Principles of Mathematical Analysis

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Contents

Preface

This book is intended to serve as a text for the course in analysis that is usually taken by advanced undergraduates or by first-year students who study mathematics.

The present edition covers essentially the same topics as the second one, with some additions, a few minor omissions, and considerable rearrangement. I hope that these changes will make the material more accessible and more attractive to the students who take such a course.

Experience has convinced me that it is pedagogically unsound (though logically correct) to start off with the construction of the real numbers from the rational ones. At the beginning, most students simply fail to appreciate the need for doing this. Accordingly, the real number system is introduced as an ordered field with the least-upper-bound property, and a few interesting applications of this property are quickly made. However, Dedekind's construction is not omitted. It is now in an Appendix to Chapter 1, where it may be studied and enjoyed whenever the time seems ripe.

The material on functions of several variables is almost completely rewritten, with many details filled in, and with more examples and more motivation. The proof of the inverse function theorem–the key item in Chapter 9–is simplified by means of the fixed point theorem about contraction mappings. Differential forms are discussed in much greater detail. Several applications of Stokes' theorem are included.

As regard a other changes, the chapter on the Riemann-Stieltjes integral has been trimmed a bit, a short do-it-yourself section on the gamma function has been added to Chapter 8, and there is a large number of new exercises, most of them with fairly detailed hints.

I have also included several references to articles appearing in the *American Mathematical Monthly* and in *Mathematics Magazine*, in the hope that students will develop the habit of looking into the journal literature. Most of these references were kindly supplied by R. B. Burckel.

Over the years, many people, students as well as teachers, have sent me corrections, criticisms, and other comments concerning the previous editions of this book. I have appreciated these, and I take this opportunity to express my sincere thanks to all who have written me.

WALTER RUDIN

CHAPTER 1

The Real and Complex Number Systems

Introduction

A satisfactory discussion of the main concepts of analysis (such as convergence, continuity, differentiation, and integration) must be based on an accurately defined number concept. We shall not, however, enter into any discussion of the axioms that govern the arithmetic of the integers, but assume familiarity with the rational numbers (i.e., the numbers of the form m/n, where m and n are integers and $n \neq 0$).

The rational number system is inadequate for many purposes, both as a field and as an ordered set. (These terms will be defined in Secs. **??** and **??**.) For instance, there is no rational p such that $p^2 = 2$. (We shall prove this presently.) This leads to the introduction of so-called "irrational numbers" which are often written as infinite decimal expansions and are considered to be "approximated" by the corresponding finite decimals. Thus the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \cdots$$

"tends to $\sqrt{2}$." But unless the irrational number $\sqrt{2}$ has been clearly defined, the question must arise: Just what is it that this sequence "tends to"?

This sort of question can be answered as soon as the so-called "real number system" is constructed.

Example 1.1. We now show that the equation

$$p^2 = 2 \tag{1.1}$$

is not satisfied by any rational p. If there were such a p, we could write $p = m/n$ where m and n are integers that are not both even. Let us assume this is done. Then 1.1 implies

$$m^2 = 2n^2 \tag{1.2}$$

this shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of 1.2 is divisible by 4. so that n^2 is even, which implies that n is even.

The assumptions that 1.1 holds thus leads to the contradiction that both m and n are even, contrary to our choice of m and n. Hence 1.1 is impossible for rational p.

If p is in A then $p^2 - 2 < 0$, **??** shows that $q > p$, and **??** shows that $q^2 < 2$. Thus q is in A.

If p is in B then $p^2 - 2 > 0$, **??** shows that $0 < q < p$, and **??** shows that $q^2 > 2$. Thus q is in B.

Remark 1.2. The purpose of the above discussion has been to show that the rational number system has certain gaps, in spite of the fact that between any two rationals there is another: If $r < s$ then $r < \frac{(r+s)}{2} < s$. The real number system fills these gaps. This is the principal reason for the fundamental role which it plays in analysis.

In order to elucidate its structure, as well as that of the complex numbers, we start with a brief discussion of the general concepts of *ordered set* and *field*. Here is some of the standard set-theoretic terminology that will be used throughout this book.

DEFINITION 1.3. If A is any set (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A.
If x is not a member of A, we write: $x \notin A$.

The set which contains no element will be called the *empty set*. If a set has at least one element, it is called *nonempty*.

