Homework 5 Solutions Fall 2011

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Note: graphs not to scale

1. If you saw the movie The Perfect Storm then you know that commercial fishing is a risky venture. In particular a fishing boat cannot be assured of finding a sufficient number of fish to make any money on the trip. Suppose that there are two fishermen, George and Mark, who are each setting off on fishing trips in the morning. Both are experienced fishermen so they each know where the fish are likely to be. However they also know that they cannot be absolutely sure that they will be successful. If they find a large school of fish then the fishing trip will earn a profit for each of them of \$50,000. However if they cannot find a large school of fish then they will earn no profits on the trip (we will assume that they can find enough fish to cover the costs of the trip). Each fisherman knows that the likelihood that he will find a large school of fish is .9 so that the probability that he will not find a large school is .1.

a) What are the expected profits of each of the fishing trips?

Answer The expected profits are

$$EV = .9\,(50000) + .1\,(0) \implies \boxed{EV = 45000}$$

b) Suppose that the night before they sail George proposes that they pool their risks. In particular he proposes that if only one of them find a large school of fish then they will share the profits of the successful trip equally (if neither finds the large school or if both find the large school then they exchange no money). What are the expected profits of each of the fishing trips? If both fishermen are risk averse then will Mark accept the proposal? Explain your answer.

Answer In this case, expected profit will be

$$EV = .81\,(50000) + .18\,(25000) + .01\,(0) = 45000$$

The expected profits are the same and there is less risk in this scenario, thus Mark will accept the proposal. The deal smooths income for the two individuals.

c) Suppose that George's preferences can be represented by the utility function

$$u\,(x) = (.002x)^{1/2}$$

What is the certainty equivalent to his original situation (i.e. where he does not pool risks with Mark)? What is the risk premium? Illustrate the certainty equivalent and the risk premium in a diagram.

Answer George's original expected utility is

$$EU = .9\,(.002 * 50000)^{1/2} + .1\,(0)^{1/2} \implies \boxed{EU = 9}$$

Therefore

$$U\,(CE) = EU \implies (.002CE)^{1/2} = 9 \implies \boxed{CE = 40500}$$

Risk premium is

$$RP = EV - CE = 45000 - 40500 \implies \boxed{RP = 4500}$$

2. Rachel has an income of \$10,000 but she has a 10% chance of suffering a loss of \$1,900. Her utility of income can be represented by the function $U\,(\$) = (\$)^{1/2}$.

a) What is her expected utility given that she has a 10% chance of a loss? What is her expected income?

Answer Her expected utility is

$$EU = .1\,(8100)^{1/2} + .9\,(10000)^{1/2} = 90 + 9 \implies \boxed{EU = 99}$$

and expected income is

$$EI = .1\,(8100) + .9\,(10000) = 810 + 9000 \implies \boxed{EI = 9810}$$

b) How much coverage would a full insurance policy give her? What is the <u>maximum</u> that she would be willing to pay for a full insurance policy? How much would an insurance company expect to make if it sold her a full insurance policy at the maximum price that she is willing to pay?

Answer Under full insurance coverage will equal loss. Hence $\boxed{C = 1900}$. The maximum an individual is willing to pay is $I - CE$. Thus she will pay up to

$$I - CE = 10000 - U^{-1}\,(99) = 10000 - (99)^2 = 10000 - 9801 = \boxed{199}$$

(you can also take risk premium minus expected loss). The insurance company will have expected profits

$$EP = .1\,(199 - 1900) + .9\,(199) = 199 - .1\,(1900) \implies \boxed{EP = 9}$$

c) What is the price per dollar of coverage for a fair insurance contract? Is the full insurance contract in part (b) a fair contract?

Answer Under fair insurance, $r = p \implies \boxed{r = .1}$. Full insurance in part (b) has a rate $199/1900 > .1$ so it is not fair.

d) If Rachel is offered only a policy that provides \$1,000 of coverage at the fair price will she take this policy (over no policy)? If she has the option of taking the \$1,000 policy at the fair price or purchasing the full insurance contract from part (b) which policy would she prefer?

