ELEC 221 Lecture 03
CT complex exponential signals and the Fourier series representation

Thursday 15 September 2022

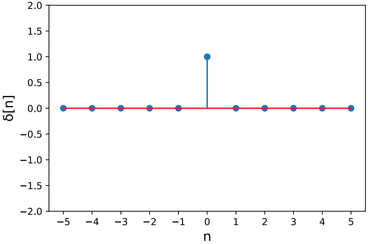
Announcements

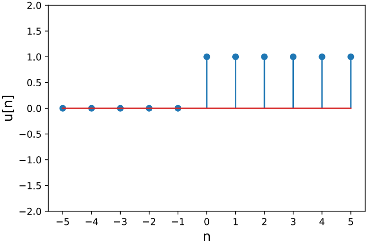
- Assignment 1 available on PrairieLearn (due next Wednesday at 23:59)
- No office hour or tutorial on Monday 19 Sept. due to holiday

Response to feedback:

- Will try to slow down and pause more before changing slides
- Review impulse response, convolution sum and integral
- Practice problems: see Oppenheim "basic problems with answers".

We defined the DT unit step and impulse functions.



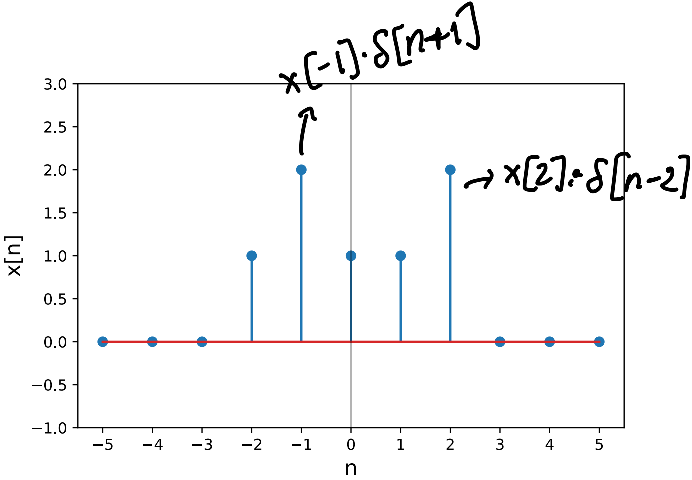


We saw how DT signals can be represented as a sum of shifted, weighted impulses:

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

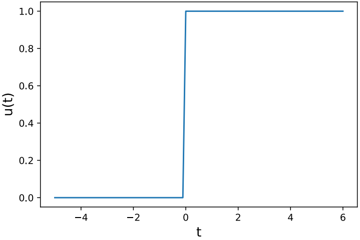
Last time



$x[-1] \cdot \delta[n+1]$
$\rightarrow x[2] \cdot \delta[n-2]$
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
x[n]
−5
−4
−3
−2
−1
0
1
2
3
4
5
n

Last time

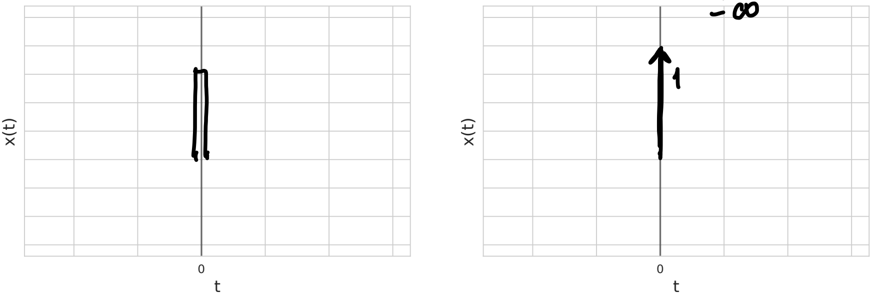
We defined the CT unit step function, $u(t)$



And the CT unit impulse function, $\delta(t)$:

$$\delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

This is how CT unit impulse should be represented: $\int_{-\infty}^{\infty} \delta(\tau)\,d\tau = 1$



We saw how CT signals can be represented as a sum of shifted, weighted impulses:

$$x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Last time

If we represent a DT signal as a sum of weighted, shifted impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

and put it through an LTI system, the output is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = x[n] * h[n]$$

where $h[n]$ is called the **impulse response**.

