ELEC 221 Lecture 02
LTI systems, impulse response and the convolution sum

Tuesday 13 September 2022

Announcements

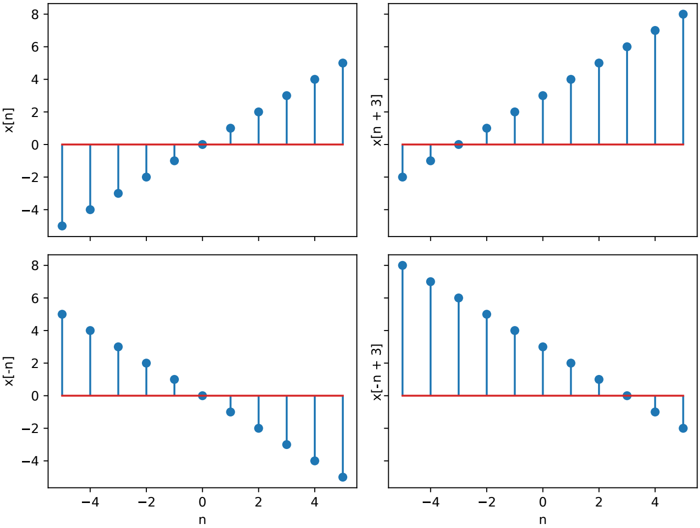
- Quiz 1 today
- Assignment 1 released tomorrow (pen and paper assignment)
- **No python** on quizzes, midterms, or exams; only for computational assignments and in-class demos.

Response to feedback:

- Improving code demos by using light mode with larger font
- Less functional aspects to programming
- Include reading suggestions for *next* class on the last slide
- Will recap the system properties (esp. causality and time invariance)

Last time

We saw continuous-time and discrete-time signals, and applied some simple transformations to them.



Last time

We introduced the idea of systems, which respond to signals, transform them, and output new signals.

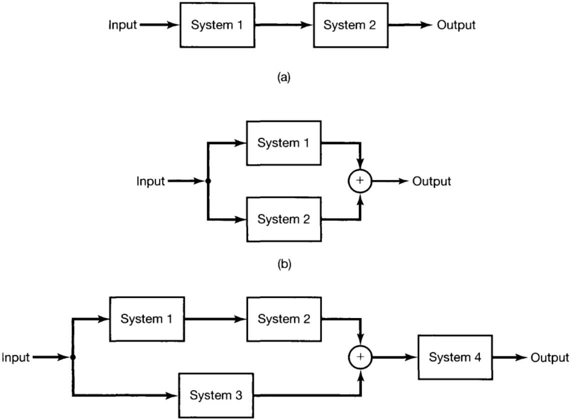


Image credits: Signals and Systems 2nd ed., Oppenheim

Last time

We explored some properties of systems:

1. Memory
2. Invertibility
3. Causality
4. Stability
5. **Linearity** $ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$
6. **Time invariance**

Last time: causality

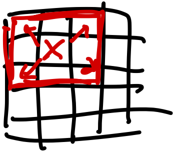
A system is **causal** if the output at **any** *n* depends only on the current or previous *n*.

(✓) Accumulator

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

(✓) Difference from past value

$$y[n] = x[n] - x[n-1]$$



(X) Difference from “future” value

$$y[n] = x[n] + x[n+1]$$

A system is **time invariant** if a time-shifted input leads to an output time-shifted by the same amount.

$$x(t) \to y(t) \qquad x(t-t_0) \to y(t-t_0)$$

Example: $x(t) \to y(t) = \sin(x(t))$ is time-invariant. Why?

$$x_1(t) \to y_1(t) = \sin(x_1(t)) \to \sin(x_1(t-t_0))$$

$$x_2(t) = x_1(t-t_0) \longrightarrow y_2(t) = \sin(x_2(t)) = \sin(x_1(t-t_0))$$

Fall 2022

Available at ca.prairielearn.com

Open book:

- Work individually
- You may use lecture slides, notes, or the textbook.
- Do not search online for answers

Once you start the quiz you will have 10 minutes to complete it.

How do we *characterize* an LTI system?

