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Master Thesis

A Numerical Perspective on Modern Koopman Operator Theory

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Figures

Figures 2.1,2.2, 3.2, 4.3, 4.6, 4.8, 4.9, 4.10, 4.11, 4.12, 4.13 have been designed by the author for this thesis.

Figure 3.1 has been designed by the author for this thesis, but is conceptually identical to the figure found in [2]. Figures 4.1, 4.2, 4.4, 4.5, 4.7 have been produced by the author for this thesis, but are conceptually similar to the ones found in [6].

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Contents

Chapter 1

Introduction

It is fundamentally important that we study and model the physical processes surrounding us. Understanding their behaviour can determine how we interact with them, and in our search for stability of systems in engineering. Motion of celestial objects, climate change, population growth, stock market are some examples of systems of interest. Most, if not all of these systems, are highly nonlinear, and also exhibit chaotic behaviour. A study of these systems can be thought of as a two step process. We first need to identify the governing equations, and then analyse them. Deriving a mathematical model is a challenge in itself, and it depends on the complexity of the system and our understanding of the underlying dynamics. Once we arrive at a mathematical description of the dynamics, we can identify the relation between different components that make up the system, and also gain insight on the time evolution of the system.

Historically, differential (difference) equations for continuous (discrete) time systems has been the language through which we described the dynamics [43]. The motion of planets [45], swinging of a pendulum, fluid flow over an object [55], all have been described in differential equations. Some of them in ordinary differential equations (ODEs), some in partial differential equations (PDEs). For the most part, the future state of a system solely depends on its current state and not its past states. Such methods can be labeled as *memoryless* [58]. Intuitively, historical data of a system should have some effect on the future state. By describing the time evolution of a system through differential equations, the history does translate to the future, but it does so with one time step in most cases. This calls for a system identification method which utilises historical data, or models with memory.

With increasing amounts of data collected, identifying governing equations of a system, has become a pursuit in system identification from gathered data on all the relevant variables. Modern system identification employs a plethora of techniques ranging from linear regression [40] to the use of neural networks [11], and choosing a technique depends on the characteristics of the problem at hand.

Another issue with using data for system identification is that often times, it is not physically possible to measure all the variables or states that describe the system completely. An intuitive way to understand the concept would be to try and predict the temperature

at a place, using only the wind speed, while being aware that there are other variables like pressure, elevation, precipitation that influence the temperature.

Such a perspective was explored by Floris Takens in 1981 [54] when he came up with his delay embedding theorem, in which under certain conditions, the historical data on a partial set of observations can be used to construct a model that is topologically equivalent to the original system where the underlying dynamics are described by the complete set of observations. Takens established that the topologically equivalent model will exhibit qualitatively similar behaviour to the original (or complete) model. Such transformations are known as state space reconstructions, and they have been widely used to study the properties of chaotic systems. Delay embeddings have been used in chaotic systems with singular system analysis (SSA) [3], in natural signals [26] just to name a few applications.

When working with ODEs, we need to understand their qualitative structure. We have to analyse them and find solutions, but solutions can be difficult to find, or even impossible. In such cases, we can still investigate the underlying mathematical structure. We look for stable and unstable regions, and we then can comment on possible solutions. Such a qualitative study of ODEs forms the subject of dynamical systems, and its formal treatment mostly originated in the 19th century through the work of Henri Poincare, during his research on the motion of planets [45].

Being over 100 years old, dynamical systems draws tools from numerous disciplines in mathematics. Topology, real analysis, functional analysis, linear algebra are all fields that are incorporated to some degree. In recent years, there has been abundance of data, and with it, data driven methods are finding their way into all fields wherever data is collected. This has given rise to a new perspective on the study of dynamical systems, and it is an operator theoretic view of such analysis. The idea itself is not new, but in the age of data, there has been a renewed interest.

Bernard Koopman in 1931 [27] introduced his perspective of dynamical systems. It was an operator theoretic perspective, and it described the evolution of measurements of Hamiltonian systems. And the following year, Koopman and von Neumann [28] proposed a Hilbert space formulation of classical mechanics.

In 2004, Igor Mezic and collaborators [38] started modern Koopman operator theory and made connections to Takens' embedding theorem. The efforts since have extended Koopman theory from Hamiltonian systems with measure preserving dynamics to dissipative and non smooth dynamics [36]. And later, Rowley et al. [48] connected the Koopman mode decomposition introduced by Mezic [37] and dynamic mode decomposition algorithm introduced by Schmid [51]. Modern Koopman theory essentially takes finite dimensional nonlinear dynamics and projects it onto an infinite dimensional Hilbert space of all possible linear functions. We then are left with the problem of solving an infinite dimensional linear system. This field has also been well documented by Brunton et. al [7].

In 2017, Brunton et al. [6] introduced the Hankel Alternative view of Koopman (HAVOK) analysis. In this, Koopman theory and Takens' embedding theorem have been used to decompose highly nonlinear systems (chaotic) into linear systems with intermittent forcing for qualitative future state prediction. Such a result is highly desirable, as we can decompose nonlinear systems into linear systems, which are completely characterized by their spectral decomposition.

In this thesis, the HAVOK analysis is approached from a theoretical, numerical and an application centered perspective. A brief introduction to dynamical systems is provided, along with a short review of chaotic systems. This is done since the Lorenz system [33] is analysed using HAVOK. Koopman operator theory is explained in brief. A comprehensive review of Takens' embedding theorem is provided, along with the prerequisites from topology and differential topology. State space reconstructions on the Lorenz system using Takens' theorem are shown, highlighting the conditions to be fulfilled for such a reconstruction. Then, the use of Hankel matrices in these reconstructions is discussed, and a connection between Koopman theory and delay embedding is explained.

The forcing signals used in the HAVOK analysis are limited to the least energy singular vectors obtained from SVD, and in this thesis, use of other forcing signals is explored, along with the use of multiple forcing signals.

HAVOK is not a tool that can be used for forecasting however, as the intermittent forcing signal is retrieved from the input data, making it an open forecasting tool. This disadvantage however can be overcome if we use neural networks to forecast the intermittent forcing signal, which would make such a hybrid method completely closed. Closed in the sense, we provide input data, and the tool forecasts future states.

Nevertheless, HAVOK is interesting to study, as it has been shown to decompose chaotic systems into linear systems, along with identifying globally stable and unstable regions. In contrast, using traditional analysis of chaotic systems only provides us with locally stable and unstable regions around fixed points of nonlinear systems.

Finally, the results obtained in this thesis suggest the use of higher energy right singular vectors as forcing terms. It has also been shown that using multiple forcing terms requires less number of right singular vectors in the intermittently forced linear system, and also reduces the reconstruction errors.

List of essential symbols

Symbol	Meaning
X	Full state space
x, y, z	State vectors of system
ϕ	Flow of the system
f, g, j	Functions
A, B	Matrices
$\mathcal{A}$	Attracting set
θ	Measurement function
Θ	Space of measurement functions
$\mathcal{K}$	Koopman operator
F	Discrete time analogue of f
$(\mathcal{X}, \tau)$	Topological space
M, N	Manifolds
U, V	Open sets
h, q	Homeomorphisms
Ψ	Embedding
H	Hankel matrix
p	Number of delays used to construct Hankel matrix
$\mathcal{V}$	Right singular vectors of H
r	Total number of right singular vectors in linear model
γ	Position of forcing term

Chapter 2

Dynamical systems

Dynamical systems is the qualitative study of ODEs [43]. One might say that in the real world, both ODEs and PDEs are used to define physical processes, yet why does dynamical system theory associate itself with the study of ODEs? It is possible because, the study of PDEs being quite complicated, requires a separate subject. And also, most of the techniques used for PDE analysis convert PDEs to ODEs, and proceed to solve them. Techniques like the finite difference method, finite volume method, finite element method (galerkin method) all find an ODE approximation of a PDE. So studying the structure of solutions to an ODE is imperative and helpful.

Firstly, a basic mathematical framework of our problem is laid out, and we then build upon this in further subsections.

2.1 Mathematical formulation of dynamical systems

In this section, the basic elements of a dynamical system are defined, such definitions can be found in standard textbooks on the subject [43]. A system can be thought of as a collection of interactions we are interested in studying. This can be climate, stock market, human body, a simple pendulum etc.

Definition 2.1.1 (State)**.** $X \in \Omega \subset \mathbb{R}^n$ is the state of a system. State of a system refers to the parameters that affect the system. Ω is an open subset of $\mathbb{R}^n$

If we consider climate as the system, the state includes but not limited to the temperature, pressure, humidity and other parameters that we have identified as affecting the climate. We use the subset Ω, as these parameters cannot take all possible values in $\mathbb{R}^n$, due to the limitations set by physical laws.

Definition 2.1.2 (Dynamical system)**.** A dynamical system on Ω is a C^1 map

$$\phi : \mathbb{R} \times \Omega \to \Omega \tag{2.1}$$

Where Ω is an open subset of $\mathbb{R}^n$ and if $\phi_t(X) = \phi(t, X)$, then ϕ_t satisfies
1. $\phi_0(X) = X$ for all $X \in \Omega$ and
2. $\phi_t \circ \phi_s(X) = \phi_{t+s}(X)$ for all $s, t \in \mathbb{R}$ and $X \in \Omega$

A dynamical system consists of a state space and a map describing the time evolution of the state. Correlations between states are by design part of the map ϕ. The range of the map is all the possible values $\phi_t(X)$ can take.

Definition 2.1.3 (Fixed point)**.** A fixed point $X_p \in \Omega$ is a point that is mapped to itself. $\phi_t(X_p) = X_p$.

Fixed points are of special interest, as they remain constant in time. If at $t = t_0$ we are at the fixed point X_p, we will always remain at X_p for $t \to \infty$. We usually shift the coordinates of the fixed point to origin with no loss of generality. The map ϕ which is also called flow, can be linear or nonlinear. Linear systems are easier to work with, and are completely characterized.

Definition 2.1.4 (Linear systems)**.** A linear system of ordinary differential equations is given by

$$\dot{X} = A(t)X \tag{2.2}$$

Where, $A(t)$ is a linear operator that depends on time.

When the linear operator $A(t)$ is independent of time, it can be written as A, and such a system is called an autonomous system. Such a distinction is important, as the theory differs for autonomous and time dependent linear systems. In this thesis, we will concern ourselves with the study of autonomous systems. The following is an important theorem for autonomous linear systems, and the proof can be found in a standard textbook on the subject [43], chapter 1.4.

Theorem 2.1.1 (The fundamental theorem for autonomous linear systems)**.** *Let A be an $n \times n$ matrix. Then for a given $X_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{aligned} \dot{X} &= AX \\ X(0) &= X_0 \end{aligned} \tag{2.3}$$

has a unique solution given by

$$X(t) = e^{At}X_0. \tag{2.4}$$

Definition 2.1.5. The fixed point (or equilibrium point) of the linear system 2.3 is X_p, where $\dot{X}_p = AX_p = 0$.

In order to solve the given linear system 2.3, we need to compute the matrix exponential e^{At}. The matrix exponential is given by its Taylor series.

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k \tag{2.5}$$

Computing the exponential using the Taylor series expansion is not straight forward and time consuming in some cases. In such cases, we analyse linear systems is using the spectral decomposition of matrix A. The reason for this is, given an uncoupled linear system, we can individually solve for each state, and computing the matrix exponential is straight forward. So, for any given system (coupled), the goal of the eigenvalue approach is to apply a linear

transformation on our original coordinate system to coordinates where the linear system is uncoupled, or where the matrix A is diagonal. Moreover, the qualitative behaviour of solutions of the system can be inferred by the spectral decomposition of A.

When our matrix A has real (or complex) and distinct eigenvalues, the corresponding eigenvectors form an orthonormal basis for $\mathbb{R}^n$ (or $\mathbb{R}^{2n}$ for the complex case). In the basis of eigenvectors, our matrix A is diagonal. Since we can shift from one basis to the other, we can solve the uncoupled system in the basis of eigenvectors, and bring back our solution to the original basis.

This can also be extended for the repeating eigenvalues case, where the eigenvectors do not form an orthonormal basis. In such cases we find generalized eigenvectors. A construction method is shown in [43], chapter 1.7.

Let $w_j = u_j + iv_j$ be an eigenvector of the (real) matrix A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$. And let $B = \{u_1, ..., u_k, u_{k+1}, v_{k+1}, ..., u_m, v_m\}$ be a basis of $\mathbb{R}^n$ with $n = 2m-k$, where we have k real valued eigenvectors corresponding to k real valued eigenvalues, and $(n-k)$ complex valued eigenvectors corresponding to $(n-k)$ complex eigenvalues.