This is the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

and in CT the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

As long as we know how a system responds to a unit impulse, we can determine its response to any other signal.

Is the unit impulse the only "basic" signal we can do this for?...

Today

Learning outcomes:

- Express CT periodic signals as linear combinations of complex exponential functions
- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems
- Express a CT signal as a Fourier series, and compute its Fourier coefficients

Get ready for MATH! $\heartsuit$

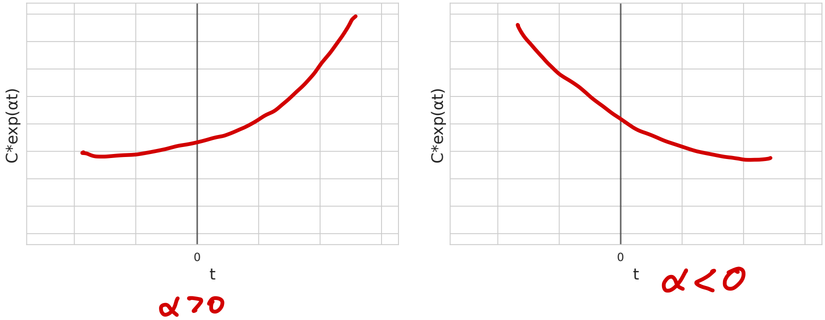
Review: complex exponential functions

Most general form:

$$x(t) = Ce^{\alpha t}$$

where α can be real or complex.

Case: both C and α are real-valued.



Case: α is complex. Recall can write any complex z in two ways:

$$z = a + bj \qquad z = |z|e^{j\theta} \qquad |z| = \sqrt{a^2 + b^2}$$

Euler's relation allows us to rewrite the second form:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

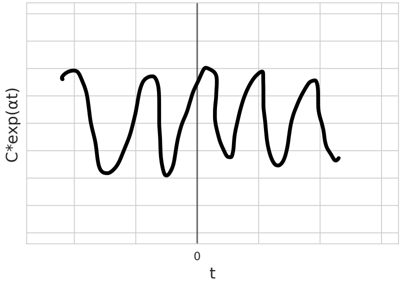
As a result,

$$\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right] \qquad \sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

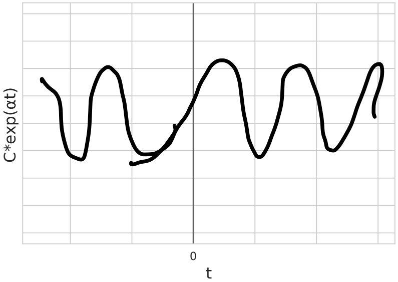
Case: α is a complex number $\alpha = j\omega$. Then,

$$x(t) = Ce^{j\omega t} = C\cos\omega t + Cj\sin\omega t$$

Re[x(t)]



Im[x(t)]



Review: complex exponential functions

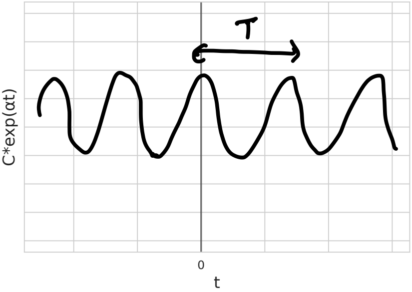
$x(t) = Ce^{j\omega t}$ is periodic:

