Assignment 4

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***Abstract*—solution for Assignment 4 - Papoulis Eercise-8.6**

Papoulis Exercise 8.6: Consider a random variable x with density $f(x) = xe^{-x}U(x)$. Predict with 95 confidence that the next value of x will be in the interval(a,b).Show that the lenght $b - a$ of this interval is minimum if a and b are such that
$f(a) = f(b)$
$P(a < x < b) = 0.95$
Solution: We shall show that if f(x) is a density with a single maximum and $P(a < x < b) = \gamma$,then b-a is not minimum if $f(a) = f(b)$.The density $xe^{-x}U(x)$ is aspecial case.It suffies to show that b-a is not minimum if $f(a) < f(b)$ or $f(a) > f(b)$
Suppose first that $f(a) < f(b)$as in fig(a).Clearly,$f'(a) > 0$ and $f'(b) < 0$,hence ,we can find two constants $\delta_1 > 0$ and $\delta_2 > 0$ such that
$P(a + \delta_1, b + \delta_2)$ and
$f(a) < f(a + \delta_1) < f(b + \delta_2)$
From this ot follows that $\delta_1 > \delta_2$,hence,the length of the new interval $(a + \delta_1, b + \delta_2)$ is smaller than b-a
If $f(a) > f(a)$,we form the interval $(a - \delta_1, b - \delta_2)$ and proceed similarly
Special case: If $f(x) = xe^{-x}$ then ,
$F(x) = 1 - e^{-x} - xe^{-x}$,hence

$$P(a < x < b) = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = 0.95$$

And since $f(a) = f(b)$,the system
$ae^{-a} = be^{-b}$
therefore,$e^{-a} - e^{-b} = 0.95$
solving we obtain approximately $a = 0.04$ and
$b = 5.75$
A numerically simpler solution results if we set
$0.025 = P(x \leq a) = F(a)$
$0.025 = P(x > b) = 1 - F(b)$
This yields the system
$0.025 = 1 - e^{-a} - ae^{-a}$
$0.025 = e^{-b} + be^{-b}$
Solving we obtain $a = 0.242$ and $b = 5.572$ However,the length $5.572 - 0.242 = 5.33$ of the resulting interval is larger than the length $4.75 - 0.04 = 4.71$ of the optimal interval.
Hence,the lenght $b - a$ of this interval is minimum if a and b are such that
$f(a) = f(b)$
$P(a < x < b) = 0.95$
hence proved