W1211 Introduction to Statistics
Lecture 4

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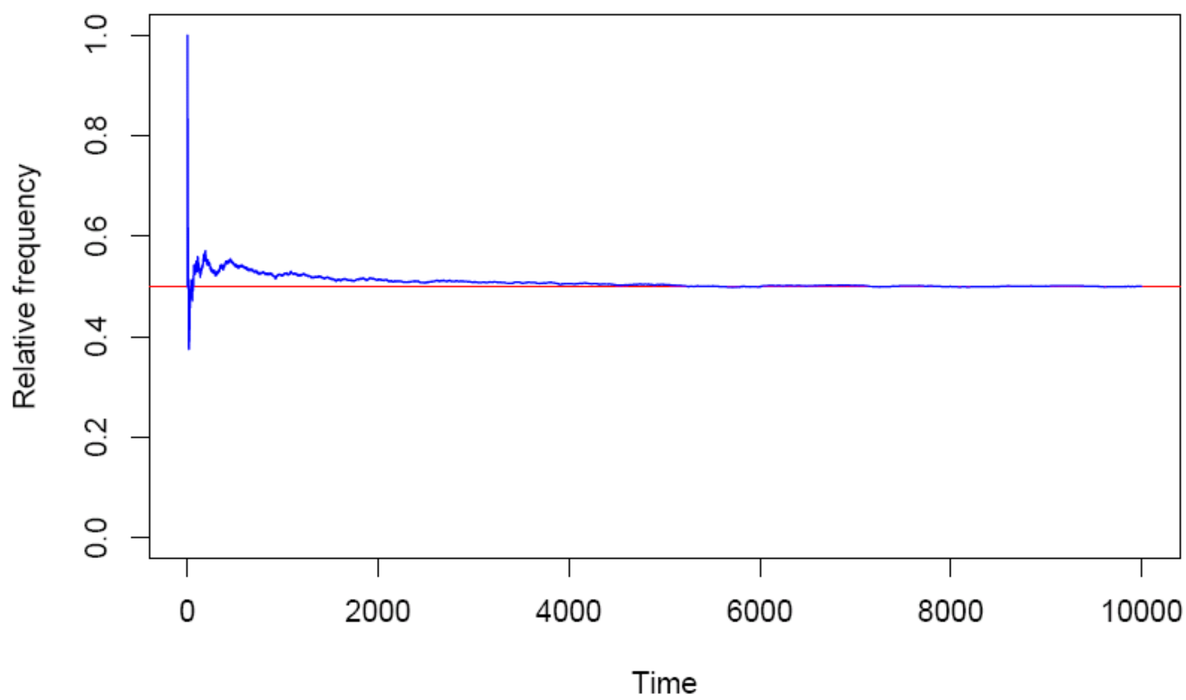
Interpreting Probability

- What does it mean when we say we have 50% chance of having a head when flipping a coin? Or what does it mean when we put P(H)=0.5?

- Probability is often treated as the *long-term relative frequency* or the *limiting relative frequency*.

Interpreting Probability

<u>Ex.</u> Flip a fair coin *n* times and calculate the proportion of heads.



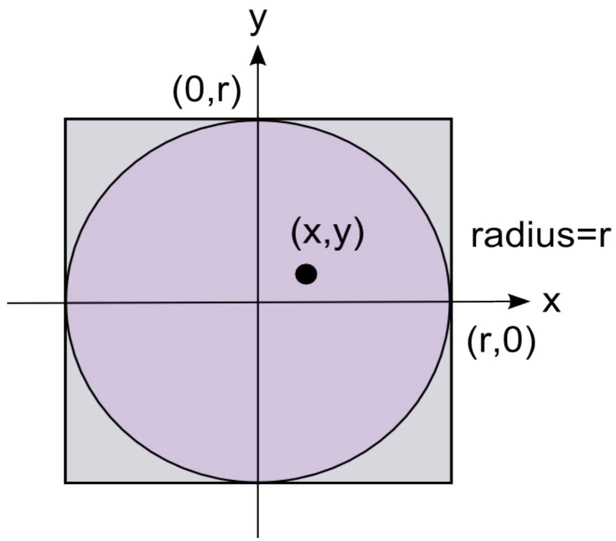
- R demo. (Function: `sample(x, size)`; `rbinom(x, size, prob)`)

