APM 523 Mittleman
Homework 1
Matt Kinsinger, Hongjun Choi, Adarsh Akkshai
Due 9-7-2017

2) Deriving the 1-D Newton Method for finding the zeros of a function: Begin with an initial value x_0 and consider the tangent line to the function at $f(x_0)$. Assuming that $f^{'}(x_0) \neq 0$, this tangent line crosses the x-axis as some point x_1. We can use this to perform iterations that under sufficient conditions will converge to a zero of the function f.

$$\begin{aligned} f^{'}(x_k) &= \frac{0 - f(x_k)}{x_{k+1} - x_k} \\ (x_{k+1} - x_k)f^{'}(x_k) &= -f(x_k) \\ x_{k+1}f^{'}(x_k) &= x_k f^{'}(x_k) - f(x_k) \\ x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \end{aligned}$$

a) Consider $f(x) = x^3(1-x)^2$ and analyze the convergence of Newton to it's zeros.

Clearly, the zeros of f are $x = 0$ and $x = 1$. We can assume the convergence of Newton's method to the zero's of f. Now we will analyze the convergence of Newton's method to it's zeros.

$$\begin{aligned} f(x) &= x^3(1-x)^2 \\ f^{'}(x) &= 3x^2(1-x)^2 - 2x^3(1-x) \\ &= x^2(1-x)(3-5x) \end{aligned}$$

Thus,

$$\begin{aligned} x_{k+1} &= x_k - \frac{x_k^3(1-x_k)^2}{x_k^2(1-x_k)(3-5x_k)} \\ &= x_k - \frac{x_k(1-x_k)}{3-5x_k} \end{aligned}$$

[case i] $\{x_k\} \to 0$.

$$\left|\frac{x_{k+1} - 0}{x_k - 0}\right| = \left|\frac{x_k - \frac{x_k(1-x_k)}{3-5x_k}}{x_k}\right| = \left|1 - \frac{1-x_k}{3-5x_k}\right|. \tag{1}$$

We can get rid of the absolute value if we can show that there exists M such that $k \geq M \Rightarrow \left|\frac{1-x_k}{3-5x_k}\right| < 1$. To this end let M be such that $k \geq M$ implies $|x_k| < \frac{1}{5}$ which is possible since $\{x_k\} \to 0$. It follows that:

i) $|1 - x_k| \leq 1 + |x_k| < 1 + \frac{1}{5} = \frac{6}{5}$.

ii) $|3 - 5x_k| \geq 3 - 5|x_k| > 3 - 5\frac{1}{5} = 2$.

Thus,

$$\frac{|1-x_k|}{|3-5x_k|} < \frac{\frac{6}{5}}{2} = \frac{3}{5} < 1.$$

Returning to (1), we have

$$\left|\frac{x_{k+1}-0}{x_k-0}\right| = \left|1-\frac{1-x_k}{3-5x_k}\right| = 1-\frac{1-x_k}{3-5x_k} < 1, \tag{2}$$

where the final inequality comes from the fact that

i) $x_k < \frac{1}{5} < 1 \Rightarrow 1-x_k > 0.$

ii) $x_k < \frac{1}{5} < \frac{3}{5} \Rightarrow 3-5x_k > 0.$

These two together show that $\frac{1-x_k}{3-5x_k} > 0$, hence (2) follows. This proves that convergence of Newton's method in this case is Q-linear.

