ELEC 221 Lecture 04
Properties of the CT Fourier series and the Gibbs phenomenon

Tuesday 20 September 2022

Announcements

- Quiz 2 today
- Assignment 1 due tomorrow at 23:59
  - no late submissions
  - include statement of contributions or you will get 0, no exceptions
- Assignment 2 (computational assignment) released tomorrow

$$e^{jk\omega t} = x(t)$$

We explored complex exponential signals, and saw how they are eigenfunctions of LTI systems.

s complex $= j\omega$

scaling factor

Given

$$x(t) = e^{st} \to y(t) = H(s)e^{st}$$

then

$$x(t) = \sum_k c_k e^{s_k t} \to y(t) = \sum_k c_k H(s_k) e^{s_k t}$$

Last time

We expressed CT signals as Fourier series, and computed their Fourier coefficients.

Fourier coefficients

Fourier synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Fourier analysis equation:

$$c_k = \frac{1}{T}\int_T e^{-jk\omega t} x(t)\, dt$$

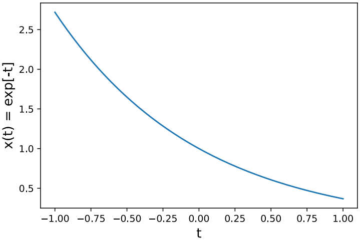
integrate over period

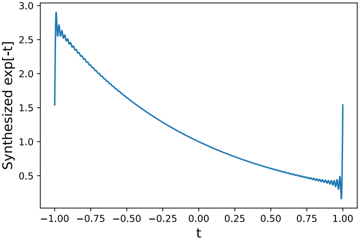
Last time

We computed the Fourier coefficients of

$$x(t) = e^{-t}, -1 < t < 1$$

but we got a funny result.





What is going on here?

Today

Learning outcomes:

- State the Dirichlet conditions for convergence of a Fourier series
- Describe the Gibbs phenomenon
- Compute the power and energy of a signal, and state Parseval's theorem for CT periodic signals

Dirichlet conditions

Can we *always* express a signal as a Fourier series?

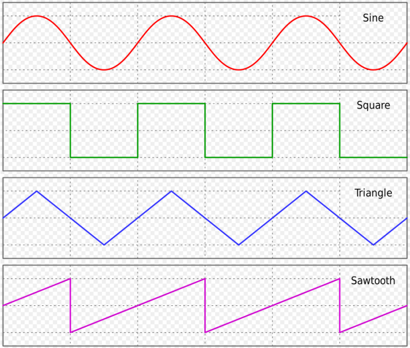


Image credit: *Sine, square, triangle, and sawtooth waveforms* (author: Omegatron)

`https://en.wikipedia.org/wiki/Triangle_wave#/media/File:Waveforms.svg` (CC BY-SA 3.0)

The **Dirichlet conditions** can help us here.

If over one period, a signal $x(t)$

1. is single-valued
2. is absolutely integrable $\int_T |x(t)|\,dt < \infty$
3. has a finite number of maxima and minima
4. has a finite number of discontinuities

then the Fourier series converges to

- $x(t)$ where it is continuous
- half the value of the jump where it is discontinuous

These conditions are **sufficient** but not necessary.

Recall the synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

Consider what happens if we truncate the series:

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega t}$$

How good of an approximation is this?

Can look at the approximation error

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} c_k e^{jk\omega t}$$

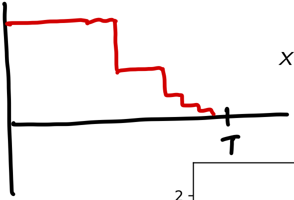
How much error is there over a given period?

$$E_N = \int_T |e_N(t)|^2$$

This should go to 0 as $N \to \infty$.