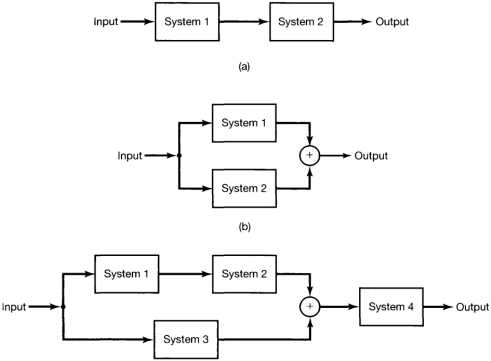


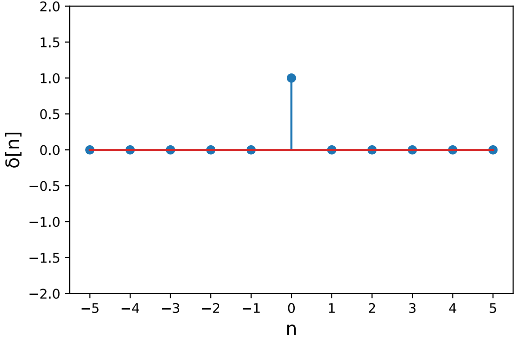
Image credits: Signals and Systems 2nd ed., Oppenheim

Today

Learning outcomes:

- Define the DT/CT unit impulse and step functions
- Define the impulse response
- Express a system using the DT convolution sum and CT convolution integrals
- Apply key properties of the convolution sum/integral to determine the response of compound LTI systems

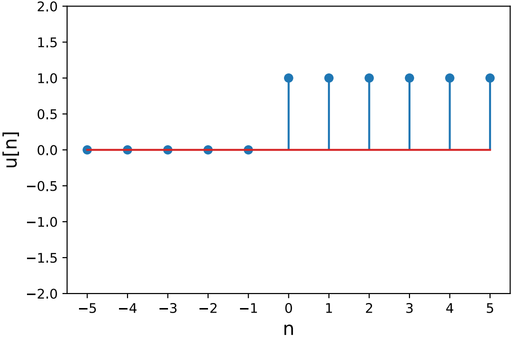
The DT unit impulse



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

The DT unit step



$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

We can express these in terms of each other:

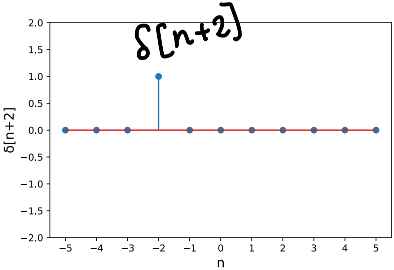
$$\delta[n] = u[n] - u[n-1]$$

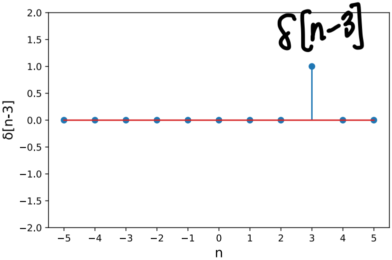
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

The unit impulse is an important tool for characterizing the behaviour of systems.

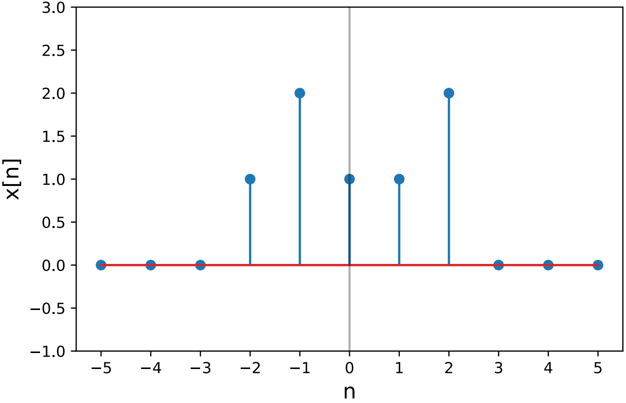
By considering unit impulses time-shifted as various points, we can pick out, or *sift* out specific parts of the signal.





