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Deep Unfolding of Wirtinger Flow Type Schemes

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TO THE MEMORY OF MOHAMMAD MAHDI ELYASI
A DOWN TO EARTH TRUE PHYSICIST AND A LEGENDARY PETROLHEAD

To be Written

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Acronyms and List of Special Symbols

$\in$	belongs to
$\notin$	does not belong to
$\subset, \supset$	inclusion signs
$\mathbb{Q}$	rational field
$<, \leq, >, \geq$	inequality signs
$\sup$	least upper bound
$\inf$	greatest lower bound
$\mathbb{R}$	real field
$+\infty, -\infty, \infty$	infinities
$\overline{z}$	complex conjugate
$\mathrm{Re}(z)$	real part
$\mathrm{Im}(z)$	imaginary part
$\sum$	summation sign
$\mathbb{R}^k$	euclidean k-space
$\mathbf{0}$	null vector
$\boldsymbol{x} \cdot \boldsymbol{y}$	inner product
$\lvert\boldsymbol{x}\rvert$	norm of vector $\boldsymbol{x}$
$\{x_n\}$	sequence
$\bigcup, \cup$	union
$\bigcap, \cap$	intersection
(a, b)	segment
$[a, b]$	interval
E^{c}	complement of E
E'	limit points of E
$\overline{E}$	closure of E
$\lim$	limit
$\to$	converges to
$\limsup$	lim sup
$\liminf$	lim inf
$g \circ f$	composition
$f(x+)$	right-hand limit
$f(x-)$	left-hand limit
$f', \boldsymbol{f}(\boldsymbol{x})'$	derivatives
$U(\boldsymbol{P}, f), U(\boldsymbol{P}, f, \alpha), L(\boldsymbol{P}, f), L(\boldsymbol{P}, f, \alpha)$	Riemann sums
$\mathcal{R}, \mathcal{R}(\alpha)$	classes of Riemann (Stieltjes) integrable functionas
$\mathcal{C}(X)$	space of continiuous functions
$\lVert\ \rVert$	norm
$\exp$	exponential function
D_N	Dirichlet kernel

$\Gamma(x)$	gamma function
$\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$	standard basis
$L(X), L(X,Y)$	spaces of linear transformation
$[\boldsymbol{A}]$	matrix
$D_J f$	partial derivative
∇f	gradient
$\mathcal{C}', \mathcal{C}''$	classes of differentiable functions
$\det[\boldsymbol{A}]$	determinant
$\boldsymbol{J}_f(\boldsymbol{x})$	Jacobian
$\frac{\partial(y_1,\cdots,y_n)}{\partial(x_1,\cdots,x_n)}$	Jacobian
$\mathbb{I}^k$	k-cell
$\mathbb{Q}^k$	k-simplex
$d\boldsymbol{x_I}$	basic k-form
$\wedge$	multiplication symbol
d	defferentiation operator
$\omega_{\boldsymbol{T}}$	transform of ω
∂	boundary operator
$\nabla \times \boldsymbol{F}$	curl
$\nabla \cdot \boldsymbol{F}$	divergence
$\mathcal{E}$	ring of elementary sets
m	Lebesgue measure
μ	measure
$\mathcal{M}_F, \mathcal{M}$	families of measurable sets
f^+, f^-	positive(negative) part of f
K_E	characteristic function
$\mathcal{L}, \mathcal{L}(\mu), \mathcal{L}^2, \mathcal{L}^2(\mu)$	classes of Lebesgue-integrable functions
WF	Wirtinger Flow
TWF	Truncated Wirtinger Flow
ITWF	Incrementally Truncated Wirtinger Flow
IMTWF	Incrementally Minibatched Truncated Wirtinger Flow
RWF	Reshaped Wirtinger Flow
IRWF	Incrementally Reshaped Wirtinger Flow
IMRWF	Incrementally Minibatched Reshaped Wirtinger Flow
$\langle \cdot, \cdot \rangle$	scalar product
$\lvert z \rvert$	absolute value

1 Introduction

1 Problem. Recover $\boldsymbol{x} \in \mathbb{R}^n/\mathbb{C}^n$ from measurements y_i given by

$$y_i = |\langle \boldsymbol{a}_i, \mathbf{x} \rangle|, \quad \text{for} \;\; i = 1, \cdots, m, \tag{1.1}$$

where $\boldsymbol{a}_i \in \mathbb{R}^n/\mathbb{C}^n$ are random design vectors (known).