[case ii] $\{x_k\} \to 1$. We need,

$$\left|\frac{x_{k+1}-1}{x_k-1}\right| = \left|\frac{x_k - \frac{x_k(1-x_k)}{3-5x_k} - 1}{x_k-1}\right| = \left|\frac{(x_k-1)\left[1+\frac{x_k}{3-5x_k}\right]}{x_k-1}\right| = \left|1+\frac{x_k}{3-5x_k}\right| < 1. \tag{3}$$

Or equivalently

$$-2 < \frac{x_k}{3-5x_k} < 0. \tag{4}$$

Since $\{x_k\} \to 1$ there exists M such that for $k \geq M$ we have $x_k > \frac{2}{3} > \frac{3}{5}$. It follows that:

$$\begin{aligned}
\frac{3}{5} &< x_k \\
3 &< 5x_k \\
3-5x_k &< 0 \\
\frac{1}{3-5x_k} &< 0 \\
\frac{x_k}{3-5x_k} &< 0.
\end{aligned}$$

and,

$$\begin{aligned}
\frac{2}{3} &< x_k \\
6 &< 9x_k \\
6-10x_k &< -x_k \\
3-5x_k &< \frac{-x_k}{2} \\
\frac{3-5x_k}{x_k} &< -\frac{1}{2} \\
\frac{x_k}{3-5x_k} &> -2.
\end{aligned}$$

Hence we have (3), and it follows that Newton's method converges Q-linearly to both zeros of f. ∎

b) This iteration is a disguised Newton's method.

$$
\begin{aligned}
x_{k+1} = \frac{1}{5}\left[4x_k - \frac{a}{x_k^4}\right] &= \frac{4x_k^5 - a}{5x_k^4} \\
&= \frac{5x_k^5 - x_k^5 - a}{5x_k^4} \\
&= x_k - \frac{x_k^5 + a}{5x_k^4}
\end{aligned}
$$

We can see that this iterartion in Newton's method for finding the zeros of

$$
\begin{aligned}
f(x) &= x^5 + a \\
f^{'}(x) &= 5x^4.
\end{aligned}
$$

Since every real number has an n^{th} root this iteration should converge to $-\sqrt[5]{a}$ for any $a \in \mathbb{R}$. We now consider the speed of convergence.

We can show that the convergence in Q super-linear when $a \neq 0$, but only Q linear when $a = 0$. Note that

$$
x_k^5 + a = (x_k + a^{\frac{1}{5}})\left(x_k^4 - x_k^3 a^{\frac{1}{5}} + x_k^2 a^{\frac{2}{5}} - x_k a^{\frac{3}{5}} + a^{\frac{4}{5}}\right). \tag{5}
$$

It follows that,

$$
\begin{aligned}
\lim_{k\to\infty} \frac{\left|x_{k+1} + a^{\frac{1}{5}}\right|}{\left|x_k + a^{\frac{1}{5}}\right|} &= \lim_{k\to\infty} \frac{\left|x_k - \frac{x_k^5+a}{5x_k^4} + a^{\frac{1}{5}}\right|}{\left|x_k + a^{\frac{1}{5}}\right|} \\
&= \lim_{k\to\infty} \frac{\left|x_k + a^{\frac{1}{5}}\right| \cdot \left|1 - \frac{\left(x_k^4 - x_k^3 a^{\frac{1}{5}} + x_k^2 a^{\frac{2}{5}} - x_k a^{\frac{3}{5}} + a^{\frac{4}{5}}\right)}{5x_k^4}\right|}{\left|x_k + a^{\frac{1}{5}}\right|} \\
&= \lim_{k\to\infty} \left|1 - \frac{\left(x_k^4 - x_k^3 a^{\frac{1}{5}} + x_k^2 a^{\frac{2}{5}} - x_k a^{\frac{3}{5}} + a^{\frac{4}{5}}\right)}{5x_k^4}\right| \\
&= \left|1 - \lim_{k\to\infty} \frac{\left(x_k^4 - x_k^3 a^{\frac{1}{5}} + x_k^2 a^{\frac{2}{5}} - x_k a^{\frac{3}{5}} + a^{\frac{4}{5}}\right)}{5x_k^4}\right| \\
&= \left|1 - \frac{\left(a_k^{\frac{4}{5}} + a_k^{\frac{3}{5}} a^{\frac{1}{5}} + a_k^{\frac{2}{5}} a^{\frac{2}{5}} + a_k^{\frac{1}{5}} a_k^{\frac{3}{5}} + a^{\frac{4}{5}}\right)}{5a_k^{\frac{4}{5}}}\right| \\
&= \left|1 - \frac{5a_k^{\frac{4}{5}}}{5a_k^{\frac{4}{5}}}\right| \\
&= 0, \qquad\qquad a \neq 0.
\end{aligned}
$$

Thus Newton's method converges **Q super-linearly** when $a \neq 0$.