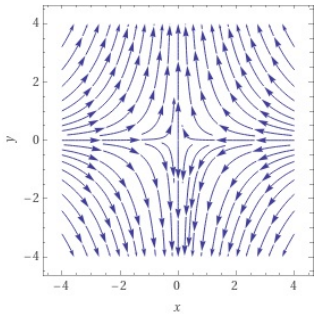
Definition 2.1.6 (Stable, unstable, and center subspaces)**.** Let $\lambda = a_j + ib_j$, $w_j = u_j + iv_j$, and B be described as above. Then,

$$\begin{aligned} E^s &= Span\{u_j, v_j | a_j < 0\} \\ E^c &= Span\{u_j, v_j | a_j = 0\} \\ E^u &= Span\{u_j, v_j | a_j > 0\} \end{aligned} \tag{2.6}$$

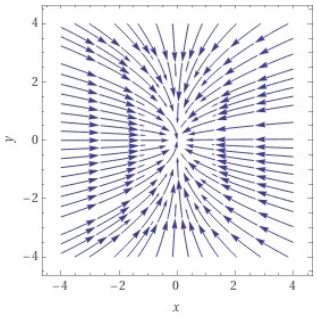
E^s, E^c, E^u are the subspaces of $\mathbb{R}^n$ spanned by the real and imaginary parts of the generalized eigenvectors w_j corresponding to eigenvalues λ_j with negative, zero, and positive real parts respectively.

E^s, E^c, E^u are named the stable, center, and unstable subspace respectively. The stable subspace is the space in which the trajectories (solution curves) approach a fixed point as $t \to \infty$, unstable subspace is the space in which the trajectories (solution curves) are unbounded as $t \to \infty$, and the center subspace is the space in which trajectories neither approach a fixed point nor unbounded as $t \to \infty$. These subspaces are usually referred to as manifolds. A more precise definition is given in chapter 3. In the two dimensional case, the subspaces or manifolds are surfaces. It is important to note that if we start in the stable or unstable manifold at $t = 0$, we will remain in the same manifold till $t \to \infty$.

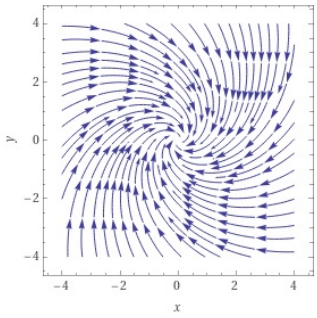
Given a system, it is helpful to find its stability, or in other words, finding the subset of all possible states that our system tends towards with evolution in time. Such stability is not guaranteed in all systems, and we want to comment on such properties. For example, we would like to know, in a weeks time, which temperatures does the system called climate tends towards, and which set of temperatures our system goes away from. For linear systems, such behaviour can be identified by computing fixed points and analysing the spectral decomposition of our matrix. Once we complete the spectral decomposition, it is helpful to study the time evolution of states using phase portraits.



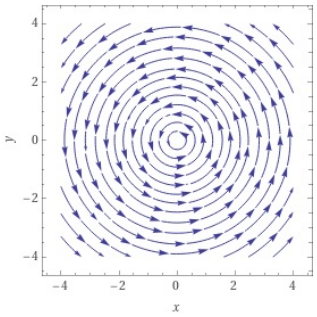
(a) Saddle



(b) Stable node



(c) Stable focus



(d) Center

Figure 2.1: Phase portraits

Definition 2.1.7 (Phase plane)**.** A coordinate plane with axes being the values of the n state variables of $X \in \mathbb{R}^n$

A phase plane is used to graphically describe time evolution of our state.

Definition 2.1.8 (Phase portrait)**.** The phase portrait of a system of ordinary differential equations such as 2.3, is a geometric representation of all trajectories (solution curves) in the phase plane $\mathbb{R}^n$.

The evolution forward in time is indicated by arrows. Once we draw the phase portrait of the autonomous linear system, we can understand time evolution of any state of the system.

For example, in the two dimensional case when $X \in \mathbb{R}^2$ and A is a 2×2 matrix independent of time (autonomous linear system), some possible phase portraits are shown in figure 2.1. We briefly discussed homogeneous linear systems, let us now look at non-homogeneous linear systems.

Definition 2.1.9. A linear system of the following type is a non-homogeneous linear system.

$$\dot{X} = AX + b(t) \tag{2.7}$$

Where, A is an $n \times n$ matrix and $b(t)$ is a continuous vector valued function.

Definition 2.1.10. Fundamental matrix solution of a homogeneous linear system 2.3 is any non singular $n \times n$ matrix function $\Phi(t)$ that satisfies

$$\dot{\Phi}(t) = A\Phi(t) \, \forall \, t \in \mathbb{R} \tag{2.8}$$

Theorem 2.1.2. *If Φ is any fundamental matrix solution of 2.3, then the solution of the non-homogeneous linear system 2.7 along with the initial condition $X(0) = X_0$ is unique and is given by*

$$X(t) = \Phi(t)\Phi^{-1}(0)X_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)b(\tau)\,d\tau \tag{2.9}$$

The theorem 2.1.2 is from [43], chapter 1.10. For linear systems, we had the linear operator as the matrix A. For nonlinear systems, we have the function f.

Definition 2.1.11. A general nonlinear system of differential equations is given by

$$\dot{X} = f(X, t, \mu) \tag{2.10}$$

Where, $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ with $\Omega \subset \mathbb{R}^n$ denotes the map, X denotes the state of the system, and $\mu \in \mathbb{R}^m$ denotes the parameters that influence our system.

When the nonlinear system of differential equations is independent of time t, it is an autonomous system $\dot{X} = f(X)$. Solutions to nonlinear systems cannot be obtained in all cases, however qualitative behaviour of such systems can be explained locally. To guarantee our problem is well posed, we must impose some conditions on f. For nonlinear autonomous systems, the fundamental existence-uniqueness theorem and theorem for smooth dependence on initial conditions can be found in book [43], in chapter 2.2-2.3.

Theorem 2.1.3 (The fundamental Existence-Uniqueness Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ containing X_0 and let us assume $f \in C^1(\Omega)$. Then there exists an $\epsilon > 0$ such that the nonlinear initial value problem*

$$\begin{aligned} \dot{X} &= f(X) \\ X(0) &= X_0 \end{aligned} \tag{2.11}$$

has a unique solution $X(t)$ in the interval $[-\epsilon, \epsilon]$

Theorem 2.1.4 (Dependence on initial condition)**.** *Let $\Omega \subset \mathbb{R}^n$ containing X_0 and assume $f \in C^1(\Omega)$. Then there exists an $\epsilon > 0$ and a $\delta > 0$ such that for all $\widetilde{X} \in N_\delta(X_0)$ the nonlinear initial value problem*

$$\begin{aligned} \dot{X} &= f(X) \\ X(0) &= \widetilde{X}_0 \end{aligned} \tag{2.12}$$

has a unique solution $u(t, \widetilde{X})$ with $u \in C^1(G)$, where $G = [-\epsilon, \epsilon] \times N_\delta(X_0) \subset \mathbb{R}^n$. Furthermore, for each $\widetilde{X}_0 \in \mathcal{N}_\delta(X_0)$, $u(t, \widetilde{X})$ is a twice continuously differentiable function of t for $t \in [-\epsilon, \epsilon]$

Definition 2.1.12. The fixed points (or equilibrium points) of the nonlinear system 2.11 is given by X_p, where $\dot{X}_p = f(X_p) = 0$.

After guaranteeing our problem is well posed locally, we can study the system locally. To do this, we first find the equilibrium points of nonlinear system 2.11, and describe the behaviour of the system in a δ neighbourhood of the equilibrium points. In the δ neighbourhood of the fixed points, we find an equivalent linear system.

Definition 2.1.13. $A = Df(X_p)$ is an equivalent linear system of the autonomous nonlinear system 2.11, in a δ neighbourhood of X_p. The equivalent linear system is given by $\dot{X} = AX$.

$Df(X_p)$ is the Jacobian matrix of f at X_p, and is given by,

Definition 2.1.14. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable function at X_p, then the Jacobian matrix of f at X_p, with respect to X is an $n \times n$ matrix of partial derivatives given by

$$Df(X_p) = \left[\frac{\partial f_i(X_p)}{\partial x_j}\right] \tag{2.13}$$

Definition 2.1.15 (Hyperbolic equilibrium point)**.** An equilibrium point X_p is called a hyperbolic equilibrium point of 2.11, if none of the eigenvalues of the matrix $Df(X_p)$ have zero real part.

There is a very important result in local theory of nonlinear differential equations known as Hartman-Grobman Theorem. The theorem tells us that the nonlinear system 2.11 near a hyperbolic equilibrium point, has the same structure as the linear system $\dot{X} = AX$, with $A = Df(X_p)$. The notion of topological equivalence is important.

Consider the two autonomous systems $\dot{X} = f(X)$ and $\dot{X} = g(X)$, and assume $f, g \in C^1(\Omega)$, $\Omega \in \mathbb{R}^n$ an open subset, and $X_p \in \Omega$ and $f(X_p) = g(X_p)$.

Definition 2.1.16 (Topological equivalence)**.** $\dot{X} = f(X)$ and $\dot{X} = g(X)$ are said to be topologically equivalent in a neighbourhood of the equilibrium points X_1, X_2 if $\exists\, h : U \to V$ a homeomorphism mapping an open set U containing X_1, X_2 onto another open set V, which maps trajectories of $\dot{X} = f(X)$ in U onto trajectories of $\dot{X} = g(X)$ in V and preserves their orientation by time.

Definition 2.1.17 (Homeomorphism)**.** A function $h : U \to V$ between two topological spaces is a homeomorphism if it has the following properties:
1. h is a bijection (one-to-one and onto).
2. h is continuous.
3. the inverse function h^{-1} exists, and it is continuous.

Theorem 2.1.5 (Hartman-Grobman Theorem)**.** *Consider the system $\dot{X} = f(X)$, where $X \in \Omega \subset \mathbb{R}^n$, $f \in C^1(\Omega)$ and the fixed point $X_p \in \Omega$. Let the flow of the system be defined by ϕ_t. Suppose X_p is a hyperbolic equilibrium point, then $\exists h : U \to V$ a homeomorphism of an open set U containing the fixed point onto an open set V containing the origin such that $\forall\, X_1 \in U$, $\exists\, I_p \subset \mathbb{R}$ an open interval containing the fixed point, and $h(\phi_t(X_1)) = e^{At}h(X_1)$, where $A = Df(X_p)$.*

In other words, it means that H maps trajectories of $\dot{X} = f(X)$ near the fixed point onto trajectories of $\dot{X} = AX$ near the fixed point and preserves the parameterization by time. For a detailed proof of the Hartman-Grobman theorem, please refer to the book [43], chapter 2.4. Local analysis of nonlinear systems is justified by the Hartman-Grobman Theorem.

Fixed points of nonlinear systems are classified in a similar way to linear systems, but there can be more patterns like saddle-nodes, cusps, center-focus and more. A very intuitive way to visualize this is to open an online weather forecasting website, and look at a map of any parameter. We can see some points where there is no wind flow, temperature gradient. These places can be thought of as fixed points, and in their neighbourhood, patterns like in figure 2.1 can be seen. This is an oversimplified example, however it is a great way to visualise behaviour of fixed points.

Global analysis of nonlinear systems is done by conducting local analysis of all the fixed points of our system 2.11. We now look at attractors, as these are of special interest and appear often in nonlinear systems.

Definition 2.1.18 (Attracting set)**.** For the system $\dot{X} = f(X)$, $f \in C^1(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is an open subset. A closed invariant set $\mathcal{A} \subset \Omega$ is called an attracting set of the system if there is some neighbourhood U of $\mathcal{A}$ such that, $\forall X \in U$, the flow $\phi_t(X) \in U \ \forall t \geq 0$ and $\phi_t(X) \to \mathcal{A}$ as $t \to \infty$

Definition 2.1.19 (Attractor)**.** An attractor, of the system mentioned above, is an attracting set $\mathcal{A}$ which contains a dense orbit.

At this point, it is helpful to differentiate between equilibrium points that are stable, attractive and asymptotically stable. There are many definitions available for them, but here are the definitions as introduced by Lyapunov, and we should also mention this when stating the definitions. Consider the solution of a differential equation $\dot{X} = f(X)$ from the initial point X_0 at time t, to be given by $\phi_t(X, X_0)$. At the initial time t_0, the solution is given by $\phi_{t_0}(X, X_0) = X_0$. By using such a notation, we intend to examine stability with respect to the initial values X_0. The following are definitions 2.1-2.3 from [21].

Definition 2.1.20 (Stability)**.** The equilibrium point X_p of the differential equation $\dot{X} = f(X)$ is stable (in the sense of Liapunov) if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\phi_t(X_0) < \epsilon \quad \forall \quad t \geq t_0 \tag{2.14}$$

is valid whenever X_0 belongs to a δ neighbourhood of the equilibrium point X_p.

Definition 2.1.21 (Attractivity)**.** Let the equilibrium point X_p be isolated, that is, the neighbourhood of X_p consists no other points for which $f(X_p) = 0$ for all t. Then, the equilibrium point X_p is called attractive, if there exists $\eta > 0$ with the property

$$\lim_{t \to \infty} \phi_t(X_0) = X_p \tag{2.15}$$

whenever X_0 belongs to an η neighbourhood of the equilibrium point X_p.