Example: the square wave

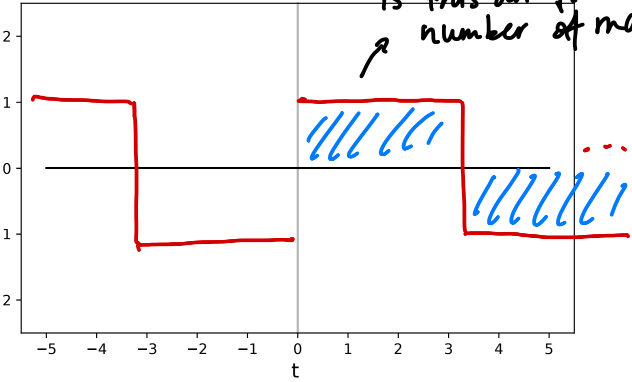
Let's consider a square wave signal with period 2π:



$$x(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi \end{cases}$$

$$\int_T |x(t)|dt < \infty$$

is this an finite number of maxima??



$f(x) = -f(-x)$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(\omega t)$$

Let's compute its Fourier coefficients.

First, note that since $x(t)$ is an odd function, we can express it using only sin terms:

$$T = 2\pi \qquad \omega = \frac{2\pi}{T} = 1$$

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega t) = \sum_{k=1}^{\infty} b_k \sin(kt)$$

You can derive that for a 2π-periodic function, the coefficients have the following form:

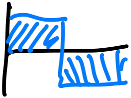
$$a_k = \frac{2}{T}\int_T x(t)\cos(k\omega t)\,dt$$

$$b_k = \frac{2}{T}\int_T x(t)\sin(k\omega t)\,dt$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x(\tau)\sin(k\tau)\,d\tau$$

Example: the square wave

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x(\tau)\sin(k\tau)d\tau$$



Since the function is symmetric, we can rewrite this:

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^{\pi} \sin(k\tau)\,d\tau \\
&= \frac{2}{\pi}\left(-\frac{1}{k}\right)\cos(kt)\Big|_0^{\pi} \\
&= -\frac{2}{k\pi}\left[\cos(k\pi) - 1\right] \\
&= -\frac{2}{k\pi}\left[(-1)^k - 1\right] \\
&= \begin{cases} 0 & \text{if } k \text{ is even} \\ 4/k\pi & \text{if } k \text{ is odd} \end{cases}
\end{aligned}$$

Example: the square wave

Putting everything together,

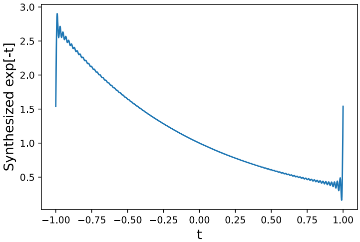
$$x(t) = \sum_{k=1}^{\infty} \frac{4}{k\pi} \sin(kt) \quad \text{only odd } k$$

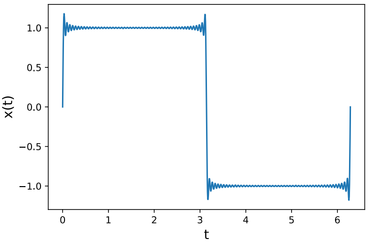
Does this really converge? Let's try some increasingly good approximations by truncating the series:

$$x_N(t) = \sum_{k=1}^{N} \frac{4}{k\pi} \sin(kt) \quad \text{only odd } k$$

The Gibbs phenomenon

This is called the Gibbs phenomenon.





Truncation to *any* N results in “ringing” at the discontinuities.

The amount of "overshoot" is about 9% of the jump of the discontinuity, no matter the size of N. With a bit of work, we can derive this!

$$\begin{aligned} x_N(t) &= \sum_{k=1}^{N} \frac{4}{\pi}\left[\frac{\sin(kt)}{k}\right] \quad \text{only odd } k \\ &= \sum_{k=1}^{N} \frac{4}{\pi} \int_0^t \cos(kx)\,dx \\ &= \frac{4}{\pi} \int_0^t \underbrace{\left[\sum_{k=1}^{N} \cos(kx)\right]}\,dx \end{aligned}$$

$e^{jx} = \cos x + j\sin x$

Let's evaluate what is in the sum...

$$\sum_{k=1}^{N} \cos(kx) = \cos(x) + \cos(3x) + \dots + \cos(Nx)$$

$$= \mathrm{Re}\left[e^{jx} + e^{3jx} + \dots + e^{Njx}\right]$$

$$= \mathrm{Re}\left[e^{jx}\underline{\left(1 + e^{2jx} + \dots + e^{(N-1)jx}\right)}\right]$$

★ In class I made an error in the bound. It is fixed going forward.