When $a = 0$ we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{\left|x_{k+1} + a^{\frac{1}{5}}\right|}{\left|x_k + a^{\frac{1}{5}}\right|} &= \lim_{k\to\infty} \frac{\left|x_k - \frac{x_k^5}{5x_k^4}\right|}{|x_k|} \\
&= \lim_{k\to\infty} \frac{\left|x_k - \frac{1}{5}x_k\right|}{|x_k|} \\
&= \lim_{k\to\infty} \left|1 - \frac{1}{5}\right| \\
&= \frac{4}{5}.
\end{aligned}$$

Thus when $a = 0$ we have **Q linear** convergence. □

c) Analyze the convergence of Newton to the zeros of a function $f(x) = x^2 + 4$.

Given a function $f(x) = x^2 + 4$ and the standard Newton's iteration, $x_{k+1} = x_k - f(x_k)/f\,'(x_k)$, we can define a function as $f(x_k) = x_k^2 + 4$ and the zero's of f are $\pm 2i$.

$$\begin{aligned}
x_{k+1} &= x_k - \frac{x_k^2 + 4}{2x_k} \\
&= \frac{2x_k^2 - x_k^2 - 4}{2x_k} \\
&= \frac{x_k^2 - 4}{2x_k}
\end{aligned}$$

We can say that Q-linear convergence is obtained if there is a constant $r \in (0, 1)$ such that

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||} \leq r, \quad \text{for all } k \text{ is sufficiently large.} \tag{6}$$

And then, we can analyze the rate of convergence at point $x^* = +2i$.

$$\begin{aligned}
\lim_{x_k\to+2i} \frac{||x_{k+1} - x^*||}{||x_k - x^*||} &= \lim_{x_k\to+2i} \frac{||x_k^2 - 4 - i4x_k||}{||2x_k(x_k - 2i)||} \\
&= \lim_{x_k\to+2i} \frac{||x_k^2 - 4 - i4x_k||}{||2x_k^2 - i4x_k||} \\
&= \lim_{x_k\to+2i} \frac{||x_k^2(1 - \frac{4}{x_k^2} - \frac{i4}{x_k})||}{||x_k^2(2 - \frac{i4}{x_k})||}, \quad [\frac{0}{0}]form
\end{aligned}$$

From above results, we can solve above limitation problems by applying L'Hopital's Rule

$$\begin{aligned}\lim_{x_k\to+2i}\frac{||2x_k(1-\frac{4}{x_k^2}-\frac{i4}{x_k})+x_k^2(8x_k^{-3}+i4x_k^{-2})||}{||2x_k(2-\frac{i4}{x_k})+x_k^2(i4x_k^{-2})||} &= \lim_{x_k\to+2i}\frac{||2x_k-i4||}{||4x_k-4i||}\\ &= 0.\end{aligned}$$

Now, where $x^*=-2i$

$$\begin{aligned}\lim_{x_k\to-2i}\frac{||x_{k+1}-x^*||}{||x_k-x^*||} &= \lim_{x_k\to-2i}\frac{||x_{k+1}+2i||}{||x_k+2i||}\\ &= \lim_{x_k\to-2i}\frac{||x_k^2-4+i4x_k||}{||2x_k(x_k+2i)||}\\ &= \lim_{x_k\to-2i}\frac{||x_k^2-4+i4x_k||}{||2x_k^2+i4x_k||}\\ &= \lim_{x_k\to-2i}\frac{||x_k^2(1-\frac{4}{x_k^2}+\frac{i4}{x_k})||}{||x_k^2(2+\frac{i4}{x_k})||}, \quad [\frac{0}{0}]form.\end{aligned}$$

