer style used in publications on a wide range of subjects such as *Computer Science, Economics, Engineering, Geosciences, Life Sciences, Medicine, Biomedicine* are [22, 23, 25, 24, 26].

1. Broy, M.: Software engineering — from auxiliary to key technologies. In: Broy, M., Dener, E. (eds.) Software Pioneers, pp. 10-13. Springer, Heidelberg (2002)
2. Dod, J.: Effective substances. In: The Dictionary of Substances and Their Effects. Royal Society of Chemistry (1999) Available via DIALOG.
   `http://www.rsc.org/dose/titleofsubordinatedocument.Cited15Jan1999`
3. Geddes, K.O., Czapor, S.R., Labahn, G.: Algorithms for Computer Algebra. Kluwer, Boston (1992)
4. Hamburger, C.: Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations. Ann. Mat. Pura. Appl. **169**, 321–354 (1995)
5. Slifka, M.K., Whitton, J.L.: Clinical implications of dysregulated cytokine production. J. Mol. Med. (2000) doi: 10.1007/s001090000086

---

[3] Make sure that all references from the list are cited in the text. Those not cited should be moved to a separate *Further Reading* section or chapter.

[4] Always use the standard abbreviation of a journal's name according to the ISSN *List of Title Word Abbreviations*, see `http://www.issn.org/en/node/344`

6. J. Dod, in *The Dictionary of Substances and Their Effects*, Royal Society of Chemistry. (Available via DIALOG, 1999), `http://www.rsc.org/dose/titleofsubordinatedocument.Cited15Jan1999`

7. H. Ibach, H. Lüth, *Solid-State Physics*, 2nd edn. (Springer, New York, 1996), pp. 45-56

8. S. Preuss, A. Demchuk Jr., M. Stuke, Appl. Phys. A **61**

9. M.K. Slifka, J.L. Whitton, J. Mol. Med., doi: 10.1007/s001090000086

10. S.E. Smith, in *Neuromuscular Junction*, ed. by E. Zaimis. Handbook of Experimental Pharmacology, vol 42 (Springer, Heidelberg, 1976), p. 593

11. Calfee, R. C., & Valencia, R. R. (1991). *APA guide to preparing manuscripts for journal publication.* Washington, DC: American Psychological Association.

12. Dod, J. (1999). Effective substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
    `http://www.rsc.org/dose/Effectivesubstances`. Cited 15 Jan 1999.

13. Harris, M., Karper, E., Stacks, G., Hoffman, D., DeNiro, R., Cruz, P., et al. (2001). Writing labs and the Hollywood connection. *J Film* Writing, 44(3), 213–245.

14. O'Neil, J. M., & Egan, J. (1992). Men's and women's gender role journeys: Metaphor for healing, transition, and transformation. In B. R. Wainrig (Ed.), *Gender issues across the life cycle* (pp. 107–123). New York: Springer.

15. Kreger, M., Brindis, C.D., Manuel, D.M., Sassoubre, L. (2007). Lessons learned in systems change initiatives: benchmarks and indicators. *American Journal of Community Psychology*, doi: 10.1007/s10464-007-9108-14.

16. Alber John, Daniel C. O'Connell, and Sabine Kowal. 2002. Personal perspective in TV interviews. *Pragmatics* 12:257–271

17. Cameron, Deborah. 1997. Theoretical debates in feminist linguistics: Questions of sex and gender. In *Gender and discourse*, ed. Ruth Wodak, 99–119. London: Sage Publications.

18. Cameron, Deborah. 1985. *Feminism and linguistic theory.* New York: St. Martin's Press.

19. Calderón, Alberto, F. M. Christ, Carlos E. Kenig and Cora Sadosky. "Harmonic analysis and partial differential equations : essays in honor of Alberto P. Calderón." (1999).

20. Dod, Jake. 1999. Effective substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
    http://www.rsc.org/dose/title of subordinate document. Cited 15 Jan 1999

21. Suleiman, Camelia, Daniel C. O'Connell, and Sabine Kowal. 2002. 'If you and I, if we, in this later day, lose that sacred fire...': Perspective in political interviews. *Journal of Psycholinguistic Research*. doi: 10.1023/A:1015592129296.

22. Brown B, Aaron M (2001) The politics of nature. In: Smith J (ed) The rise of modern genomics, 3rd edn. Wiley, New York

23. Dod J (1999) Effective Substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
    `http://www.rsc.org/dose/titleofsubordinatedocument.Cited15Jan1999`

24. Slifka MK, Whitton JL (2000) Clinical implications of dysregulated cytokine production. J Mol Med, doi: 10.1007/s001090000086

25. Smith J, Jones M Jr, Houghton L et al (1999) Future of health insurance. N Engl J Med 965:325–329

26. South J, Blass B (2001) The future of modern genomics. Blackwell, London

Part II
(The Elements of Lebesgue Measure)

Appendix A
Chapter Heading

All's well that ends well

Use the template *appendix.tex* together with the Springer document class SVMono (monograph-type books) or SVMult (edited books) to style appendix of your book.

A.1 Section Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations.

A.1.1 Subsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. A.1.

For multiline equations we recommend to use the `eqnarray` environment.

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}$$
$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \tag{A.1}$$

A.1.1.1 Subsubsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Furtheron please use the LaTeX automatism for all your cross-references and citations as has already been described in Sect. A.1.1.

Fig. A.1 Please write your
figure caption here
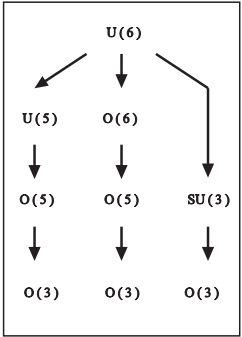
Please note that the first line of text that follows a heading is not indented, whereas
the first lines of all subsequent paragraphs are.

Table A.1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA^a	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

^{a} Table foot note (with superscript)

Acronyms and Abbreviations

Here you can see a list of important acronyms.

ANSI	American National Standards Institute
ASCII	American Standard Code for Information Interchange
CPU	Central Processing Unit
CUDA	Compute Unified Device Architecture
DRAM	Dynamic Random Access Memory
GNU	GNU's Not Unix
GPU	Graphics Processing Unit
grep	g lobal(ly) search r egular e xpression p rint
NVRAM	Non-Volatile Random Access Memory
pip	Pip Installs Packages
RAM	Random Access Memory
SDRAM	Static Random Access Memory
TPU	Tensor Processing Unit

Glossary

Use the template *glossary.tex* together with the Springer document class SVMono (monograph-type books) or SVMult (edited books) to style your glossary in the Springer layout.

GNU GNU is not UNIX

glossary term Write here the description of the glossary term. Write here the description of the glossary term. Write here the description of the glossary term.

glossary term Write here the description of the glossary term. Write here the description of the glossary term. Write here the description of the glossary term.

abs datenwise

glossary term Write here the description of the glossary term. Write here the description of the glossary term. Write here the description of the glossary term.

References

Solutions

Problems of Chapter ??

1.1 The solution is revealed here.

1.2 Problem Heading
(a) The solution of first part is revealed here.
(b) The solution of second part is revealed here.

Index