Homework 5 Solutions Fall 2011

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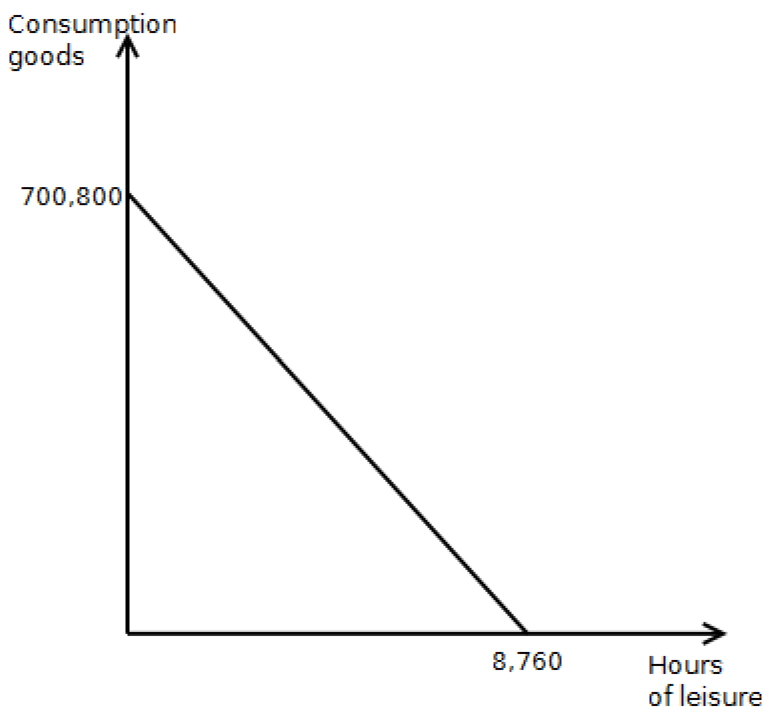
Note: graphs not to scale

1. There are 8760 hours in a (non-) leap year. Assume that a typical upper middle income worker earns \$80 per hour and has no other source of income. The worker can use his income to purchase a consumption good priced at \$1 per unit.

a) Illustrate the budget set of a typical upper middle income worker.

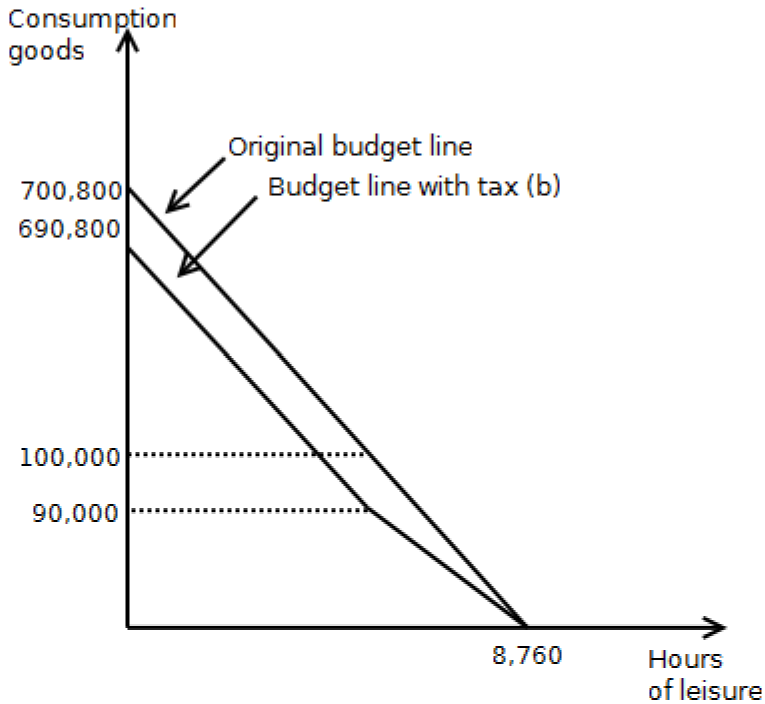
Answer We have $w = 80$ so a worker can consume a maximum of $80 \cdot 8760 = 700{,}800$ consumption goods or 8760 hours of labor.

Suppose that the government levies a Social Security tax on wage income. The tax is levied only on the first \$100,000 of income. The marginal tax rate is 10% so the worker pays \$.10 on every dollar earned up to 100,000. There are no taxes levied on income earned over \$100,000.



b) In your diagram for part (a) illustrate the effect of the tax system on the worker's budget set.

