Bayesian System Identification for Structural Health Monitoring

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Abstract
Structural health monitoring (SHM) is an emerging tool with significant socio-economic benefits. It can save lives and provide long-term economic gains to various public and private organizations that deal with civil infrastructure. System identification is an extensively used technique for SHM, where the objective is to identify the parameters of the assumed mathematical model that best represent the actual structure. Probabilistic system identification methods based on Bayesian inference are state-of-the-art techniques for SHM and have shown significant promise. This is because most structural engineering problems are replete with uncertainties (material properties, connections, input excitation), and Bayesian methods explicitly treat such uncertainties in their formulation. Furthermore, Bayesian methods convert the inverse problem associated with model updating to a forward problem and thus, eliminate the usual difficulties that come with inverse problems such as illconditioning. This article is intended to be a straightforward exposition to Bayesian methods, particularly Bayesian finite element model updating, for practicing engineers with basic exposure to probability and structural dynamics. In order to comprehend and appreciate the advantages of Bayesian methods, one numerical example for the identification of a single-degree-of-freedom system and one experimental implementation of damage detection on a four-story shear building model is included in this article. It may be noted that the Bayesian methods discussed in this article apply to a broader spectrum of structural/civil engineering applications too.

Keywords
Bayesian FEMU; system identification; BAYOMA; probabilistic structural health monitoring

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1. Introduction

The last few decades have witnessed tremendous growth in research and applications of structural health monitoring (SHM). In SHM, the current health of a structural system is assessed through the information gained from various measurements. The four objectives of SHM are 1) identifying the existence of damage, 2) locating the damage, 3) quantifying damage severity, and 4) prognosis of damage. The objectives are mentioned in the increasing order of difficulty. After a natural or manmade disaster, SHM can be used to check if the structural system can continue to give its intended performance or if it is safe for re-occupation. It can also be used for monitoring heritage and other structures that had passed their service lives. In such cases, SHM helps to decide whether the structure is safe to be in service or needs repair. It even gives scope for utilizing the over-strength in structures.

The methods used for SHM can be broadly classified into two: 1) qualitative and 2) quantitative. Qualitative methods mainly include the ones based on visual inspection, while quantitative methods include non-destructive testing, static-based and vibration-based methods. Among the various methods mentioned previously, vibration-based methods [1, 2] are superior. The basic premise in vibration-based SHM is that damage changes the properties of a system, and these changes can be detected through the measured dynamic data (accelerations, strains, modal parameters). This is accomplished with the help of a mathematical model. Even if the damage is not visible to the naked eye or if the damaged location is inaccessible, these methods can provide a good assessment of-structural health conditions. Vibration-based methods can be further classified into data-driven methods and physics-based methods. Artificial neural networks and finite element model updating are examples of data-driven and physics-based methods, respectively. In data-driven methods, the parameters of the mathematical model involved generally do not have any physical interpretations, while in physics-based methods, the parameters have physical interpretations. Data-driven methods generally only achieve the first two objectives of SHM. On the contrary, physics-based methods can accomplish the first three objectives, and research is still ongoing for achieving the fourth objective.

Finite element model updating (FEMU) [3] is a system identification technique in which we tune the parameters of a finite element (FE) model by using measured data from the actual structure such that the output of the model reflects the measured data and thus, the model represents the true structural behaviour. The assumption here is that, if the actual structure sustains damage, it will be reflected in the measured data and subsequently, changes in the parameters of the FE model can be captured post model updating. Often these model parameters are stiffness parameters, and any damage to the actual structure is manifested as a reduction in these stiffness parameters. Further, these model parameters show significant variations due to uncertainty, which can be mistaken for damage. Conventional FEMU methods like inverse sensitivity updating do not consider these uncertainties and try to find one solution, which is the best solution. When multiple solutions exist, these conventional methods often fail. Bayesian FEMU [4, 5] is a probabilistic method that explicitly considers uncertainty in its formulation. Contrary to the conventional methods, Bayesian methods find a set of solutions due to uncertainty for a given the model class. These methods remain robust in presence of noise and/or possibilities of multiple solutions.