Definition 2.1.22 (Asymptotic stability)**.** The equilibrium point X_p is asymptotically stable (in the sense of Liapunov) if it is both stable and attractive.

2.2 Chaotic systems

Chaotic systems are dynamical systems that exhibit chaos, that is systems that are highly sensitive to the initial conditions. Chaos theory was first summarized by Edward Lorenz [33]. Systems where minuscule changes in initial condition gives rise to a completely different future. Such systems however exhibit order, contradictory to their name as they are completely deterministic. Chaos theory is the branch of mathematics focused on the study of such systems that are highly sensitive to the initial conditions, and the name comes from the fact that once, such deviation was considered disorderly.

Lorenz used the term butterfly effect in a lecture he gave in December 1972 to summarise the sensitive dependence on initial conditions. He said "A butterfly flapping its wings in Brazil can produce a tornado in Texas". Lot of systems in nature exhibit chaotic behaviour, some examples of chaos are turbulence, motion of planets on larger time scales, weather and climate, human behaviour.

In 1963, Edward Lorenz [33], simplified the Navier-Stokes equations for fluid flow, into a set of three ordinary coupled differential equations known as the Lorenz system. The Lorenz system was developed to be a simplified mathematical model for atmospheric convection.

This system has since been the face of chaos theory, and the Lorenz system is treated as a starting point in analysing chaotic systems. The system is given by

$$\begin{aligned} \dot{x} &= \sigma(x - y) \\ \dot{y} &= x(\rho - z) \\ \dot{z} &= xy - \beta z \end{aligned} \tag{2.16}$$

The equations describe the behaviour of a two dimensional fluid layer warmed uniformly from below and cooled from above. The quantities x, y and z describe the rate of convection, horizontal temperature variation and vertical temperature variation respectively. And the constants σ, ρ, and β are chosen to represent the Prandtl number, Rayleigh number and the physical dimensions of the space occupied by the fluid respectively.

By choosing the values $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$ the Lorenz system can be visualised as shown in figure 2.2. There is an attractor present, and is known as the Lorenz attractor, with two lobes.

Fixed point analysis of the Lorenz system gives us 3 fixed points, at the origin $(0, 0, 0)$, $(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$, and at $(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)$. These are marked in red in the above figure. The attractor has two lobes surrounding the two fixed points excluding the origin.

The Lorenz attractor is a strange attractor. An attractor is called strange if it has a fractal structure. A system is said to have a fractal structure if it shows similar patterns across

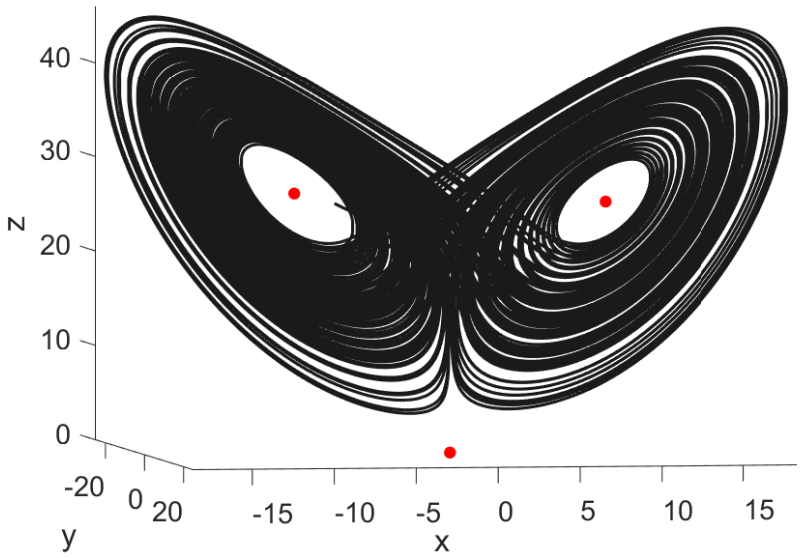


Figure 2.2: Lorenz Attractor

different scales. This often occurs when dynamics are chaotic. The term strange attractor was first used by Floris Takens and David Ruelle [49]. Other examples of such attractors are the Rössler attractor, Hènon attractor. Throughout this thesis, the Lorenz attractor is analysed using methods which will be described in further sections.

Now that we have briefly gone through an overview of dynamical systems, and a short note on chaotic systems including some properties of the Lorenz attractor, we continue to look at a data driven dynamical systems.

2.3 Data oriented view of dynamical systems

Physical phenomena are complex, and not all of them have their governing equations well defined. In the age of data, even if we do not have the governing equations, we have measurements of states, recorded over a time period. If the governing equations are known, we can compliment them with observed data, if not it has become our job to find the governing equations, so as to understand time evolution and interdependence of states.

Prediction of future states from a time series of data is one of the most commonly encountered problems in data driven discovery. Nonlinear time series forecasting methods have been built on the pioneering work of Takens [54], where the concept of state space reconstructions from sampled data was introduced. Since Takens' embedding theorem, there have been many methods developed for nonlinear and chaotic systems time series analysis [19], [8], [39].

Modelling and prediction of complex systems using data has become increasingly sought after with our advancements in computing power and data storage. And along with these

advancements, the resolution of data we simulate or observe has also increased proportionally. We are able to record data for massive number of parameters that affect a system. We can in theory develop complex techniques that utilize all the information, however it is desirable to develop simple models using a limited number of observations. This approach is useful not only to better understand importance of a parameter to the system, but it is also advantageous to run simpler models on smaller processors.

This has served as a motivation to the development of reduced order modelling (ROM) techniques. One of the most important and widely used model order reduction technique is the Singular Value Decomposition (SVD). Other techniques such as Principle Component Analysis (PCA), Proper Orthogonal Decomposition (POD) are built using the SVD. With advancements in machine learning techniques, autoencoders [17], neural networks [15] have also been used for model order reduction.

Some methods built on ROM that take in high dimensional data to make predictions based on identified model include the dynamic mode decomposition (DMD) [51], [31], eigensystem realization algorithm (ERA) [25], empirical dynamic modelling [59] and neural networks [46], [18].

When governing equations are believed to exist, data driven techniques can help identify the best fit pre-defined equation to the observed data [52]. There have been advancements in compressed sensing, and methods like sparse identification of nonlinear dynamics (SINDy) [4] have been used to pick the best fit function from a library of candidate models to identify the dynamical system. Recently, completely data driven techniques such as DMD, complimented by Koopman operator theory [36], have been used for system identification and modelling.

Recent research in data oriented dynamical systems is being influenced heavily by machine learning algorithms, neural networks in particular [32]. Neural networks have been used to make predictions for a number of different time series. They have also been used in tandem with traditional physics based methods [47], [10]. One of the first uses of neural networks for dynamical systems analysis was in the 1990s [41], and a very important method called reservoir computing for predicting dynamical systems using neural networks was developed by Herbert Jaeger et.al. [24] to forecast the Mackey-Glass dynamical system.

Advancements in machine learning have seen innovative methods for system identification and time series forecasting such as recurrent neural networks (RNNs) [1] and long short-term memory networks (LSTM) [22], [56]. Other techniques from machine learning include the use of deep neural networks (DNNs) [46], ordinary differential equation networks (ODENet) [10], deep Koopman methods [34], [60]. These approaches build a neural network model that captures and imitates the system dynamics only using recorded data.

Building models purely based on observed data can be aided by incorporating physical laws. This can be done by modifying the neural network loss function, and penalising the network if physical laws are not obeyed [47]. There have also been neural network architectures

that have been developed which incorporate underlying physical laws of the desired system in the neural network topology for improved training for the system it has been trained for. One such example is the Hamiltonian neural networks [20] for learning Hamiltonian systems.

As we can see, to use data for dynamical systems analysis, we have a plethora of techniques to choose from, and it depends on the system we want to solve, and the nature of data available.

2.4 Koopman operator theory

Global analysis for nonlinear systems is a fundamental challenge in the field of dynamical systems. However, there is no method for complete characterization of such systems. Linear systems on the other hand have been completely characterized by the spectral decomposition of the linear operator aided by basic linear algebra. We are able to obtain global solutions for linear systems. Analysis of nonlinear systems has strong foundations only for local analysis around equilibrium points, and the quest for global representations of nonlinear systems is still a challenge.

2.4.1 An overview

An alternative perspective to traditional dynamical systems comes in the form of the Koopman operator theory for dynamical systems. The Koopman operator is an infinite-dimensional linear operator, advancing measurement functions of the system. Spectral decomposition of the Koopman operator can be used to completely characterize the system, in other words, it is possible to study the global behaviour of nonlinear systems. The problem then becomes finding finite-dimensional approximations of the infinite-dimensional Koopman operator, which can be thought of as finding coordinate transformations in which the nonlinear dynamics become linear.

Mezič and collaborators [38], [37] pioneered recent work on Koopman operator theory, which was first introduced by Koopman in 1931 [29] to describe dynamics of Hamiltonian systems. Later Koopman and von Neumann generalized the theory to systems with continuous eigenvalue spectrum [30]. The central proofs of the ergodic theorem by von Neumann and Birkhoff were built upon Koopman's 1931 paper.

Let us consider an autonomous nonlinear systems of the form $\dot{X}(t) = f(X(t))$, where $X \in \Omega \subseteq \mathbb{R}^n$ is the state of the system, and f is the nonlinear function. In general the dynamics may also depend on time t, parameters μ and control input $u(t)$, but we omit such considerations for simplicity.

Finding coordinates in which the dynamics are simplified can be helpful in analysing any system. Such coordinate are sought even for linear systems, where we look for coordinates in which the linear operator, is a diagonal matrix. So, for nonlinear systems, we look for

coordinates in which the dynamics are ideally linearized or in a broad sense simplified.

In traditional geometric analysis, such a simplification of nonlinearities is done by finding homeomorphic maps, which are not always guaranteed to exist, and they are difficult to find globally. By lifting the dynamics into a higher dimensional space, we do not face such an issue.

2.4.2 Mathematical framework

The Koopman operator advances measurements functions of the state of a dynamical systems. Consider the autonomous ordinary differential equation 2.11 on $\Omega \subseteq \mathbb{R}^n$. Let the nonlinear function $f^t : \Omega \to \Omega$ take our initial condition X_0 forward along the trajectory by a time t. The evolution of trajectories follows

$$X(t) = f^t(X_0) \tag{2.17}$$

Definition 2.4.1 (Measurement functions)**.** $\theta : \Omega \to \mathbb{C}$ is a measurement function otherwise known as an observable function. θ corresponds to our observation of the state and $\Theta(\Omega)$ is the set of measurement functions θ.

The space of functions $\Theta(\Omega)$ is loosely specified. Hilbert space such as the $\mathcal{L}^2$ space is commonly used in modern applications, although Banach spaces such as $C^1(\Omega)$, and $\mathcal{L}^1$ spaces have also been used. In all cases, $\Theta(\Omega)$ is significantly higher in dimension that Ω.

Definition 2.4.2. The family of Koopman operators, parameterized by t, are written as $\mathcal{K}^t : \Theta(\Omega) \to \Theta(\Omega)$, and are given by

$$\mathcal{K}^t\theta(X) = \theta(f^t(X)) \tag{2.18}$$

Equation 2.18 can be rewritten as $\theta_t := \mathcal{K}^t\theta$, with initial measurement as $\theta_0 := \theta$, which would indicate that the family of Koopman operators is advancing the measurement θ at time $t = 0$ to θ_t at time t.

One of the most important and interesting properties of the Koopman operator is that when the function space $\Theta(\Omega)$ is linear, the Koopman operator is linear.

$$\begin{aligned} \mathcal{K}^t(\alpha_1\theta_1(X) + \alpha_2\theta_2(X)) &= \alpha_1\theta_1(f^t(X)) + \alpha_2\theta_2(f^t(X)) \\ &= \alpha_1\mathcal{K}^t\theta_1(X) + \alpha_2\mathcal{K}^t\theta_2(X) \end{aligned} \tag{2.19}$$

This property holds true even when $f^t : \Omega \to \Omega$ is nonlinear. In essence, finite dimensional nonlinear dynamics in f^t have been traded for infinite dimensional linear dynamics in $\mathcal{K}^t$, by switching from the finite dimensional space Ω to the infinite dimensional function space $\Theta(\Omega)$

For discrete time autonomous systems, where $t \in \mathbb{N}$, f^t is applied to the state vector at every time step. so at t^{th} time step, $f^t(X) = f(f(...(f(X))))$. Likewise, $\mathcal{K}$ represents the Koopman operator that advances the measurement by one time step.