Answer At fair price, Rachel will insure up to loss, so she will purchase \$1,000 of coverage. Her expected utility with full insurance in part (b) is

$$EU = .1\,(8100 + 1900 - 199)^{1/2} + .9\,(10000 - 199)^{1/2} = (10000 - 199)^{1/2} = 99$$

and with the \$1,000 at fair price the expected utility is

$$EU = .1\,(10000 + 1000 - 100)^{1/2} + .9\,(10000 - 100)^{1/2} = 99.04$$

Since expected utility is greater under the new plan, she will buy \$1,000 of coverage at .1.

3. Recall Scarlett the OB-GYN from class. Suppose the probability that she is sued is .1 and her income is 10,000. In the case she is sued she will lose 7,500 in the settlement. She may purchase malpractice insurance at a rate of $\$r$ per \$1 of coverage. Finally assume that her utility of income is $U(\$) = (\$)^{1/2}$. What is Scarlett's demand curve for insurance (that is find her demand for insurance for all $r \geq .1$)?

Answer Demand for insurance will be the optimal level of coverage for any rate r. Thus it will solve the utility maximization problem. We begin with

$$\max_C EU = .9\,(10000 - rC)^{1/2} + .1\,(10000 - 7500 - rC + C)^{1/2}$$

With FOC

$$-.45r\,(10000 - rC)^{-1/2} + .05\,(1-r)\,(10000 - 7500 - rC + C)^{-1/2} = 0$$

Solving for C,

$$\begin{aligned}
9r\,(10000 - rC)^{-1/2} &= (1-r)\,(2500 - rC + C)^{-1/2} \\
(1-r)^2\,(10000 - rC) &= 81r^2\,(2500 + C\,(1-r)) \\
10000\,(1-r)^2 - rC\,(1-r)^2 &= 2500\left(81r^2\right) + C\,(1-r)\,81r^2 \\
C\,(1-r)\,81r^2 + rC\,(1-r)^2 &= 10000\,(1-r)^2 - 2500\left(81r^2\right) \\
C\left((1-r)\,81r^2 + r\,(1-r)^2\right) &= 10000\,(1-r)^2 - 2500\left(81r^2\right) \\
C &= \frac{10000\,(1-r)^2 - 2500\left(81r^2\right)}{(1-r)\,81r^2 + r\,(1-r)^2}
\end{aligned}$$

This is her demand for insurance.

4. In the state of Oklahoma the Sooner Insurance Company has noticed the following two differences between men and women drivers:

1. Men are more likely to have driving accidents: the probability that a man has a car accident is 25% and the probability that a woman has an accident is 19%.
2. When a man has an accident the damage is likely to be higher: the average damage to a 15,000 car driven by a man is $7,500 (so the car is only worth $7,500 after the accident) and only $3,600 if driven by a woman (so the car is worth only $11,400).

In addition the firm knows that men are willing to pay at most $2,343.75for a full insurance policy. Women are willing to pay at most $745.50 for a full insurance policy. Drivers are equally divided between men and women.

a) What is the cost of providing full fair insurance for men? Given the information provided above will men purchase this policy? What is the cost of providing full fair insurance for women? Will women purchase this policy?

Answer At fair insurance $r = p = .25$. The cost of full insurance is thus $.25\,(7500) = \boxed{1875}$. This costs less than the maximum men are willing to pay, thus they will purchase it. For women the cost is $.19\,(3600) = \boxed{684}$, thus they will purchase the policy.

b) What is the expected value of a $15,000 car driven by a man? Given their willingness to pay for full insurance what is the certainty equivalent value of the car? What is the risk premium that they are willing to pay? Illustrate the expected value, the certainty equivalent and the risk premium in a diagram.