1.1 Related work

- Citations can be done with `biblatex`. Here is how to cite a book [1], an article [2], a proceedings paper [3] and a technical report [4]. In this example, the actual references are defined in the database file `thesis.bib`, which is included in the header of this document.
- References should be done using `\cref`. This way, the type of the object we are referencing is automatically added. For instance, "`\cref{chap:Introduction}`" leads to "Chapter 1".
- Plots, even from data files, can be done with TikZ, cf. Figure 1.1
- Values with units should preferably printed with the `siunitx` package, e.g. 1 m is the result of "`\SI{1}{\metre}`". This works both in text and in math mode.

$|z|$

sdf fsarg The style defines multiple mathematical environments. All the environments allow to specify a name as optional parameter, as exemplified in Theorem 1.1.1.

1.1.1 Theorem (Theorem Name, optional)**.** *Here goes the actual theorem description.*

Proof. Here goes the proof of the theorem. This environment automatically puts a QED square at its end. Sometimes, the automatic placement is not optimal. In this case, `\qedhere` allows to place the symbol at a specific position, for instance in an equation:

$$a^2 + b^2 = c^2 \qquad \square$$

1.1.2 Example. Example of an example.

1.1.3 Definition. This is a definition.

1.1.4 Proposition. *This is a proposition.*

Equivalence proofs where each direction is shown separately can be formatted using the `\itemize` environment with custom labels. If the proof starts with this environment, put a `\leavevmode` before the environment to ensure that the first direction starts on a new line.

Proof.

"$\Rightarrow$": First direction.

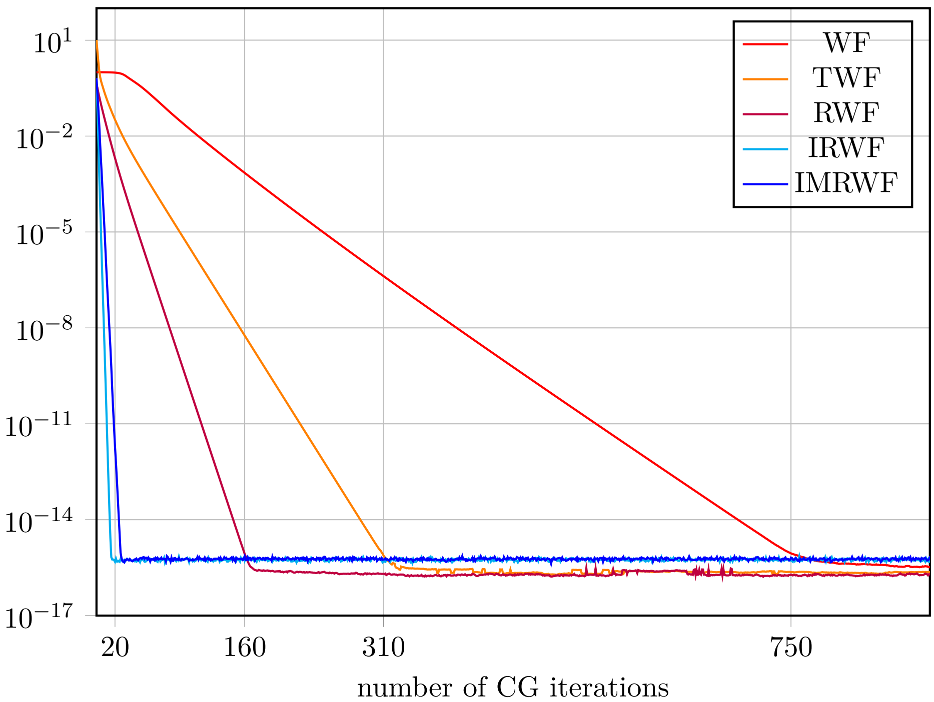

Figure 1.1: TikZ can create beautiful plots directly from data files. These plots use vector graphics and their fonts are fully consistent with the fonts of the document.

