ELEC 221 Lecture 05
The discrete-time Fourier series

Thursday 22 September 2022

- Assignment 2 available (computational); due next Wednesday
- **Make sure to complete the statement of contributions or you will get 0, no exceptions.**
- Reminder: lowest assignment grade and lowest quiz grade will be dropped.

Clarification request: invertibility and inverse systems

Oppenheim provides two definitions for system invertibility:

1. "A system is said to be *invertible* if distinct inputs lead to distinct outputs"
2. "If a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system."

Sometimes you need to try more than one way to check this: maybe you can construct the inverse system, or find two input signals that lead to the same output.

Let's do some examples from Oppenheim problem 1.30

Clarification request: invertibility and inverse systems

Example: $y(t) = \sin(x(t))$.

Initial thought: inverse system should be $y(t) = \arcsin(x(t))$.

But consider the following:

$$\begin{aligned} x_1(t) \quad &\rightarrow \quad y_1(t) = \sin(x_1(t)) \\ x_2(t) = x_1(t) + 2\pi \quad &\rightarrow \quad y_2(t) = \sin(x_2(t)) = \sin(x_1(t)) \end{aligned}$$

Different input signals led to the same output signal.

This system is not invertible.

Clarification request: invertibility and inverse systems

Example: $y(t) = \frac{dx(t)}{dt}$

Initial thought: integrate it!

But consider the following:

$$\begin{aligned} x_1(t) = 3 \quad &\rightarrow \quad y_1(t) = 0 \\ x_2(t) = 4 \quad &\rightarrow \quad y_2(t) = 0 \end{aligned}$$

This system is not invertible.

Clarification request: invertibility and inverse systems

Example: $y[n] = x[4n]$.

Initial thought: inverse system is $y[n] = x[n/4]$

One way to think of this: you lose information (downsampling) after a signal goes through the first system; so you cannot invert it.

Another way: find a counter example.

$$
\begin{aligned}
x_1[n] = \delta[n] \quad &\rightarrow \quad y_1[n] = \delta[4n] = \delta[n] \\
x_2[n] = \delta[n] + \delta[n-1] \quad &\rightarrow \quad y_2[n] = \delta[4n] + \delta[4n-1] = \delta[n]
\end{aligned}
$$

This system is not invertible.

Clarification request: invertibility and inverse systems

Example:

$$y[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

This one actually is invertible!

$$y[n] = x[2n]$$

Last time

We introduced the Dirichlet conditions, which tell us whether a CT signal can be expanded as a Fourier series.

If over one period, a signal

1. is single-valued
2. is absolutely integrable
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

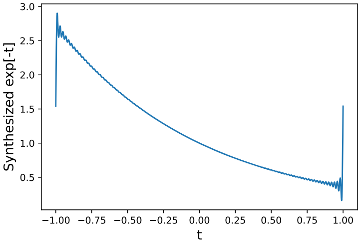
then the Fourier series converges to

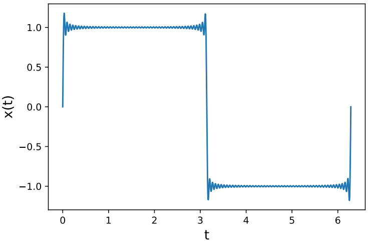
- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

Last time

We encountered the Gibbs phenomenon, a ringing that occurs when you truncate the Fourier series representation.

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega t}$$





Last time

We learned about energy and power of a signal

$$E = \int_T |x(t)|^2 dt, \quad P = \frac{1}{T}\int_T |x(t)|^2 dt$$

and derived Parseval's relation between Fourier coefficients and signal power

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Today

We are shifting over to **discrete time**.

Learning outcomes:

- Compute the fundamental period and frequency of a DT signal
- Express DT periodic signals as Fourier series and compute the Fourier coefficients
- Compute the frequency response of a system, and determine how it responds to a complex exponential signal

DT complex exponential signals

Recall our continuous-time representation of complex exponential signals:

$$x(t) = Ce^{\alpha t}$$

where α could be real or complex.

In DT, we instead often write

$$x[n] = C\beta^n \quad (= Ce^{\alpha n})$$

where β can be real or complex.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is real.