Definition 2.4.3 (Koopman Operator)**.** The generator $\mathcal{K}$ of the countable composition semigroup is called the Koopman operator. It results in the infinite dimensional linear dynamical system,

$$\theta_{k+1} = \mathcal{K}\theta_k \tag{2.20}$$

which is analogous to the finite dimensional nonlinear discrete time autonomous system

$$X_{k+1} = F(X_k) \tag{2.21}$$

Definition 2.4.4 (Koopman eigenfunction)**.** A Koopman eigenfunction corresponding to an eigenvalue λ is given by,

$$\varphi(X_{k+1}) = \mathcal{K}\varphi(X_k) = \lambda\varphi(X_k) \tag{2.22}$$

Even when working with real valued functions $F(X)$ and state space Ω, the eigenvalues and eigenfunctions may be complex valued.

To understand the relationship between Koopman eigenfunctions and eigenvalues of a linear operator, consider the finite dimensional discrete time linear system $X_{n+1} = AX_n$ in the state space Ω, and its analogous infinite dimensional system $\theta_{n+1} = \mathcal{K}\theta_n$ in the function space $\Theta(\Omega)$. The left eigenvector ξ of A, and the corresponding Koopman eigenfunction are given by

$$\begin{aligned} \xi^T A &= \lambda\xi^T \\ \varphi(X) &= \xi^T X \\ \mathcal{K}\varphi(X) = \varphi(AX) = \xi^T AX &= \lambda\xi^T X = \lambda\varphi(X) \end{aligned} \tag{2.23}$$

Definition 2.4.5 (Koopman modes)**.** The right eigenvectors of A give rise to time invariant directions in the state space Ω are known as Koopman modes or dynamic modes. The left eigenvectors give rise to Koopman eigenfunctions which are similarly time invariant directions in the function space $\Theta(\Omega)$.

The level sets or invariant sets obtained from the spectral analysis of the Koopman operator enable us to analyse the phase portrait of a dynamical systems.

Chapter 3

On using memory in dynamical systems

Memory refers to the past states that the dynamical system had taken. While describing a dynamical system in terms of an ODE, we imply that the state of the system at $t + 1$ depends on the state of the system at t. Or in other words, the future state of the system depends on its current state. When we use such a representation, state at $t + k$ does indeed depend on the state at t, but the dependency translates forward in time by one time step.

Such a translation of historical states is however limited to some representations of the system, namely the ODE. Lot of Data driven approaches however takes into account past states to predict the future. Some prominent techniques in time series analysis like linear regression, auto regressive integrated moving average (ARIMA) and its variants all take into account the historical data, and determine how much a past state influences the future state.

Such methods work well in most cases where they have been traditionally used, i.e. linear systems like the ones found in forecasting sales, product pricing etc. But for nonlinear and chaotic systems, a more rigorous framework is required to use memory.

Another intriguing fact in the study of dynamical systems theory is that phase space of simple nonlinear systems may contain strange attractors. For relatively simple systems where we can measure all the parameters influencing the system, such as pendula, the existence of strange attractors can be experimentally investigated. But in general, it might not be possible to measure all the variables that influence the system, and in some cases, the variables influencing the system might not be known. The branch of mathematics concerned with extraction of information about features of a dynamical systems from a time series of measurements is referred to as embedology by Sauer, Yorke and Casdagli [50].

One of the fundamental theorems in embedology was given by Floris Takens in 1981 when he gave his embedding theorem [54], which discusses the conditions under which certain properties of a dynamical system can be reconstructed from a scalar valued observable. The method of delays described by Takens can be applied to any time series and has since been used in a wide variety of time series data.

As we can see, partial information modelling and the use of memory go hand in hand. Partial information modelling refers to the use of only certain variables of a dynamical system to infer its behaviour, and not the use of partial available data on a single variable. Such techniques are called state space reconstructions, as the original state space which includes all known and observed variables is reconstructed from a subset of known and observed variables.

Consider a thought experiment: Let us say we want to build a project, where we can forecast the temperature on a mountain top. We set up our measurement devices, and measure temperature, wind speed, and pressure. But there are a lot of other parameters that influence the temperature, like precipitation, humidity, amount of sunlight, presence of pollutants etc. If we could measure all of them, we know that there are climate forecast models which utilise these measurements and forecast temperature. However, we might not have the capacity to measure the complete set of observations. In this case, it is helpful to construct a model which utilises data gathered on the partial set of observations, and forecast temperature. And since we have historical data on the partial set of observations, we can utilise this information in our model too.

Takens' theorem is written in the language of differential topology, so we will briefly discuss it in the following section. For a more thorough knowledge of the subject, introductory texts in differential topology [14] can be referred to.

3.1 Prerequisites in topology

Some basic definitions are given. The essence of this section is to understand the topological notion of genericity, which is important to access Takens' embedding theorem.

Definition 3.1.1 (Topological space)**.** Let $\mathcal{X}$ be a non-empty set. A set τ of subsets of $\mathcal{X}$ is said to be a topology on $\mathcal{X}$ if
(1) $\mathcal{X}$ and the empty set Φ belong to τ.
(2) The union of any (finite or infinite) number of sets in τ belongs to τ
(3) The intersection of any two sets in τ belongs to τ
The pair $(\mathcal{X}, \tau)$ is called a *topological space*.

A *Hausdorff* space is a topological space where any two distinct points have neighbourhoods that are disjoint from one another.

A topological space $(\mathcal{X}, \tau)$ is said to be *locally Euclidean*, if there exists a positive integer n such that each point in $\mathcal{X}$ has an open neighbourhood homeomorphic to an open ball about the origin in $\mathbb{R}^n$.

Definition 3.1.2 (Manifold)**.** A Hausdorff locally Euclidean space is said to be a *topological manifold*. Every point in an n-dimensional manifold, or an n-manifold has an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

A pair (U, h), where $U \subset N$ is open and $h : U \to \mathbb{R}^n$ is a homeomorphism (see definition 2.1.17) onto its range is called a *chart*. U is the *chart domain*, and h is the *coordinate function*.

A collection of charts whose domains cover the manifold N is called an *atlas*.

Consider two charts (U, h) and (V, q). If they have overlapping domains, the coordinate transformations of the intersecting part can be written as

$$\begin{aligned} hg^{-1} &: q(U \cap V) \to \mathbb{R}^n \\ qh^{-1} &: h(U \cap V) \to \mathbb{R}^n \end{aligned} \tag{3.1}$$

These coordinate transformations are functions from open subsets of $\mathbb{R}^n$ to $\mathbb{R}^n$.

Remark. *The manifold M is said to be smooth, if the all its charts have smooth coordinate transformations 3.1. In this thesis, we will only deal with smooth manifolds.*

Two such charts are *C^r-related* if both the coordinate transformations are continuously differentiable r times. And we say an atlas is *C^r -differentiable* if all its charts are *C^r-related* to their overlapping neighbours. The set of all charts which are C^r-related is an atlas itself, and is called a *differential structure*.

Definition 3.1.3 (Differentiable Manifold)**.** A manifold with a differential structure is *differentiable*, and is called a *C^r manifold*

Let M and N be n dimensional C^r manifolds. A function $j : M \to N$ is a *C^s-differentiable* function with $s \leq r$, if for every point $p \in M$ there are charts (U, h) in M, and (V, q) in N, with $p \in U$ and $j(p) \in V$, such that the coordinate transformation $qjh^{-1} : h(U \cap j^{-1}V) \to \mathbb{R}^n$ is s times continuously differentiable at $h(p)$.

Definition 3.1.4 (Immersion)**.** An *immersion* of M into N is a differentiable function $j : M \to N$ such that $dj_p : T_pM \to T_kN$ is injective at each $p \in M$, where T_pM is the tangent space to M at $p \in M$, T_kN is the tangent space to N at $k \in N$ and $j(p) = k$.

It should be noted that, an immersion simply means that the tangent spaces are mapped injectively. If the differentiable function j is injective and an immersion, then we call j an *injective immersion*.

Definition 3.1.5 (Embedding)**.** A function $j : M \to N$ is an embedding if it is both an injective immersion and proper. An immersion which carries a differentiable manifold M homeomorphically onto its image is an *embedding*.

A proper function is a function for which the preimage of every compact set is compact. For any compact manifold, an injective immersion defined over it is automatically proper, so it is an embedding.

Remark. *Since an embedding $j : M \to N$ is an injective immersion, it preserves the full structure of the topological space it is embedding M, in the embedded space N. An embedding does not allow for self intersections.*

If M is an m-manifold with $m \leq n$ and $M \subset N$, then M is called a *submanifold* of N, if at every point of M, we can get a chart from (V, q) of N, by restricting q from $V \cap M$.

Definition 3.1.6 (Diffeomorphism)**.** Let M and N be manifolds, and $j : M \to N$ be an embedding, then $j(M)$ is a submanifold of N, and $j : M \to j(M)$ is a *diffeomorphism*. A diffeomorphism is a differentiable function with a differentiable inverse.

A diffeomorphism between two manifolds can mean that apart from a smooth change of coordinates, the manifolds are considered to be the same. For the purposes of our study, an embedding will allow us to take our phase space to a diffeomorphic subset of $\mathbb{R}^n$ without any loss of the dynamics in the phase space. By investigating this subset, we seek to gain knowledge of the dynamics in the original phase space.

Definition 3.1.7 (*Diff*^{r}). The set of C^r functions from M to itself, which are also diffeomorphisms, i.e., have C^r inverses are denoted by *Diff*^{r}(M).

Now we look at some function spaces equipped with topology.

Definition 3.1.8 (C^s*topology*)**.** Let $C^r(M, N)$ be the set of C^r maps from M to N and $j \in C^r(M, N)$, with (U, h) a chart on M and (V, q) a chart on N; assume $Q \subset U$ to be a compact set such that $j(Q) \subset V$, and let $\epsilon \in (0, \infty]$. The C^s*topology* with $s \leq r$ on $C^r(M, N)$ consists of the set $\mathcal{N}^s(j; (U, h); (V, q), Q, \epsilon)$, whose elements are the functions $\tilde{j} \in C^r(M, N)$ for which $\tilde{j}(Q) \subset V$ and

$$\|D^k q\tilde{j}h^{-1}(x) - D^k qjh^{-1}(x)\| < \epsilon \tag{3.2}$$

In other words, to check if two functions are close together, the domain M is broken up into pieces which can be transformed to subsets of $\mathbb{R}^m$. Here we can use the Euclidean norm.

A property of points of topological space $(\mathcal{X}, \tau)$ is generic if the property holds for all points on an open and dense subset of $\mathcal{X}$. This can be applied to function spaces.

As mentioned in [23], to show that a set of functions (say immersions) is *generic* in the C^s topology on $C^r(M, N)$, we need to show that they are open and dense. To show that a set of functions is open in the C^s topology on $C^r(M, N)$, we need to find, for each member j of the set, charts (U_i, h_i) and (V_i, q_i), compact sets Q_i and numbers ϵ_i, so that all members of $\cap_i \mathcal{N}^s(j; (U_i, h_i); (V_i, q_i); Q_i; \epsilon_i)$ are members of the set.

And to show that the set of functions is dense, we need to show that for every $j \in C^r(M, N)$, every neighbourhood of j contains a member of the set.

The term generic in C^s topology of maps would mean open and dense in the C^s topology of maps. And a generic embedding would mean that around every function that is an embedding, there is a ball with positive radius whose elements are also embeddings (openness), and for every function that is not an embedding, it only takes an infinitesimal perturbation to arrive at an embedding (denseness). The term generic can be thought of as "almost all" in some sense. This would certainly mean that there can be exceptions.

Acquainted with some basic knowledge of differential topology and the notion of genericity, we can now look at embedding theorems.

3.2 Takens' embedding theorem

Before we look at Takens' theorem, let us look at a much older theorem by Whitney in 1936 [57].

Theorem 3.2.1 (Whitney 1936)**.** *Let M be an m dimensional smooth and compact manifold, then the set of functions $j : M \to \mathbb{R}^{2m+1}$ that are an embedding is generic.*

Whitney's theorem ensures that given a smooth and compact manifold M, an arbitrarily chosen smooth $(2m+1)$ dimensional mapping will be an embedding. In other words, almost all (generic) smooth mappings that are sufficiently high dimensional, are embeddings of the manifold. A proof of the theorem can be found in [44].

Takens' embedding theorem builds upon Theorem 3.2.1. In particular, Takens talks about the dynamics of a system in a manifold M, and gives a specific choice of a map that would be an embedding. Consider the discrete time nonlinear system 2.21, where the system is advanced one step at a time by F. Let the $\theta : M \to \mathbb{R}$ be a scalar valued observable function (measurement function). With this setting, we can now look at the embedding theorem from Takens' paper [54].

Theorem 3.2.2 (Takens' embedding theorem)**.** *Let M be a compact manifold of dimension m. For pairs (F, θ), with $F \in \mathrm{Diff}^2(M)$, with $\theta \in C^2(M, \mathbb{R})$, it is a generic property that the map $\Psi_{F,\theta} : M \to \mathbb{R}^{2m+1}$, defined by*

$$\Psi_{F,\theta}(x) = (\theta(x), \theta(F^{-1}(x)), ..., \theta(F^{-2m}(x))) \tag{3.3}$$

is an embedding.