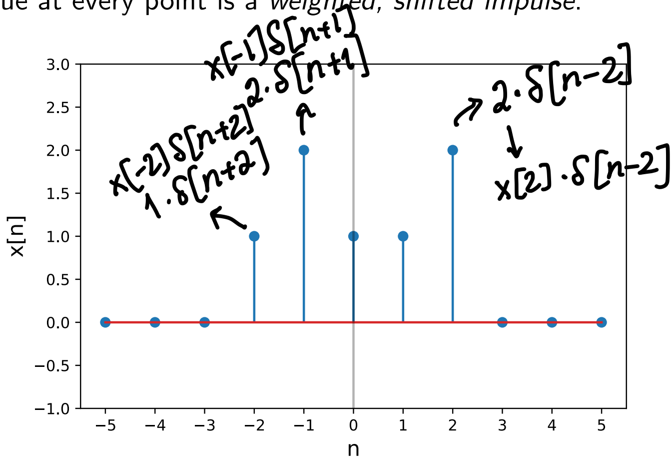
The sifting property

Consider the following signal:



The sifting property

The value at every point is a *weighted, shifted impulse*.



This has important consequences...

The unit impulse as a sampler

Multiplying the signal by a shifted impulse picks out the value of the signal at that point:

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k]$$

This allows us to write any signal as a **superposition of weighted impulses.**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$$

Given a signal

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}\,\underbrace{\delta[n-k]}$$

how does an LTI system respond to it?

$z_1[n] \to w_1[n]$

We know that for a **linear** system $z[n] \to w[n]$, $z_2[n] \to w_2[n]$

$$a z_1[n] + b z_2[n] \longrightarrow a w_1[n] + b w_2[n]$$

More generally,

$$\sum_k \underbrace{a_k}\,\underbrace{z_k[n]} \longrightarrow \sum_k a_k w_k[n]$$

The impulse response

If we know how a **linear** system responds to the unit impulse, we can learn how it responds to **any other signal**!

Suppose the system sends $\delta[n-k] \to h_k[n]$. Then

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h_k[n] = y[n]$$

$h_k[n]$ is called the **impulse response**.

We took advantage of linearity. What if the system is also time invariant? Recall time invariance mean that

$$x[n] \rightarrow y[n] \qquad x[n-k] \rightarrow y[n-k]$$

Then

$$h_k[n]$$

$$\delta[n-k] \rightarrow h_k[n] = h_0[n-k] = h[n-k]$$

$$x[n] \to y[n] \qquad \delta[n-k] \to h[n-k]$$

Our expression becomes

$$x[n] = \sum_{-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

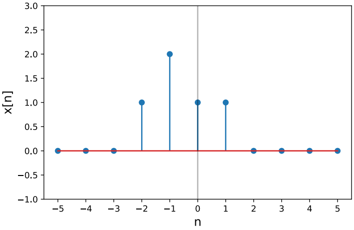
This is the **convolution sum**. We are "convolving" the sequences $x[n]$ and $h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

convolution operator

Example: convolution sum

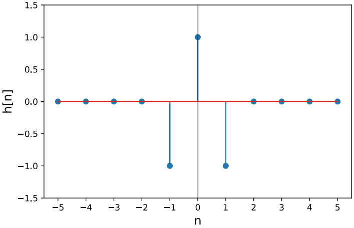
Consider the signal



$x[n]$

$x[n] * h[n]$

input to a system with impulse response

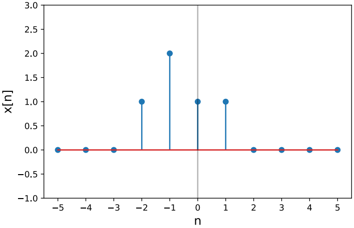


$h[n]$

Example: convolution sum

To learn the system output, we must consider the contribution of each weighted impulse response:

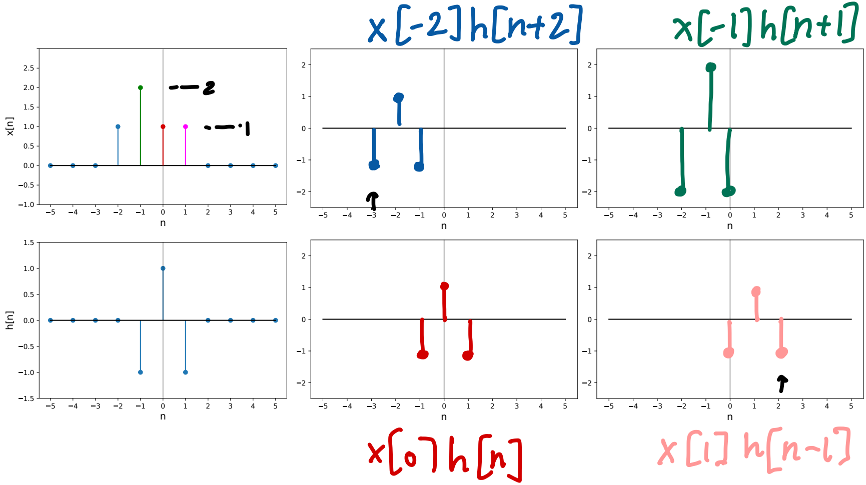
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