Let's make use of an identity for complex numbers:

$$1 + e^{2jx} + e^{4jx} + \dots + e^{(N-1)jx}$$

$$z^0 \quad z^1 \quad z^2 \quad \dots \quad z^{\frac{N-1}{2}jx}$$

$$\sum_{k=0}^{M} z^k = \frac{1 - z^{M+1}}{1 - z} \qquad z \neq 1$$

The Gibbs phenomenon

$e^{jx} = \cos x + j\sin x$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$z = e^{2jx}$ apply identity

$\Rightarrow \left(1-e^{2jx\cdot\left(\frac{N-1}{2}+1\right)}\right)/\left(1-e^{2jx}\right)$

Let's evaluate what is in the sum...

$$\sum_{k=1}^{N} \cos(kx) = Re\left[e^{jx}\left(1+e^{2jx}+\dots+e^{j(N-1)x}\right)\right]$$

$$= Re\left[e^{jx}\left(\frac{1-e^{jx(N-1)+2jx}}{1-e^{2jx}}\right)\right] \qquad = e^{jx(N+1)}$$

$$= Re\left[\cancel{e^{jx}}\,\frac{e^{j\frac{N+1}{2}x}\left(e^{-j\frac{N+1}{2}x}-e^{+j\frac{N+1}{2}x}\right)}{\cancel{e^{jx}}\left(e^{-jx}-e^{jx}\right)}\right]$$

$$= Re\left[\underbrace{e^{j\frac{N+1}{2}x}}_{\cos Nx + j\sin Nx}\;\underbrace{\frac{\sin\frac{N+1}{2}x}{\sin x}}_{\text{real}}\right]$$

$$= \Re e\left[e^{jNx}\frac{\sin(Nx)}{\sin(x)}\right]$$

$$\sum_{k=1}^{N}\cos(kx) = \mathfrak{Re}\left[e^{j\overset{\frac{N+1}{2}}{N}x}\frac{\sin(\overset{\frac{N+1}{2}}{\cancel{N}}x)}{\sin(x)}\right]$$

Apply Euler's relation and apply some trig identities...

$$\sum_{k=1}^{N}\cos(kx) = \frac{\cos(\frac{N+1}{2}x)\sin(\frac{N+1}{2}x)}{\sin x}$$

$$= \frac{1}{2}\frac{\sin((N+1)x)}{\sin x}$$

Let's put this back in:

$$x_N(t) = \frac{4}{\pi}\int_0^t \left[\sum_{k=1}^{N} \cos(kx)\right] dx$$

$$= \frac{4}{\pi}\int_0^t \frac{1}{2}\,\frac{\sin((N+1)x)}{\sin x}\, dx$$

$$= \frac{2}{\pi}\int_0^t \frac{\sin((N+1)x)}{\sin x}\, dx$$

Now, we want to find t where the most amount of ring occurs; take the derivative!

$$\frac{dx_N(t)}{dt} = \frac{d}{dt}\left[\frac{2}{\pi}\int_0^t \frac{\sin((N+1)x)}{\sin x}\,dx\right]$$
$$= \frac{2}{\pi}\,\frac{\sin((N+1)t)}{\sin t}$$
$$= 0$$

when is it 0?

This is true when $t = \frac{\pi}{N+1}$

Evaluate the sum at this point:

$$x_N\left(\frac{\pi}{N+1}\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{N+1}} \frac{\sin((N+1)x)}{\sin x}\,dx$$

Let's make a change of variables: $\alpha = (N+1)x$

$$x_N\left(\frac{\pi}{N+1}\right) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(\alpha)\,d\alpha}{\boxed{\sin\left(\frac{\alpha}{N+1}\right)}\cdot N+1}$$

We care about the case where $N \to \infty$.