$$x(t+T) = Ce^{j\omega(t+T)} = Ce^{j\omega t} e^{j\omega T} = 1$$

$$e^{2\pi j \cdot n} = 1$$

$$j\omega T = 2\pi j$$

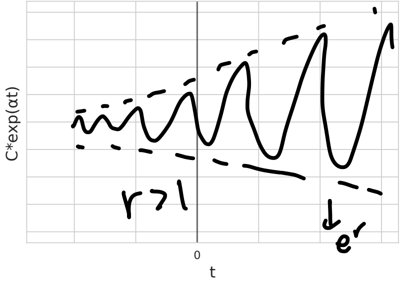
$$T = \frac{2\pi}{\omega}$$

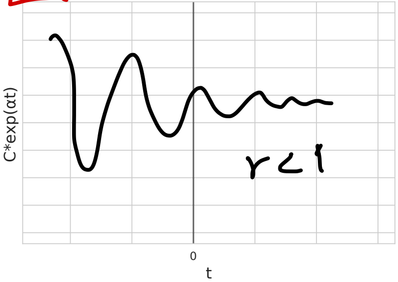


Case: α is a complex number $\alpha = r + j\omega$. Then,

$$x(t) = Ce^{\alpha t} = Ce^{rt}\cos(\omega t) + jCe^{rt}\sin(\omega t)$$

$$x(t) = Ce^{\alpha t} = Ce^{r}e^{j\omega t}$$
$$= C\boxed{e^{r}}\cos\omega t + Cj\boxed{e^{r}}\sin\omega t$$





Recall the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

What happens when our signal $x(t)$ is a complex exponential function?

Write $x(t) = e^{st}$. Then:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{st}e^{-s\tau}h(\tau)d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau$$

$$y(t) = e^{st} \cdot H(s)$$

e^{st}: eigenfunc

$H(s)$: eigenvalue

$A\vec{v} = \lambda\vec{v}$

λ: eigenvalue

$\vec{v}$: eigenvector

To summarize:

$$x(t) = e^{st} \rightarrow y(t) = \underline{H(s)} \cdot e^{st}$$

Complex exponentials are **eigenfunctions** of LTI systems.

$H(s)$ is called the **system function**, or *frequency response*, of an LTI system.

Recall that LTI system sends superpositions of inputs to superpositions of outputs. If $x(t) \to y(t)$,

$$x(t) = \sum_i a_i x_i(t) \to y(t) = \sum_i a_i y_i(t)$$

...so what?

If all the $x_i(t)$ are complex exponential functions and

$$x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$$

then

$$x(t) = \sum_k c_k e^{s_k t} \rightarrow y(t) = \sum_k c_k \cdot H(s_k) e^{s_k t}$$

The response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

How can we express arbitrary signals as superpositions of complex exponentials?

Let's consider a special set of signals[1]:

$$x(t) = e^{st} = e^{j\omega t}$$

This signal has frequency ω and period $T = 2\pi/\omega$.

We write its system function as $H(j\omega)$.

[1] We will see the general case at the end of the course.

It also has an infinite number of cousins:

$$x_k(t) = e^{j\omega k t} = e^{jk\frac{2\pi t}{T}}, \quad k = 0, \pm 1, \pm 2$$

These are called **harmonics**; ω is the fundamental frequency.

Harmonics

Yes, these harmonics.

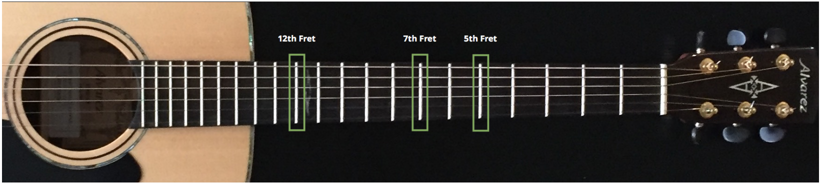


Image credit: `https://musiciantuts.com/guitar-harmonics/`

We can create a superposition of all possible harmonics k:

$$\sum_{k=-\infty}^{\infty} c_k e^{jkwt}$$

This signal is also periodic with period $T = 2\pi/\omega$.

The Fourier series

The representation of a periodic signal like

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

is called the **Fourier series**.

The c_k are the **Fourier coefficients**.

Usually we are dealing with a signal $x(t)$ which is always *real*.

$$x(t) = x^*(t)$$

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega t}$$

This means that $c_{-k} = c_k^*$

We can leverage this fact to express $x(t)$ in a different way.