Answer For the first \$100,000, there will be a 10% tax, totaling \$10,000. Income after tax will be 700,800−10,000 = 690,800. Furthermore, the number of hours that used to pay \$100,000 will only earn \$90,000.



Given the wage rate and the tax system the typical worker chooses to work 40 hours per week for 50 weeks in the year for a total of 2,000 hours per year. You may assume that leisure and consumption goods are not perfect complements.

c) How much does the typical worker pay in taxes? How much does he earn after taxes? What is his MRS at his best bundle?

Answer At \$80/hour, in a year the typical worker would make $80 \cdot 2{,}000 = 160{,}000$, of which \$10,000 will be taken in taxes. Thus annual income will be \$150,000. This places the worker on the steeper part of the new budget line, with slope $\frac{w}{p} = 80$ and thus MRS$= 80$.

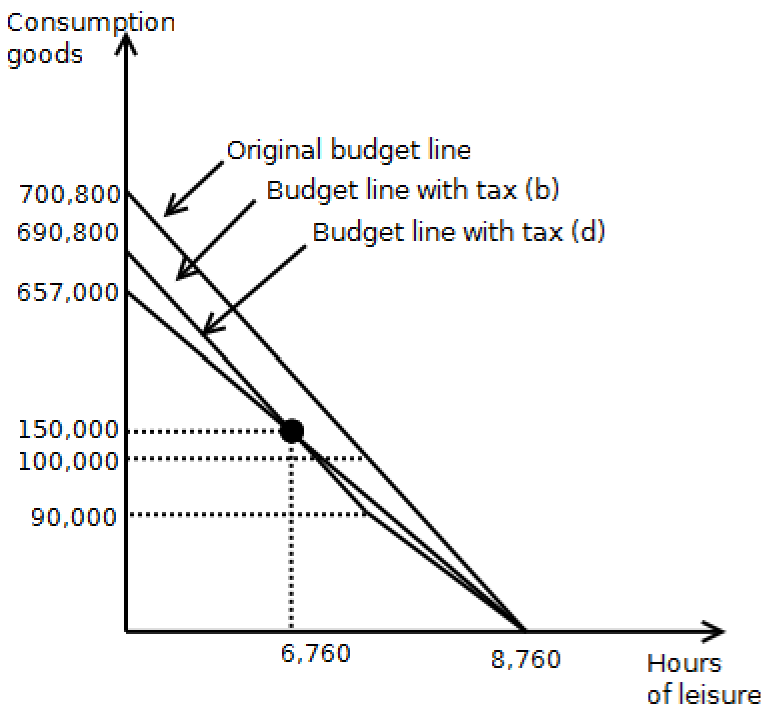
The government proposes to change the tax code. The government will lower the marginal tax rate to 6.25% (.0625). Simultaneously the government will tax every dollar earned (including income earned above \$100,000). In other words the worker will pay \$.0625 in taxes on every dollar earned.

d) Show algebraically that the bundle chosen by the worker under the old tax scheme is still available under the new tax scheme. Illustrate the new budget line in your diagram in (a).

Answer The bundle in part (c) is $(6760, 15000)$. If the typical worker were to work 2,000 hours after the new tax, income would be

$$2{,}000\,(80 \cdot .9375) = 150{,}000$$

Thus the previous bundle is still available.



e) Will the new tax scheme encourage or discourage more work? Briefly explain your answer. Will the new tax scheme raise more money, less money or do you need more information to answer this question. Briefly explain your answer.

Answer To determine whether the new tax scheme will encourage or discourage more work we look at the MRS and price ratio. At the point $(6760, 150000)$ on the budget line in part (b), the price ratio and thus the MRS is 80. At that point on the new budget line the price ratio is

$$\frac{p_x}{p_y} = \frac{80\,(.9375)}{1} = 75 < \mathrm{MRS}$$

When price ratio is less than MRS, you move down and to the right along the budget line. Revenue can be seen geometrically by the space in between the new budget line and the original budget line, therefore the new tax scheme will raise less money.

2. Joe is a risk-averse farmer who owns a 1000-acre farm in South Dakota. He can plant either wheat or corn or any combination of wheat and corn on his farm. At the end of the summer Joe is certain that he can sell his wheat at \$5 per bushel and his corn at \$4 per bushel. However the yield, from his planting (the amount of crops produced) will depend on the weather in the upcoming growing season. The summer season may be either dry or wet and the yields of the two crops will vary depending on the amount of rain. The yields are summarized in the following table.