Takens mentioned genericity, i.e., open and dense in the C^1 topology. The map $\Psi_{F,\theta}$ involves a single distinct observation function θ, and thus represents a small subset in the larger set of all possible mappings $j : M \to \mathbb{R}^{2m+1}$ which was discussed by Whitney. The Theorem 3.2.2 states that using pairs (F, θ), we can construct a map $\Psi_{F,\theta}$ such that it is a generic embedding (almost all such maps are embeddings).

J.P. Huke described Theorem 3.2.2 in great detail and also provided its proof [23]. One of the conditions assumed for the theorem is that F should not have periodic orbits with period less than or equal to 2m. To understand this condition, let us consider the one dimensional ($m = 1$) discrete time system 2.21, and let us say that there is a periodic orbit of period 1. This would mean that $F(x_k) = F(x_{k+1})$, and the embedding would become $\Psi_{F,\theta}(x_k) = (\theta(x_k), ..., \theta(x_k))$. As all the points in the embedding are the same, the dynamics of the original system will not be carried forward. In a more complicated line of reasoning, this condition also holds true for periodic orbit of period 2 in $\mathbb{R}$, which translates to $2m$ in the m dimensional case.

Here is an illustration of the dynamics in Figure 3.1 as depicted in [2]. Whitney's Theorem ensured that generic choices of $(2m+1)$ dimensional functions are embeddings, whereas Takens' Theorem narrows down the choice of embedding to a specific set of functions from

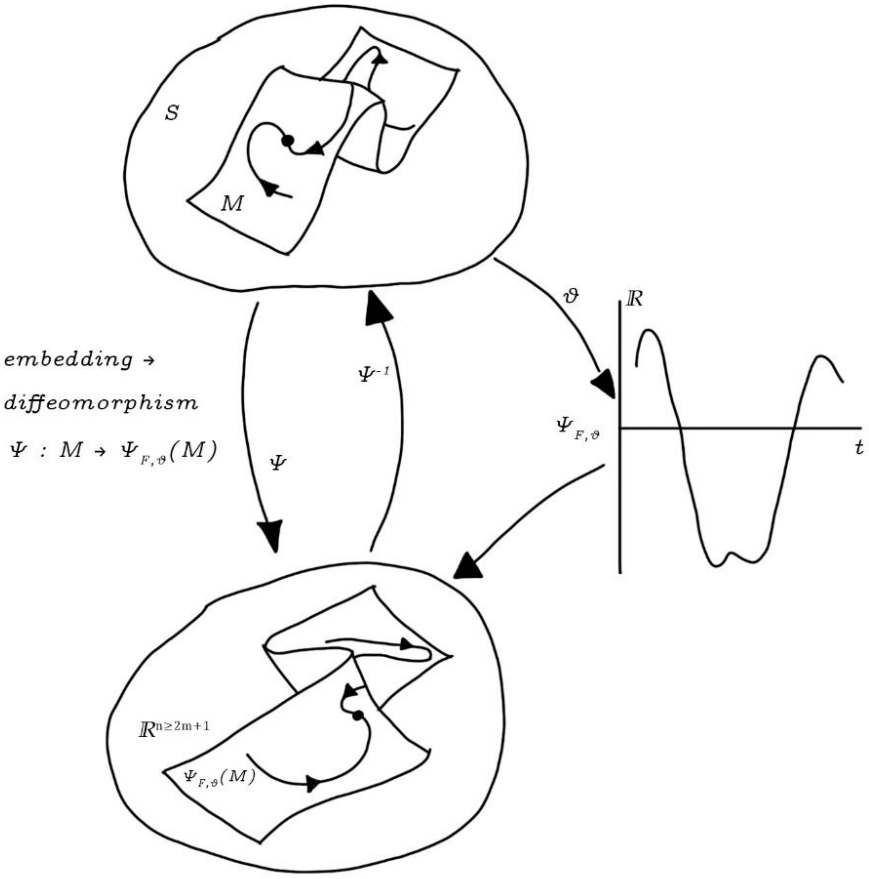


Figure 3.1: Embedded dynamics

the $(2m+1)$ dimensional functions, while still retaining genericity, meaning almost all such maps are embeddings, and there can be cases when a map constructed following the theorem might fail to be one. In 2011, a more generalized theorem for nonlinear state space reconstructions was given by E.R. Deyle and G. Sugihara [13], in which multiple scalar variables were considered to construct such embeddings.

Equipped with Takens' theorem, we can now reconstruct qualitative properties of a dynamical systems only using a single scalar valued observable. We have imposed conditions on the embedding that will preserve the dynamics. So properties like the number and nature of fixed points, Lyapunov numbers, attractors are all preserved in the embedded dimension.

3.3 Takens' theorem in practice

Consider the discrete time system 2.21, with the function $F : M \to M$, which takes the state at time t_i to time t_{i+1}, and also consider that this function is diffeomorphic. Let us say that this is the Lorenz system 2.16. Let us observe the dynamical system, and record the state of the system at each time step. We just need to assume that the measurement at time t_i directly depends on the state at t_i, and that this dependence is smooth.

Let the measurement be described by a differentiable function $\theta : M \to \mathbb{R}$. In our case, the measurements are obtained by simulating the Lorenz system using Ode45 in MATLAB [53], which uses a Runge-Kutta method.

If the conditions imposed on F and θ in Takens' Theorem are met, the Theorem tells us that if we take a time series of measurements and construct n-vectors from it by selecting n consecutive elements ($n > 2(dim(M))$), these vectors lie on a subset of $\mathbb{R}^n$ which is an embedding of M. In general we need to have sufficient well sampled data of the system.

For the Lorenz system, the original dynamics lie on a 3 dimensional manifold. From Takens' Theorem, an embedding dimension of 7 or more should be enough to create an embedding from the observations. However, the Lorenz attractor is a strange attractor. In 1991, work of Sauer, Yorke and Casdagli [50] showed that the embedding dimension in Takens' embedding Theorem 3.2.2 can be reduced when working with strange attractors by using box-counting dimension as the dimension of the manifold, instead of the Euclidean dimension. It has been shown for the Lorenz system [12] [42], that an embedding dimension of 3 is enough for the delay coordinate maps of x and y coordinates to give an embedding. This highlights the fact that the embedding theorems only give a sufficient dimension for a generic mapping to be an embedding, but the embedding dimension may be lower.

Let us now apply this to the Lorenz system. Let us take the embedding $\Psi_{F,\theta}(x(t)) = (x(t), x(t-\tau), x(t-2\tau))$ using only the x coordinates, where $x(t-\tau)$ is equivalent to $F^{-1}(x(t))$. Our choices of the embedding satisfy the requirements of Takens' embedding theorem, namely $F^{-1} \in Diff^2(M)$ (Lorenz system), $\theta \in C^2(M, \mathbb{R})$. Then the Theorem 3.2.2 tells us that it is a generic property (almost all) the a map $\Psi_{F,\theta}$ constructed using (F, θ) is

an embedding. If, our choice of $\Psi_{F,\theta}$ is not an embedding, then by making an infinitesimal perturbation, we should arrive at an embedding.

Hence, an embedding of the form $\Psi_{F,\theta}(x(t)) = (x(t), x(t-\tau), x(t-2\tau))$ satisfies the conditions, and can be used to create so called x-shadow attractor as shown in figure 3.2b. The same holds for the y coordinate too. The y-shadow attractor can be constructed by considering $\Psi_{F,\theta}(y(t)) = (y(t), y(t-\tau), y(t-2\tau))$ as shown in figure 3.2c.

However, the same cannot be done using the z coordinates. This is due to the fact that by using the z coordinate, we cannot distinguish between the two solutions $(x(t), y(t), z(t))$ and $(-x(t), -y(t), z(t))$ of the Lorenz system. And, this makes the map of the form $\Psi_{F,\theta}(z(t)) = (z(t), z(t-\tau), z(t-2\tau))$ an immersion which is not an injective (one-to-one) mapping, and hence not an embedding. As shown in figure 3.2d, the two fixed points (excluding origin) of the Lorenz system are mapped to 1 in the z-shadow attractor. Since it still is an immersion, it still reproduced a single lobe. In the complete formulation of the Takens' theorem, a coordinate that produces such a symmetry should not be considered.

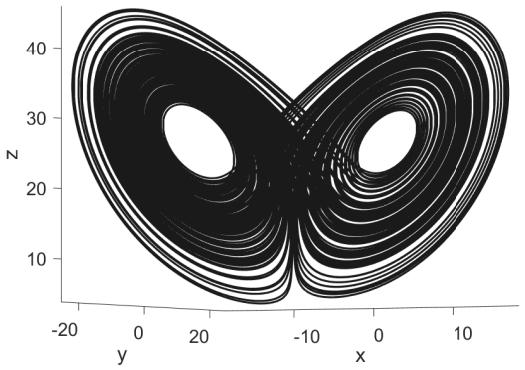
The original Lorenz attractor and its x, y and z shadow attractors are shown in figure 3.2. Thus, by choosing an appropriate measurement function and coordinate, we are able to use Takens' embedding Theorem on the Lorenz system and reconstruct the dynamics.

3.4 Method of delays - Hankel state space reconstructions

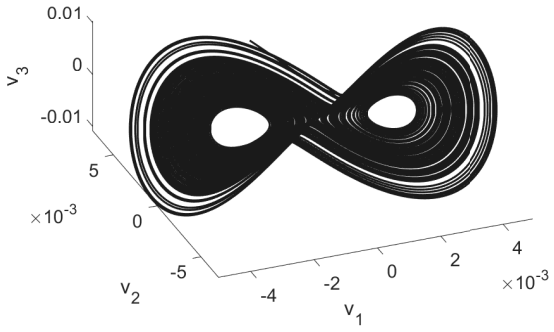
Takens in his embedding theorem never mentioned the process of measurement. In previous section, we chose single measurements $x(t)$, $x(t-\tau)$ and $x(t-2\tau)$ as elements of the embedding, but this does not have to be the case. As shown by D. S. Broomhead and G. P. King in 1986 [2] in their "Method of delays", the embedding created in the Singular System Analysis is done so by computing the SVD of a Hankel matrix H. A p-window on the time series is chosen to construct H as shown in equation 3.4.

$$H = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_p \\ x_2 & x_3 & x_4 & \dots & x_{p+1} \\ x_3 & x_4 & x_5 & \dots & x_{p+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{3.4}$$

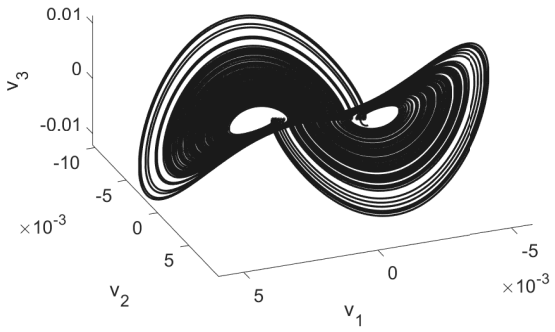
Such a method also works very well when our observation function has noise, as SVD is implemented. The SVD of H gives us $\mathcal{U}\Sigma\mathcal{V}^T$, whose structures are shown in equation 3.5 The first three right singular vectors of the SVD output, which when constructed using suitable length and number of delays in the Hankel matrix H, have the same dimensions as our original system and can be used as the embedding as shown in equation 3.6. Ψ_1 in equation 3.6 refers to $\Psi(x(t))$ at $t = t_1$. Using such an embedding still produces reconstructed



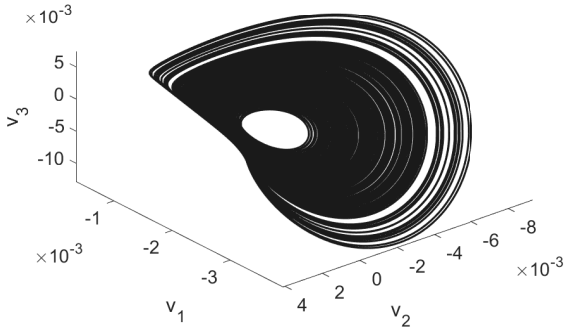
(a) Original Lorenz Attractor



(b) x-Shadow Attractor



(c) y-Shadow Attractor



(d) z-Shadow Attractor

Figure 3.2: Shadow Attractors

attractors as shown in figure 3.2. So it is advantageous to use the method of delays and the SVD as shown in SSA to create our embedding.

$$\underbrace{\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \dots & \mathbf{u}_r\end{bmatrix}}_{\text{Col}\, H} \quad \underbrace{\begin{bmatrix}\mathbf{u}_{r+1} & \dots & \mathbf{u}_m\end{bmatrix}}_{\text{Nul}\, H^T} \begin{bmatrix}\sigma_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix}\mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \\ \dots \\ \mathbf{v}_r^T \\ \mathbf{v}_{r+1}^T \\ \dots \\ \mathbf{v}_n^T \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right\} \text{Row}\, H \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \text{Nul}\, H \end{matrix} \tag{3.5}$$

$$\begin{bmatrix}\Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \\ v_{4,1} & v_{4,2} & v_{4,3} \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{3.6}$$

In practice, we do not need to compute the full SVD of the Hankel matrix H. We can compute the truncated SVD or T-SVD to save computational costs. Alternatively, we can also set a hard threshold for singular values as mentioned by Gavish et. al. [16].