Apply Euler's relation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} = \sum_{k=-\infty}^{\infty} c_k \left[\cos(k\omega t) + j\sin(k\omega t)\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(k\omega t) + \sum_{k=-\infty}^{\infty} j c_k \sin(k\omega t)$$

$$c_{-k}\cos(-k\omega t) = c_{-k}\cos(k\omega t) = c_k^*\cos(k\omega t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \cos(k\omega t) + \sum_{k=-\infty}^{\infty} jc_k \sin(k\omega t)$$

Leverage the fact that sin is odd and cos is even:

$$x(t) = \sum_{k=0}^{\infty} \underbrace{(c_k + c_k^*)}_{2\cdot Re(c_k)} \cos(k\omega t) + \sum_{k=0}^{\infty} j\underbrace{(c_k - c_k^*)}_{-2\cdot Im(c_k)} \sin(k\omega t)$$

$$= \sum_{k=0}^{\infty} (c_{-k} + c_k)\cos(k\omega t) + \sum_{k=0}^{\infty} j(c_k - c_{-k})\sin(k\omega t)$$

$$\downarrow \qquad\qquad \downarrow$$

$$a+bj+(a-bj) = 2a \qquad a+bj-(a-bj) = 2bj$$

$$x(t) = \sum_{k=0}^{\infty} (c_{-k} + c_k)\cos(k\omega t) + \sum_{k=0}^{\infty} j(c_k - c_{-k})\sin(k\omega t)$$

we double counted 0

Use the fact that $c_{-k} = c_k^*$

$$x(t) = \sum_{k=0}^{\infty} 2\,\mathrm{Re}(C_k)\cos(k\omega t) - \sum_{k=0}^{\infty} 2\,\mathrm{Im}(C_k)\sin(k\omega t)$$

$$= \sum_{k=0}^{\infty} 2\Re(c_k)\cos(k\omega t) - \sum_{k=0}^{\infty} 2\Im(c_k)\sin(k\omega t)$$

$$x(t) = \sum_{k=0}^{\infty} \underbrace{2\mathfrak{Re}(c_k)}_{a_k} \cos(k\omega t) \underbrace{- \sum_{k=0}^{\infty} 2\mathfrak{Im}(c_k)}_{b_k} \sin(k\omega t)$$

Separate out the $k = 0$ term and relabel:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega t) + b_k \sin(k\omega t))$$

Note: sometimes a_0 is written $a_0/2$ for convenience

Let's explore what these look like!

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k \cos k\omega t + b_k \sin k\omega t \right)$$

Recall: the response of LTI systems to superpositions of complex signals can be expressed as a superposition **of those same signals**.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$$

As long as we know how a system responds to complex exponentials, and we know c_k, we can determine the response to the full signal.

Evaluating Fourier coefficients

Given an arbitrary signal, how do we compute the c_k[2]?

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

We leverage the fact that the $e^{jk\omega t}$ are **basis functions** and have **orthogonality** relations w.r.t. integration.

[2] Note that we are assuming that this is actually possible. We will see the conditions for this next class.

Example: let's compute c_m.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Multiply on both sides by the conjugate of the basis function:

$$e^{-jm\omega t} x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(k-m)\omega t}$$

$$e^{-jm\omega t}x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} e^{-jm\omega t}$$

Integrate over a period:

$$\frac{1}{T}\int_0^T e^{-jm\omega t} x(t)\,dt = \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} c_k e^{j(k-m)\omega t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\cdot c_k \int_0^T e^{j(k-m)\omega t}\,dt$$

Evaluating Fourier coefficients

Evaluate the integral

$$\frac{1}{T}\int_0^T e^{j(k-m)\omega t}\,dt$$

Case 1: $k = m$

$$\frac{1}{T}\int_0^T 1\,dt = 1$$

Case 2: $k \neq m$

$$\frac{1}{T}\int_0^T e^{j(k-m)\omega t}\,dt = \frac{1}{T}\int_0^T \cos((k-m)\omega t)\,dt + \frac{j}{T}\int_0^T \sin((k-m)\omega t)\,dt$$

$$= 0$$

Evaluating Fourier coefficients

From here:

$$\frac{1}{T}\int_0^T e^{-jm\omega t}x(t)dt = \sum_{k=-\infty}^{\infty} c_k \cdot \frac{1}{T}\int_0^T e^{j(k-m)\omega t}dt$$