	Dry Season		**Wet Season**	
	Bushels per acre	\$ per acre	Bushels per acre	\$ per acre
Wheat	80	\$400	100	\$500
Corn	160	\$640	110	\$440

Note that the dollars per acre is simply the product of the price per bushel and the bushels per acre. The Farmers' Almanac predicts that the probability of a wet growing season will be .6 (60%) and the probability of the dry growing season will be .4 (40%). Joe always believes what he reads in the Farmers' Almanac so he believes these probabilities to be accurate.

a) Let α represent the proportion of acreage that Joe plants with wheat. If Joe would like to have a sure income (that is an income that does not vary with the seasons) then what α should he choose?

Answer Equate the income in both states:

$$\begin{aligned} 400\alpha + 640(1-\alpha) &= 500\alpha + 440(1-\alpha) \\ 100\alpha &= 200(1-\alpha) \\ 300\alpha &= 200 \\ \alpha &= \frac{2}{3} \end{aligned}$$

b) Set up Joe's expected utility maximization problem as a function of α and write the first order necessary condition.

Answer The utility maximization problem is

$$\begin{aligned} \max_{\alpha} EU &= .4U(400{,}000\alpha + 640{,}000(1-\alpha)) + .6U(500{,}000\alpha + 440{,}000(1-\alpha)) \\ &= .4U(-240{,}000\alpha + 640{,}000) + .6U(60{,}000\alpha + 440{,}000) \end{aligned}$$

The FOC is

$$\left(\frac{d}{d\alpha}\right) EU = 0$$

$$(-240{,}000)(.4)U'(-240{,}000\alpha + 640{,}000) + (60{,}000)(.6)U'(60{,}000\alpha + 440{,}000) = 0$$

$$\implies 8U'(-240{,}000\alpha + 640{,}000) = 3U'(60{,}000\alpha + 440{,}000)$$

c) Use your first order condition to show that Joe will NOT choose the α that you determined in part (a). Note that you cannot solve for α because you do not have the functional form of his utility function but you can determine that it is not the value from part (a).

Answer If Joe were to use the α from part (a) then we plug in $\frac{2}{3}$ for α in the FOC

$$\begin{aligned}
8U'\left(-240{,}000\left(\frac{2}{3}\right)+640{,}000\right) &= 3U'\left(60{,}000\left(\frac{2}{3}\right)+440{,}000\right)\\
8U'\left(-180{,}000+640{,}000\right) &= 3U'\left(40{,}000+440{,}000\right)\\
8U'\left(480{,}000\right) &= 3U'\left(480{,}000\right)
\end{aligned}$$

Which will not hold in general, and will definitely not be true in the risk averse case.

d) ***Bonus***: Is the α that you determined in part (a) too big or too small relative to the expected utility maximizing?

Answer At the optimal level

$$\begin{aligned}
8U'\left(-240{,}000\alpha+640{,}000\right) &= 3U'\left(60{,}000\alpha+440{,}000\right)\\
\Longrightarrow \frac{8}{3}U'\left(-240{,}000\alpha+640{,}000\right) &= U'\left(60{,}000\alpha+440{,}000\right)
\end{aligned}$$

Recognize that $-240{,}000\alpha+640{,}000$ is the income in the dry season and $60{,}000\alpha+440{,}000$ is the income in the wet season. So the slope of the utility function (U') has a lower slope in the dry season implying income is higher for the dry season by the concavity of the utility function $\left(\frac{d^2U}{d\$^2}<0\right)$ so at higher income the slope is less. Thus income will not be equal between the two states, it will be higher in the dry season. Hence

$$\begin{aligned}
400\alpha+640\left(1-\alpha\right) &> 500\alpha+460\left(1-\alpha\right)\\
240 &> 300\alpha\\
\frac{2}{3} &> \alpha
\end{aligned}$$

3. Suppose that Simon has \$100. He can invest any fraction α of his money in real estate (a risky asset). If housing prices are up then he will receive a return of \$2 on every dollar invested (for a total of \$3). If housing prices are down he will lose all of his investment. Simon believes that it is twice as likely that housing prices will be up than down. The remaining fraction of his wealth, $1-\alpha$, is placed in the bank at no interest, that is if he puts a dollar into the bank then he can withdraw only a dollar. Assuming that Simon's utility of income is given by the utility function $.5\ln(\$)$ how much will Simon invest in real estate?