3.5 Hankel delay embeddings and Koopman theory

The work of Mezic and collaborators [38], which started off Modern Koopman theory established a connection between delay embeddings and Koopman operator, where a stochastic Koopman operator was defined and a statistical version of Takens' theorem was proven. Low rank approximation of the time delayed measurements can be used to construct an intrinsic coordinate system that forms a Koopman invariant subspace. If the conditions imposed by Takens' theorem are satisfied, we have seen that these coordinates (or observation functions) provide an embedding. The Hankel matrix used to create the embedding as in equation 3.4, can be rewritten under the action of a Koopman operator $\mathcal{K}$ as in equation 3.7. The state x_2 can be thought of as $\mathcal{K}(x_1)$.

$$H = \begin{bmatrix} x_1 & \mathcal{K}(x_1) & \mathcal{K}^2(x_1) & ... & \mathcal{K}^{p-1}(x_1) \\ \mathcal{K}(x_1) & \mathcal{K}^2(x_1) & \mathcal{K}^3(x_1) & ... & \mathcal{K}^{p}(x_1) \\ \mathcal{K}^2(x_1) & \mathcal{K}^3(x_1) & \mathcal{K}^4(x_1) & ... & \mathcal{K}^{p+1}(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{3.7}$$

In equation 3.7, each column can be obtained by applying the Koopman operator to the previous column. Using Koopman operator provides us a shift of perspective. We are not mapping a value of the observable from one time step to another, but we are mapping the observable function (the column of H) from one time step to the other. As mentioned earlier, the goal of using the Koopman operator is to study a given system in terms of the eigenfunctions and eigenmodes of the Koopman operator instead of using eigenvalues and eigenvectors of a linear approximation of the system.

For sufficient volume of data, the columns of H become approximately linearly dependent. In this case, we can construct an approximately Koopman invariant subspace by computing the SVD of H as mentioned in [5] (page 59). The first r columns of $\mathcal{U}$ and $\mathcal{V}$ provide a low rank approximation of H.

Each column from the matrix H from equation 3.7 can be thought of as a trajectory of x, then the matrix H is formed by stack shifted trajectories of the variable x. By computing the SVD, the matrix $\mathcal{V}$ can be thought of as the most prominent directions of the trajectories. This is approximately invariant, and can be used to model the system.

Chapter 4

Hankel Alternative View Of Koopman (HAVOK) analysis

We now know that the Hankel matrix of scalar observables as constructed in equation 3.4 can be used as an embedding, and also that it forms an approximately linear subspace in the right eigenvectors $\mathcal{V}$. It has been shown [9] for weakly nonlinear systems that a linear system of the form 4.1 models the dynamics. But for chaotic systems, such a model is not able to capture the qualitative dynamics of the system.

$$\dot{\mathcal{V}}(t) = A\mathcal{V}(t) \tag{4.1}$$

Instead, a linear system forced by a right eigenvector as shown in equation 4.2 has been shown to accurately model a chaotic system by Brunton and collaborators in their method called Hankel Alternative view of Koopman (HAVOK) [6].

$$\dot{\mathcal{V}}(t) = A\mathcal{V}(t) + Bv_r(t) \tag{4.2}$$

Where v_r is the r^{th} right eigenvector of the Hankel matrix H, and $\mathcal{V}$ is the first $r-1$ right eigenvectors of H. The statistics of the forcing term v_r are non Gaussian and give high values which correspond to rare events such as lobe switching for the Lorenz system.

4.1 The model described

The application of the HAVOK model on the Lorenz system is first seen. Using the x coordinate of the Lorenz system we first reconstruct the attractor as shown in figure 4.1, and model the chaotic dynamics of Lorenz system using an intermittently forced linear system as shown in figure 4.2.

Let r denote the total number of right singular vectors considered for the intermittently forced linear system, and γ denote the position of right singular vector considered for forcing. In the paper [6], $r = \gamma = 15$ was considered. The matrix A was observed to have a sparse structure regardless of the value of r, and for $r = 11$, $\gamma = 11$ the sparse structure is shown in figure 4.5. Existence of such a sparse structure is intriguing and a topic of ongoing research.

