ELEC 221 Lecture 03
CT complex exponential signals and the Fourier series representation

Thursday 15 September 2022

Announcements

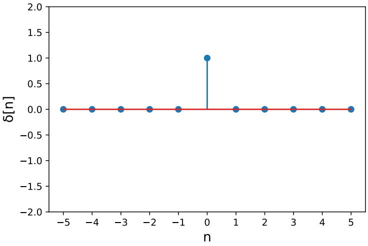
- Assignment 1 available on PrairieLearn (due next Wednesday at 23:59)
- No office hour or tutorial on Monday 19 Sept. due to holiday

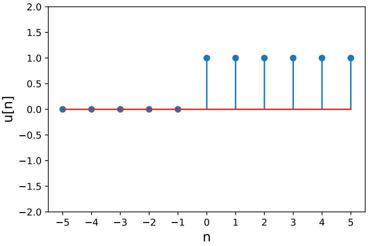
Response to feedback:

- Will try to slow down and pause more before changing slides
- Review impulse response, convolution sum and integral
- Practice problems: see Oppenheim "basic problems with answers".

Last time

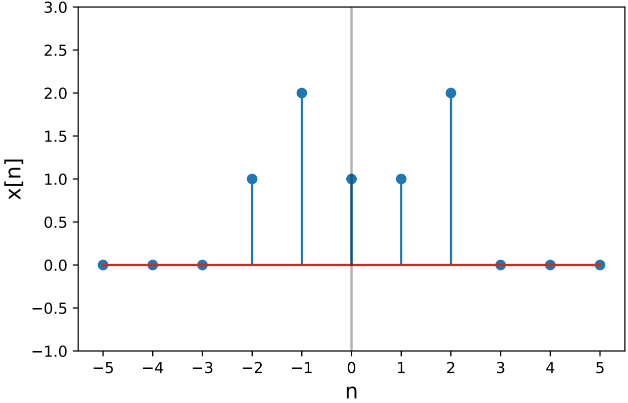
We defined the DT unit step and impulse functions.





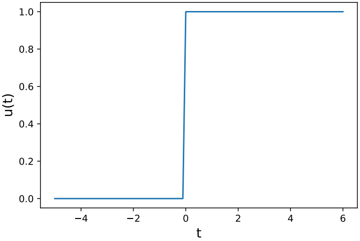
We saw how DT signals can be represented as a sum of shifted, weighted impulses:

Last time



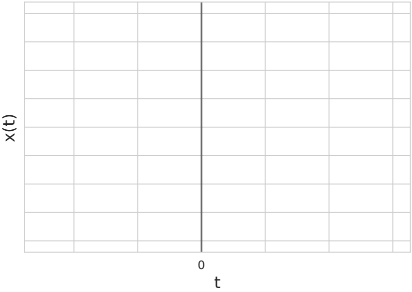
Last time

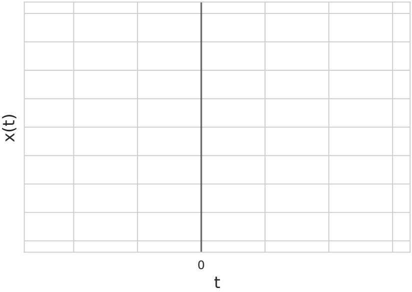
We defined the CT unit step function, $u(t)$



And the CT unit impulse function, $\delta(t)$:

This is how CT unit impulse should be represented:





We saw how CT signals can be represented as a sum of shifted, weighted impulses:

Last time

If we represent a DT signal as a sum of weighted, shifted impulses

and put it through an LTI system, the output is

where $h[n]$ is called the **impulse response**.

This is the convolution sum

and in CT the convolution integral,

As long as we know how a system responds to a unit impulse, we can determine its response to any other signal.

Is the unit impulse the only "basic" signal we can do this for?...

Today

Learning outcomes:

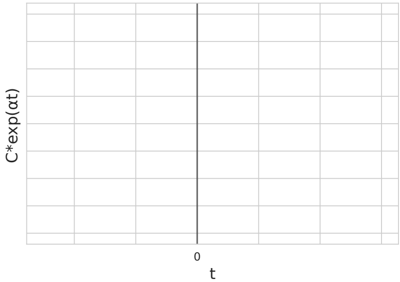
- Express CT periodic signals as linear combinations of complex exponential functions
- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems
- Express a CT signal as a Fourier series, and compute its Fourier coefficients

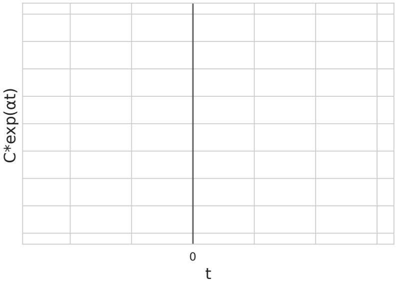
Get ready for MATH! ♡

Most general form:

where α can be real or complex.