Answer Simon will choose α that maximizes expected utility. Say the probability prices go up is p and the probability prices go down is q. Then we know $p+q=1$ and $p=2q$ so $3p=1 \implies p=\frac{1}{3}$. Income in the two states of the world are

	Income	Probability
Housing prices go up	$100\alpha(3)+100(1-\alpha)(1)$	$\frac{2}{3}$
Housing prices go down	$100\alpha(0)+100(1-\alpha)(1)$	$\frac{1}{3}$

His maximization problem is

$$\max_{\alpha} EU = \frac{2}{6}\ln\left(200\alpha+100\right)+\frac{1}{6}\ln\left(100-100\alpha\right)$$

The FOC is

$$\frac{dEU}{d\alpha} = \frac{2}{6} \cdot \frac{200}{200\alpha + 100} - \frac{1}{6} \cdot \frac{100}{100 - 100\alpha} = 0$$

The FOC gives us the optimal level of α (which is the level he will choose)

$$\begin{aligned} \frac{2}{6} \cdot \frac{200}{200\alpha + 100} &= \frac{1}{6} \cdot \frac{100}{100 - 100\alpha} \\ \frac{4}{2\alpha + 1} &= \frac{1}{1 - \alpha} \\ 4 - 4\alpha &= 2\alpha + 1 \\ 3 &= 6\alpha \\ \implies \alpha &= \frac{1}{2} \end{aligned}$$

4. Emmitt is a risk averse investor whose preferences over dollar wealth are given by the utility function $u(\$) = \ln(\$)$.Emmitt has the option to buy a portfolio of precious metals. If there is high inflation in the future then the portfolio will be worth $\$10,000$. If there is low inflation then the portfolio will be worth only \$6400. Emmitt considers high and low inflation to be equally likely.

a) What is the expected value of the portfolio of precious metals? What is the certainty equivalent to this portfolio? What is the risk premium?

Answer Expected value is

$$EV = \left(\frac{1}{2}\right) 10{,}000 + \left(\frac{1}{2}\right) 6{,}400 = 8{,}200$$

Expected utility is

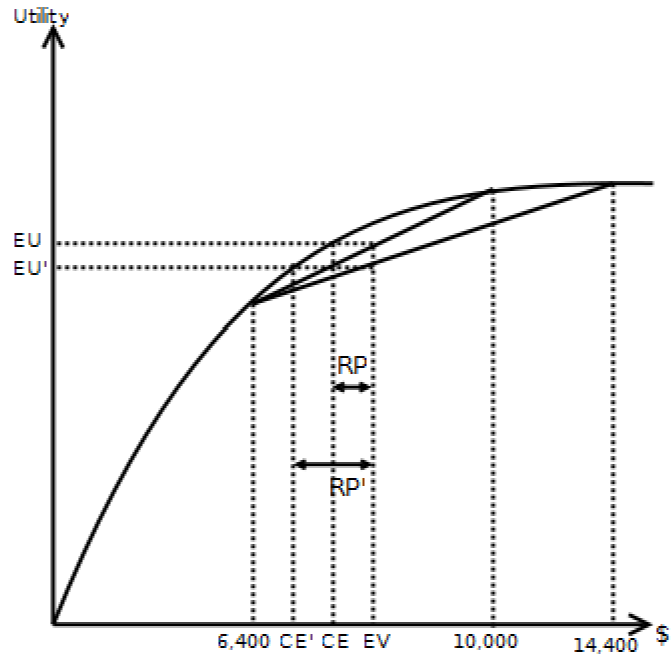
$$EU = \left(\frac{1}{2}\right) \ln(10{,}000) + \left(\frac{1}{2}\right) \ln(6{,}400) \approx 8.99$$

To find the certainty equivalent take the inverse of the utility function at the expected utility level.

$$U(CE) = 9 \implies U^{-1}(9) = e^{8.99} \approx 8{,}000$$

The risk premium is $EV - CE = 8{,}200 - 8{,}000 = 200$.

b) Illustrate the expected utility of this portfolio in a diagram. Be sure to label the expected value, expected utility, certainty equivalent and risk premium.