The rest of the article is organized as follows. The next section shows the motivation behind SHM and Bayesian system identification. After that, the methodology of Bayesian approaches is elaborated, followed by numerical and experimental examples and finally the conclusions are drawn.

2. Motivation and objectives

2.1 Relevance of SHM

The 2021 Report Card of America's Infrastructure (https://infrastructurereportcard.org/), prepared by the American Society of Civil Engineers, has given an overall grade of C- to the US's infrastructure. This grade shows much room for improving infrastructure in an economically and technologically advanced nation like the US. To the authors' knowledge, a similar detailed and large-scale grading has not been done for the infrastructure of India. However, if a study were to be done, our infrastructure would likely score worse. The condition of so many infrastructures might be severe. Many of them might be nearing the end of their service lives, and many might have passed their service lives. Fig. 1 shows the status of railway infrastructure viz. railway bridges in India. These bridges are listed according to their year of construction and importance. The category highlighted in red consists of relatively old bridges (>80 years old) or whose construction year is unknown. Such bridges require maximum attention from the SHM point of view. A significant number (around 53%) of the railway bridges fall in this category. So, the information from this figure highlights the need for and importance of SHM in our country, at least for railway bridges. The condition is likely to be the same for other infrastructure (dams, highways, ports). SHM is a desideratum for socio-economic gains.

Period of Construction	Important Category	Major Category	Minor Category	Total	% w.r.t to total
Prior to 1900	254	2917	33679	36850	28.98
1901-1920	85	1284	17950	19319	15.19
1921-1940	31	712	10647	11390	8.96
1941-1960	67	871	11726	12664	9.96
1960-1980	193	2293	22160	24646	19.38
1981-2002	83	1606	14713	16402	12.90
N/A	18	552	5313	5883	4.63
Total	731	10235	116188	127154	

Maximum attention
Moderate attention
Relatively less attention

Courtesy: CAG of India Report, 2002

Figure 1. Status of railway bridges in India

2.2 Bayesian FEMU as a tool for SHM

SHM schemes based on FEMU compare the identified parameters of a model representative of the healthy and damaged states. However, these identified parameters can vary significantly in civil engineering applications due to uncertainties (material properties, sensor noise, input excitation, ambient conditions). If these uncertainties are not accounted, the variations in parameters can bemisinterpreted as damage. Bayesian FEMU is an excellent tool for handling uncertainties. The Bayesian framework can 1) fuse information from various sources, 2) incorporate additional information based on engineering judgements, 3) handle nonlinearities, and 4) use partial information for localizing damage. The primary objective of this article is to demonstrate and disseminate the advantages of Bayesian FEMU in SHM to the practicing engineers. It is envisioned that this introductory article can be a starting point for practicing engineers interested in delving into SHM. SHM of real-life structures in India is still an untapped field and requires collaborative efforts from consulting engineers and researchers.

3. Methodology

3.1 Bayesian probability

The Bayes' theorem was conceived in its incipient form by Reverend Thomas Bayes in the 1700s. His work was published posthumously by Richard Price. Later, Pierre-Simon Laplace propounded the modern form of the Bayes' theorem in 1774. The world of statistics had shunned this theorem for many decades. It was only towards the end of the 20th century that Bayes' theorem started to become widely accepted. Mcgrayne[6] gives a good account of the history and development of the Bayes' theorem. The systematic development of Bayesian system identification in structural engineering is primarily attributed to James Leslie Beck [7, 8, 9] and his group from Caltech.

Two schools of thought exist for probability: 1) frequentist probability and 2) Bayesian probability. In frequentist probability, the probability of an event, A, is defined as the relative frequency of the event over a large number of trials. This definition is formally expressed as:

$$p(A) = \lim_{N_t \to \infty} \frac{N_A}{N_t} \tag{1}$$

Here, N_A is the number of occurrences of the event, A, in N_t number of trials. The caveat of such a definition is that the probability can only be defined for an event for which trials can be performed. This is the most common interpretation of probability.

