ELEC 221 Lecture 05
The discrete-time Fourier series

Thursday 22 September 2022

Announcements

- Assignment 2 available (computational); due next Wednesday
- **Make sure to complete the statement of contributions or you will get 0, no exceptions.**
- Reminder: lowest assignment grade and lowest quiz grade will be dropped.

Clarification request: invertibility and inverse systems

Oppenheim provides two definitions for system invertibility:

1. "A system is said to be *invertible* if distinct inputs lead to distinct outputs"
2. "If a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system."

Sometimes you need to try more than one way to check this: maybe you can construct the inverse system, or find two input signals that lead to the same output.

Let's do some examples from Oppenheim problem 1.30

Clarification request: invertibility and inverse systems

Example: $y(t) = \sin(x(t))$.

Initial thought: inverse system should be .

But consider the following:

Different input signals led to the same output signal.

This system is not invertible.

Clarification request: invertibility and inverse systems

Example:

Initial thought: integrate it!

But consider the following:

This system is not invertible.

Example: .

Initial thought: inverse system is

One way to think of this: you lose information (downsampling) after a signal goes through the first system; so you cannot invert it.

Another way: find a counter example.

This system is not invertible.

Clarification request: invertibility and inverse systems

Example:

This one actually is invertible!

We introduced the Dirichlet conditions, which tell us whether a CT signal can be expanded as a Fourier series.

If over one period, a signal

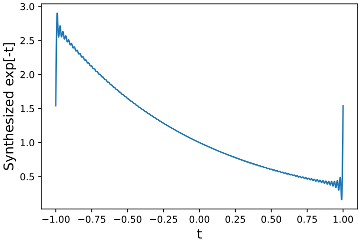
1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

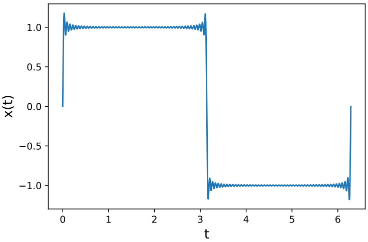
then the Fourier series converges to

- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Last time

We encountered the Gibbs phenomenon, a ringing that occurs when you truncate the Fourier series representation.





Last time

We learned about energy and power of a signal

and derived Parseval's relation between Fourier coefficients and signal power

Today

We are shifting over to **discrete time**.

Learning outcomes:

- Compute the fundamental period and frequency of a DT signal
- Express DT periodic signals as Fourier series and compute the Fourier coefficients
- Compute the frequency response of a system, and determine how it responds to a complex exponential signal

DT complex exponential signals

Recall our continuous-time representation of complex exponential signals:

where α could be real or complex.

In DT, we instead often write

where β can be real or complex.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is real.

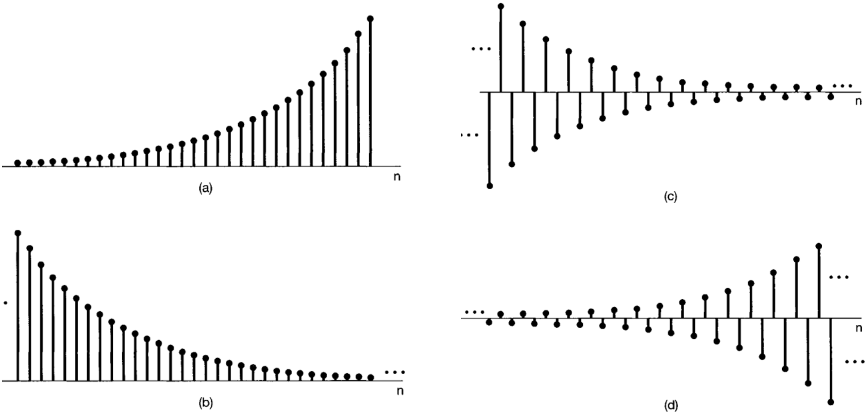


Image credit: Oppenheim chapter 1.3.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is purely complex:

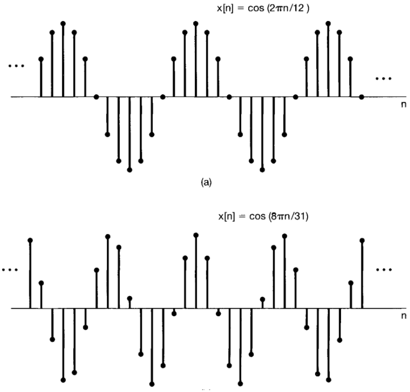


Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

While these might look similar to their discrete counterpoints, there is a **very important difference** relating to frequency.

In CT,

This has frequency ω is periodic with period $T = 2\pi/\omega$.

The bigger ω gets, the faster it oscillates!

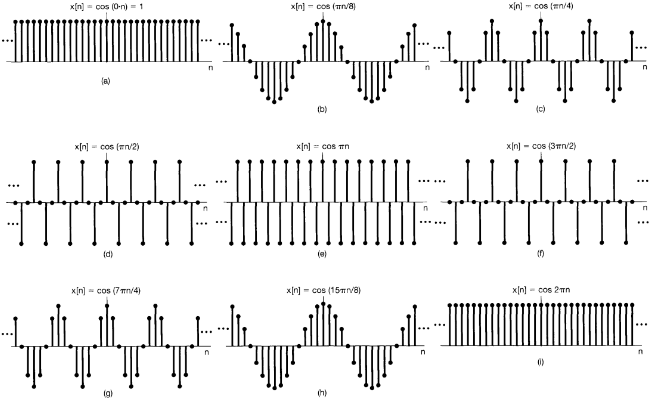
What happens in DT? Imagine we have some starting ω:

Suppose that we slowly increase ω. Eventually we reach $\omega + 2\pi$:

Frequency and period of DT complex exponential signals

For a DT signal with frequency ω, the signals with frequencies

are the same!