Applying L'Hopital's rules to above results, we have

$$\begin{aligned}\lim_{x_k\to-2i}\frac{||2x_k(1-\frac{4}{x_k^2}+\frac{i4}{x_k})+x_k^2(8x_k^{-3}-i4x_k^{-2})||}{||2x_k(2+\frac{i4}{x_k})+x_k^2(i4x_k^{-2})||} &= \frac{2x(-2i)+4i}{4x(-2i)+12i}\\ &= 0\end{aligned}$$

Thus, it follows that given function $f(x)$ converges Q-super linear to both zeros $x^*=\pm 2i$

Further, we can determine that it is Q-quadratic convergence with power order of 2 at points $x^*=+2i$.

$$\begin{aligned}\lim_{x_k\to+2i}\frac{||x_k-\frac{x_K^2+4}{2x_k}-2x_ki||}{||(x_k-2i)^2||} &= \lim_{x_k\to+2i}\frac{||2x_k^2-x_k^2-4-4ix_k||}{||2x_k(x_k^2-4x_ki+4)||}\\ &= \lim_{x_k\to+2i}\frac{||x_k^2-4-4x_ki||}{||2x_k(x_k^2-4x_ki+4)||}, \quad [\frac{0}{0}]form.\end{aligned}$$

Applying L'Hopital's rules to above results, we have

$$\begin{aligned}\lim_{x_k\to+2i}\frac{||2x_k-4i||}{||2(x_k^2-4x_ki+4)+2x_k(2x_k-4i)||} &= \lim_{x_k\to+2i}\frac{||2||}{||12x_k-16i||}\\ &= \frac{2}{24i-16i}\\ &= \frac{1}{4i}.\end{aligned}$$

Now, where $x^* = -2i$, it follows that

$$\lim_{x_k \to -2i} \frac{||x_k - \frac{x_K^2+4}{2x_k} + 2x_k i||}{||(x_k+2i)^2||} = \lim_{x_k \to -2i} \frac{||2x_k^2 - x_k^2 - 4 + 4ix_k||}{||2x_k(x_k^2 + 4x_k i - 4)||}$$
$$= \lim_{x_k \to -2i} \frac{||x_k^2 - 4 + 4x_k i||}{||2x_k(x_k^2 + 4x_k i - 4)||}, \quad [\frac{0}{0}]form.$$

Applying L'Hopital's rules to above results, we have

$$\lim_{x_k \to -2i} \frac{||2x_k + 4i||}{||2(x_k^2 + 4x_k i - 4) + 2x_k(2x_k + 4i)||} = \lim_{x_k \to -2i} \frac{||2||}{||12x_k + 16i||}$$
$$= \frac{2}{24i + 16i}$$
$$= -\frac{1}{4i}.$$

Hence, this function is not a Q-quadratic convergence at points $x^* = \pm 2i$ because of $\pm\frac{1}{4i} \notin (0, \infty)$.

As a result, **Q-super linear convergence** of function, $f(x) = x^2 + 4$, can be found at points $x^* = \pm 2i$

3) The classical definition of a stationary point (critical point) is found by equating $f^{'}(x) = 0$. In case of multivariate equations, we equate the corresponding first order partial differentials to zeros, namely $\frac{\delta f(x_1,x_2)}{\delta x_1}$ and $\frac{\delta f(x_1,x_2)}{\delta x_2} = 0$.