In Bayesian probability, the probability of an event is always conditioned on some available information. Probability is the degree of belief of an event in Bayesian philosophy. The probability of an event, A, conditioned on information, B, is given as:

$$p(A/B) = \frac{p(A)p(B/A)}{p(B)} \tag{2}$$

where $p(A)$ is the probability of A before acquiring the information B; $p(B/A)$ is the probability of B given A and $p(B)$ is the probability of B. Unlike frequentist probability, no trials are required to get $p(A/B)$. Once the information of B is obtained, the mathematical model that correlates A and B can be used to get $p(A/B)$. In other words, $p(A)$ is updated to $p(A/B)$ using the information B. Equation 2 is referred to as the Bayes' theorem.

3.2 Bayesian finite element model updating

Two types of problems exist in Bayesian system identification. In the first problem, the objective is to identify the parameters of a given model from the acquired data and is known as the parametric identification problem. The second problem involves finding the best model from a suite of candidate models from the data. This is known as the model identification problem. To keep the article concise and straightforward, only the first problem is considered.

In Bayesian FEMU, the parameters of a FE model, which is representative of the structural system, is identified by using the data measured from the actual system. Equation 2 can be rewritten in the form probability density functions (PDFs) as:

$$p(\boldsymbol{\theta}/\mathscr{D},\mathscr{M}) = \frac{p(\boldsymbol{\theta}/\mathscr{M})p(\mathscr{D}/\boldsymbol{\theta},\mathscr{M})}{p(\mathscr{D}/\mathscr{M})} \tag{3}$$

In the above equation, $\boldsymbol{\theta} \subset \boldsymbol{\Theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}$ are the parameters of the FE model; $\mathscr{D}$ represents the measured data (available information); and $\mathscr{M}$ reflects the model class. $N_{\boldsymbol{\theta}}$ is the total number of parameters of considered in the model. A model class constitutes all the modeling assumptions and the probability distributions used while forming Equation 3. In Equation 3, $p(\boldsymbol{\theta}/\mathscr{D},\mathscr{M})$ is known as the posterior PDF of $\boldsymbol{\theta}$; $p(\boldsymbol{\theta}/\mathscr{M})$ is the prior PDF of $\boldsymbol{\theta}$; $p(\mathscr{D}/\boldsymbol{\theta},\mathscr{M})$ is the likelihood function; $p(\mathscr{D}/\mathscr{M})$ is the probability of the data given the model, evaluated as:

$$p(\mathscr{D}/\mathscr{M}) = \int_{\boldsymbol{\Theta}} p(\boldsymbol{\theta}/\mathscr{M})p(\mathscr{D}/\boldsymbol{\theta},\mathscr{M})d\boldsymbol{\theta} \tag{4}$$

In parametric identification, evaluation of Equation 4 is not necessary. If $\boldsymbol{\theta}$ are the stiffness parameters of an FE model, ones objective is to calculated the posterior PDF of these parameters from measurable data like modal data. A Bayesian framework is appropriate here because measuring the stiffness of a building or a bridge can be extremely strenuous, if not impossible. After forming the numerator of Equation 3, techniques like Laplace's method or Markov Chain Monte Carlo (MCMC) can be used to evaluate the moments of posterior PDF. Laplace's method will be used in the numerical and experimental implementations demonstrated in Sections 4 and 5, respectively.

3.3 Stiffness parameter updating using modal data

The objective is to update the stiffness matrix parameters formed through FE modeling using the estimated natural frequencies and mode shapes. These modal parameters may be obtained from dynamic data of the structural system through experimental or operational modal analysis. The measured natural frequencies and mode shapes are assumed to be related to parameters of the FE model as:

$$f_i^m = f_i(\boldsymbol{M},\boldsymbol{K}(\boldsymbol{\theta})) + \varepsilon_{f_i^m} \tag{5}$$

$$\boldsymbol{\phi}_i^m = \boldsymbol{\phi}_i(\boldsymbol{M},\boldsymbol{K}(\boldsymbol{\theta})) + \boldsymbol{\varepsilon}_{\boldsymbol{\phi}_i^m} \tag{6}$$