Only need to consider ω in the range 0 to 2π (or $-\pi$ to π).

Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

Another consequence: there are additional criteria for periodicity.

Suppose the period is N:

This implies that

must be rational for the signal to be periodic.

Frequency and period of DT complex exponential signals

This can be counterintuitive.

Example: consider the CT signal

This is periodic with period $T = 2\pi/3$.

What about the DT equivalent,

The signal

is *not periodic*.

n can only be an integer, so we will never get to the "time" of $2\pi/3$ where the signal will start to repeat.

Frequency and period of DT complex exponential signals

Example: what is the fundamental period of

For the CT equivalent, $T = 2\pi/\omega = 12/5$, but this is not a valid n.

But after 5 periods, we are at $n = 12$ which is valid so the fundamental period is $N = 12$.

Harmonics of DT complex exponential signals

What about harmonics?

In CT we had an infinite number of these. What about DT?

Note that, once we reach $k = N$, we get back to our original signal... so have only N distinct harmonics:

Recall that in CT, complex exponentials are eigenfunctions of LTI systems. What about in DT?

Consider a system with impulse response $h[n]$ and DT signal $x_m[n] = e^{jm\omega n}$. Use the convolution sum:

So in DT, if we know how the system responds to complex exponential signals, we can figure out what it does to any signal that is expressed in terms of them.

We need a Fourier series representation of DT signals:

Just like in CT: how do we find the c_k?

DT Fourier coefficients

Option 1: solve a linear system of equations.

This gives the system:

(This may look familiar!)

Option 2: obtain closed-form expressions like the CT case.

Leverage the following identity about complex numbers:

We will multiply on both sides, and sum.

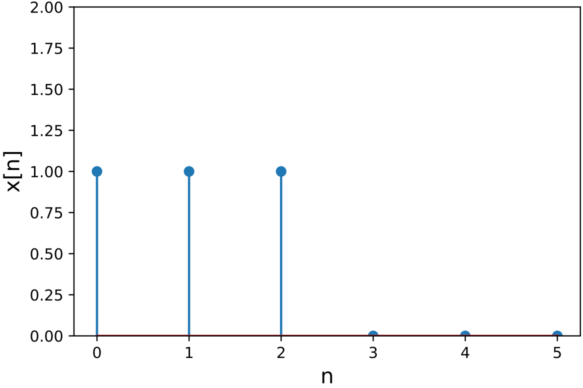
DT Fourier coefficients

So the DT synthesis and analysis equations are

and

Example: the DT square wave

Let's compute the Fourier coefficients of this signal:



System functions

Let's recap what we've done the previous three or so lectures.

Back in lectures 2 and 3, we talking about LTI systems and impulse response. We showed a couple of important things...

Any signal can be expressed as a combination of weighted, shifted impulses.

System functions

If we know what the system does to a unit impulse, i.e., the impulse response $h(t)$ or $h[n]$, we can learn what it does to any signal. This was the convolution integral and sum:

We now know many periodic signals, in both CT and DT, can be expressed as Fourier series (linear combinations of complex exponential signals):

And we know that these complex exponential signals are eigenfunctions of LTI systems:

In general, $H(s)$ is called the **system function**; when we are limiting to $H(j\omega)$ we term this the **frequency response**.

In other words, when we input a complex exponential signal at a particular frequency, how does the system respond?

Example: computing the frequency response

Consider an LTI system with impulse response:

What happens when we put in the following signal?

Example: computing the frequency response

Three steps:

1. Determine the Fourier coefficients of the signal
2. Determine the frequency response
3. Use 1 and 2 to obtain Fourier coefficients of the output signal

Example: computing the frequency response

Step 1: what are the Fourier coefficients of

Expand sin and cos as complex exponentials:

Then we see that:

Example: computing the frequency response

Step 2: what is the frequency repsonse of the system with

Integrate!

Example: computing the frequency response

$$H(j\omega) = \int_{-\infty}^{0} e^{(4-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(4+j\omega)\tau} d\tau$$

Continue...

Example: computing the frequency response

We have

Step 3: get the Fourier coefficients of the output signal.
In general, we know that

Two sets of coefficients to deal with:

and

Example: computing the frequency response

So our original signal was

Our final signal is

Recap

Today's learning outcomes were:

- Compute the fundamental period and frequency of a DT signal
- Express DT periodic signals as Fourier series and compute the Fourier coefficients
- Compute the frequency response of a system, and determine how it responds to a complex exponential signal

What topics did you find unclear today?

For next time

We are going to learn how to *design* system functions to manipulate signals in certain ways:

- Frequency-shaping filters
- Frequency-selective filters

Action items:

1. Assignment 2 is due next Wednesday
2. Quiz 3 is on Tuesday

Recommended reading:

- From today's class: Oppenheim 1.3, 3.6-3.8
- For next class: Oppenheim 3.9