Suppose that the likelihood of high inflation falls to 22.5% and that the value of the portfolio of precious metals rises to $14,400 in the high inflation state (the value in the low inflation state remains unchanged at $6,400).

c) Show that the expected value of the portfolio has not changed but that both the risk premium has risen? Illustrate the expected utility of the portfolio under the new circumstances in your diagram for part (b). Be sure to indicate the new certainty equivalent and the new risk premium.

Answer The expected value of the portfolio is now

$$EV' = (.225)\,14{,}400 + (.775)\,6{,}400 = 8{,}200$$

which is the same as before. The new expected utility is

$$EU' = (.225)\ln(14{,}400) + (.775)\ln(6{,}400) \approx 8.95$$

The certainty equivalent is

$$CE' = U^{-1}(EU) = e^{8.95} = 7{,}700$$

The new risk premium is

$$RP' = 8{,}200 - 7{,}700 = 500$$

5. Martha is a risk averse investor who is considering purchasing stock in a small biotech company, Gintech. Currently Gintech's stock is selling at \$1 per share. Gintech has a new chemotherapy drug that is awaiting FDA approval. If the FDA approves of the drug then the price of Gintech's stock will rise to \$1.5 per share. However, if the FDA does not approve the drug then the company will go bankrupt and the shares will be worth nothing (\$0). Martha has \$100 to allocate between shares of Gintech stock and cash (the only other asset). Every \$1 that she keeps in cash will be worth \$1 in the future regardless of whether or not the FDA approves Gintech's drug. The probability that the FDA will approve the drug is .8.

a) Would Martha like to purchase any shares of the stock (you may assume that she may purchase fractional shares)? Briefly explain your answer.

Answer The expected value of the bet is $.8\,(.5) + .2\,(-1) = .2$ so as a risk averse investor she will buy some of it. Diagrammatically, this is because expected utility will be along the line segment, higher than the utility of \$100 with certainty.

For the remainder of the question assume that Martha's preferences over dollar wealth by the utility function

$$u\,(\$) = (\$)^{\frac{1}{2}}$$

One possible portfolio is the portfolio consisting only of shares in Gintech (100 shares of Gintech).

b) What is the expected value of the portfolio of 100 shares of stock? What is the certainty equivalent to this portfolio? What is the risk premium associated with this portfolio?

Answer The expected value of this portfolio is

$$EV = .8\,(150) + .2\,(0) = 120$$

The expected utility will be

$$EU = .8\,(150)^{1/2} + .2\,(0)^{1/2} = .8\,(150)^{1/2} \approx 9.8$$

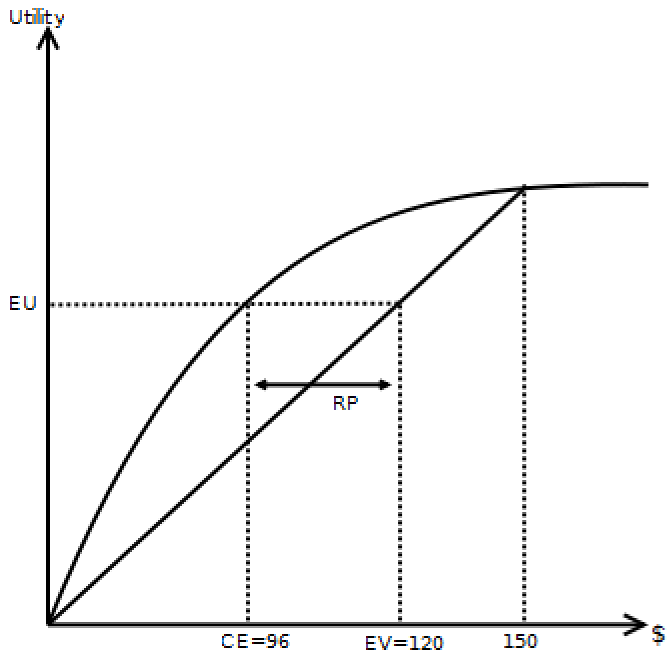
The certainty equivalent is the inverse of the utility function at the expected utility level

$$CE = U^{-1}\left(.8\,(150)^{1/2}\right) = \left(.8\,(150)^{1/2}\right)^2 = .64 \cdot 150 = 96$$