Here, $f_i^m \in \mathbb{R}$ and $\boldsymbol{\phi}_i^m \in \mathbb{R}^{N_D \times 1}$ are the estimated natural frequency and mode shape of the i^{th} mode; $f_i \in \mathbb{R}$ and $\boldsymbol{\phi}_i \in \mathbb{R}^{N_D \times 1}$ are the functions (undamped eigenvalue problem) that map a given set of values of $\boldsymbol{\theta}$ to the respective modal parameters; and $\varepsilon_{f_i^m} \in \mathbb{R}$ and $\boldsymbol{\varepsilon}_{\boldsymbol{\phi}_i^m} \in \mathbb{R}^{N_D \times 1}$ represents the uncertainties in each case. In Equations 5 and 6, $\boldsymbol{M} \in \mathbb{R}^{N_D \times N_D}$ and

$\boldsymbol{K}(\boldsymbol{\theta}) \in \mathbb{R}^{N_D \times N_D}$ are the mass and stiffness matrices coming from the FE model. N_D is the degree-of-freedoms of the FE model. $\varepsilon_{f_i^m}$ and $\boldsymbol{\varepsilon}_{\boldsymbol{\phi}_i^m}$ are usually modeled as Gaussian probability distributions: $\varepsilon_{f_i^m} \sim N(0, \sigma_i)$ and $\boldsymbol{\varepsilon}_{\boldsymbol{\phi}_i^m} \sim N(\mathbf{0}, \boldsymbol{C}_{\boldsymbol{\phi}_i^m})$. The Principal of Maximum Entropy [10] can be used as the justification for selecting Gaussian distributions here.

Finding the modal parameters for a given set of $\boldsymbol{\theta}$ is the forward problem. However, the objective here is to find the probability distribution of $\boldsymbol{\theta}$ from the estimated values of modal parameters, which is an inverse problem. From Equation 3, one can see that the Bayes' theorem converts the inverse problem to a forward problem and eliminates the difficulties in an inverse problem like ill-conditioning and ill-posedness. The goal is to find $p(\boldsymbol{\theta}/\mathscr{D}, \mathscr{M})$ from $p(\boldsymbol{\theta})$ and $p(\mathscr{D}/\boldsymbol{\theta}, \mathscr{M})$. In the current context, both $p(\boldsymbol{\theta}/\mathscr{M})$ and $p(\mathscr{D}/\boldsymbol{\theta}, \mathscr{M})$ are assumed to have a multivariable Gaussian form. The prior distribution, $p(\boldsymbol{\theta}/\mathscr{M}) \sim N(\boldsymbol{\theta}_0, \boldsymbol{C}_{\boldsymbol{\theta}})$, and can be expressed as:

$$p(\boldsymbol{\theta}/\mathscr{M}) = c_1 exp\left[-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \boldsymbol{C}_{\boldsymbol{\theta}}{}^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}_{\mathbf{0}})\right] \tag{7}$$

Here, $\boldsymbol{\theta}_0$ and $\boldsymbol{C}_{\boldsymbol{\theta}}$ are selected subjectively or come from a previous posterior estimation of $\boldsymbol{\theta}$. The prior term makes sure that the posterior estimates of $\boldsymbol{\theta}$ are close to $\boldsymbol{\theta}_0$. The degree of closeness depends on $\boldsymbol{C}_{\boldsymbol{\theta}}$. For N_m modes and independent modal data, the likelihood function is:

$$p(\mathscr{D}/\boldsymbol{\theta}, \mathscr{M}) = \prod_{i=1}^{N_m} p(f_i/\boldsymbol{\theta}, \mathscr{M}) p(\boldsymbol{\phi}_i/\boldsymbol{\theta}, \mathscr{M}) \tag{8}$$

where

$$p(f_i/\boldsymbol{\theta}, \mathscr{M}) = c_2 exp\left[-\frac{1}{2}\left(\frac{f_i^m - f_i(\boldsymbol{\theta})}{\sigma_i}\right)^2\right] \tag{9}$$

$$p(\boldsymbol{\phi}_i/\boldsymbol{\theta}, \mathscr{M}) = c_3 exp\left[-\frac{1}{2}(\boldsymbol{\phi}_i^m - \boldsymbol{\phi}_i(\boldsymbol{\theta}))^T \boldsymbol{C}_i^{-1}(\boldsymbol{\phi}_i^m - \boldsymbol{\phi}_i(\boldsymbol{\theta}))\right] \tag{10}$$

