What is so great about Black & Scholes (B-S) pricing formula? Why is it so popular? Contributions of Robert Merton to it?

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I. Black & Scholes pricing formula

A. *On Applications*

As per Duffie (1998):

1. Option markets which were sparse and thinly traded before the advent of B-S pricing formula became very active and exponentially grew in size. This is because it provides a benchmark for valuation of options and also a method for replicating or hedging options positions (through a risk-less portfolio built on no arbitrage argument).
2. B-S pricing formula/partial differential equation (PDE) is not just used to price put and call options but also to value executive stock compensation plans, real production options, warrants, convertible, securities, debt etc., by corporations.
3. B-S approach can be extended to a wide variety of instruments with embedded options such as caps, floors, collars, collateralized mortgage obligations, knockout options, swaptions, look-back options, barrier options, compound options and the list goes on.

B. *On assumptions, derivations and generalization*

The Black-Scholes-Merton model assumes that

1. Underlying asset prices follow a lognormal distribution based on the principle that asset prices cannot take a negative value.
2. The options can only be exercised on its expiration or maturity date (T).
3. Underlying asset prices follow geometric brownian motion (GBM).

$$ds = \mu\, s\, dt + \sigma\, s\, dw \tag{1}$$

where s = underlying asset price, dw is increment in weiner process = $\epsilon\sqrt{dt}$, where ϵ = N(0,1), dt is increment in time, μ and σ are the mean and volatility of the underlying asset.

4. There are no financial frictions: No transaction costs, including commission and brokerage.
5. r is the constant risk-free interest rate at which money can be borrowed or lent.
6. Stock returns are normally distributed which means the volatility of the market is constant over time.
7. There is no arbitrage. It avoids making a risk less profit.

There are 3 approaches to B-S pricing formula derivation

- No arbitrage portfolio with continuous dynamic hedging
- Equilibrium approach
- Risk-neutral approach

Among these, the first method of derivation of the model is widely used.

B.1. No arbitrage portfolio with continuous dynamic hedging

Black and Scholes applied the Capital Asset Pricing Model (CAPM) at each instant of time, for investments over an infinitesimally small period of time. But Bob Merton suggested that if you assume continuous trading in the option or underlying asset, you can maintain a hedged position between them (No arbitrage condition) that is literally risk-less. This helped generalize B-S pricing formula. With this addition, option pricing formula does not have to rely on the mean term (μ), risk aversion of investors, or the market equilibrium condition of pricing models such as CAPM, but only the no-arbitrage condition would suffice.

Form a risk-less/arbitrage-free portfolio:
$c_s = \partial c/\partial$s (is a hedge ratio) = number of shares of underlying asset needed to hedge 1 option written:

Position taken	Intital CF
Write 1 call	+c
Buy c_s shares	$-c_s s$
Borrow difference	$c_s s - c$
Net of the positions	0

Let's assume portfolio value is p. Initial value of p is zero (zero net investment). Now look at the change in the value of the portfolio over the next instant, dt. Also assume no dividend payments.

$$dp = -dc \quad + c_s ds \quad - r\,(c_s\,s - c)\,dt \tag{2}$$

Now I examine each term in eq(2):

$$\begin{aligned}
-dc \quad &= -\left[c_s\,\alpha\,s + c_t + 1/2\,c_{ss}\,\sigma^2\,s^2\right] dt - c_s\,\sigma\,s\,dw \text{ (from Ito's lemma)} \\
+c_s\,ds \quad &= c_s\,\mu\,s\,dt + c_s\,\sigma\,s\,dw \text{ (from eq(1))} \\
-r\,(c_s s - c)\,dt &= -r\,(c_s s - c)\,dt
\end{aligned}$$

Apply the above three terms in eq(2):

$$dp = \left[-c_t - 1/2 c_{ss}\sigma^2 s^2 + rc - rsc_s\right] dt \tag{3}$$

As we know, the arbitrage-free portfolio is self-financing and is worth zero at every point in time (dp=0). This allows us to derive a B-S PDE eq(4) for the option price $c(s,t)$ that would apply at any time $t < T$ and at any underlying asset price s for that time.

$$c_t(s,t) + c_s(s,t)rx + \frac{1}{2}c_{ss}(s,t)\sigma^2 s^2 - rc(s,t) = 0 \tag{4}$$

where subscripts indicate partial derivatives, with the boundary condition: $c(s,T) = max(s-K, 0)$

Since the PDE eq(4) did not involve μ (mean term), any expected return for the underlying asset would generate the same option price. Nowhere have we specified that this is a call option. Hence the same PDE is satisfied by put option, forward contract on the stock, warrant on the stock, etc.

B.2. Equilibrium approach

An equilibrium pricing model, e.g., CAPM or Arbitrage pricing theory (APT), provides a risk premium for c.
By definition:

$$E(dc/c) = (r + RP_c)\,dt$$

But for any equilibrium model, it must be true that $RP_c = RP_s \times$ elasticity $= RP_s \times (c_s s/c)$

$$\Rightarrow \quad \underbrace{E(dc/c)}_{\substack{\text{actually expected return}\\ Math: e.g., Ito's lemma]}} \quad = \quad \underbrace{[r + RP_s\,(c_s s/c)]}_{\substack{\text{fair expected return}\\ Finance: equilibrium condition}} \quad dt$$

$$\Rightarrow \quad c_s\mu s + c_t + 1\frac{1}{2}c_{ss}\sigma^2 s^2 = rc + RP_s c_s s \,\text{(from Ito's lemma)} \tag{5}$$

Rearranging the above eq:

$$c_s s\,(\mu - RP_s) + c_t + 1/2 c_{ss}\sigma^2 S^2 = rc$$

Finally, for a traded asset paying no dividends,

$$\mu = r + RP_s \Rightarrow \mu - RP_s = r \tag{6}$$

By substituting this in eq(5), we obtain the same B-S PDE as eq(4)

B.3. Risk-neutral approach

In a risk-neutral economy, there are no investor risk preferences and hence the risk premiums are 0. Therefore in eq(6), μ = r, and we would get same PDE as in (4) with the same boundary conditions.

REFERENCES

Duffie, Darrell, 1998, Black, merton and scholes: Their central contributions to economics, *The Scandinavian Journal of Economics* 100, 411–423.