Case: both C and α are real-valued.





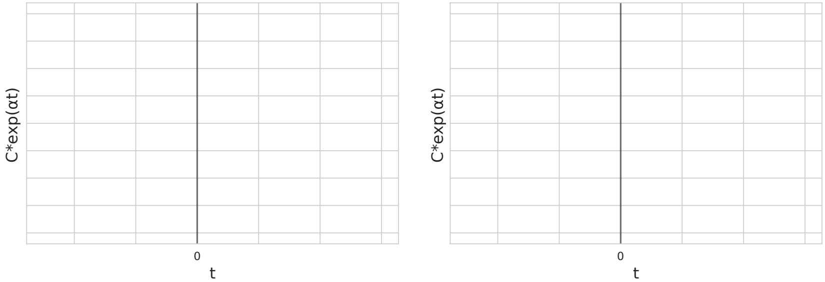
Case: α is complex. Recall can write any complex z in two ways:

Euler's relation allows us to rewrite the second form:

As a result,

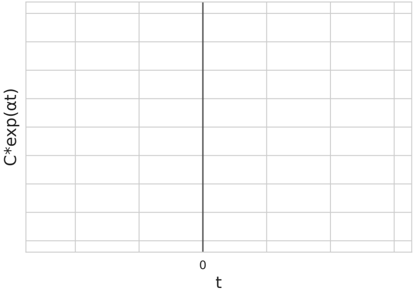
Review: complex exponential functions

Case: α is a complex number $\alpha = j\omega$. Then,

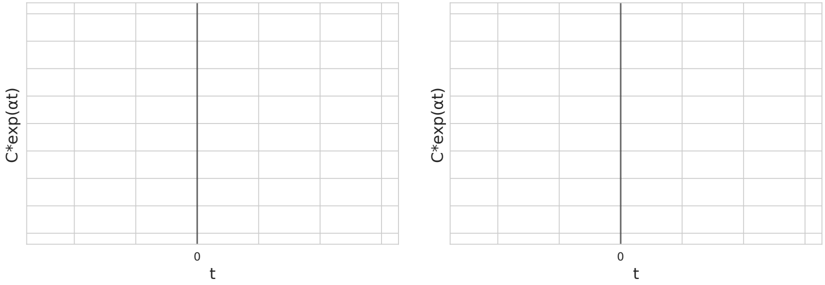


Review: complex exponential functions

$x(t) = Ce^{j\omega t}$ is periodic:



Case: α is a complex number $\alpha = r + j\omega$. Then,



LTI systems and complex exponential functions

Recall the convolution integral:

What happens when our signal $x(t)$ is a complex exponential function?

LTI systems and complex exponential functions

Write $x(t) = e^{st}$. Then:

To summarize:

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

...so what?

Recall that LTI system sends superpositions of inputs to superpositions of outputs. If $x(t) \rightarrow y(t)$,

...so what?

If all the $x_i(t)$ are complex exponential functions and

then

The response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

How can we express arbitrary signals as superpositions of complex exponentials?

Let's consider a special set of signals[1]:

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

[1] We will see the general case at the end of the course.

The Fourier series

It also has an infinite number of cousins:

These are called **harmonics**; ω is the fundamental frequency.

Harmonics

Yes, these harmonics.

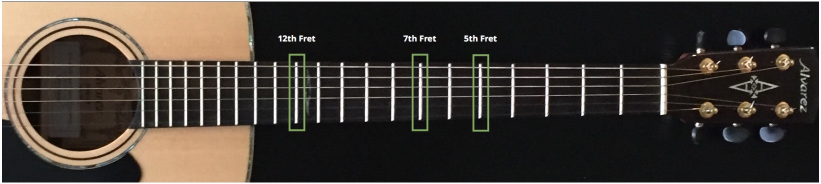


Image credit: `https://musiciantuts.com/guitar-harmonics/`

The Fourier series

We can create a superposition of all possible harmonics k:

This signal is also periodic with period $T = 2\pi/\omega$.

The Fourier series

The representation of a periodic signal like

is called the **Fourier series**.