Answer The expected value is

$$EV = .25\,(7500) + .75\,(15000) \implies \boxed{EV = 13{,}125}$$

Since the maximum willingness to pay is $I - CE$ then

$$CE = 15000 - 2345.75 \implies \boxed{CE = 12{,}656.25}$$

$$RP = EV - CE = 13125 - 12656.25 \implies \boxed{RP = 468.75}$$

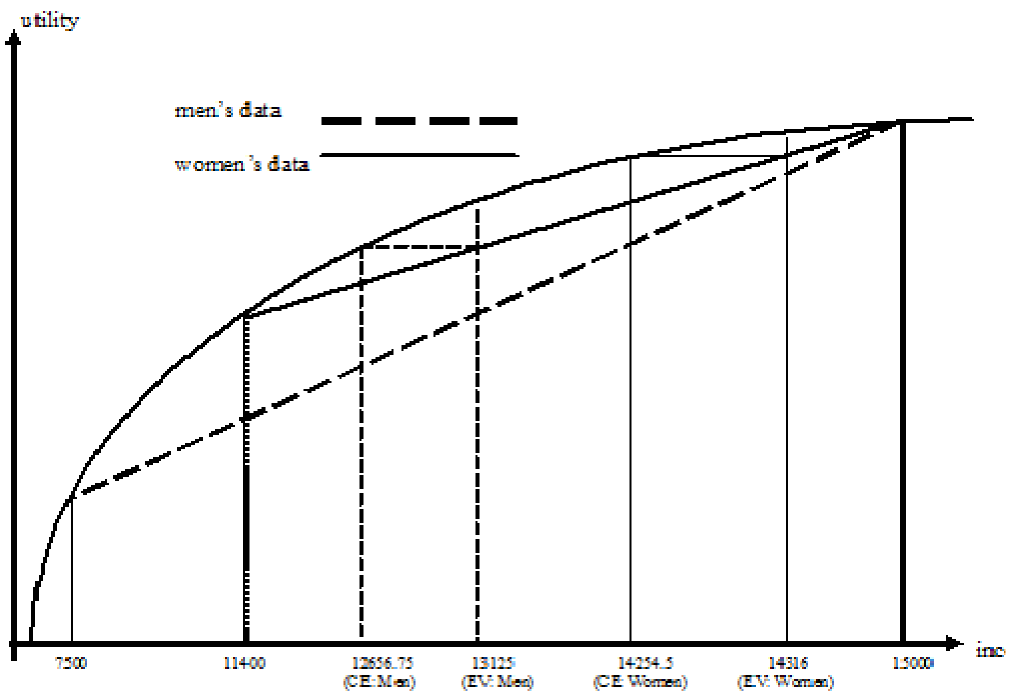
c) What is the expected value of a $15,000 car driven by a woman? Given their willingness to pay for full insurance what is the certainty equivalent value of the car? What is the risk premium that they are willing to pay? Illustrate the expected value, the certainty equivalent and the risk premium in a diagram.

Answer Similarly to men,

$$EV = .19\,(11400) + .81\,(15000) \implies \boxed{EV = 14{,}316}$$

$$CE = 15,000\text{–}745.50 \implies \boxed{CE = 14{,}254.5}$$

$$RP = EV\text{–}CE = 14{,}316 - 14{,}254.5 \implies \boxed{RP = 61.5}$$



Suppose that the state government passes anti-discrimination legislation and requires that insurance companies offer policies at the same price per dollar of coverage to both men and women.

d) Suppose that the firm offers only a single price per dollar of coverage and allows individuals to buy insurance to fully insure against their losses. Show that the firm cannot offer a single price for insurance that does not lose money and is acceptable to both men and women.

Answer If the firm only offers one price and it must be acceptable to both men and women then it will charge the maximum women are willing to pay. The expected profit for women will be their risk premium, 61.5. This price $r = 745.50/3600 \approx .2$ will lead men to fully insure, thus expected profit for the firm from men is

$$EP = .2\,(7500) - .25\,(7500) = -375$$

Since men and women are represented equally in the population, it suffices to observe that since the loss with men is greater than the profit from women, the firm will lose money overall.

e) Given your answer to (c) what policy will the firm offer? Who will be covered men or women?

