STAT W1211.001 Mid-term Examination

Practice Problems

Part I: Multiple Choice

1. Which of the following is true for events *M* and *N*?

    A. $(M \cup N)' = M' \cap N'$

    B. $(M \cap N)' = M' \cup N'$

    C. Neither *A* nor *B* is true
    D. Both *A* and *B* are true

    ANSWER: D

2. Which of the following statements are true?

    A. The unit for the standard deviation is the same as the unit for the data values.
    B. The average deviation of $x_1, x_2, \ldots\ldots, x_n$ from the mean $\bar{x}$ may be positive or negative.

    C. The average absolute deviation of $x_1, x_2, \ldots\ldots, x_n$ from the mean $\bar{x}$ is always zero.

    D. We use σ^2 (square of the lowercase Greek letter sigma) to denote the sample variance.

    ANSWER: A

3. How many permutations of size 3 can be constructed from the set (A, B, C, D, E)?

    A. 60
    B. 20
    C. 15
    D. 8

    ANSWER: A

4. If $P(A)$ = .30, $P(B)$ = .10 and events A and B are independent, then $P(A' \cap B')$ is

A. .63
B. .70
C. .90
D. .40

ANSWER: A

5. Which of the following statements are correct if $y_i = x_i + 5 \; \forall i = 1, 2, \ldots\ldots, n$?

A. $\bar{y} = \bar{x}$

B. $\bar{y} = \bar{x} + 5$

C. $s_y^2 = s_x^2 + 5$

D. $s_y^2 = s_x^2 + 25$

E. None of the above

ANSWER: B

6. Which of the following assumptions does not lead to the hypergeometric distribution?

A. The population or set to be sampled consists of N individuals, objects, or elements (a finite population).
B. Each individual can be characterized as a success (S) or a failure (F), and there are M successes in the population.
C. A sample of n individuals is selected without replacement in such a way that each subset of size n is equally likely to be chosen.
D. All of the above are assumptions leading to the hypergeometric distribution.

ANSWER: D

7. The cumulative distribution function $F(x)$ of a discrete random variable X is given by $F(0) = .30$, $F(1) = .70$, $F(2) = .90$, and $F(3) = 1.0$, then the value of the probability mass function $p(x)$ at $x = 1$ is

A. .30
B. .40
C. .20
D. .80

ANSWER: B

8. If the probability density function of a continuous random variable X is

$$f(x) = \begin{cases} .5x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

then, $P(1 \le x \le 1.5)$ is

A. .5625
B. .3125
C. .1250
D. .4375

ANSWER: B

9. Which of the following statements about the z_α notation used in your book is correct?

A. α of the area under the standard normal curve lies to the left of z_α.

B. $1-\alpha$ of the area under the standard normal curve lies to the right of z_α.

C. z_α is the $100(1-\alpha)th$ percentile of the standard normal distribution

D. z_α is the $100\alpha th$ percentile of the standard normal distribution

ANSWER: C

10. If X is a normally distributed random variable with a mean of 10 and standard deviation of 4, then the probability that X is between 6 and 16 is

A. .9332
B. .7745
C. .6587
D. .0668
E. .8413

ANSWER: B

11. A continuous random variable X is uniformly distributed over the interval [12, 18]. The variance of X is

A. 15
B. 6
C. 3
D. 9

ANSWER: C

Part II: Structured Questions

1. Let X denote the amount of space occupied by an article placed in a $1-\text{ft}^3$ packing container. The pdf of X is

$$f(x) = \begin{cases} 90x^8(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Obtain the cdf of *X*..

b. What is $P(X \le .5)[i.e., F(.5)]$?

c. What is $P(.25 < X \le .5)$?

d. Compute $E(X)$ and σ_x.

e. What is the probability that X is within 1 standard deviation of its mean value?

ANSWER:

a. $F(x) = \int_{-\infty}^{x} f(y)dy = \int_{0}^{x} 90y^8(1-y)dy = 90\int_{0}^{x}(y^8 - y^9)dy$

$$= 90\left(\frac{1}{9}y^9 - \frac{1}{10}y^{10}\right)\Big]_0^x = 10x^9 - 9x^{10}$$

Therefore,

$$F(x) = \begin{cases} 0 & \text{for } x \le o \\ 10x^9 - 9x^{10} & \text{for } 0 < x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

b. $F(.5) = 10(.5)^9 - 9(.5)^{10} \approx .0107$

c. $P(.25 \le X \le .5) = F(.5) - F(.25) \quad \approx .0107 - [10(.25)^9 - 9(.25)^{10}]$

$$\approx .0107 - .0000 \approx .0107$$

d. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{0}^{1} x \cdot 90x^8(1-x)dx = 90\int_{0}^{1} x^9(1-x)dx$

$$= 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = \frac{9}{11} \approx .8182$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_{0}^{1} x^2 \cdot 90x^8(1-x)dx = 90\int_{0}^{1} x^{10}(1-x)dx$$

$$= \frac{90}{11}x^{11} - \frac{90}{12}x^{12}\Big]_0^1 \approx .6818$$

$V(X) \approx .6818 - (.8182)^2 = .0124, \qquad \sigma_x = .11134.$

e. $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068)$

$$= .8465 - .1602 = .6863$$

2. Seventy percent of all vehicles examined at a certain emissions inspection station pass the inspection. Assuming that successive vehicles pass or fail independently of one another, calculate the following probabilities.