"$\Leftarrow$": Second direction. □

1.1.5 Lemma. *This is a lemma.*

1.1.6 Corollary. *This is a corollary.*

sdf

2 Problem.

2 Algorithms

write something about the history of the other algorithms and the introduce the main 5 algorithms

- Wirtinger FLow suggested by [5]
- Truncated Wirtinger Flow suggested by [6]
- Incrementally Truncated Wirtinger Flow suggested by [7]
- Reshaped Wirtinger Flow and Incrementally Reshaped Wirtinger FLow suggested by [8]

Input: $\boldsymbol{y} = \{y_i\}_{i=1}^m$, $\{\boldsymbol{a}_i\}_{i=1}^m$;
Parameters: Lower and upper thresholds α_l, α_u for truncation in initialization, step size μ;
Initialization: Let $\boldsymbol{z}^{(0)} = \lambda_0 \tilde{\boldsymbol{z}}$, where $\lambda_0 = \frac{mn}{\sum_{i=1}^m \|\boldsymbol{a}_i\|_1} \cdot \left(\frac{1}{m}\sum_{i=1}^m y_i\right)$ and $\tilde{\boldsymbol{z}}$ is the leading eigenvector of

$$\boldsymbol{Y} := \frac{1}{m}\sum_{i=1}^m y_i \boldsymbol{a}_i \boldsymbol{a}_i^* \mathbf{1}_{\{\alpha_l \lambda_0 < y_i < \alpha_u \lambda_0\}}. \tag{2.1}$$

Gradient loop: for $t = 0 : T - 1$ do

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} - \frac{\mu}{m}\sum_{i=1}^m \left(\boldsymbol{a}_i^* \boldsymbol{z}^{(t)} - y_i \cdot \frac{\boldsymbol{a}_i^* \boldsymbol{z}^{(t)}}{|\boldsymbol{a}_i^* \boldsymbol{z}^{(t)}|}\right) \boldsymbol{a}_i. \tag{2.2}$$

Output $\boldsymbol{z}^{(T)}$.

Algorithm 1: Reshaped Wirtinger Flow suggested by [8]

Input: $\boldsymbol{y} = \{y_i\}_{i=1}^m$, $\{\boldsymbol{a}_i\}_{i=1}^m$;
Initialization: Same as in RWF (Algorithm 1);
Parameters: Lower and upper thresholds α_l, α_u for truncation in initialization, step size μ;
Gradient loop: for $t = 0 : T-1$ do
Choose i_t uniformly at random from $\{1, 2, \ldots, m\}$, and let

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} - \mu \left(\boldsymbol{a}_{i_t}^* \boldsymbol{z}^{(t)} - y_{i_t} \cdot \frac{\boldsymbol{a}_{i_t}^* \boldsymbol{z}^{(t)}}{|\boldsymbol{a}_{i_t}^* \boldsymbol{z}^{(t)}|} \right) \boldsymbol{a}_{i_t}, \tag{2.3}$$

Output $\boldsymbol{z}^{(T)}$.
Algorithm 2: Incremental Reshaped Wirtinger Flow (IRWF) suggested by [8]

Input: $\boldsymbol{y} = \{y_i\}_{i=1}^m$, $\{\boldsymbol{a}_i\}_{i=1}^m$;
Initialization: Same as in RWF (Algorithm 1);
Gradient loop: for $t = 0 : T-1$ do
Choose Γ_t uniformly at random from the subsets of $\{1, 2, \ldots, m\}$ with cardinality k, and let

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} - \mu \cdot \boldsymbol{A}_{\Gamma_t}^* \left(\boldsymbol{A}_{\Gamma_t} \boldsymbol{z}^{(t)} - \boldsymbol{y}_{\Gamma_t} \odot \mathrm{Ph}(\boldsymbol{A}_{\Gamma_t} \boldsymbol{z}^{(t)}) \right), \tag{2.4}$$

where $\boldsymbol{A}_{\Gamma_t}$ is a matrix stacking $\boldsymbol{a}_i^*$ for $i \in \Gamma_t$ as its rows, $\boldsymbol{y}_{\Gamma_t}$ is a vector stacking y_i for $i \in \Gamma_t$ as its elements, $\odot$ denotes element-wise product, and $\mathrm{Ph}(\boldsymbol{z})$ denotes a phase vector of $\boldsymbol{z}$.
Output $\boldsymbol{z}^{(T)}$.
Algorithm 3: Minibatch Incremetnal Reshaped Wirtinger Flow (minibatch IRWF) suggested by [8]

3 Unrolling

talam

4 UNROLLED WIRTINGER FLOWS

5 Results

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