Answer Since the firm can only offer one policy, it will sell it to men at \$1,875 per policy. Women will not be covered.

5. Lloyd's Limited sells boat insurance. A typical insurance policy costs \$r per dollar of coverage which is paid out if an accident occurs. The average probability of a boating accident is 20%.

a) Write the expected payout of an insurance policy. What is the value of r such that the expected payout of the policy is 0?

Answer The expected payout of an insurance policy is

$$EP = .8\,(-rC) + .2\,(C - rC) \implies \boxed{EP = .2C - rC}$$

The value of r leading to expected payout of 0 is fair insurance thus $r = p \implies \boxed{r = .2}$.

There are two kinds of individuals. 75% of boaters are low risk who have only a 10% chance of an accident. 25% of boaters are high risk who have a 50% chance of an accident. Lloyds cannot tell whether someone is high or low risk, and it sells the insurance policy at $r = .20$ per dollar of coverage to all individuals. For simplicity, assume that all boaters have boats valued at \$120,000 and that the damage from an accident is \$90,000 (leaving the value of the boat at only \$30,000 in the event of an accident). Finally assume that all boaters have a utility function $u\,(\$) = \ln\,(\$)$ where \$ represent wealth which is the value of the boat net insurance payments (i.e. subtract the cost of insurance and add the payout from the policy if applicable).

<u>High Risk</u>
Luna is a high risk boater so her probability of having an accident is 50%.

b) Assume that she does not have insurance. What is her expected wealth? What is her expected utility?

Answer Luna's expected wealth is

$$EW = .5\,(120000) + .5\,(30000) \implies \boxed{EW = 75000}$$

Her expected utility is

$$EU = .5\ln(120000) + .5\ln(30000) = \ln(120000)^{1/2} + \ln(30000)^{1/2} = \ln(3600000000)^{1/2} = \ln(60000) \approx 11$$

c) What is the certainty equivalent to her situation without insurance? What is the risk premium associated with her situation without insurance?

Answer The certainty equivalent is

$$U(CE) = EU \implies \ln(CE) = EU \implies CE = e^{11} \implies \boxed{CE = 60000}$$

Risk premium

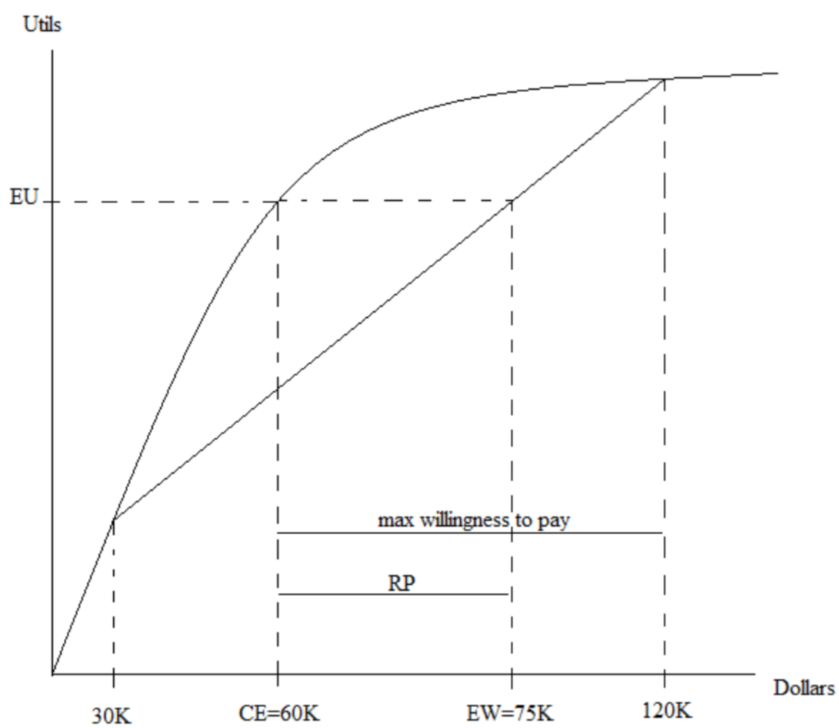
$$RP = EV - CE = 75000 - 60000 \implies \boxed{RP = 15000}$$

d) What is the maximum that Luna would be willing to pay for a full insurance policy?