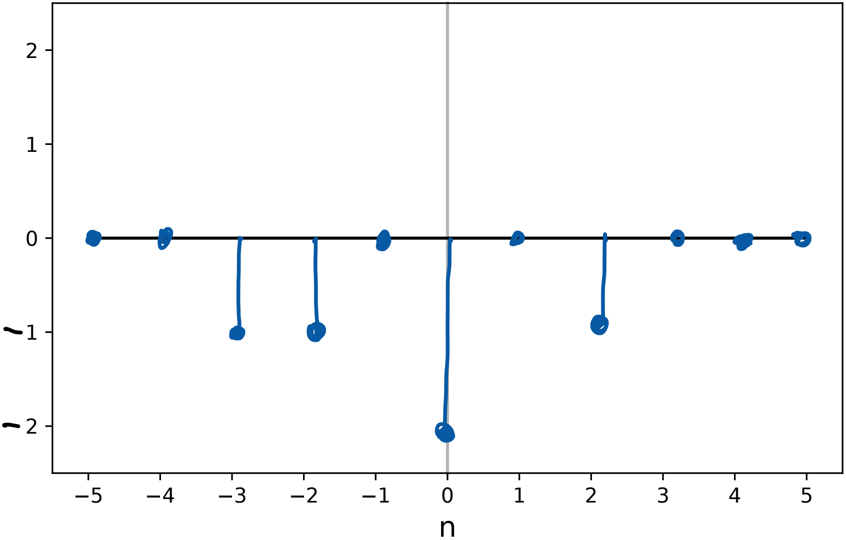
Only $x[k] \neq 0$ only for $k \in \{-2, -1, 0, 1\}$. So need to determine $x[k]h[n-k]$ for these cases, and sum them.

Example: convolution sum

$\sum^{1} x[k]\,h[n-k] \qquad k = -2, -1, 0, 1$



Example: convolution sum



If we know $h[n]$, we can determine the system response for any signal $x[n]$, and more.

$= x[n] * h_2[n] * h_1[n]$

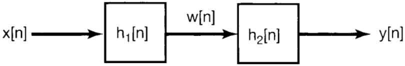
Convolution is:

- Associative: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Commutative: $x[n] * h[n] = h[n] * x[n]$
- Distributive: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

This enables us to more easily determine the response of systems connected in series and parallel.

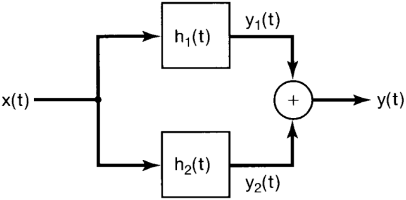
Properties of convolutions

Associativity:



$$x[n] \longrightarrow \boxed{h_1[n] * h_2[n]} \longrightarrow y[n]$$

Distributivity:



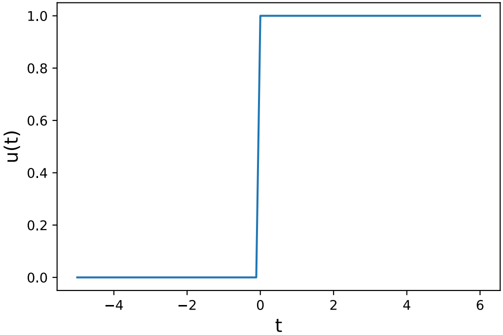
$$x[n] \rightarrow \boxed{h_1[n] + h_2[n]} \rightarrow y[n]$$

Image credits: Signals and Systems 2nd ed., Oppenheim

Live code

Let's implement everything we just did to see how it works.

