Lab 9 Report
Inverted Pendulum

Palaskar Adarsh Mahesh

B20EE087

October 24, 2022

1 Objective

To design and implement a state-feedback control system that will balance the pendulum in the upright, vertical position, and implement a energy-based swing up control.

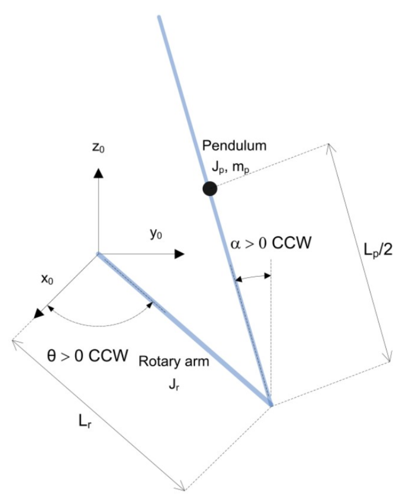

Figure 1: Rotary inverted pendulum modeling

2 Non-Linear Equations of Motion

The Euler-Lagrange method is used to find out the equations of motion. The obtained general equations can be given by:

$$\frac{\partial^2 L}{\partial t \partial q_i} - \frac{\partial L}{\partial q_i} = Q_i$$
$$q(t) = \begin{bmatrix} \theta(t) \\ \alpha(t) \end{bmatrix}$$
$$L = T - V$$

The generalized equations for the force acting on the rotary arm and on the pendulum are given by:

$$Q_1 = \tau - B_r\dot{\theta}$$
$$Q_2 = -B_p\dot{\alpha}$$

The obtained non linear equations of motion are:

$$\begin{aligned}
\left(m_p L_r^2 + \frac{1}{4}m_p L_p^2 - \frac{1}{4}m_p L_p^2 \cos(\alpha)^2 + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_p L_p L_r \cos(\alpha)\right)\ddot{\alpha} \\
+ \left(\frac{1}{2}m_p L_p^2 \sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_p L_p L_r \sin(\alpha)\right)\dot{\alpha}^2 &= \tau - B_r\dot{\theta} \\
-\frac{1}{2}m_p L_p L_r \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_p L_p^2\right)\ddot{\alpha} - \frac{1}{4}m_p L_p^2 \cos(\alpha)\sin(\alpha)\dot{\theta}^2 \\
-\frac{1}{2}m_p L_p g \sin(\alpha) &= -B_p\dot{\alpha}.
\end{aligned}$$

3 Liner State-Space Model

Linearizing the non-linear equations of motion, we have:

$$\frac{dx}{du} = Ax + Bu$$
$$y = Cx + Du$$

where x is the state, u is the control input, A, B, C, and D are state-space matrices.

$$x = \begin{bmatrix} 0 \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix}, y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 80.3 & -45.8 & -0.93 \\ 0 & 122 & -44.1 & -1.4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 83.4 \\ 80.3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4 Balance Control:

Balance Control system will be used to maintain the erect position of the pendulum. Pole placement to the desired locations can be performed as follows:

1. Using inbuilt place function:

Dominant Poles -

$p_1 = -\sigma_1 + j\omega_d$

$p_2 = -\sigma_1 - j\omega_d$

$p_3 = -30$

$p_4 = -40$

$\sigma = \xi\omega_n = 0.7 \times 4 = 2.8$

$\omega_d = \omega_n\sqrt{1-\xi^2} = 4\sqrt{1-(0.7)^2}$

$\therefore \omega_d = 2.85$

$\therefore D.P = [-2.8-2.85j \quad -2.8+2.85j \quad -30 \quad -40]$

```
poles = [-2.8+2.85i -2.8-2.85i -30 -40]

A = [0 0 1 0;
     0 0 0 1;
     0 80.3 -45.8 -0.93                              ;
     0 122 -44.1 -1.4]

B = [0; 0; 83.4; 80.3]

K = place(A,B, poles)
```

2. From scratch using controllability matrix and characteristic polynomials:

```
poles = [-2.8+2.85i -2.8-2.85i -30 -40];

A = [0 0 1 0;
     0 0 0 1;
     0 80.3 -45.8 -0.93                      ;
     0 122 -44.1 -1.4];

B = [0; 0; 83.4; 80.3];

% Controllability matrix Q is given by
Qc = [B A*B (A^2)*B (A^3)*B];
det(Qc)
%since Qc != 0 system is controllable

ch_eqn_coeff = charpoly(A);
ch_eqn_coeff_2_4 = ch_eqn_coeff(1,2:5);

poles_eqn_coeff = poly(poles);
poles_eqn_coeff_2_4 = poles_eqn_coeff(1,2:5);

P1 = [0 0 0 1] / (Qc);

Pc = [P1; P1*A; P1*A^2; P1*A^3];

ch_eqn_coeff_2_4_rev = flip(ch_eqn_coeff_2_4);

poles_eqn_coeff_2_4_rev = flip(poles_eqn_coeff_2_4);

K2 = (poles_eqn_coeff_2_4_rev - ch_eqn_coeff_2_4_rev)*Pc;
```