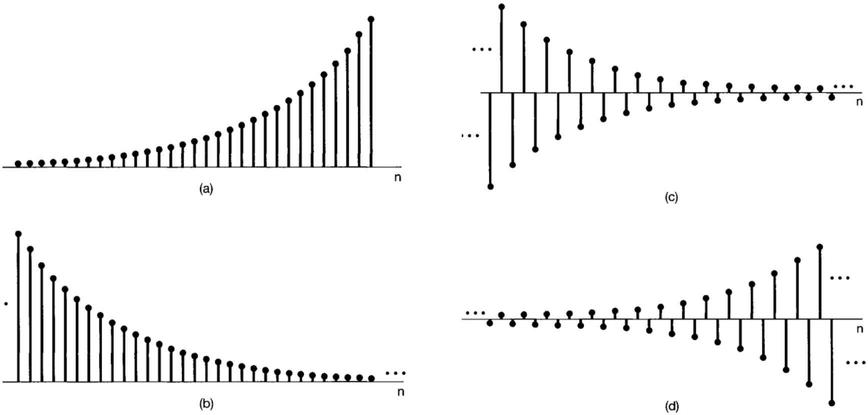


Image credit: Oppenheim chapter 1.3.

DT complex exponential signals

Case: $x[n] = C\beta^n$, where β is purely complex:

$$x[n] = Ce^{j\omega n} = C\cos(\omega n) + Cj\sin(\omega n)$$

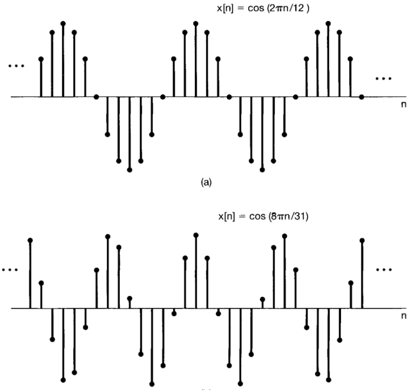


Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

While these might look similar to their discrete counterpoints, there is a **very important difference** relating to frequency.

In CT,

$$x(t) = A\cos(\omega t + \theta), \quad x(t) = Ce^{j\omega t}$$

This has frequency ω is periodic with period $T = 2\pi/\omega$.

The bigger ω gets, the faster it oscillates!

Frequency and period of DT complex exponential signals

What happens in DT? Imagine we have some starting ω:

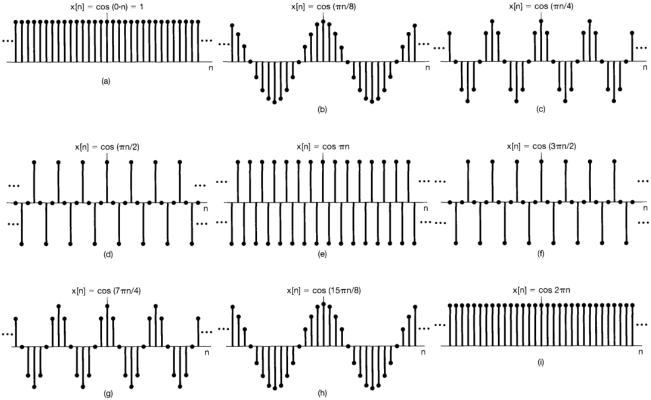
$$x[n] = e^{j\omega n}$$

Suppose that we slowly increase ω. Eventually we reach $\omega + 2\pi$:

$$\begin{aligned} x[n] &= e^{j(\omega+2\pi)n} \\ &= e^{j\omega n} e^{j2\pi n} \\ &= e^{j\omega n} \end{aligned}$$

Frequency and period of DT complex exponential signals

For a DT signal with frequency ω, the signals with frequencies $\omega \pm 2n\pi$ *are the same!*



Only need to consider ω in the range 0 to 2π (or $-\pi$ to π).

Image credit: Oppenheim chapter 1.3.

Frequency and period of DT complex exponential signals

Another consequence: there are additional criteria for periodicity.

Suppose the period is N:

$$\begin{aligned} x[n] &= e^{j\omega(n+N)} \\ &= e^{j\omega n} e^{j\omega N} \end{aligned}$$

This implies that

$$e^{j\omega N} = 1 \quad \rightarrow \quad \omega N = 2\pi m, \;\; m \in \mathbb{Z}$$

$\omega/2\pi$ must be rational for the signal to be periodic.

This can be counterintuitive.

Example: consider the CT signal

$$x(t) = \cos(3t)$$

This is periodic with period $T = 2\pi/3$.

What about the DT equivalent,

$$x[n] = \cos(3n)$$

Frequency and period of DT complex exponential signals

The signal

$$x[n] = \cos(3n)$$

is *not periodic*.

n can only be an integer, so we will never get to the "time" of $2\pi/3$ where the signal will start to repeat.

Example: what is the fundamental period of

$$x[n] = \cos(5\pi n/6)$$

For the CT equivalent, $T = 2\pi/\omega = 12/5$, but this is not a valid n.

But after 5 periods, we are at $n = 12$ which is valid so the fundamental period is $N = 12$.

Harmonics of DT complex exponential signals

What about harmonics?

$$x_k[n] = e^{jk\omega n} = e^{jk\frac{2\pi n}{N}}, \quad k \in \mathbb{Z}$$

In CT we had an infinite number of these. What about DT?