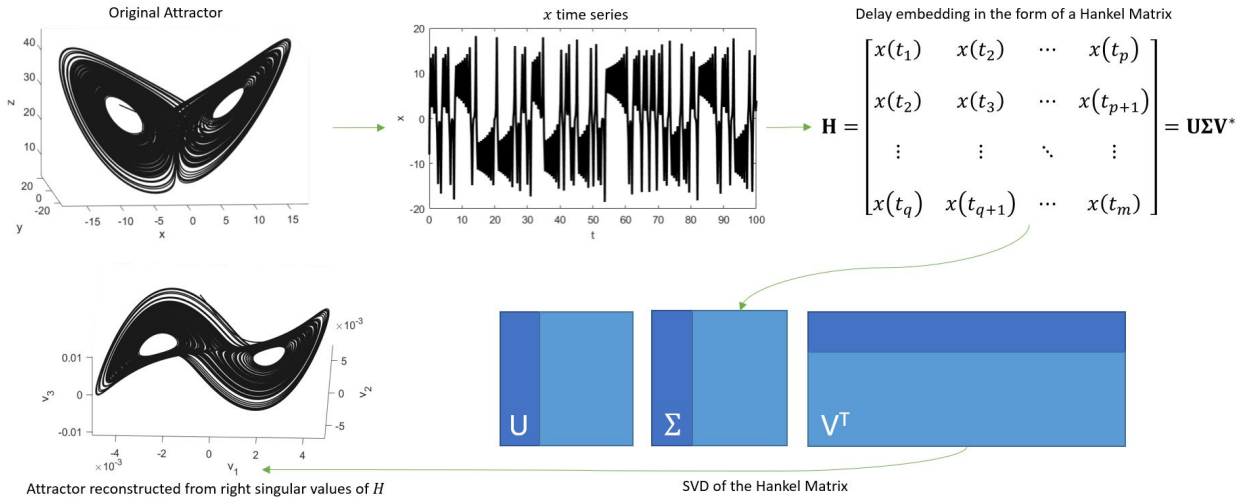


Figure 4.1: Hankel embedded reconstruction

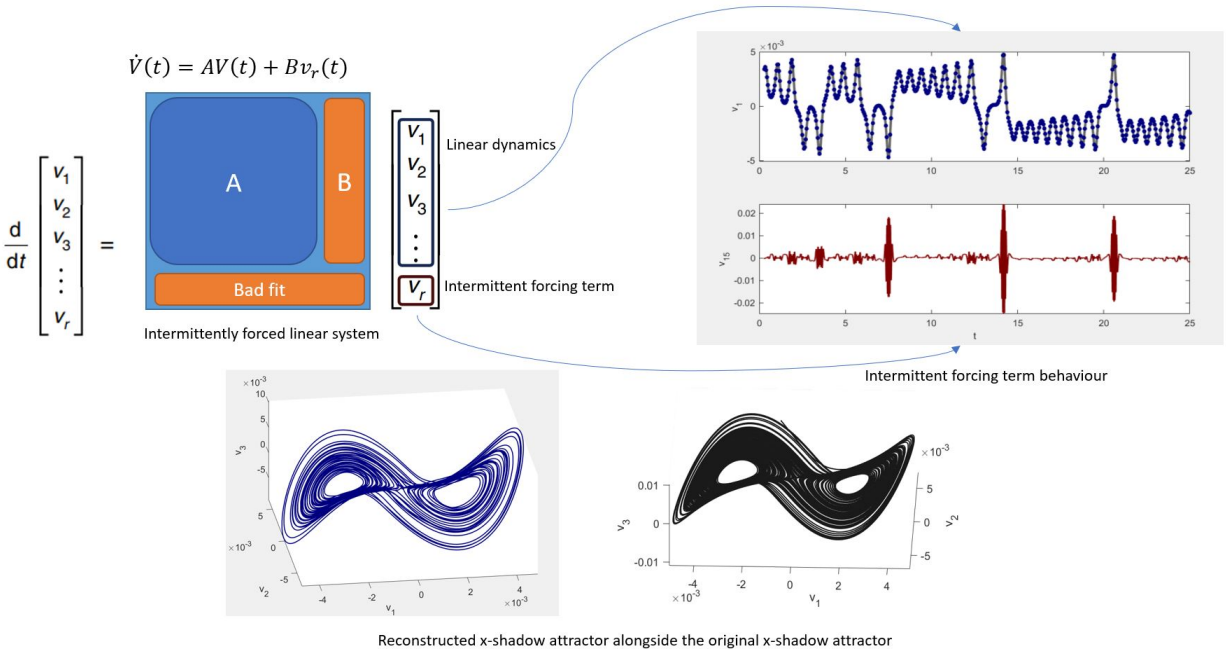


Figure 4.2: Intermittently forced linear system

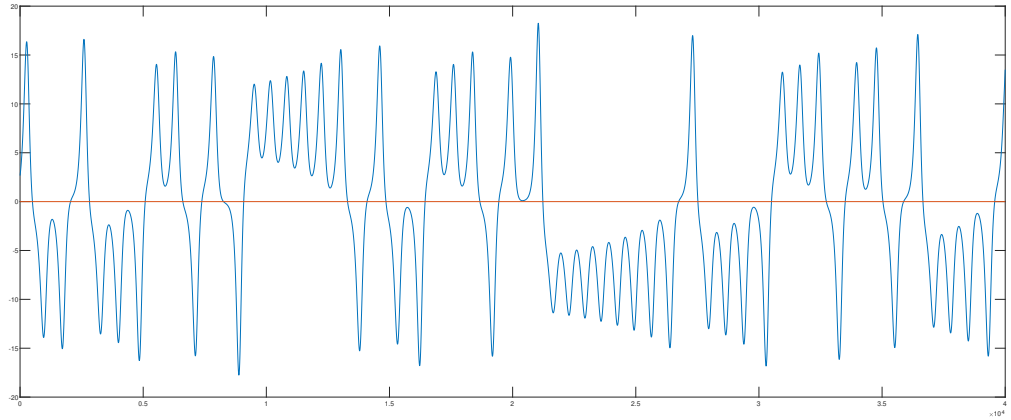


Figure 4.3: Actual Lobe switching from x-coordinate

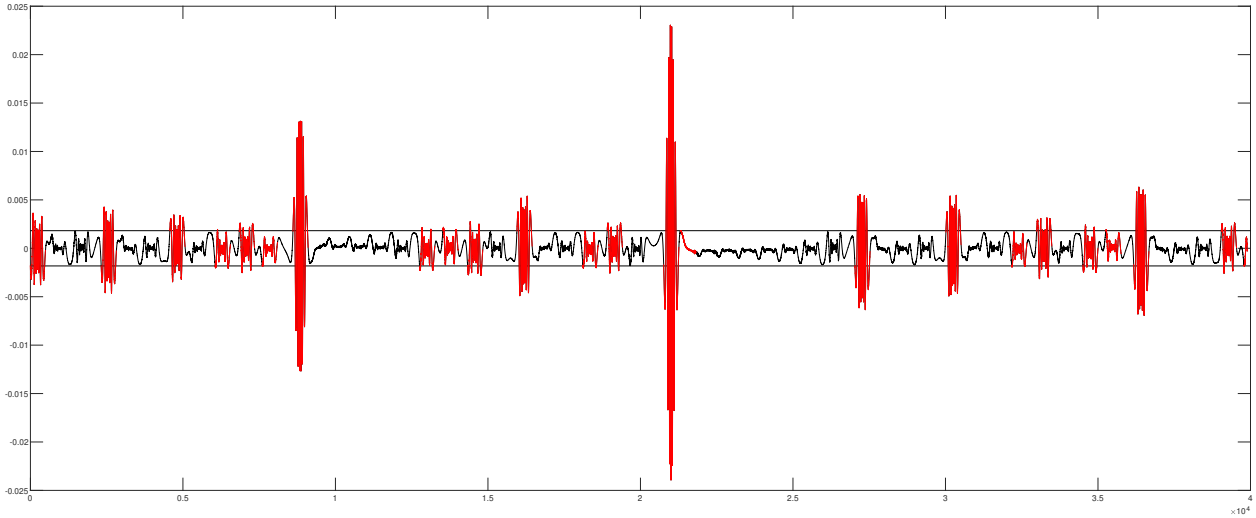
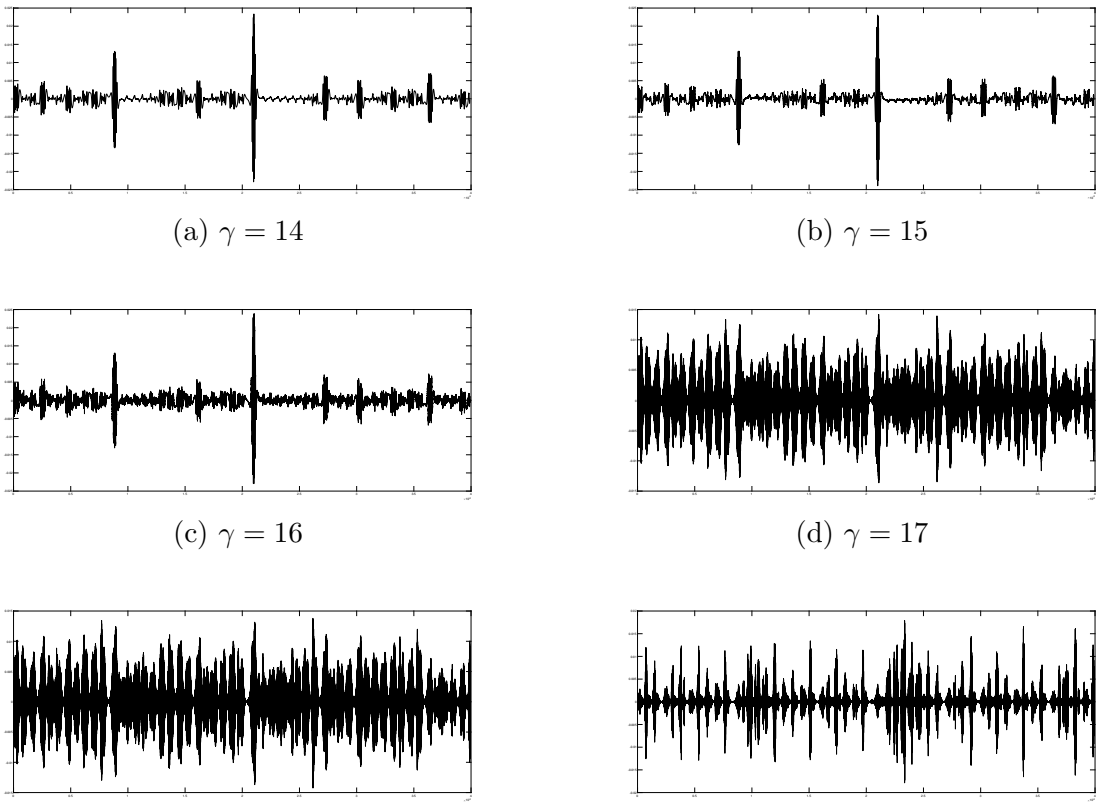


Figure 4.4: Lobe switching prediction from the forcing term $\gamma = 15$

	Linear Matrix										
	A										B
	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10	v11
dv1	0.029387	5.212133	0	0.125745	0	0	0	0	0	0	0
dv2	-5.08085	-0.22659	-9.91079	0	-0.1926	0	0	0	0	0	0
dv3	0	9.230823	-0.3315	-14.2501	0	0.262906	0	0	0	0	0
dv4	0	-0.86844	13.08953	-0.82489	-18.551	0	0.331261	0	0	0	0
dv5	0	-0.55629	-0.77215	17.22859	-0.45622	24.27987	0	0.425581	0	0	0
dv6	0	0.63059	0.54112	1.144432	-23.3713	-0.40963	-28.0998	0	-0.49804	0	0
dv7	0	0.18013	0.152102	0	0.17762	27.87235	0	-34.418	0	-0.60503	0
dv8	0	0	0	-0.20121	-0.58698	0	34.45364	0	-38.6568	0	-0.69867
dv9	0	-0.72257	-0.9017	-1.33168	-1.018	1.428912	0.236964	38.53056	-0.52817	-44.6223	0
dv10	0	-0.97289	-1.23364	-1.7934	-1.39637	1.261407	0.921561	-0.20637	43.22801	-0.96963	-50.3075

Figure 4.5: Sparse structure when $\gamma = r$



(a) $\gamma = 14$ (b) $\gamma = 15$

(c) $\gamma = 16$ (d) $\gamma = 17$

(e) $\gamma = 18$ (f) $\gamma = 19$

Figure 4.6: Various forcing terms

The matrices A and B in the linear system 4.2 can be learned by Sparse identification of nonlinear dynamics (SINDy) [4], or by least squares linear regression.

The time step considered was $dt = 0.001$, and the model is trained using 200 time units of data. As for lobe switching, the original system switches lobes whenever the value of x crosses the line $x = 0$. This is shown for time units $200 - 240$, where 24 instances of lobe switching have been identified as shown in figure 4.3. To pick the lobe switching, a cutoff on the forcing term is used. In the paper, a cutoff of 0.002 was used for $r = \gamma = 15$ for a prediction window of 1000 time units. But for 40 time units, a cutoff of 0.00182 was sufficient as shown in figure 4.4.

In the paper, $r = \gamma = 15$ has been chosen, as from $r = \gamma \geq 16$, there are high reconstruction errors, which have been attributed to the corresponding forcing term $\gamma \geq 16$. Figure 4.6 shows forcing terms $\gamma = 14 - 19$, and here it can be seen that $\gamma \geq 16$ is not a good forcing term as the statistics are Gaussian. The attractor reconstruction is shown in figure 4.7, regions where forcing is active is coloured in red. When forcing is active, lobe switching is predicted. As we can see, in the outer regions of each lobe the forcing term is higher than set threshold.

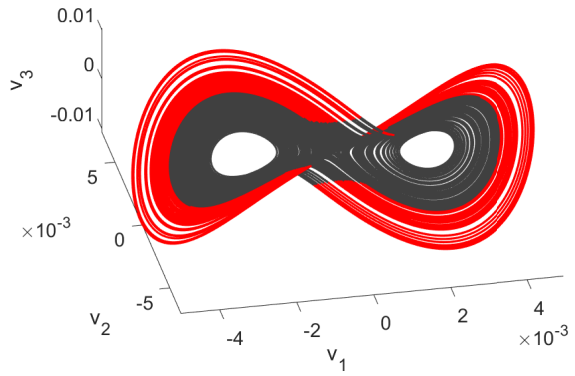


Figure 4.7: $r = \gamma = 15$ reconstruction colored when forcing term set threshold

4.2 Modifications and improvements

4.2.1 (r, γ) Analysis

In each case of r, the least energy right singular value has been used as the intermittent forcing term. In theory, other forcing terms can be used. For example, for $r = 15$, forcing term $\gamma = 15$ is used in the paper, but $\gamma = 10$ or $\gamma = 20$ can also be used to force the linear system. In this section, novel work done by the author of this thesis is presented. The parameters considered for the analysis are shown below.

p	length(H)	dt	training set	test set
100	199900	0.001	1-200	200-240

Where, p denotes the number of columns of H and dt is the data sampling interval. First let us consider the case when the forcing term used is not the least energy forcing term. We use three main reconstruction errors.

1. v_1 error is given by $\|v_{1actual} - v_{1predicted}\|_2$.

2. $v_1 - v_3$ error is given by $\|(v_{1actual}, v_{2actual}, v_{3actual}) - (v_{1predicted}, v_{2predicted}, v_{3predicted})\|_2$.

3. $v_1 - v_3$ error Frobenius norm is given by the Frobenius norm of the matrix $[(v_{1actual}, v_{2actual}, v_{3actual}) - (v_{1predicted}, v_{2predicted}, v_{3predicted})]$. This is also known as the element wise L_2 norm of the matrix.

The reconstruction error for v_1 is computed for different values of r and γ in figure 4.8. We can see 2 main patterns here.

1. Whenever $\gamma > r$, we have high reconstruction error.

2. When $\gamma >= 16$, we have high reconstruction error.

r	γ	v_1 error	$v_1 - v_3$ error	$v_1 - v_3$ error Frobenius norm
29	14	0.019213	0.027148	0.036941
17	14	0.020937	0.027164	0.038083
27	14	0.027333	0.027335	0.042882
14	12	0.033202	0.035922	0.044873
15	12	0.035284	0.038012	0.047483
12	12	0.036543	0.039559	0.048305

Table 4.1: (r, γ) reconstruction errors

Point 1 can be explained by looking at the sparse structure. The sparse structure for $r = 11$ and $\gamma = 12$ is shown in figure 4.9. As we can see, the matrix B does not have high values, and this could mean that it is insufficiently small to properly force the linear system. But when $r = 11$ and $\gamma = 9$, the matrix B does have significant values as seen in figure 4.10, and can be used to force the system. This result can bee seen in the figure 4.8, where we have low reconstruction errors for $\gamma \leq r$.

Point 2 has already been noted by Brunton et. al. [6] when using $\gamma = r \geq 16$. Our results match this pattern. Whenever we take forcing term $\gamma \geq 16$ for any value of r, we get high reconstruction errors due to the fact that right singular vectors more than 16 are not Gaussian.

Other than these two patterns, another pattern observed is when $\gamma = 13$, regardless of our choice of r, we get high reconstruction errors. Looking at sparse structures does not give us any indication as to why this happens. This can mean that $r = 13$ does not contain information about the lobe switching event, but this has not been verified in any other way.

Table 4.1 shows six (r, γ) combinations with the lowest reconstruction errors. For $r = 15$, the reconstruction errors are shown in figure 4.11, and we can see that for higher energy forcing terms, the reconstruction errors are lower. Figure 4.12 shows the reconstructions for some good and bad (r, γ) combinations. In general, we get low reconstruction errors when we consider a higher energy forcing term $\gamma \leq r$.

4.2.2 Multiple intermittent forcing signals

It has also been mentioned by Brunton et. al. [6] that in theory, multiple forcing terms in combination can be used. This has been implemented when considering 2 consecutive right singular vectors as forcing terms. When implemented, we immediately see low errors for low values of r which were not seen previously. Table 4.2 shows (r, γ_1, γ_2) combinations for which lowest error was obtained.

The reconstruction errors are similar to $r = \gamma = 15$ which was implemented in the paper [6]. Overall, a pattern like observed in (r, γ) analysis cannot be seen. Some combinations

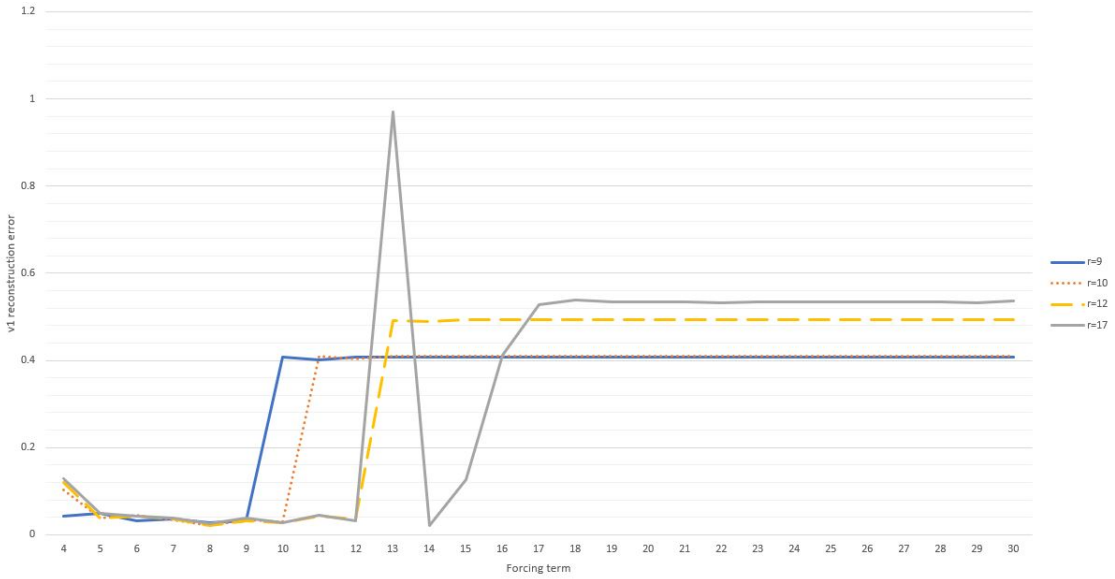


Figure 4.8: v_1 reconstruction error by varying r and γ

	Linear Matrix										
	A										B
	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10	v12
dv1	0.029387	5.212133	0	0.125745	0	0	0	0	0	0	0
dv2	-5.08085	-0.22659	-9.91079	0	-0.1926	0	0	0	0	0	0
dv3	0	9.230823	-0.3315	-14.2501	0	0.262906	0	0	0	0	0
dv4	0	-0.86844	13.08953	-0.82489	-18.551	0	0.331261	0	0	0	0
dv5	0	-0.55629	-0.77215	17.22859	-0.45622	24.27987	0	0.425581	0	0	0
dv6	0	0.63059	0.54112	1.144432	-23.3713	-0.40963	-28.0998	0	-0.49804	0	0
dv7	0	0.18013	0.152102	0	0.17762	27.87235	0	-34.418	0	-0.60503	0
dv8	0	0	0	-0.19766	-0.58428	0	34.45296	0	-38.6541	0	0
dv9	0	-0.72474	-0.90414	-1.33556	-1.02083	1.431673	0.237659	38.53022	-0.53107	-44.6263	-0.80081
dv10	1.477915	-0.75891	-1.68138	-1.37917	-1.56567	0.880863	1.12793	-0.35975	43.6533	-0.81886	0.273287

