ELEC 221 Lecture 05
The discrete-time Fourier series

Thursday 22 September 2022

Announcements

- Assignment 2 available; due next Wednesday

Last time

We introduced the Dirichlet conditions, which are sufficient conditions for a CT signal to be expanded as a Fourier series.

If over one period, a signal

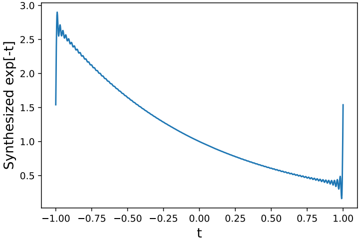
1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

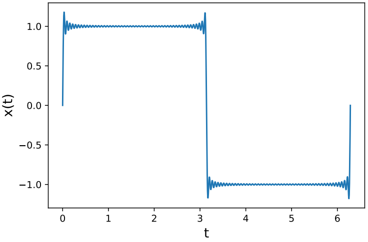
then the Fourier series converges to

- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Last time

We encountered the Gibbs phenomenon, a ringing that occurs when you truncate the Fourier series representation.





We derived that the amount of overshoot was about 9% the height of the discontinuity, no matter how large you make N.

Today

We are shifting over to **discrete time**.

Learning outcomes:

- Compute the fundamental period and frequency of a DT signal
- Express DT periodic signals as Fourier series and compute the Fourier coefficients
- Compute the frequency response of a system, and determine how it responds to a complex exponential signal

DT complex exponential signals

Recall our continuous-time representation of complex exponential signals:

where α could be real or complex.

In DT, we instead often write

where β can be real or complex.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is real.

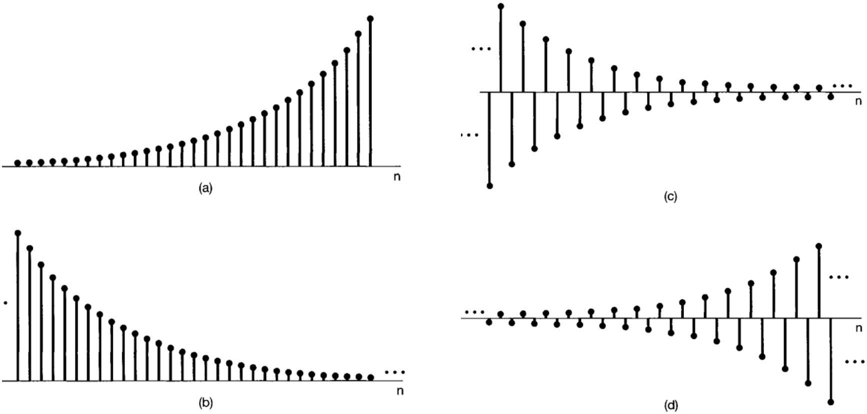


Image credit: Oppenheim chapter 1.3.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is purely complex:

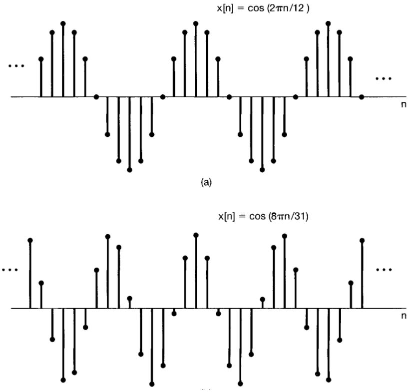


Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

While these might look similar to their discrete counterpoints, there is a **very important difference** relating to frequency.

In CT,

This has frequency ω is periodic with period $T = 2\pi/\omega$.

The bigger ω gets, the faster it oscillates!

Frequency and period of DT complex exponential signals

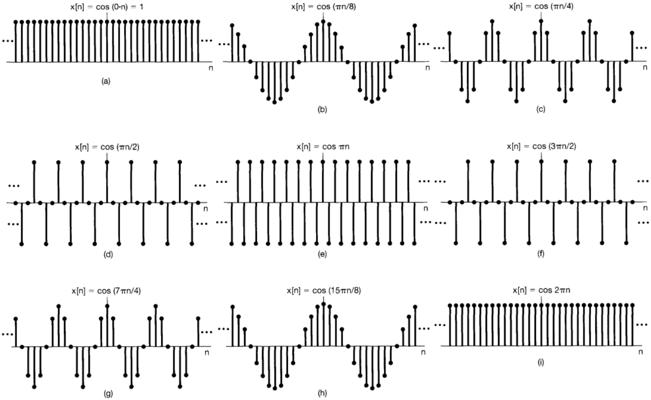
What happens in DT? Imagine we have some starting ω:

Suppose that we slowly increase ω. Eventually we reach $\omega + 2\pi$:

Frequency and period of DT complex exponential signals

For a DT signal with frequency ω, the signals with frequencies

are the same!



Only need to consider ω in the range 0 to 2π (or $-\pi$ to π).

Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

Another consequence: there are additional criteria for periodicity.

Suppose the period is N:

This implies that

must be rational for the signal to be periodic.

Frequency and period of DT complex exponential signals

This can be counterintuitive.

Example: consider the CT signal

This is periodic with period $T = 2\pi/3$.

What about the DT equivalent,

Frequency and period of DT complex exponential signals

The signal

is *not periodic*.

n can only be an integer, so we will never get to the "time" of $2\pi/3$ where the signal will start to repeat.

Example: what is the fundamental period of

For the CT equivalent, $T = 2\pi/\omega = 12/5$, but this is not a valid n.

But after 5 periods, we are at $n = 12$ which is valid so the fundamental period is $N = 12$.

Harmonics of DT complex exponential signals

What about harmonics?

In CT we had an infinite number of these. What about DT?

Note that, once we reach $k = N$, we get back to our original signal... so have only N distinct harmonics:

Recall that in CT, complex exponentials are eigenfunctions of LTI systems. What about in DT?

Consider a system with impulse response $h[n]$ and DT signal $x_m[n] = e^{jm\omega n}$. Use the convolution sum:

DT Fourier series

So in DT, if we know how the system responds to complex exponential signals, we can figure out what it does to any signal that is expressed in terms of them.

We need a Fourier series representation of DT signals:

Just like in CT: how do we find the c_k?

DT Fourier coefficients

Option 1: solve a linear system of equations.

This gives the system:

(This may look familiar!)

DT Fourier coefficients

Option 2: obtain closed-form expressions like the CT case.

Leverage the following identity about complex numbers:

We will multiply on both sides, and sum.

DT Fourier coefficients

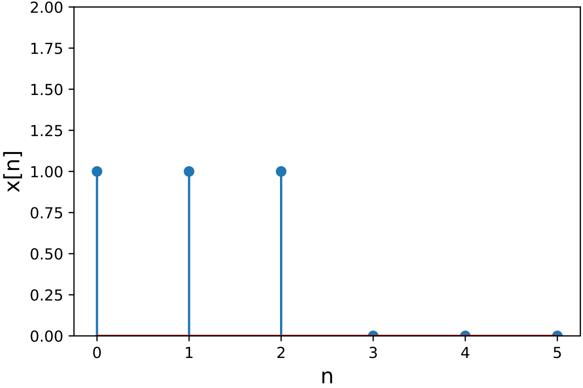
DT Fourier coefficients

So the DT synthesis and analysis equations are

and

Example: the DT square wave

Let's compute the Fourier coefficients of this signal:



Let's recap what we've done the previous three or so lectures.

Back in lectures 2 and 3, we talking about LTI systems and impulse response. We showed a couple of important things...

Any signal can be expressed as a combination of weighted, shifted impulses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau, \quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

If we know what the system does to a unit impulse, i.e., the impulse response $h(t)$ or $h[n]$, we can learn what it does to any signal. This was the convolution integral and sum:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

We now know many periodic signals, in both CT and DT, can be expressed as Fourier series (linear combinations of complex exponential signals):

System functions

And we know that these complex exponential signals are eigenfunctions of LTI systems:

In general, $H(s)$ is called the **system function**; when we are limiting to $H(j\omega)$ we term this the **frequency response**.

In other words, when we input a complex exponential signal at a particular frequency, how does the system respond?

Example: computing the frequency response

Consider an LTI system with impulse response:

What happens when we put in the following signal?

Example: computing the frequency response

Three steps:

1. Determine the Fourier coefficients of the signal
2. Determine the frequency response
3. Use 1 and 2 to obtain Fourier coefficients of the output signal

Step 1: what are the Fourier coefficients of

Expand sin and cos as complex exponentials:

Then we see that:

Step 2: what is the frequency repsonse of the system with

Integrate!

Example: computing the frequency response

$$H(j\omega) = \int_{-\infty}^{0} e^{(4-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(4+j\omega)\tau} d\tau$$

Continue...

(What does this tell us about our system?)

We have

Step 3: get the Fourier coefficients of the output signal.
In general, we know that

Example: computing the frequency response

Two sets of coefficients to deal with:

and

Example: computing the frequency response

So our original signal was

Our final signal is

Recap

Today's learning outcomes were:

- Compute the fundamental period and frequency of a DT signal
- Express DT periodic signals as Fourier series and compute the Fourier coefficients
- Compute the frequency response of a system, and determine how it responds to a complex exponential signal

What topics did you find unclear today?

For next time

We are going to learn how to *design* system functions to manipulate signals in certain ways:

- Frequency-shaping filters
- Frequency-selective filters

Action items:

1. Assignment 2 is due next Wednesday
2. Quiz 3 is on Tuesday

Recommended reading:

- From today's class: Oppenheim 1.3, 3.6-3.8
- For next class: Oppenheim 3.9