$\frac{\alpha}{N+1}$

$\sin(x) \approx x$

x small

$\sin\left(\frac{\alpha}{\text{really big}}\right)$

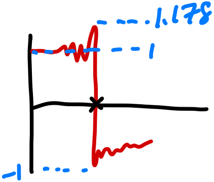
The Gibbs phenomenon

Recall that for small x, $\sin x \approx x$:

$$x_N\left(\frac{\pi}{2N}\right) \approx \frac{2}{\pi}\int_0^{\pi} \frac{\sin(\alpha)}{\alpha/\cancel{N+1}} \cdot \frac{d\alpha}{\cancel{N+1}}$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin\alpha}{\alpha}\, d\alpha$$

We can get a numerical value for the integral...



$$x_N\left(\frac{\pi}{2N}\right) \approx \frac{2}{\pi} \cdot 1.85194\ldots$$

$$\approx 1.178$$

The highest peak of the ring is 0.17898 higher than it should be.
This is $\approx 9\%$ of the jump!

Break time

We stopped at this point during class. I've filled in details on the subsequent slides for reference.

Go to `menti.com`

Signal power and energy

As long as you truncate, the ringing doesn't go away. However, we saw earlier that

$$\lim_{N\to\infty} E_N = \lim_{N\to\infty} \int_t |x(t) - x_N(t)|^2 = 0$$

This quantity here is actually the **energy** of the error.

Signal power and energy

Energy of a signal $x(t)$ over a single period:

$$E = \int_T |x(t)|^2 dt$$

Power of a signal over a period:

$$P = \frac{1}{T} \int_T |x(t)|^2 dt$$

We can relate this back to the Fourier coefficients...

Signal power and energy

$$
\begin{aligned}
P &= \frac{1}{T}\int_T |x(t)|^2 dt \\
&= \frac{1}{T}\int_T \left|\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}\right|^2 dt \\
&= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}\right)\left(\sum_{k=-\infty}^{\infty} c_m^* e^{-jm\omega t}\right) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_k c_m^* \int_T e^{j(k-m)\omega t} dt \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2
\end{aligned}
$$

only $k=m$ survives

Signal power and energy

This is called **Parseval's relation**

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

The average power of a signal is the sum of the average powers contributed from each harmonic component.

Properties of Fourier series

Fourier coefficients have some really useful properties that help us evaluate them.

What happens when we apply the following transformations to the Fourier series representations of our signals?

- Superposition
- Time shift / scale / reversal
- Multiplication

Properties of Fourier series

Fourier coefficients combine linearly.

Suppose we have two signals $x(t), y(t)$ with period T,

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}, \\
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t},
\end{aligned}
$$

Then the signal $z(t) = Ax(t) + By(t)$ has the form

$$
z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}, \quad c_k = Aa_k + Bb_k
$$

Time shift $x(t) \to x(t - t_0)$:

$$
\begin{aligned}
c_k' &= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega t}\, dt \\
&= \frac{1}{T}\int_T x(\tau)\, e^{-j\omega(\tau+t_0)}\, d\tau \\
&= e^{-jk\omega t_0} \cdot \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega \tau}\, d\tau \\
&= e^{-jk\omega t_0} \cdot c_k
\end{aligned}
$$

set
$\tau = t - t_0$

Time scale $x(t) \to x(\alpha t)$.

If original period was T, new period is T/α, new fundamental frequency is $\omega\alpha$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} c_k' e^{jk\omega\alpha t}$$

where

$$c_k' = \frac{1}{T/\alpha} \int_T x(\alpha t) e^{-jk\omega\alpha t} \, dt = c_k$$

same coeffs, but different series since $\omega \to \omega\alpha$

Multiplication leads to convolution of the coefficients:

Given

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t},$$
$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t},$$

Then the signal $z(t) = x(t)y(t)$ has the form

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}, \quad c_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

Today's learning outcomes were:

- State the Dirichlet conditions for convergence of a Fourier series
- Describe the Gibbs phenomenon
- Compute the power and energy of a signal, and state Parseval's theorem for CT periodic signals

What topics did you find unclear today?

For next time

Content:

- The discrete-time Fourier series representation

Action items:

1. Assignment 1 is due **tomorrow**
2. Assignment 2 (computational assignment) will be released tomorrow

Recommended reading:

- From today's class:
  - Oppenheim 3.4-3.5,
  - Derivation of Gibbs phenomenon: https://www.youtube.com/watch?v=MvXY8MXSit4
- For next class: Oppenheim 1.3, 3.6-3.8