Answer The maximum Luna is willing to pay is $I - CE = 120000 - 60000 = \boxed{60000}$

e) Illustrate her expected utility, expected wealth, certainty equivalent, the risk premium and her maximum willingness to pay for a full insurance policy.

Answer



f) At the price of .2 per dollar of coverage, would she be willing to fully insure? Briefly explain your answer

Answer The price $r = .2$ is less than her fair price of .5, thus she will fully insure. Similarly, full insurance at .2 costs 45000 while she is willing to pay 60000 for fair insurance.

Low Risk
Helios is a low risk boater so his probability of an accident is 10%.

g) Write his expected utility if he purchases $\$C$ of coverage at a price of .2 per dollar of coverage

Answer His expected utility is

$$EU = .1\ln(30000 + C - .2C) + .9\ln(120000 - .2C)$$

h) Using the expected utility from part (h) write the first order conditions that define the coverage C that maximizes his expected utility.

Answer His problem is

$$\max_C EU = .1\ln(30000 + C - .2C) + .9\ln(120000 - .2C)$$

with FOC

$$\frac{.08}{(30000 + .8C)} - \frac{.18}{(120000 - .2C)} = 0$$

i) Show that full insurance does NOT solve the FOC above.

Answer At full insurance, $C = 60000$. Considering the FOC,

$$\begin{aligned}\frac{8}{(30000 + .8(60000))} - \frac{18}{(120000 - .2(60000))} &= \frac{8}{112000} - \frac{18}{112000} \\ &\neq 0\end{aligned}$$

j) Solve the FOC to determine how much insurance that Helios will buy at a price of .2 per dollar of coverage.

Answer Solving the FOC leads to

$$\begin{aligned}\frac{4}{(30000+.8C)} &= \frac{9}{(120000-.2C)}\\ 4\,(120000-.2C) &= .9\,(30000+.8C)\\ 480000-.8C &= 270000+7.2C\\ 210000 &= 8C\\ \implies C &= 26250\end{aligned}$$

A second insurance company, Greenwich LLC , offers another insurance policy. The price of Greenwich's policy is only .10 per dollar of coverage. However, the maximum coverage that Greenwich will sell is $30,000.

k) Will Helios purchase a policy from Greenwich? Briefly explain.

Answer This is Helios's fair price, thus he will purchase as much as possible (30000). This is greater than what he would purchase as Lloyds and cheaper still. Thus he will purchase the Greenwich policy.

l) Will Luna purchase a policy from Greenwich? Briefly explain.

Answer Luna's expected utility with insurance from Lloyds is

$$EU = u\,(120000-18000) = \ln(102000)$$

meanwhile from Greenwich she will purchase 30000 because it is cheaper than fair price and her expected utility will be

$$EU = .5\ln(120000-3000) + .5\ln(30000+27000) = \ln(117000*57000)^{1/2} = \ln(81663)$$

which is less than insurance from Lloyds. Thus she will purchase insurance from Lloyds.

m) Given the answers above would you Lloyds to continue selling the $r = .2$ policy? Briefly explain.

Answer Lloyds will not. At .2 the firm is losing money from high risk and making money on low risk, but since all low risk will move to Greenwich, Lloyds will charge more and only take business from high risk individuals.

6. Suppose Sheila's utility functions is $U_S(I) = I^{1/2}$ and Bruce's is $U_B(I) = I^{1/4}$.

a) Use the Arrow-Pratt measure of risk aversion (as defined in the text) to show that Bruce is more risk averse than Sheila.