Lastly, the risk premium is

$$RP = EV - CE = 120 - 96 = 24$$

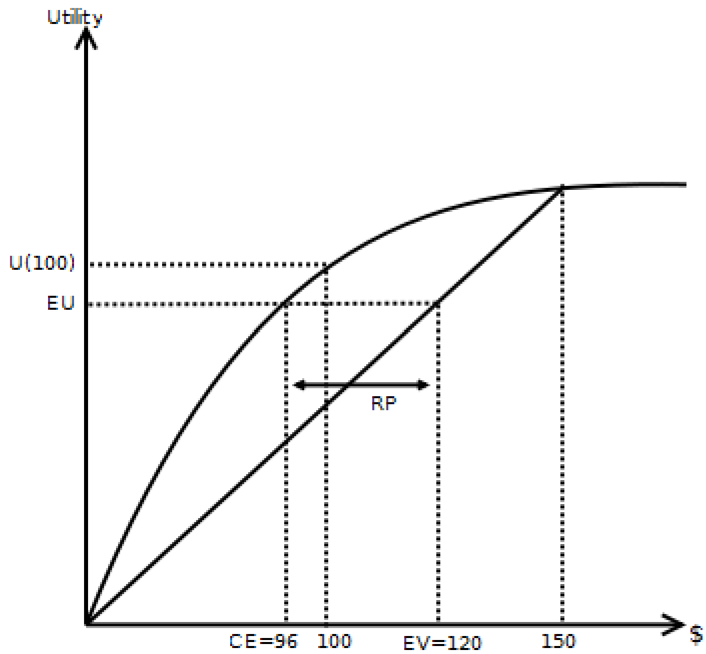
c) Illustrate her expected utility of the portfolio of 100 shares of stock.



Answer

d) If she had only the two options of the portfolio of 100 shares of stock and the portfolio of all cash which will she choose? Illustrate the expected utility of all cash in your diagram for part (c).

Answer She would choose all cash. Since the certainty equivalent ($96) is less than the cash ($100), Martha will prefer having all cash.



Another possible portfolio is one consisting of 25 shares of stock and \$75in cash.

e) What is the expected value of this portfolio? What is the certainty equivalent to this portfolio? What is the risk premium associated with this portfolio?