DEFINITION 1.4. Throughout Chap. 1, the set of all rational numbers will be denoted by $\mathbb{Q}$.

1.1. Ordered Sets

DEFINITION 1.5. Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < x$, then $x < z$.

The statement "$x < y$" may be read as "x is less than y" or "x is smaller than y" or "x precede y".

It is often convenient to write $y > x$ in place of $x < y$.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these is to hold. In other words, $x \leq y$ is the negation of $x > y$

DEFINITION 1.6. An *ordered set* is a set S in which an order is defined. For example, $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Fields

This is an example of an unnumbered first-level heading.

The Real Field

This is an example of an unnumbered first-level heading.

The Extended Real Number System

This is an example of an unnumbered first-level heading.

The Complex Field

This is an example of an unnumbered first-level heading.

Euclidean Spaces

This is an example of an unnumbered first-level heading.

Appendix

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Exercises

Exercise 1.7. This is an example of the `xca` environment. This environment is used for exercises which occur within a section.

1.2. Some more list types

This is an example of a bulleted list.

- $\mathcal{J}_g$ of dimension $3g-3$;
- $\mathcal{E}^2_g = \{$Pryms of double covers of $C = \square$ with normalization of C hyperelliptic of genus $g-1\}$ of dimension $2g$;
- $\mathcal{E}^2_{1,g-1} = \{$Pryms of double covers of $C = \square^H_{P^1}$ with H hyperelliptic of genus $g-2\}$ of dimension $2g-1$;
- $\mathcal{P}^2_{t,g-t}$ for $2 \le t \le g/2 = \{$Pryms of double covers of $C = \square^{C'}_{C''}$ with $g(C') = t-1$ and $g(C'') = g-t-1\}$ of dimension $3g-4$.

This is an example of a 'description' list.

Zero case: $\rho(\Phi) = \{0\}$.
Rational case: $\rho(\Phi) \ne \{0\}$ and $\rho(\Phi)$ is contained in a line through 0 with rational slope.
Irrational case: $\rho(\Phi) \ne \{0\}$ and $\rho(\Phi)$ is contained in a line through 0 with irrational slope.

[1].

Lemma 1.8. *Let $f, g \in A(X)$ and let E, F be cozero sets in X.*

(1) If f is E-regular and $F \subseteq E$, then f is F-regular.
(2) If f is E-regular and F-regular, then f is $E \cup F$-regular.
(3) If $f(x) \ge c > 0$ for all $x \in E$, then f is E-regular.

The following is an example of a proof.

Proof. Set $j(\nu) = \max(I \backslash a(\nu)) - 1$. Then we have

$$\sum_{i \notin a(\nu)} t_i \sim t_{j(\nu)+1} = \prod_{j=0}^{j(\nu)} (t_{j+1}/t_j).$$

Hence we have

$$\begin{aligned} \prod_\nu \Bigl(\sum_{i \notin a(\nu)} t_i \Bigr)^{|a(\nu-1)|-|a(\nu)|} &\sim \prod_\nu \prod_{j=0}^{j(\nu)} (t_{j+1}/t_j)^{|a(\nu-1)|-|a(\nu)|} \\ &= \prod_{j \ge 0} (t_{j+1}/t_j)^{\sum_{j(\nu)\ge j}(|a(\nu-1)|-|a(\nu)|)}. \end{aligned} \tag{1.3}$$

[1]Here is an example of a footnote. Notice that this footnote text is running on so that it can stand as an example of how a footnote with separate paragraphs should be written.

And here is the beginning of the second paragraph.

FIGURE 1. This is an example of a figure caption with text.

FIGURE 2

By definition, we have $a(\nu(j)) \supset c(j)$. Hence, $|c(j)| = n - j$ implies (5.4). If $c(j) \notin a$, $a(\nu(j))c(j)$ and hence we have (5.5). □

> This is an example of an 'extract'. The magnetization M_0 of the Ising model is related to the local state probability $P(a) : M_0 = P(1) - P(-1)$. The equivalences are shown in Table 1.