Answer Arrow-Pratt measure of risk aversion is given byh

$$A = -\frac{U''(I)}{U'(I)}$$

Bruce's measure of risk aversion is,

$$U_B' = .25W^{-.75}$$
$$U_B'' = -.1875W^{-1.75}$$
$$A_B = -\frac{-.1875W^{-1.75}}{.25W^{-.75}} = .75W^{-1}$$

Sheila's is

$$U_S' = .5W^{-.5}$$
$$U_S'' = -.25W^{-1.5}$$
$$A_S = -\frac{-.25W^{-1.5}}{.5W^{-.5}} = .5W^{-1}$$

Thus Bruce is more risk averse than Sheila.

Let g be a gamble that pays off w_1 with probability p and w_2 with probability $1-p$.

b) Write the expressions for the certainty equivalents to this gamble for both Sheila and Bruce (assume that their only income will be the payouts of the gamble).

Answer The certainty equivalent for Sheila will be the value of certain money that gives the same expected utility as the gamble. Hence

$$\begin{aligned} U_S(CE_S) &= pw_1^{.5} + (1-p)\,w_2^{.5} \\ (CE_S)^{.5} &= pw_1^{.5} + (1-p)\,w_2^{.5} \\ CE_S &= \left(pw_1^{.5} + (1-p)\,w_2^{.5}\right)^2 \end{aligned}$$

The certainty equivalent for Bruce is calculated with Bruce's utility function

$$\begin{aligned} U_B(CE_B) &= pw_1^{.25} + (1-p)\,w_2^{.25} \\ (CE_B)^{.25} &= pw_1^{.25} + (1-p)\,w_2^{.25} \\ CE_B &= \left(w_1^{.25} + (1-p)\,w_2^{.25}\right)^4 \end{aligned}$$

c) Show that Bruce's utility of Sheila's certainty equivalent is at least as large as Bruce's utility of his own certainty equivalent. Hint: $f(x) = x^{1/2}$ is a concave function so we know that it satisfies $f(px + (1-p)y) \geq pf(x) + (1-p)f(y)$.

Answer The utility of Bruce's certainty equivalent to Bruce is

$$CE_B^{.25} = \left(pw_1^{.25} + (1-p)\, w_2^{.25}\right)$$

Similarly, Bruce's utility of Sheila's certainty equivalent is

$$CE_S^{.25} = \left(pw_1^{.5} + (1-p)\, w_2^{.5}\right)^{.5}$$

Since $f(x) = x^{.5}$ is a concave function. As described in the problem set, it satisfies the inequality $f(px + (1-p)\, y) \geq pf(x) + (1-p)\, f(y)$. Therefore

$$CE_S^{.25} = \left(pw_1^{.5} + (1-p)\, w_2^{.5}\right)^{.5} \geq pw_1^{.25} + (1-p)\, w_2^{.25} = CE_B^{.25}$$

The equation above shows that the utility Bruce gets from Sheila's certainty equivalent is greater than or equal to (at least as large as) the utility he receives from his own certainty equivalent.

d) Use your answer in part (c) to show that Bruce's certainty equivalent is less than or equal to Sheila's certainty equivalent. Moreover show that Bruce's risk premium is higher than Shelia's. So more risk averse individuals require greater risk premia.

Answer Because Bruce gets more utility from more money, that he gets at least as much utility from Sheila's CE implies that her CE is greater than or equal to his. Mathematically from the equation above

$$\begin{aligned} CE_S^{.25} &\geq CE_B^{.25} \\ CE_S &\geq CE_B \end{aligned}$$

Furthermore, since risk premium equals expected value minus certainty equivalent and the expected value of the gamble is the same for both people, we see

$$\begin{aligned} EV - CE_S &\leq EV - CE_B \\ RP_S &\leq RP_B \end{aligned}$$

This equation shows that Bruce's risk premium is larger than Sheila's.