Using these definitions we compute the derivatives for the given function;

$$f(x_1, x_2) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 1) \tag{7}$$

$$\frac{\delta f}{\delta x_1} = 6x_1^2 - 6x_1 - 12x_1x_2 + 6x_2^2 + 6x_2$$
$$\frac{\delta f}{\delta x_1} = 6\big[x_1^2 + x_2^2 - 2x_1x_2 - x_1 + x_2\big]$$

$$\frac{\delta f}{\delta x_1} = 6\big[(x_1 - x_2)^2 - x_1 + x_2\big] \tag{8}$$

Similarly,

$$\frac{\delta f}{\delta x_2} = -6x_1^2 + 12x_1x_2 + 6x_1$$
$$\frac{\delta f}{\delta x_2} = -6\big[x_1^2 - 2x_1x_2 - x_1\big]$$

$$\frac{\delta f}{\delta x_2} = -6x_1\big[x_1 - 2x_2 - 1\big] \tag{9}$$

Equating both 8 and 9 to zeros we find the solutions, i.e. $-6x_1 = 0$ or $x_1 = 2x_2 + 1$ If $x_1 = 0$, substituting, we have; $x_2 = 0$ or $x_2 = -1$ This is one of the possible stationary points The other stationary point can be found at $x_1 = 2x_2 + 1$, substituting this we have; $x_2 = 0$ or $x_2 = -1$, in turn having $x_1 = 1$ or $x_1 = -1$

Now to classify if the given points are either maximum or minimum points we have the criteria that if $f_{x_1x_1} < 0$ and $f_{x_2x_2} < 0$ at the stationary point (x_1, x_2) then we have a **maximum point**, else if, $f_{x_2x_2} > 0$ and $f_{x_2x_2} > 0$ then we have a **minimum point**. Computing the second order partial derivatives of f we have

$$\frac{\delta^2 f}{\delta x_1^2} = 12x_1 - 6 - 12x_2 \tag{10}$$

$$\frac{\delta^2 f}{\delta x_2^2} = 12x_1 \tag{11}$$

After substitution of stationary points to the above equations we have the **minimum** of the function at $(1, 0)$ Similarly, we also have the **maximum** of the function at (-1,-1).

In addition there exists saddle point if $f_{x_1x_1} = 0$ and $f_{x_2x_2} = 0$. Thus, the equation has **saddle points** at (0,0), (0,-1).

4) The problem requires us to classify the given set of LP/IQP problems and the solvers required to be used for them.

For the problems given in **models.zip**, we have a common objective function to minimize.

$$\text{minimize } x^2 - y^2 \tag{12}$$

This objective function in itself is non-linear, rather it is the equation of a hyperbola. Let us analyze this function by the various methods possible through the NEOS solvers, while we also argue geometrically is it is possible to minimize this function.

The equation 12 can also be viewed as setting $|y| >> |x|$ or as setting $x^2 = 0$ with the max possible y^2

In case one (conv1.mod), we impose a linear constraint on 12, say

$$s.t. \quad x + y <= 1$$

This equation is the equation of a line, when we analyze this geometrically we could say that the minimum point where the set of lines intersect the hyperbola, we have then minimized the function. If we further limit x and y to be bounded then we could form a closer minimized value over a limited interval , i.e $x \in [-2.5, \infty]$ and $y \in [-3, \infty]$. Solution of this means $x < y \leq 1 - x$,

Now for the second problem we have the constraints as $x < y \leq \sqrt{2 - x^2}$, we can view this as a circle intersecting the hyperbola.

Models	CPLEX
conv1.mod	This solution is at x = 0, y = -3 when converted to conv1.lp format
conv2.mod	Diagonal QP Hessian has elements of the wrong sign.
conv3.mod	Diagonal QP Hessian has elements of the wrong sign.
conv4.mod	Diagonal QP Hessian has elements of the wrong sign.
(Integer)conv1.mod	The solution is at x = 0 and y = -3.0
(Integer)conv2.mod	QP Hessian is not positive semi-definite.
(Integer)conv3.mod	QP Hessian is not positive semi-definite.
(Integer)conv4.mod	QP Hessian is not positive semi-definite.