Note that, once we reach $k = N$, we get back to our original signal... so have only N distinct harmonics:

$$x_k[n] = e^{jk\frac{2\pi n}{N}}, \quad k \in 0, \ldots N - 1$$

Recall that in CT, complex exponentials are eigenfunctions of LTI systems. What about in DT?

Consider a system with impulse response $h[n]$ and DT signal $x_m[n] = e^{jm\omega n}$. Use the convolution sum:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{jm\omega(n-k)}h[k] \\ &= e^{jm\omega n} \sum_{k=-\infty}^{\infty} e^{-jm\omega k}h[k] \\ &= x_m[n] \cdot H(e^{jm\omega}) \end{aligned}$$

DT Fourier series

So in DT, if we know how the system responds to complex exponential signals, we can figure out what it does to any signal that is expressed in terms of them.

We need a Fourier series representation of DT signals:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\omega n}$$

Just like in CT: how do we find the c_k?

DT Fourier coefficients

Option 1: solve a linear system of equations.

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\omega n}$$

$$\begin{aligned}
x[0] &= \sum_{k=0}^{N-1} c_k = c_0 + c_1 + \ldots + c_{N-1} \\
x[1] &= \sum_{k=0}^{N-1} c_k e^{jk\omega} = c_0 + c_1 e^{j\omega} + \ldots + c_{N-1} e^{j(N-1)\omega} \\
&\vdots \\
x[N-1] &= \sum_{k=0}^{N-1} c_k e^{jk\omega(N-1)} = c_0 + c_1 e^{j\omega(N-1)} + \ldots + c_{N-1} e^{j(N-1)^2\omega}
\end{aligned}$$

DT Fourier coefficients

This gives the system:

$$\begin{bmatrix} x[0] \\ x[1] \\ \dots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\omega} & \cdots & e^{j(N-1)\omega} \\ \vdots & \ddots & \ddots & \vdots \\ 1 & e^{j\omega(N-1)} & \cdots & e^{j(N-1)^2\omega} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{N-1} \end{bmatrix}$$

(This may look familiar!)

DT Fourier coefficients

Option 2: obtain closed-form expressions like the CT case.

Leverage the following identity about complex numbers:

$$\sum_{k=0}^{N-1} e^{jk\frac{2\pi n}{N}} = \begin{cases} N \text{ if } k = 0, \pm N, \pm 2N, \ldots \\ 0 \text{ otherwise} \end{cases}$$

We will multiply on both sides, and sum.

DT Fourier coefficients

$$
\begin{aligned}
x[n] &= \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}} \\
\sum_{n=0}^{N-1} e^{-jm\frac{2\pi n}{N}} x[n] &= \sum_{n=0}^{N-1} e^{-jm\frac{2\pi n}{N}} \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}} \\
&= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi n}{N}} \\
&= c_m N
\end{aligned}
$$

DT Fourier coefficients

So the DT synthesis and analysis equations are

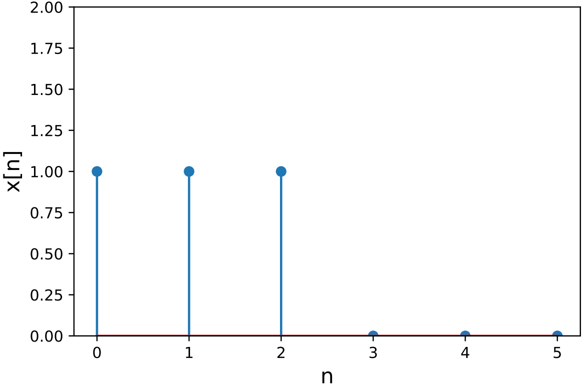
$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}}$$

and

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi n}{N}}$$

Example: the DT square wave

Let's compute the Fourier coefficients of this signal:



Let's recap what we've done the previous three or so lectures.

Back in lectures 2 and 3, we talking about LTI systems and impulse response. We showed a couple of important things...

Any signal can be expressed as a combination of weighted, shifted impulses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau, \quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

If we know what the system does to a unit impulse, i.e., the impulse response $h(t)$ or $h[n]$, we can learn what it does to any signal. This was the convolution integral and sum:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k]
\end{aligned}
$$

We now know many periodic signals, in both CT and DT, can be expressed as Fourier series (linear combinations of complex exponential signals):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}, \quad c_k = \frac{1}{T}\int_T e^{-jk\omega t} x(t) dt$$

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}}, \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi n}{N}}$$

System functions

And we know that these complex exponential signals are eigenfunctions of LTI systems:

$$
\begin{aligned}
x(t) \rightarrow y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \\
x[n] \rightarrow y[n] &= \sum_{k=0}^{N-1} c_k H(e^{jk\omega}) e^{jk\omega n}
\end{aligned}
$$

In general, $H(s)$ is called the **system function**; when we are limiting to $H(j\omega)$ we term this the **frequency response**.