From both the methods, we get the gain matrix,

$$K = \begin{bmatrix} -5.1399 & 28.0233 & -2.7117 & 3.1701 \end{bmatrix}$$

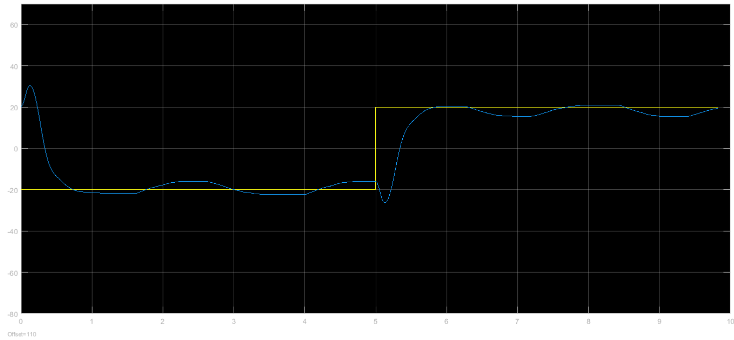

Figure 2: Rotary Arm Angle(θ)

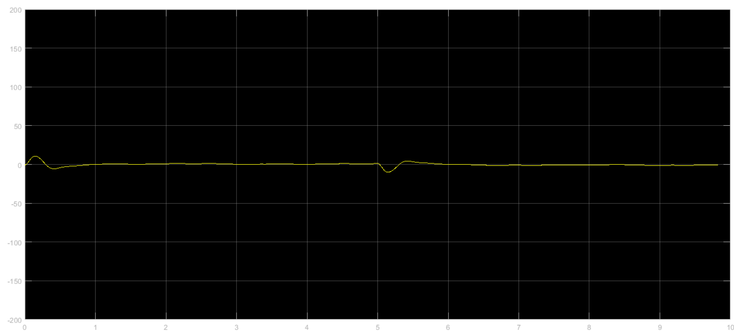

Figure 3: Inverted Pendulum Angle(α)

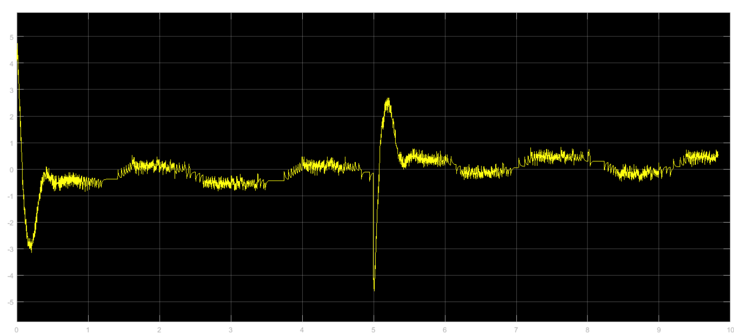

Figure 4: Motor Voltage(V)

5 Swing-up Control:

Energy based control scheme is developed to swing the pendulum up from its hanging, downward position. Once erect, balanced control can be used to maintain the vertical position of the pendulum.

$$E = \frac{1}{2} J_p \dot{\alpha}^2 + \frac{1}{2} m_p g L_p (1 - \cos\alpha)$$

$$\tau = \frac{\eta_g K_g \eta_m k_t}{R_m} (V_m - K_g k_m \dot{\theta})$$

$V_m - K_g k_m \dot{\theta} = 5$

$\eta_g = 0.69$

$K_g = 70$

$\eta_m = 0.9$

$k_t = 0.00768$

$R_m = 2.6$

$\therefore \tau = 0.0642 \ Nm$

$$\mu_{max} = \frac{\tau}{m_r L_r}$$

$m_r = 0.257$

$L_r = 0.216$

$$= \frac{0.0642}{(0.257)(0.216)}$$

$\mu_{max} = \cancel{1.1565} \ 11.6 \ m/s^2$

Upward Energy $\Rightarrow \alpha = 0$

$\therefore$ Upward Energy $= \frac{1}{2} J_p \dot{\alpha}^2 = 0.$

Downward Energy $\Rightarrow$ $\alpha = 180^\circ$

$$\text{Downward Energy} = \underbrace{\frac{1}{2} J_p \dot{\alpha}^2}_{0} + \frac{1}{2} \times (0.127)(9.8)(0.337)(1-(-1))$$

$$= 0.4194$$

The energy values above are in Joules(J).