Law of Large Numbers

- The law of large numbers says that the long-run relative frequency of repeated independent events gets closer and closer to the true relative frequency as the number of trials increases.
- $$\frac{\text{Number of Occurence of Event } A}{\text{Number of Trials}} \to P(A)$$
  as number of trials $n \to \infty$

How to calculate Pi Stochastically

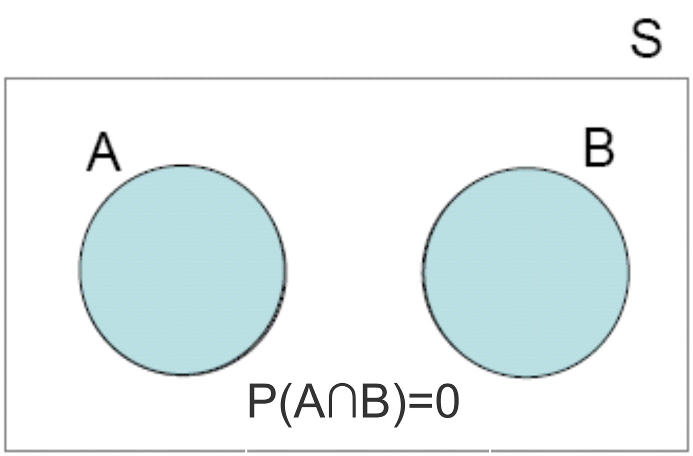
- An interesting application of Law of Large Numbers is to calculate Pi through simulations.
- If we spread a large quantity of seeds randomly but evenly on this square, what percentage of the seeds will lie inside the circle?

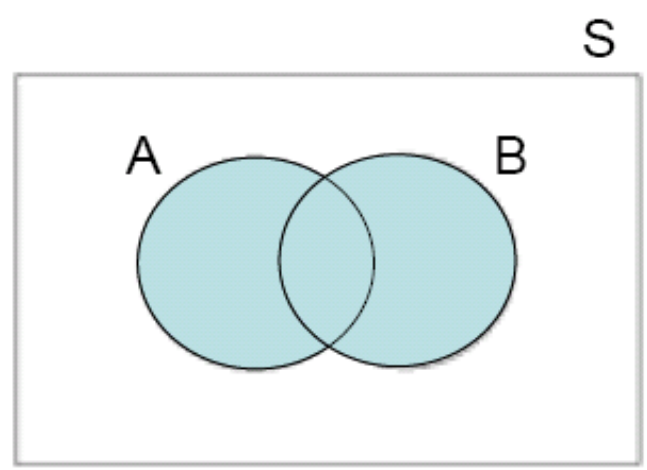


- R Demo.
- This type of simulation-based methods has a fancy name: Monte Carlo methods.

More Probability Properties

- Consider an experiment whose sample space is *S*. For each event A (B) in *S*, we assume that a number P(A) is defined and satisfies the following rules:
  1. $0 \leq P(A) \leq 1$.
  2. $P(S)=1$.
  3. $P(A^c)=1-P(A)$.
  4. If A and B are disjoint, then $P(A \cup B)=P(A)+P(B)$.
  5. For any two events A and B, $P(A \cup B)=P(A)+P(B)-P(A \cap B)$.





Example

Ex. A store accepts either VISA or Mastercard. 50% of the stores customers have VISA, 30% have Mastercard and 10% have both. What is the probability that a customer has a credit card the store accepts?

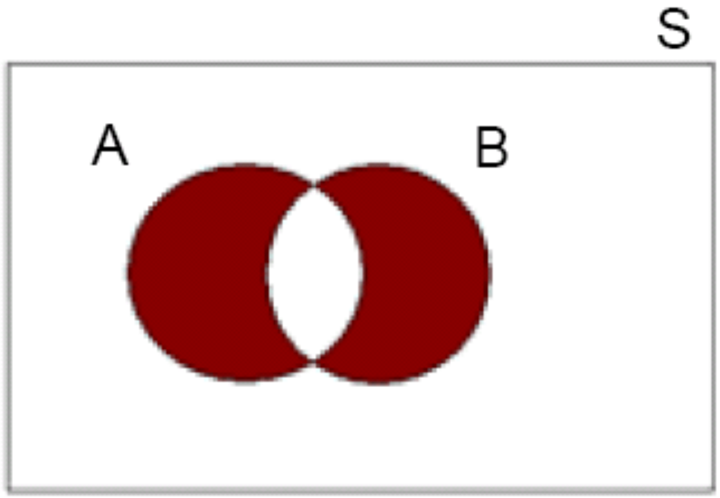
A = customers has VISA

B = customers has Mastercard

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.5 + 0.3 - 0.1 = 0.7$$

Example cont.