TABLE 1

	$-\infty$	$+\infty$
$f_+(x,k)$	$e^{\sqrt{-1}kx} + s_{12}(k)e^{-\sqrt{-1}kx}$	$s_{11}(k)e^{\sqrt{-1}kx}$
$f_-(x,k)$	$s_{22}(k)e^{-\sqrt{-1}kx}$	$e^{-\sqrt{-1}kx} + s_{21}(k)e^{\sqrt{-1}kx}$

DEFINITION 1.9. This is an example of a 'definition' element. For $f \in A(X)$, we define

$$\mathcal{Z}(f) = \{E \in Z[X] : f \text{ is } E^c\text{-regular}\}. \tag{1.4}$$

REMARK 1.10. This is an example of a 'remark' element. For $f \in A(X)$, we define

$$\mathcal{Z}(f) = \{E \in Z[X] : f \text{ is } E^c\text{-regular}\}. \tag{1.5}$$

EXAMPLE 1.11. This is an example of an 'example' element. For $f \in A(X)$, we define

$$\mathcal{Z}(f) = \{E \in Z[X] : f \text{ is } E^c\text{-regular}\}. \tag{1.6}$$

Some extra text before the `xcb` head. The `xcb` environment is used for exercises that occur at the end of a chapter. Here it contains an example of a numbered list.

Here is an example of a cite. See [**?**]. [**?**]

THEOREM 1.12. *This is an example of a theorem.*

THEOREM 1.13 (Marcus Theorem). *This is an example of a theorem with a parenthetical note in the heading.*

Exercises

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

(1) If r is rational ($r \notin 0$) and x is irrational, prove that $r+x$ and rx are irrational.
(2) Prove that there is no rational number whose square is 12.
(3) Prove Proposition **??**.
(4) Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.
(5) Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

(6) Fix $b > 1$,
   (a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

   $$(b^m)^{l/n} = (b^p)^{l/q}.$$

   Hence it makes sense to define $b^r = (b^m)^{1/n}$.
   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
   (c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

   $$b^r = \sup B(r)$$

   when r is rational. Hence it makes sense to define

   $$b^x = \sup B(x)$$

   for every real x.
   (d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.
(7) satisfied
(8) Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.
(9) Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is ca11ed a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?
(10) Suppose $z = a + bi$, $w = u + vi$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prive that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.
(11) If z is a complex number, prove that there exist an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?
(12) if $z_1, \cdots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

(13) If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

(14) If z is a complex number such that $|z| = 1$, that is, such that $z\overline{z} = 1$, compute

$$|1+z|^2 + |1-z|^2 .$$

(15) Under what conditions does equality hold in the Schwarz inequality?

(16) Suppose $k \geq 3$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$, $|x - y| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\boldsymbol{z} \in \mathbb{R}^k$ such that

$$|\boldsymbol{z} - \boldsymbol{x}| = |\boldsymbol{z} - \boldsymbol{y}| = r.$$

(b) If $2r = d$, there is exactly one such $\boldsymbol{z}$.

(c) If $2r < d$, there is no such $\boldsymbol{z}$.

How must these statements be modified if k is 2 or 1?

(17) Prove that

$$|\boldsymbol{x}+\boldsymbol{y}|^2 + |\boldsymbol{x}-\boldsymbol{y}|^2 = 2\,|\boldsymbol{x}|^2 + 2\,|\boldsymbol{y}|^2$$

if $\boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

(18) If $k \geq 2$ and $\boldsymbol{x} \in \mathbb{R}^k$, prove that there exists $\boldsymbol{y} \in \mathbb{R}^k$ such that $\boldsymbol{y} \neq \mathbf{0}$ but $\boldsymbol{x} \cdot \boldsymbol{y} = 0$. Is this also true if $k = 1$?

(19) Supposea $\boldsymbol{a} \in \mathbb{R}^k, \boldsymbol{b} \in \mathbb{R}^k$. Find $\boldsymbol{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\boldsymbol{x} - \boldsymbol{a}| = 2\,|\boldsymbol{x} - \boldsymbol{b}|$$

if and only if $|\boldsymbol{x} - \boldsymbol{c}| = r$.

(*Solution*: $3\boldsymbol{c} = 4\boldsymbol{b} - \boldsymbol{a}, 3r = 2\,|\boldsymbol{b} - \boldsymbol{a}|$)

(20) With reference to the Appendix, suppose that property **??** were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms **??** to **??** (with a slightly different zero-element!) but that **??** fails.