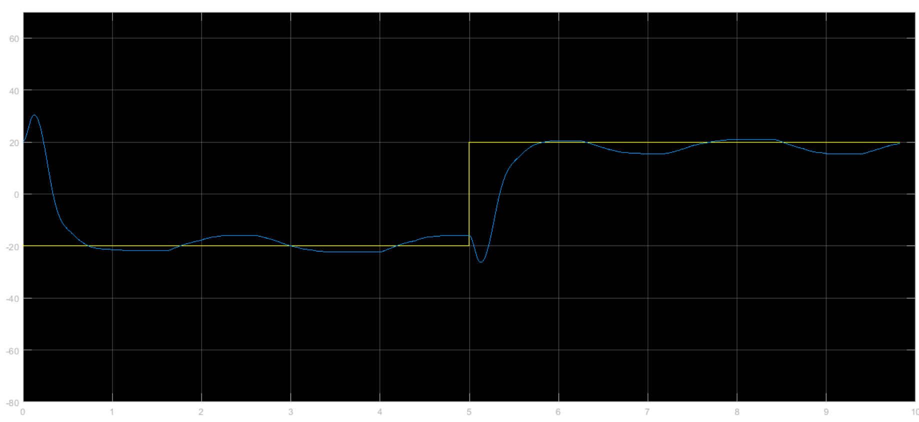

Figure 5: Rotary Arm Angle(θ)

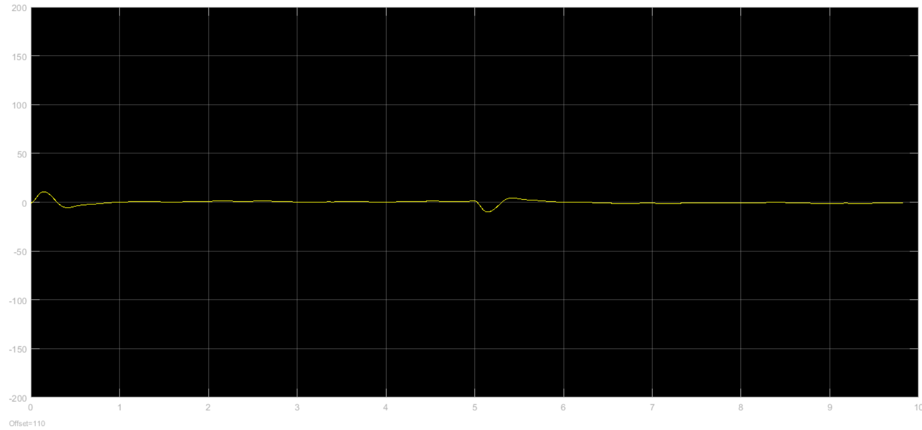

Figure 6: Inverted Pendulum Angle(α)

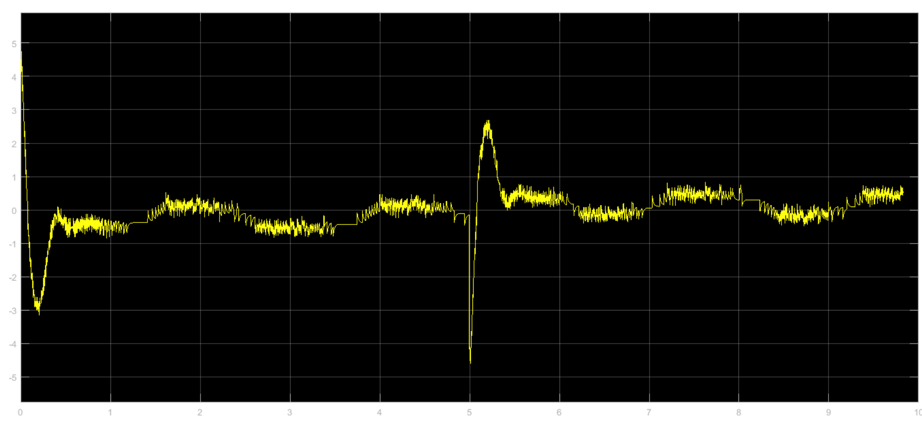

Figure 7: Motor Voltage(V)