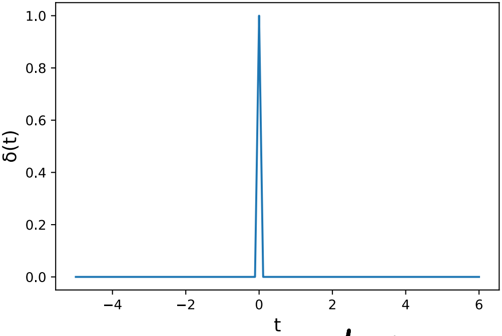
The CT unit step



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

The CT unit impulse



$$\delta(t) = \frac{du(t)}{dt}$$

$$\delta(t) = \frac{du(t)}{dt}$$

The convolution integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

The analogous of the convolution sum for CT signals and systems is the **convolution integral**. impulse response $h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

It has the same properties as the DT case (commutative, associative, distributive).

To reiterate: the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

and convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

show that as long as we know how a system responds to a unit impulse, we can determine its response to any other signal.

Knowledge of the impulse response also allows us to determine key properties of systems such as causality, invertibility, and stability.

Impulse response and memory

For reference:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

A system is memoryless if the output depends only on the input at the same time. This implies $h[n] = 0$ for $n \neq 0$, meaning

$$h[n] = K\delta[n]$$

(And analogous for CT case)

If a system is invertible, it has an inverse system.
Suppose impulse response of a system is $h(t)$. Then

$$x(t) \rightarrow y(t) = h(t) * x(t)$$

not a derivate; just some other impulse response

To send this back to $x(t)$, need to find $h'(t)$ such that

$$y(t) \rightarrow h'(t) * y(t) = \underbrace{h'(t) * h(t)}_{h'(t) * h(t) = \delta(t)} x(t)$$

$$= (h'(t) * h(t)) * x(t)$$

$$= \delta(t) * x(t)$$

$$\Downarrow$$

$$x(t) \cdot \delta(t)$$

(And analagous for DT case. We will see this later in the course.)

Impulse response and stability

Suppose we have a signal $x(t)$ with a bounded input, $|x(t)| \leq B$.
If the system is stable, the output should be bounded.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \right| \\
&= \left| \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \right| \\
&\leq \int_{-\infty}^{\infty} |x(t-\tau)| \cdot |h(\tau)|\, d\tau \\
&\leq B \int_{-\infty}^{\infty} |h(\tau)|\, d\tau
\end{aligned}$$

(And analogous for DT case)

As long as

$$\int_{-\infty}^{\infty} h(\tau)\,d\tau$$

is bounded (i.e., $h(t)$ is absolutely integrable), then the system will be stable.

(And analogous for DT case)

Example: stability

Is the system with impulse response $h(t) = e^{-4t}u(t-2)$ stable?

Integrate by parts...

$m = u(\tau-2)$ $\quad$ $n = -\frac{1}{4}e^{-\tau}$

$dm = \delta(\tau-2)$ $\quad$ $dn = e^{-4\tau}$

$$\int_{-\infty}^{\infty} h(\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-4\tau}u(\tau-2)\,d\tau$$

$$= \cancel{-\frac{1}{4}u(\tau-2)e^{-4\tau}}\Big|_{-\infty}^{\infty} - \left(-\frac{1}{4}\right)\int_{-\infty}^{\infty} \underbrace{e^{-4\tau}\delta(\tau-2)}_{\text{only } \tau=2 \text{ part survives}}\,d\tau$$

0 at $-\infty$ $\quad$ 0 at ∞

$$= \frac{1}{4}\cdot e^{-4\cdot 2}$$

$$= \frac{1}{4}e^{-8}$$

absolutely integrable

$\Rightarrow$ stable

Recap

Today's learning outcomes were:

- Define the DT/CT unit impulse and step functions
- Define the impulse response
- Express a system using the DT convolution sum and CT convolution integrals
- Apply key properties of the convolution sum/integral to determine the response of compound LTI systems

What topics did you find unclear today?

For next time

Content:

- (Complex) exponential signals
- The continuous-time Fourier series representation

Action items:

1. Start assignment 1

Recommended reading:

- From today's class: Oppenheim 1.4, 2.1-2.3
- For next class: Oppenheim 1.3, 3.0-3.3