Models	XPRESS
conv1.mod	The quadratic objective is not convex
conv2.mod	The quadratic objective is not convex
conv3.mod	The quadratic objective is not convex
conv4.mod	The quadratic objective is not convex
(Integer)conv1.mod	The solution is found at x = 0, y = -3 and objective = -9
(Integer)conv2.mod	The solution is found at x = 0 y = -1 with objective = -1
(Integer)conv3.mod	Issues warning The quadratic part of row 'R1' defines a nonconvex region. Please check your model or use Xpress-SLP, but no solution found
(Integer)conv4.mod	The solution is found at x = 0 and y =1 with objective = -1

The third problem is interesting as we have two constraints for this

$$y > x$$
$$\sqrt{1-x^2} \leq y \leq \sqrt{3-x^2}$$

We view this problem as two concentric circles intersecting with the hyperbola.

Lastly, the fourth problem we have the constraints set as $x < y \leq \sqrt{3-x^2}$ this is again as seen previous a circle intersecting a hyperbola, but with different bounds.

While most set of equations can be solved in a continuous manner, with fractional increments, we could also say we require only integers and solve accordingly. Below are the tables and their corresponding solutions or errors as listed.

Models	GUROBI
conv1.mod	Quadratic objective is not positive definite and hence no solution
conv2.mod	Quadratic objective or constraint is not positive definite
conv3.mod	Quadratic objective or constraint is not positive definite
conv4.mod	Quadratic objective or constraint is not positive definite
(Integer)conv1.mod	The solution is at x = 0 y = -3 and the objective = -9
(Integer)conv2.mod	The solution is at x = 0 and y = 1 with objective = -1
(Integer)conv3.mod	Quadratic objective or constraint is not positive definite
(Integer)conv4.mod	Solution found at x = 0 y = 1 and objective = -1

Models	KNITRO
conv1.mod	The solution was found at $x = -9.11785e^{-13}$ and $y = -3$ and objective $= -9$
conv2.mod	The solution was found at $x = 7.31248e^{-10}$ and y = 1.41422 with objective = -2.00001
conv3.mod	The solution was found at $x = 9.66462e^{-10}$ and y = 1.73205 with objective = -3.00001
conv4.mod	The solution was found at $x = 2.34256e^{-10}$ and y = 1.73205 with objective = -3
(Integer)conv1.mod	The solution was found at $x = 0$ $y = -3$ and objective $= = -9$
(Integer)conv2.mod	The solution was found at x = 0 y = 1 with objective = -1
(Integer)conv3.mod	The solution was found at x = 0 y = -1 with objective = -1
(Integer)conv4.mod	The solution was found at x = 0 y = -1 and objective = -1

Models	SCIP
conv1.mod	The solution was found at x = 0 and y = -3 with objective = -9
conv2.mod	The solution is found at $x = -1.46664e^{-09}$ and y = -1.41421 with objective = -2
conv3.mod	The solution is found at x = 0 and y = 1.73205 with objective = -3
conv4.mod	The solution is found at $x = -5.12146e^{-09}$ and y = 1.73205 with objective = -3
(Integer)conv1.mod	The solution was found at x = 0 and y = -3 with objective = -9
(Integer)conv2.mod	The solution was found at x =0 y = -1 with objective = -1
(Integer)conv3.mod	The solution is found at x = 0 and y = -1 with objective = -1
(Integer)conv4.mod	The solution was found at x = 0 y = -1 with objective = -1

A point to note here is that when we set the solvers to be integers, we approximate to the nearest possible integer. From the Neos solver documentation we learn which kinds of problems the sovlers are designed for:

Solver	Types of problems able to solve
CPLEX	LP, MILP, SOCP
GUROBI	LP, QP, QCP, MILP, MIQP, MIQCP
SCIP	GLOBAL OPT., MILP, MINCO
XPRESS	LP, MILP, SOCP
KNITRO	COMPLIMENTARY PROBLEMS, EQUILIBRIUM CONSTRAINTS, MINCO, NCO