3.1

CHAPTER 2

Basic Topology

CHAPTER 3

Numerical Sequences and Series

3.1. title

3.2. saefg

$$sDg \tag{3.1}$$

CHAPTER 4

Continuity

CHAPTER 5

Differentiation

CHAPTER 6

The Riemann-Stieltjes Integral

CHAPTER 7

Sequences and Series of Functions

CHAPTER 8

Some Special Functions

CHAPTER 9

Functions of Several Variables

CHAPTER 10

Integration of Differential Forms

CHAPTER 11

The Lebesgue Theory

[**ARTIN1964**] [**BOAS1960**] [**BUCK1962**] [**BUCK1965**] [**BURKILL1951**] [**DIEUDONNÉ1960**] [**FLEMING1965**] [**GRAVES1956**] [**HALMOS1950**] [**HALMOS1958**] [**HARDY1947**] [**HARDY1950**] [**HERSTEIN1964**] [**HEWITT1965**] [**KELLOGG1940**] [**KNOPP1928**] [**LANDAU1951**] [**MCSHANE1944**] [**NIVEN1956**] [**ROYDEN1974**] [**RUDIN1974**] [**SIMMONS1963**] [**SINGER1967**] [**SMITH1971**] [**SPIVAK1965**] [**THURSTON1956**]

List of Special Symbols

$\in$	belongs to
$\notin$	does not belong to
$\subset, \supset$	inclusion signs
$\mathbb{Q}$	rational field
$<, \leq, >, \geq$	inequality signs
$\sup$	least upper bound
$\inf$	greatest lower bound
$\mathbb{R}$	real field
$+\infty, -\infty, \infty$	infinities
$\overline{z}$	complex conjugate
$\mathrm{Re}(z)$	real part
$\mathrm{Im}(z)$	imaginary part
$\lvert z \rvert$	absolute value
$\sum$	summation sign
$\mathbb{R}^k$	euclidean k-space
$\mathbf{0}$	null vector
$\boldsymbol{x} \cdot \boldsymbol{y}$	inner product
$\lvert \boldsymbol{x} \rvert$	norm of vector $\boldsymbol{x}$
$\{x_n\}$	sequence
$\bigcup, \cup$	union
$\bigcap, \cap$	intersection
(a, b)	segment
$[a, b]$	interval
E^{c}	complement of E
$E^{'}$	limit points of E
$\overline{E}$	closure of E
$\lim$	limit
$\rightarrow$	converges to
$\limsup$	lim sup
$\liminf$	lim inf
$g \circ f$	composition
$f(x+)$	right-hand limit
$f(x-)$	left-hand limit
$f', \boldsymbol{f}(\boldsymbol{x})'$	derivatives
$U(\boldsymbol{P}, f), U(\boldsymbol{P}, f, \alpha), L(\boldsymbol{P}, f), L(\boldsymbol{P}, f, \alpha)$	Riemann sums
$\mathcal{R}, \mathcal{R}(\alpha)$	classes of Riemann (Stieltjes) integrable functionas
$\mathcal{C}(X)$	space of continiuous functions

$\|\| \ \|\|$	norm
exp	exponential function
D_N	Dirichlet kernel
$\Gamma(x)$	gamma function
$\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$	standard basis
$L(X), L(X, Y)$	spaces of linear transformation
$[\boldsymbol{A}]$	matrix
$D_J f$	partial derivative
∇f	gradient
$\mathcal{C}', \mathcal{C}''$	classes of differentiable functions
$\det[\boldsymbol{A}]$	determinant
$\boldsymbol{J}_f(\boldsymbol{x})$	Jacobian
$\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}$	Jacobian
$\mathbb{I}^k$	k-cell
$\mathbb{Q}^k$	k-simplex
$d\boldsymbol{x}_{\boldsymbol{I}}$	basic k-form
$\wedge$	multiplication symbol
d	defferentiation operator
$\omega_{\boldsymbol{T}}$	transform of ω
∂	boundary operator
$\nabla \times \boldsymbol{F}$	curl
$\nabla \cdot \boldsymbol{F}$	divergence
$\mathcal{E}$	ring of elementary sets
m	Lebesgue measure
μ	measure
$\mathcal{M}_F, \mathcal{M}$	families of measurable sets
f^+, f^-	positive(negative) part of f
K_E	characteristic function
$\mathcal{L}, \mathcal{L}(\mu), \mathcal{L}^2, \mathcal{L}^2(\mu)$	classes of Lebesgue-integrable functions

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