Figure 4.9: $r = 11$ with $\gamma = 12$

	Linear Matrix										
	A								B	A	
	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10	v11
dv1	0.029387	5.212133	0	0.125745	0	0	0	0	0	0	0
dv2	-5.08085	-0.22659	-9.91079	0	-0.1926	0	0	0	0	0	0
dv3	0	9.230823	-0.3315	-14.2501	0	0.262906	0	0	0	0	0
dv4	0	-0.86844	13.08953	-0.82489	-18.551	0	0.331261	0	0	0	0
dv5	0	-0.55629	-0.77215	17.22859	-0.45622	24.27987	0	0.425581	0	0	0
dv6	0	0.63059	0.54112	1.144432	-23.3713	-0.40963	-28.0998	0	-0.49804	0	0
dv7	0	0.18013	0.152102	0	0.17762	27.87235	0	-34.418	0	-0.60503	0
dv8	0	0	0	-0.20121	-0.58698	0	34.45364	0	-38.6568	0	-0.69867
dv10	0	-0.97289	-1.23364	-1.7934	-1.39637	1.261407	0.921561	-0.20637	43.22801	-0.96963	-50.3075
dv11	0	-1.29696	-1.42378	-2.36773	-1.68563	1.68305	0.330474	0.444893	-1.88146	47.91343	-1.6597

Figure 4.10: $r = 11$ with $\gamma = 9$

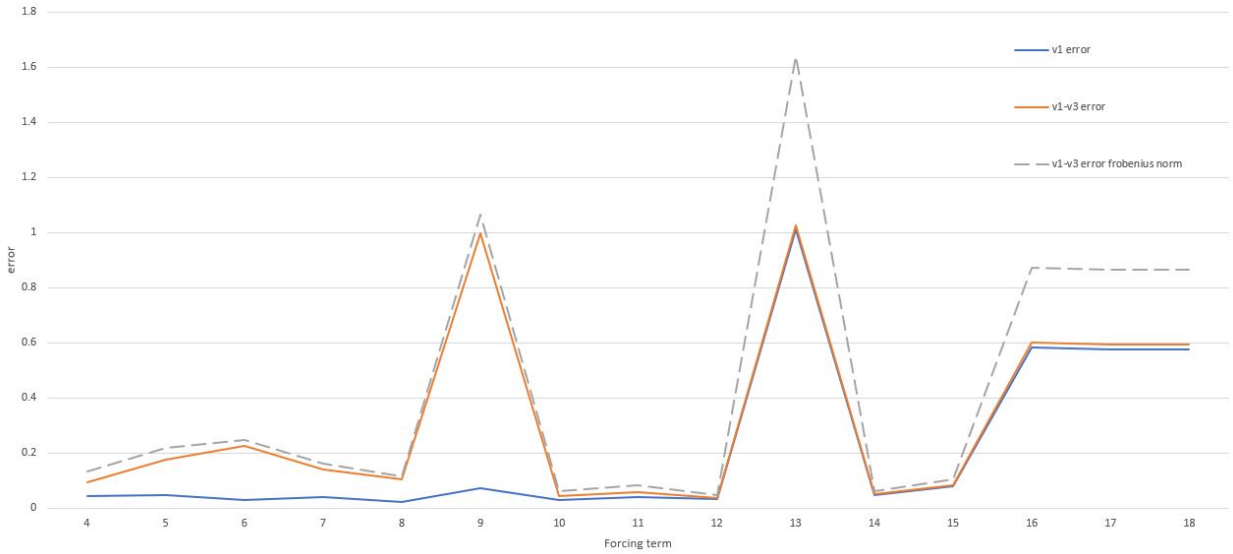


Figure 4.11: errors for $r = 15$ varying γ

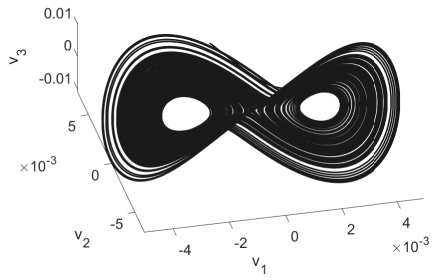
r	γ_1	γ_2	v_1 error	$v_1 - v_3$ error	$v_1 - v_3$ error Frobenius norm
5	4	5	0.017473	0.028236	0.041895
19	4	5	0.022282	0.030155	0.041977
21	15	16	0.025887	0.02733	0.045208
11	10	11	0.029267	0.029415	0.045135
9	8	9	0.01968	0.04071	0.048658
12	11	12	0.034494	0.035017	0.052595

Table 4.2: (r, γ_1, γ_2) reconstruction errors

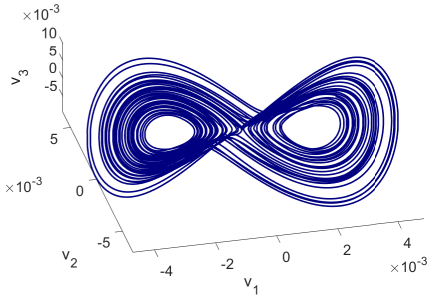
of forcing terms seem to produce low reconstruction errors, but no pattern can be distinctly identified.

It is quite strange and remarkable to note that $r = 5$ and $\gamma_1 = 4, \gamma_2 = 5$ produces the least reconstruction error. This is also verified by plotting the reconstructed attractor. Figure 4.13 shows the reconstructed attractor for a few different combinations of (r, γ_1, γ_2).

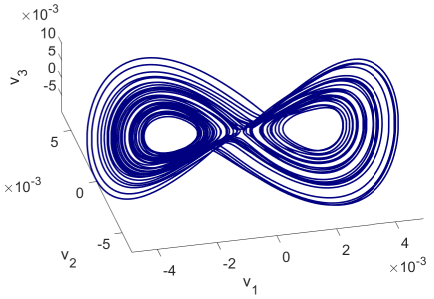
As for prediction of lobe switching using multiple forcing terms, it can be done in a number of ways. We cab use different combinations of the two forcing signals to produce a new signal, and set a threshold to predict the lobe switching. By using linear combinations such as $(\gamma_1 + \gamma_2)$, $(\gamma_1 - \gamma_2)$, the resulting signal is similar in structure to 4.4. A threshold must be set for each combination of (γ_1, γ_2).



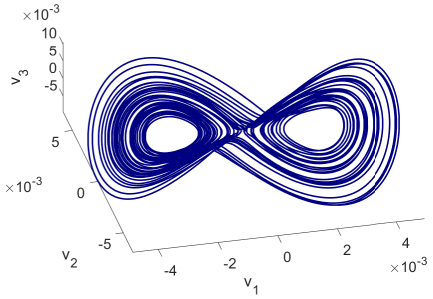
(a) Original embedded attractor



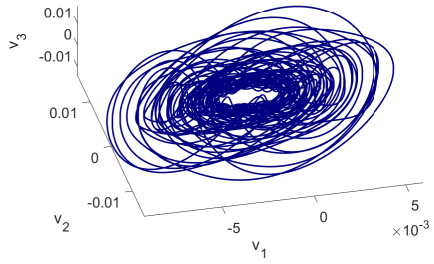
(b) $r = \gamma = 15$



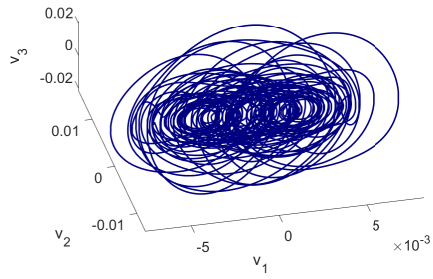
(c) $r = 29, \gamma = 16$



(d) $r = 17, \gamma = 14$

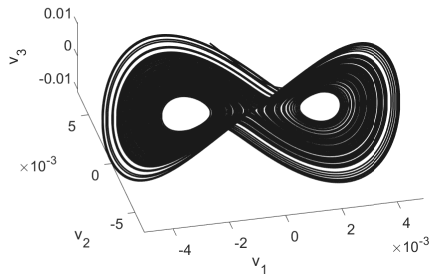


(e) $r = 12, \gamma = 14$

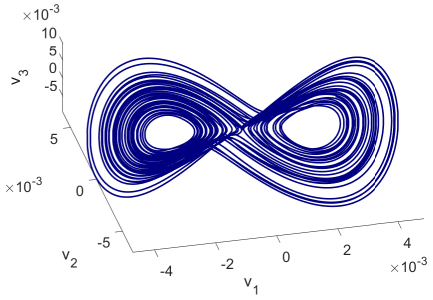


(f) $r = 26, \gamma = 19$

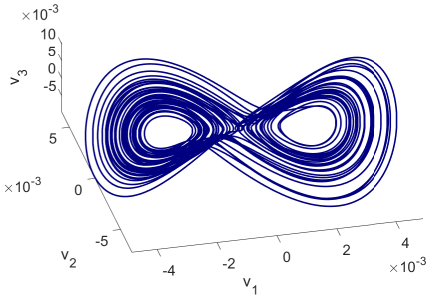
Figure 4.12: Reconstructions varying (r, γ)



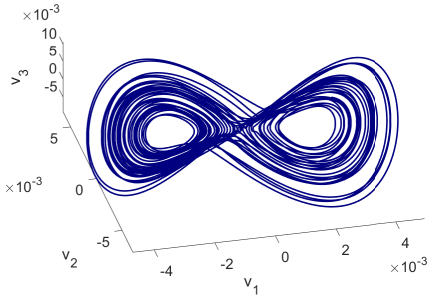
(a) Original embedded attractor



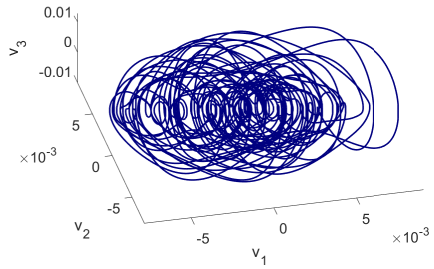
(b) $r = \gamma = 15$



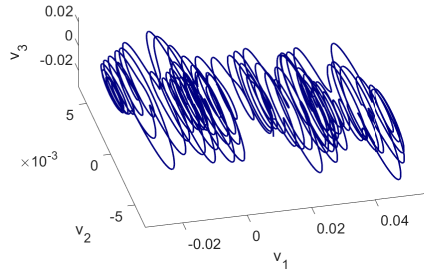
(c) $r = 5, \gamma_1 = 4, \gamma_2 = 5$



(d) $r = 19, \gamma_1 = 4, \gamma_2 = 5$



(e) $r = 13, \gamma_1 = 11, \gamma_2 = 12$



(f) $r = 20, \gamma_1 = 10, \gamma_2 = 11$

Figure 4.13: Reconstructions varying (r, γ_1, γ_2)

Chapter 5

Summary

Complex systems modelling is really interesting field with lots of new methods emerging every year. The Hankel Alternative View of Koopman (HAVOK) analysis proposed a method to fit an intermittently forced linear system to a chaotic one. The HAVOK model uses a delay embedding in the form of a Hankel matrix formed by stacking time shifted trajectories of the x coordinate of the system. By computing the SVD, and the right singular vectors, the argument is that they form an approximately invariant subspace in which nonlinear dynamics may appear linear. Such an argument works for weakly nonlinear system, but for chaotic systems it fails. In such a case, the least energy right singular vector was used as a forcing term to predict the singular vectors preceding it.

In this thesis, use of other singular vectors as forcing was explored. In theory higher energy singular vectors should work the same way. And by checking for reconstruction errors, and plotting reconstructions, it has been shown that when using higher energy singular vectors, the intermittently forced linear system gives a lower reconstruction error.

Additionally, another contribution of this thesis, the use of multiple forcing terms, which has been suggested but not implemented. It has been shown that when using two right singular vectors as forcing terms, we get good reconstructions as should be. But a remarkable result is that when using as low as a total of 5 right singular vectors, with the $4^{th}, 5^{th}$ right singular vector used for forcing, we get similar reconstruction errors to when a single forcing term with even 15 right singular vectors. Using such a low amount of singular vectors will also enhance the computational speed of the model.

This in theory can be extended to 3 or more forcing signals. An optimal ratio of forcing signals to the number of right singular vectors remains unknown and can be looked into. Furthermore, using neural networks in combination with HAVOK method can be explored. Long Short Term Memory networks (LSTM) can be used to predict the forcing term, and HAVOK can take the output of the LSTM to model other right singular vectors. Such a setting can create a closed model that can forecast a chaotic system.

All the codes were written in MATLAB [35], and have been provided along with the thesis.

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