Only the $k = m$ term survives so

$$c_m = \frac{1}{T}\int_0^T e^{-jm\omega t}x(t)\,dt$$

Since all that matters is that we are integrating over one period:

$$c_m = \frac{1}{T}\int_T e^{-jm\omega t} x(t)\,dt$$

The Fourier coefficients provide a measure of *how much* each harmonic contributes to the total signal.

Note that c_0 is a constant offset:

$$c_0 = \frac{1}{T}\int_T x(t)\,dt$$

Similar integrals can be used to determine the coefficients for the sin and cos representation:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

	$m \neq k$	$m = k \neq 0$	$m = k = 0$
$\frac{1}{T}\int_0^T \sin m\omega t \sin k\omega t dt$	0	0.5	0
$\frac{1}{T}\int_0^T \cos m\omega t \cos k\omega t dt$	0	0.5	1

Fun with math: try deriving them yourself!

To recap, we have two important expressions. They have names.

Fourier synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Fourier analysis equation:

$$c_k = \frac{1}{T}\int_T e^{-jk\omega t} x(t)\, dt$$

Example

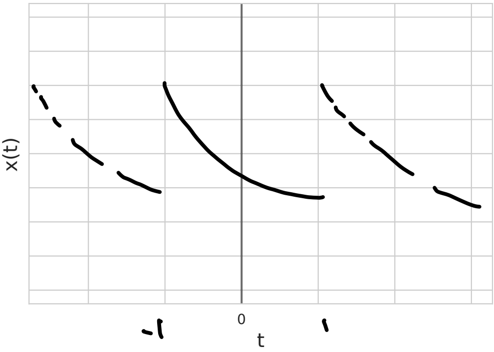
Suppose we have the following signal with period $T = 2$:

$$x(t) = e^{-t} \qquad -1 < t < 1$$

How do we express this as a Fourier series?

Example

$$x(t) = e^{-t} \quad -1 < t < 1$$



Start with the analysis equation:

$$c_k = \frac{1}{T}\int_T e^{-jk\omega t}x(t)\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-jk\omega t}e^{-t}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t(1+jk\omega)}\,dt$$

$$= \frac{1}{2}\cdot -\frac{1}{1+jk\omega}e^{-t(1+jk\omega)}\Big|_{-1}^{1}$$

$$= -\frac{1}{2(1+jk\omega)}\left[e^{-(1+jk\omega)} - e^{(1+jk\omega)}\right]$$

$$= \frac{1}{2(1+jk\omega)}\left[e^{1+jk\omega} - e^{-(1+jk\omega)}\right]$$

Example

$$c_k = \frac{1}{2(1+jk\omega)}\left[e^{(1+jk\omega)} - e^{-(1+jk\omega)}\right]$$

Note that:

$$T = 2 = 2\pi/\omega \Rightarrow \omega = \pi$$

Then we can simplify:

$$e^{jk\pi}$$

$$c_k = \frac{1}{2(1+jk\pi)}\left[e^{1+jk\pi} - e^{-(1+jk\pi)}\right]$$

$$= \frac{(-1)^k}{2(1+j\pi k)}\left[e - e^{-1}\right]$$

Let's check that this works!

Today's learning outcomes were:

- Express CT periodic signals as linear combinations of complex exponential functions
- Use the convolution sum/integral to show that complex exponentials are eigenfunctions of LTI systems
- Express a CT signal as a Fourier series, and compute its Fourier coefficients

What topics did you find unclear today?

For next time

Content:

- Convergence results for CT Fourier series
- Properties and features of CT Fourier series

Action items:

1. Work on assignment 1
2. Quiz 2 will be on Tuesday

Recommended reading:

- From today's class: Oppenheim 1.3, 3.0-3.3
- For next class: Oppenheim 3.4-3.5