The c_k are the **Fourier coefficients**.

Usually we are dealing with a signal $x(t)$ which is always *real*.

This means that .

We can leverage this fact to express $x(t)$ in a different way.

Apply Euler's relation:

The Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \cos(k\omega t) + \sum_{k=-\infty}^{\infty} jc_k \sin(k\omega t)$$

Leverage the fact that sin is odd and cos is even:

The Fourier series

$$x(t) = \sum_{k=0}^{\infty} (c_{-k} + c_k) \cos(k\omega t) + \sum_{k=0}^{\infty} j(c_k - c_{-k}) \sin(k\omega t)$$

Use the fact that $c_{-k} = c_k^*$

The Fourier series

$$x(t) = \sum_{k=0}^{\infty} 2\mathfrak{Re}(c_k)\cos(k\omega t) - \sum_{k=0}^{\infty} 2\mathfrak{Im}(c_k)\sin(k\omega t)$$

Separate out the $k = 0$ term and relabel:

Note: sometimes a_0 is written $a_0/2$ for convenience

Live code: playing with Fourier coefficients

Let's explore what these look like!

Evaluating Fourier coefficients

Recall: the response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

As long as we know how a system responds to complex exponentials, and we know c_k, we can determine the response to the full signal.

Evaluating Fourier coefficients

Given an arbitrary signal, how do we compute the c_k[2]?

We leverage the fact that the $e^{jk\omega t}$ are **basis functions** and have **orthogonality** relations w.r.t. integration.

[2]Note that we are assuming that this is actually possible. We will see the conditions for this next class.

Evaluating Fourier coefficients

Example: let's compute c_m.

Multiply on both sides by the conjugate of the basis function:

Evaluating Fourier coefficients

$$e^{-jm\omega t}x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} e^{-jm\omega t}$$

Integrate over a period:

Evaluating Fourier coefficients

Evaluate the integral

Case 1: $k = m$

Case 2: $k \neq m$

Evaluating Fourier coefficients

From here:

$$\frac{1}{T}\int_0^T e^{-jm\omega t}x(t)dt = \sum_{k=-\infty}^{\infty} c_k \cdot \frac{1}{T}\int_0^T e^{j(k-m)\omega t}dt$$

Only the $k = m$ term survives so

Evaluating Fourier coefficients

Since all that matters is that we are integrating over one period:

The Fourier coefficients provide a measure of *how much* each harmonic contributes to the total signal.

Note that c_0 is a constant offset:

Similar integrals can be used to determine the coefficients for the sin and cos representation:

	$m \neq k$	$m = k \neq 0$	$m = k = 0$
$\frac{1}{T}\int_0^T \sin m\omega t \sin k\omega t dt$	0	0.5	0
$\frac{1}{T}\int_0^T \cos m\omega t \cos k\omega t dt$	0	0.5	1

Fun with math: try deriving them yourself!

To recap, we have two important expressions. They have names.

Fourier synthesis equation:

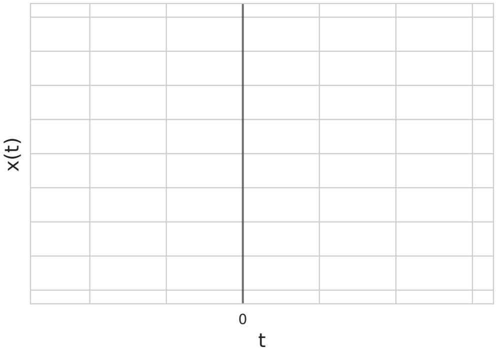
Fourier analysis equation:

Example

Suppose we have the following signal with period $T = 2$:

How do we express this as a Fourier series?

Example



Example

Start with the analysis equation:

Example

$$c_k = \frac{1}{2(1+jk\omega)}\left[e^{(1+jk\omega)} - e^{-(1+jk\omega)}\right]$$

Note that:

Then we can simplify:

Let's check that this works!

Today's learning outcomes were:

- Express CT periodic signals as linear combinations of complex exponential functions
- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems
- Express a CT signal as a Fourier series, and compute its Fourier coefficients

What topics did you find unclear today?

For next time

Content:

- Convergence results for CT Fourier series
- Properties and features of CT Fourier series

Action items:

1. Work on assignment 1
2. Quiz 2 will be on Tuesday

Recommended reading:

- From today's class: Oppenheim 1.3, 3.0-3.3
- For next class: Oppenheim 3.4-3.5