What is the probability that a customer has either a VISA or MC, but not both?



$$P(A \text{ or } B \text{ but not both}) = P(A) + P(B) - 2P(A \cap B)$$
$$= 0.5 + 0.3 - 0.2 = 0.6$$

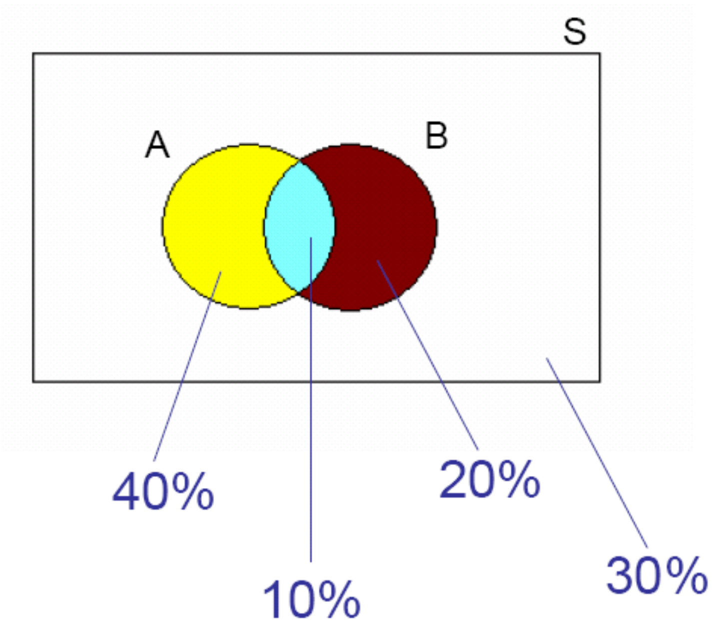
Example Cont.

What is the probability that a customer has a VISA but no MC?

$$P(\text{A but not both}) = P(A) - P(A \cap B)$$
$$= 0.5 - 0.1 = 0.4$$

What is the probability that a customer has a MC but no VISA?

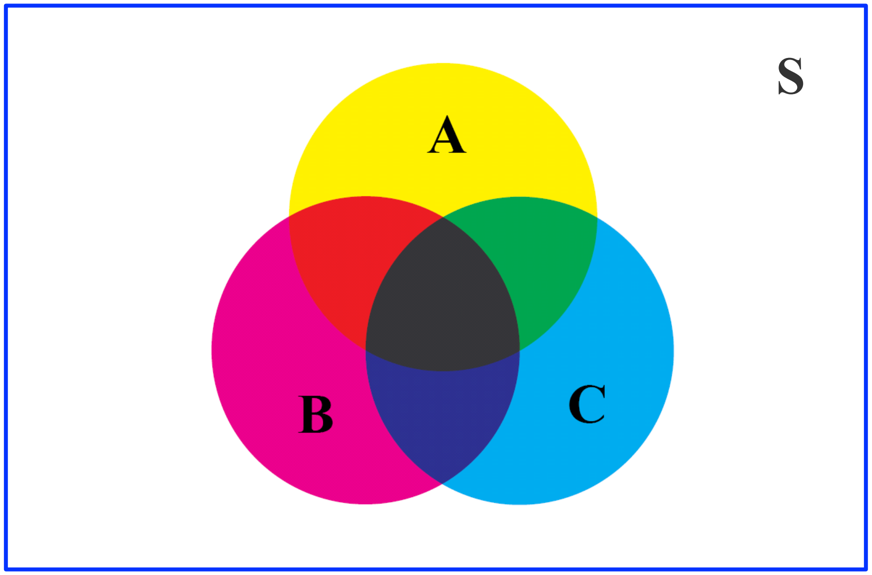
$$P(\text{B but not both}) = P(B) - P(A \cap B)$$
$$= 0.3 - 0.1 = 0.2$$



Three Events

- For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



Assigning Probabilities

- The assignment of probabilities can often be derived from the physical set-up of an experiment.

- Suppose we have *N* outcomes in our sample space, each equally likely to occur. The each has a probability of 1/*N*, and the probability of any event A is,

$$P(A) = \frac{\text{number of outcomes in A}}{N}$$

Ex. Roll a fair die. $S=\{1,2,3,4,5,6\}$. Our sample space consists of 6 points, each of which is equally likely to occur.