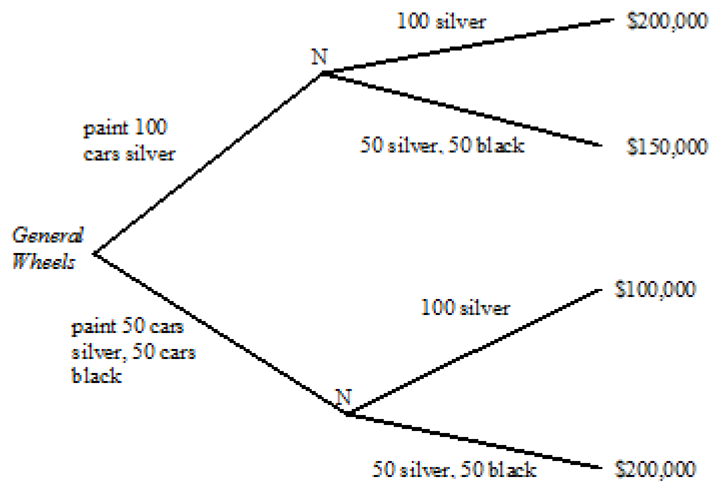
7. *General Wheels* is an auto manufacturer. Each month General Wheels readies 100 cars to be shipped to the dealers. *General Wheels* is certain that it can sell 100 cars per month at a price of \$10,000 per car. Each car costs \$7,000 to manufacture and there are \$100,000 in overhead costs (fixed costs) per month.

Just before the cars are shipped General Wheels must decide what color to paint the cars. It can choose one of two colors, silver or black. When the color choice is made however, General Wheels does not know which color consumers will demand. All 100 consumers may want to purchase silver cars (so none of them demand black cars). Alternatively, 50 customers may choose black cars and 50 choose silver cars. Each event is equally likely.

Hence General Wheels must choose whether to paint all 100 cars silver or to paint 50 of them silver and 50 of them black. If General Wheels paints all of them silver and consumers demand 50 black cars then 50 silver cars will have to be repainted black at a cost of \$1,000 per car. If General Wheels paints 50 of them black and consumers demand only silver cars then the 50 black cars must be repainted silver at a cost of \$2,000 per car.

a) Illustrate the decision problem of *General Wheels* in a decision tree.

Answer



b) How many cars painted black are *ex ante* optimal? Is that many always *ex post* optimal? If not then under what circumstances is it not ex post optimal?

Answer It is ex ante optimal for General Wheels to paint all cars silver, i.e. for them to paint 0 cars black. The expected profit from painting all cars silver is

$$EV = .5\,(200,000) + .5\,(150,000) = 175000$$

which is higher than the expected profit from painting 50 of the cars black,

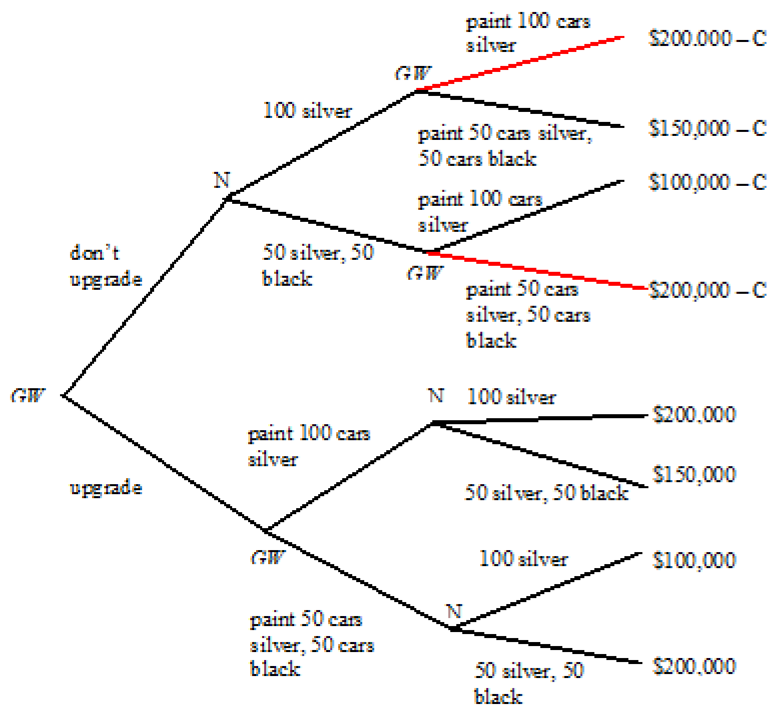
$$EV = .5\,(100,000) + .5\,(200,000) = 150000$$

Painting all cars silver is not always *ex post* optimal – if it turns out that customers wanted 50 black cars, then General Wheels would have been better off painting 50 cars silver and the rest black.