   a. P(all of the next three vehicles inspected pass)
   b. P(exactly one of the next three inspected passes)
   c. P(at most one of the next three vehicles inspected passes)
   d. Given that at least one of the next three vehicles passes inspection, what is the probability that all three pass?

ANSWER: P(pass) = .70

a. P(three pass) = (.70)(.70)(.70) = .343
b. P(exactly one passes) = (.70)(.30)(.30) + (.30)(.70)(.30) + (.30)(.30)(.70) = .189

c. P (at most one passes) = P(0 passes) + P(one passes) = $(.3)^3$ + .189 = .216

d. P(3 pass | 1 or more pass) = $\frac{P(3 \text{ pass} \cap \geq 1 \text{ pass})}{P(\geq 1 \text{ pass})} = \frac{P(3 \text{ pass})}{P(\geq 1 \text{ pass})} = \frac{.343}{.973} = .353$

3. a. A 10 meter by 10 meter plot of land, is divided into a grid of 100 squares. 300 seeds are scattered on this plot. Assume that each seed falls at random, so that it is equally likely to fall on any of the 100 squares. Consider the square in the upper left hand corner. What is the probability that exactly 4 seeds fall in it? What is the probability that 0 seeds fall in it?

b. A certain nucleus contains 10^6 genes. Each gene has probability 3.2×10^{-6} of being mutated. What is the probability that the nucleus contains at most 2 mutated genes?

c. How many people are needed so that the probability that at least one of them has the same birthday as you is greater than 0.5?

ANSWER:

a. To see that this is a binomial experiment, think of this as dropping 300 seeds, one after the other, and recording whether the seed falls into the upper left hand square or not. "Success" means falling into the square, and that happens with probability 1/100 = .01 (since there are 100 squares.) So, n = 300, and p = .01. The expected number of successes is μ = 500(.01) = 3. By the Poisson approximation,

$$P(\text{exactly 4 seeds}) \approx \frac{3^4}{4!} e^{-3} \approx .17$$

$$P(0 \text{ seeds}) \approx e^{-3} \approx .05$$

b. This is binomial, where P(mutated) = .0000032 , P(not mutated) = .9999968 , and we run 10^6 trials. The expected number of mutated cells is

$$\mu = 1 \times 10^6 \left(3.2 \times 10^{-6}\right) = 3.2$$

To find the probability of at most 2 mutated cells, we take P(0) + P(1) + P(2) , where

$$\begin{aligned} P(0 \text{ mutated cells}) &\approx e^{-3.2} \approx .0408 \\ P(\text{exactly 1 mutated cell}) &\approx 3.2\, e^{-3.2} \approx .1304 \\ P(\text{exactly 2 mutated cells}) &\approx \frac{(3.2)^2}{2!} e^{-3.2} \approx .2087 \\ \text{So, } P(\text{at most 2 mutated cells}) &\approx .3799 \end{aligned}$$

c. The number of people in a random collection of size n that have the same birthday as yourself is approximately Poisson distributed with mean n/365. Hence, the probability that at least one person has the same birthday as you is approximately $1 - e^{-n/365}$. Now, $e^{-x} = 1/2$ when $x = \log(2)$. Thus, $1 - e^{-n/365} \geq 1/2$ when $n/365 \geq \log(2)$. That is, there must be at least 365 log(2) people.

4. Suzanne is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. She figures that if she has an "on" day, then each of her examiners will pass her, independently of each other, with probability 0.8, whereas if she has an "off" day, this probability will be reduced to 0.4. Suppose that Suzanne will pass the examination if a majority of the examiners pass her. If she feels that she is twice as likely to have an "off" day as she is to have an "on" day, should she request an examination with three examiners or with five examiners?

ANSWER:

with 3: $P\{\text{pass}\} = \frac{1}{3}\left[\binom{3}{2}(.8)^2(.2) + (.8)^3\right] + \frac{2}{3}\left[\binom{3}{2}(.4)^2(.6) + (.4)^3\right]$

$= .533$

with 5: $P\{\text{pass}\} = \frac{1}{3}\sum_{i=3}^{5}\binom{5}{i}(.8)^i(.2)^{5-i} + \frac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(.4)^i(.6)^{5-i}$

$= .3038$