In other words, when we input a complex exponential signal at a particular frequency, how does the system respond?

Example: computing the frequency response

Consider an LTI system with impulse response:

$$h(t) = e^{-4|t|}$$

What happens when we put in the following signal?

$$x(t) = \frac{1}{2}\cos(\omega t) + \sin(3\omega t)$$

Three steps:

1. Determine the Fourier coefficients of the signal
2. Determine the frequency response
3. Use 1 and 2 to obtain Fourier coefficients of the output signal

Example: computing the frequency response

Step 1: what are the Fourier coefficients of

$$x(t) = \frac{1}{2}\cos(\omega t) + \sin(3\omega t)$$

Expand sin and cos as complex exponentials:

$$x(t) = \frac{1}{2}\frac{e^{j\omega t} + e^{-j\omega t}}{2} + \frac{e^{3j\omega t} - e^{-3j\omega t}}{2j}$$

Then we see that:

$$c_1 = c_{-1} = \frac{1}{4}, \quad c_3 = \frac{1}{2j}, \quad c_{-3} = c_3^* = -\frac{1}{2j}$$

Example: computing the frequency response

Step 2: what is the frequency repsonse of the system with

$$h(t) = e^{-4|t|}$$

Integrate!

$$\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} e^{-4|\tau|} e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{0} e^{-4|\tau|} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-4|\tau|} e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{0} e^{(4-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(4+j\omega)\tau} d\tau
\end{aligned}$$

Example: computing the frequency response

$$H(j\omega) = \int_{-\infty}^{0} e^{(4-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(4+j\omega)\tau} d\tau$$

Continue...

$$\begin{aligned} H(j\omega) &= \frac{1}{4-j\omega} e^{(4-j\omega)\tau}\Big|_{-\infty}^{0} + \frac{1}{-(4+j\omega)} e^{-(4+j\omega)\tau}\Big|_{0}^{\infty} \\ &= \frac{1}{4-j\omega} + \frac{1}{4+j\omega} \\ &= \frac{8}{16+\omega^2} \end{aligned}$$

Example: computing the frequency response

We have

$$
\begin{aligned}
x(t) &= \frac{1}{2}\frac{e^{j\omega t}+e^{-j\omega t}}{2}+\frac{e^{3j\omega t}-e^{-3j\omega t}}{2j} \\
H(j\omega) &= \frac{8}{16+\omega^2}.
\end{aligned}
$$

Step 3: get the Fourier coefficients of the output signal.
In general, we know that

$$
\begin{aligned}
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \to y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \\
&= \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{jk\omega t}
\end{aligned}
$$

Example: computing the frequency response

Two sets of coefficients to deal with:

$$c_1 = c_{-1} = \frac{1}{4} \quad \rightarrow \quad \text{New } \tilde{c}_1 = \tilde{c}_{-1} = \frac{2}{16+\omega^2}$$

and

$$c_3 = c_{-3}^* = \frac{1}{2j} \qquad\qquad H(3j\omega) = \frac{8}{16+9\omega^2}$$

$$\text{New} \quad \tilde{c}_3 = \tilde{c}_{-3}^* = \frac{4}{j(16+9\omega^2)}$$

Example: computing the frequency response

So our original signal was

$$
\begin{aligned}
x(t) &= \frac{1}{2}\cos(\omega t) + \sin(3\omega t) \\
&= \frac{1}{4}(e^{j\omega t} + e^{-j\omega t}) + \frac{1}{2j}(e^{3j\omega t} - e^{-3j\omega t})
\end{aligned}
$$

Our final signal is

$$
\begin{aligned}
y(t) &= \frac{2}{16+\omega^2}(e^{j\omega t} + e^{-j\omega t}) + \frac{4}{j(16+9\omega^2)}(e^{3j\omega t} - e^{-3j\omega t}) \\
&= \frac{4}{16+\omega^2}\cos(\omega t) + \frac{8}{16+9\omega^2}\sin(3\omega t)
\end{aligned}
$$

Recap

Today's learning outcomes were:

- Compute the fundamental period and frequency of a DT signal
- Express DT periodic signals as Fourier series and compute the Fourier coefficients
- Compute the frequency response of a system, and determine how it responds to a complex exponential signal

What topics did you find unclear today?

For next time

We are going to learn how to *design* system functions to manipulate signals in certain ways:

- Frequency-shaping filters
- Frequency-selective filters

Action items:

1. Assignment 2 is due next Wednesday
2. Quiz 3 is on Tuesday

Recommended reading:

- From today's class: Oppenheim 1.3, 3.6-3.8
- For next class: Oppenheim 3.9