P(roll a 1) = 1/6.

Let A = roll a 4 or less = {1,2,3,4}. P(A) = 4/6.

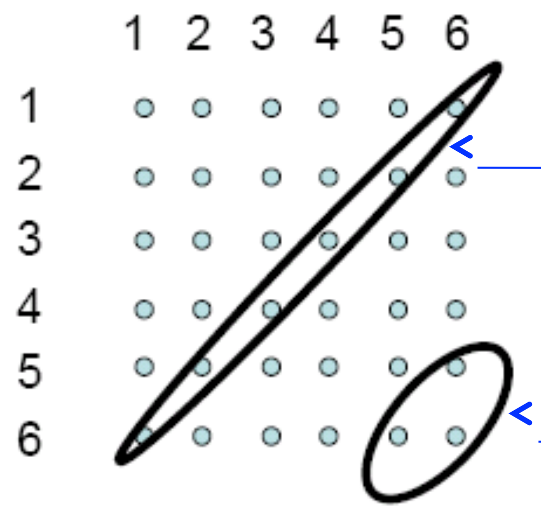
Let B = roll an even number = {2,4,6}. P(B) = 3/6.

Example

<u>Ex.</u> Roll two fair dice.

There are 36 possible outcomes: {(1,1),(1,2),(1,3),…,(6,5),(6,6)}.

Let A = sum of two rolls is 7; B = sum of two rolls is 11 or more. What are P(A) and P(B)?



P(A) = 6/36 =1/6

P(B) = 3/36 =1/12

- In the settings where the sample space is composed of finite number of outcomes that are equally likely to occur, the problem of probability boils down to the problem of counting.

$$P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcome in Sample Space S}}$$

- Counting Techniques are essential to efficiently calculate the numerator and denominator. Specifically, we will talk about Permutations and Combinations.
- Again, remember that in the language of Probability, sets and events are synonymous.

Product Rule

- A general situation is that a set consists of ordered pairs of objects and we wish to count the numbers of such pairs.
- If the first object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second objects can be selected in n_2 ways, then the number of pairs is $n_1 n_2$
- This rule applies when we have multiple stages.
- Here the key is that the stages are independent of each other.

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- There is a more general way to count. Think about positions of the ordered pair one at a time.
- Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters then shows that there are

$$n(n-1)(n-2)\ldots 3\cdot 2\cdot 1 = n!$$

  different permutations of the n objects.

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- **Solution:** There are $9! = 362880$ possible batting orders.

Example 2

- Ms. Davis has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are physics books, 2 are literature books and 1 is a language book. Ms. Mortimer wants to arrange her books so that all the books dealing with the same subject are together on her shelf. How many different arrangements are possible?

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- **Solution:** There are $4!3!2!1!$ arrangements such that the mathematics books are first in line, then the physics books, then the history books, and then the language books. Similarly, for each possible ordering of the subjects, there are $4!3!2!1!$ possible arrangements. Hence, as there are $4!$ possible ordering of the subjects, the desired answer is $4!4!3!2!1! = 6912$.

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- In particular, $P_{n,n} = n!$ and $P_{1,n} = n$

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- The denominator is $P_{9,20}$, what is the numerator?

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- There are thus $P_{3,5} = 5 \times 4 \times 3$ ways of selecting a group of 3 when the order in which the items are selected is relevant.
- But every group is counted $P_{3,3} = 3 \times 2 \times 1$ times. So there are

  $$\frac{P_{3,5}}{3!} = 10$$

  groups.

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- $\binom{n}{k} = P_{k,n}/k!$
- In particular, $\binom{n}{n} = 1$ and $\binom{n}{1} = n$.

Example 1

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- **Solution:** There are $\binom{20}{3} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140$ possible committees.

Example 2

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- **Solution:** As there $\binom{5}{2}$ possible groups of 2 women and $\binom{7}{3}$ possible groups of 3 men, if follows that there are $\binom{5}{2}\binom{7}{3} = 350$ possible committees consisting of 2 women and 3 men.

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- **Solution:** Now suppose that 2 of them refuse to serve together, because a total of $\binom{2}{2}\binom{5}{1}$ out of the $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are $35 - 5 = 30$ groups that do not contain both of the feuding men. Because there are still $\binom{5}{2}$ ways to choose 2 women, there are $30 \cdot 10 = 300$ possible committees in this case.

Example 3 Contd

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- **Solution:** $\frac{350-300}{350} = 1/7$