Answer The expected value of this portfolio is

$$EV = .8\,(1.5)\,(25) + 75 = 105$$

Expected utility is

$$EU = .8\,(112.5)^{1/2} + .2\,(75)^{1/2} \approx 10.2$$

The certainty equivalent is

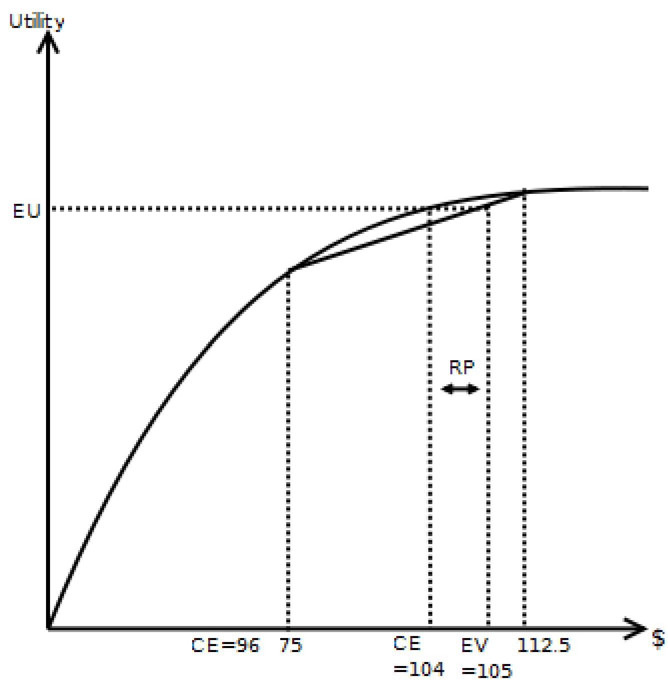
$$CE = U^{-1}\,(10.2) \approx 104$$

Lastly, the risk premium is

$$RP = 105 - 104 = 1$$

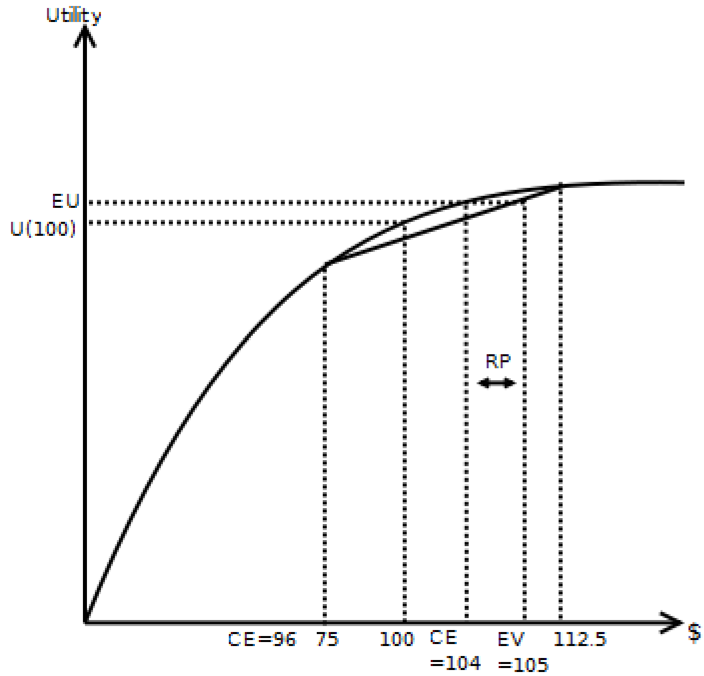
f) Illustrate her expected utility of this portfolio in a new diagram.

Answer



g) If she had only the two options of the portfolio of 25 shares of stock and $75 in cash and the portfolio of all cash which will she choose? Illustrate the expected utility of all cash in your diagram for part (f).

Answer She will choose the portfolio of 25 share and $75 in cash because the certainty equivalent is greater than $100.



6. Recall Leona the tax evader from class. Let's suppose that she has I of taxable income and she must decide how much of that income to report (R) on her taxes. She faces a marginal tax rate of t so if she reports R on her taxes then her after tax income is $I – tR$. If she under reports her income (so $R < I$) then if she is audited she will not only have to pay her back taxes, $t(I - R)$, but she will have to pay a fine on her under reported income of $f(I - R)$. The probability that she is audited is p. The table below summarizes her income as a function of her report R and whether or not she is audited:

No Audit	Audit
$I - tR$	$I - tR - t(I - R) - f(I - R) = I(1 - t) - f(I - R)$

a) Write Leona's expected utility maximization problem as a function of R.

Answer Leona's maximization problem is

$$\max_R EU = (1 - p)\, U(I - tR) + pU(I(1 - t) - f(I - R))$$

b) Write the first order condition for the expected utility maximization problem.

Answer The FOC is $\frac{\partial EU}{\partial R} = 0$. Thus

$$-t(1-p)U'(I - tR) + f \cdot pU'(I(1-t) - f(I-R)) = 0$$
$$\Longrightarrow\ t(1-p)U'(I - tR) = f \cdot pU'(I(1-t) - f(I-R))$$

c) The government would like the solution to her maximization problem to be $R = I$. Use Leona's first order condition to derive the relationship between t, f and p that must hold so that $R = I$ is the solution to her maximization problem.

Answer Setting $R = I$,

$$\begin{aligned}
t(1-p)U'(I - tI) &= f \cdot pU'(I(1-t) - f(I-I)) \\
t(1-p)U'(I(1-t)) &= f \cdot pU'(I(1-t) - f(0)) \\
t(1-p)U'(I(1-t)) &= f \cdot pU'(I(1-t)) \\
t(1-p) &= f \cdot p
\end{aligned}$$

7. Fred has \$$W$ to invest. He is considering investing in a defense company, Zee Co. If Zee Co is awarded a new defense contract then the value of any investment in Zee Co will double. However, if it does not receive a new contract then the value of the investment will be cut in half. Fred believes that it is equally likely that Zee Co will receive the new contract. Finally assume that his preferences over his investment (x) can be represented by the utility function $u(x) = \ln x$.

For parts (a) - (c) Assume that Fred has put all of his money in Zee Co (so he has invested \$$W$ in Zee Co).

a) What is the expected value of his investment (as a function of W)? What is his expected utility of his investment?

Answer The expected value of the investment is

$$EV = .5(2W) + .5(.5W) = 1.25W$$

Expected utility is

$$EU = .5\ln(2W) + .5\ln(.5W)$$

b) What is the certainty equivalent to his investment (as a function of W)? What is the risk premium?