By upgrading its communications and inventory systems General Wheels can streamline its delivery system and actually paint a car after it has been sold. In particular General Wheels can wait to find out what color the consumer would like to purchase before painting the car.

c) Illustrate the decision problem of *General Wheels* when it can choose whether or not to invest in the upgrade in a decision tree. In your tree let $C =$ the cost of the upgrade per month.

Answer



d) What is the highest price (C) that the firm would pay for the upgrade?

Answer If General Wheels chooses to upgrade, it will know in advance the preferences of its customers, so that it will paint 100 cars silver if customers want 100 silver cars, and paint 50 cars silver and 50 cars black if customers want 50 silver cars and 50 black cars. Hence if General Wheels chooses to upgrade, its profit will always be 200000–C, where C is the cost of the upgrade.

If General Wheels does not upgrade, then it will select its *ex ante* optimal choice of painting all cars silver, which has an expected profit of

$$EV = .5\,(200,000) + .5\,(150,000) = 175000$$

General Wheels will upgrade if

$$200000–C > 175,000 \implies \boxed{C \leq 25,000}$$

Bonus: Suppose that an individual has the opportunity to buy an asset that is worth either x_1 or x_2 and the probability of each value is p and $(1-p)$ respectively. The price of the asset is \$1 per unit. In addition assume that the expected value of the asset is $px_1 + (1-p)x_2 > 1$ so that purchasing the asset is not a fair bet. Show that every risk averse investor will purchase a positive number of units of this asset.

Answer Suppose the individual starts out with some level of income I and the individual invests a portion $\alpha \in [0,1]$ in this asset, then wealth in the two states of the world will be

$$\begin{aligned} w_1 &= I\alpha x_1 + (1-\alpha) I \\ w_2 &= I\alpha x_2 + (1-\alpha) I \end{aligned}$$

The individual's expected utility is thus

$$EU = pU(w_1) + (1-p) U(w_2)$$

The individual's problem is

$$\max_{\alpha} EU = pU(w_1) + (1-p) U(w_2)$$

which leads to FOC

$$p(Ix_1 - I) U'(w_1) + (1-p)(Ix_2 - I) U'(w_2) = 0$$

Rearranging the FOC,

$$\begin{aligned} p(Ix_1 - I) U'(w_1) &= (p-1)(Ix_2 - I) U'(w_2) \\ \frac{U'(w_1)}{U'(w_2)} &= \frac{(p-1)}{p} \cdot \frac{(Ix_2 - I)}{(Ix_1 - I)} \\ \frac{U'(w_1)}{U'(w_2)} &= \frac{(p-1)}{p} \cdot \frac{x_2 - 1}{x_1 - 1} \end{aligned}$$

From the assumption about the bet, we know that $x_1 \neq x_2$ as$px_1 + (1-p)x_1 \ngtr 1$. Thus without loss of generality we can assume $x_1 > x_2$. Also, either $x_1, x_2 \geq 1$ in which case the individual will purchase the bet because income is always greater or $x_2 < 1$ (which is what we'll now consider). If this is the case then for $\alpha > 0$, $w_1 > w_2$ and thus

$$\frac{U'(w_1)}{U'(w_2)} < 1 \implies \frac{(p-1)}{p} \cdot \frac{x_2 - 1}{x_1 - 1} < 1 \implies px_1 + (1-p)x_2 > 1$$

Which is fine. However if $\alpha = 0$ then $w_1 = w_2$ and

$$\frac{U'(w_1)}{U'(w_2)} = 1 \implies px_1 + (1-p)x_2 = 1$$

which is a contradiction. Therefore α will be positive in order to satisfy the FOC and the individual will purchase the asset.