Answer The certainty equivalent is

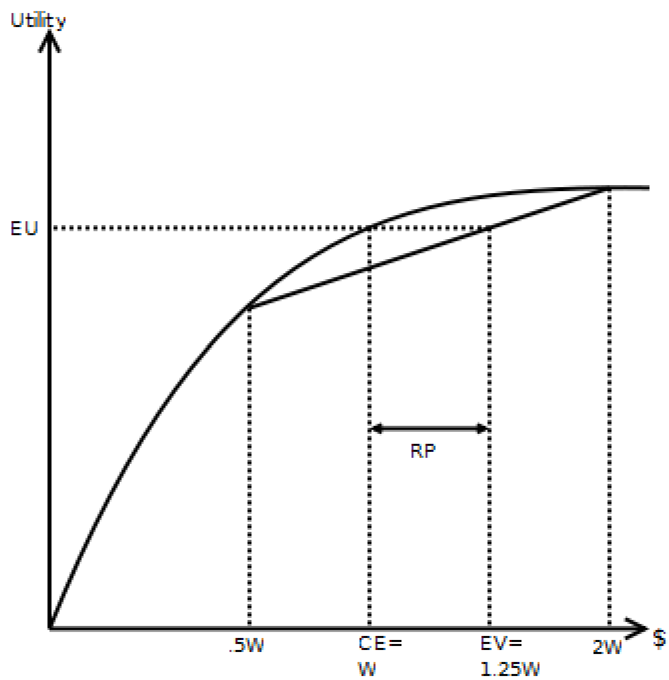
$$\begin{aligned}
CE &= U^{-1}(EU) \\
&= e^{.5\ln(2W) + .5\ln(.5W)} \\
&= e^{.5\ln(2W)}e^{.5\ln(.5W)} \\
&= e^{\ln(2W)^{.5}}e^{\ln(.5W)^{.5}} \\
&= (2W)^{.5}(.5W)^{.5} \\
&= W
\end{aligned}$$

Thus the risk premium is

$$RP = 1.25W - W = .25W$$

c) Illustrate his utility function and the expected utility of his investment in a diagram. Label the expected value, the certainty equivalent and the risk premium.

Answer



d) If his only options are investing all of his money in Zee Co or keeping his money in cash (where it will have a certain value of $\$W$) which (if any) would he prefer?

Answer Since the certainty equivalent equals the amount of cash he would have, he is indifferent between investing and keeping his money in cash.

For parts (e) - (g) below assume that Fred may put any fraction, α, of his $\$W$ in Zee Co. The remaining fraction, $(1-\alpha)W$, will be kept in cash. Assume that the returns on money put in Zee Co and the associated probabilities are as above.

e) Write his expected utility function as a function of α. Write the FOC associated with Fred's expected utility maximization problem.

Answer His expected utility will be

$$EU = .5\ln\left(2W\alpha + W\left(1-\alpha\right)\right) + .5\ln\left(.5W\alpha + W\left(1-\alpha\right)\right)$$

The maximization problem is

$$\max_{\alpha} EU = .5\ln\left(\alpha W + W\right) + .5\ln\left(W - .5\alpha W\right)$$

with FOC

$$\frac{dEU}{d\alpha} = .5\frac{W}{\alpha W + W} - .5\frac{.5W}{W - .5\alpha W} = 0$$

f) Show that at $\alpha = 0$ the first derivative of expected utility is positive (so 0 does not solve the FOC). Briefly explain why this is not a surprising result.

Answer Evaluating the first derivative at $\alpha = 0$ gives

$$\begin{aligned} \left.\frac{dEU}{d\alpha}\right|_{\alpha=0} &= .5\frac{W}{\left(0\right)W + W} - .5\frac{.5\left(0\right)W}{W - .5\left(0\right)W} \\ &= .5 \end{aligned}$$

This is not surprising because the bet has limited downside and the expected utility line segment will be above the one when there is no cash kept (parts a-c). This decreased variance will raise the certainty equivalent which used to be W. Thus with $CE >$ \$cash, he will invest some money

g) What is the value of α that maximizes his expected utility?

Answer Solving the FOC

$$\begin{aligned} .5\frac{W}{\alpha W + W} &= .5\frac{.5W}{W - .5\alpha W} \\ \frac{1}{\alpha W + W} &= \frac{.5}{W - .5\alpha W} \\ W - .5\alpha W &= .5\alpha W + .5W \\ .5W &= \alpha W \\ \Longrightarrow \alpha &= .5 \end{aligned}$$