In Equations 9 and 10, the dependence of fi and $\boldsymbol{\phi}_i$ on the FE matrices have not been shown for conciseness. Once the numerator of Equation 3 has been formed, the moments of $p(\boldsymbol{\theta}/\mathscr{D}, \mathscr{M})$ can be estimated using the methods previously mentioned in Section 3.2. If the moments corresponding to the damaged and undamaged states are estimated, the probability of damage can be calculated [11]. This procedure will be discussed in Section 5 where an attempt made to estimate the probabilistic damage measure values for a laboratory model. It should be further noted that multiple possibilities of $\boldsymbol{\theta}$ exist, and the relative plausibility of each of these possibilities, given the data and model class, is given by the probabilities in the posterior PDF.

Only a glimpse of Bayesian FEMU had been shown here. Some of the other nuances are as follows 1) in practice, partial mode shapes are used instead of full mode shapes, 2) the mass and damping matrices may also be parameterized in addition to the stiffness matrix, 3) other forms of dynamic data such as accelerations and displacements may also be used as $\mathscr{D}$, 4) σ_i and $\boldsymbol{C}_i$ may also be included in $\boldsymbol{\theta}$ as parameters to be estimated, and 5) more complicated models relating to various measured data can be incorporated.

4. Numerical example of Bayesian system identification

Bayesian system identification of a numerical single-degree-of-freedom (SDOF) model (Fig. 1) is demonstrated here. The mathematical model used for identification is:

$$m\ddot{x}(t) + c\dot{x}(t) + k_s x(t) = f(t) \tag{11}$$

$$y(t) = \ddot{x}(t) + e(t) \tag{12}$$

Equations 11 and 12 are the equation of motion and measurement equation, respectively. In Equation 11, m, c, k_s, $x(t)$, and $f(t)$ represent the mass, damping, stiffness, displacement of mass, and input force, respectively, while in Equation 12, $y(t)$ are the noise-contaminated acceleration measurements. The values of the model parameters and the degree of freedom are shown in Figs. 2a and 2b, respectively. In Equations 11 and 12, the input force, $f(t)$, and the channel noise, $e(t)$, are stochastic signals and are responsible for the uncertainty. These signals are assumed to be zero-mean Gaussian white noise. The spectral densities used for the $f(t)$ and $e(t)$ are $4N^2/Hz$ (S_f) and $1(\mu g)^2/Hz$ (S_e), respectively. The spectral density can be considered a parameter that characterizes the stochastic signal, like a random variable's mean or standard deviation. From the noise-contaminated acceleration data obtained from the forward problem, the aim is to identify the possible values that m, c, k_s, S_f, and S_e can take. The sampling used was 100 Hz, and 600s of data were simulated.

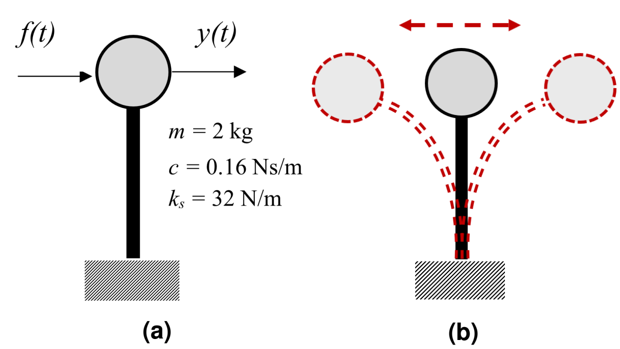

Figure 2. The numerical SDOF model used for the example. (a) Model parameters and (b) degree of freedom

4.1 Details of the formulation

The Bayesian system identification formulation used here is a modified version of the formulation given in [12]. An improper prior was used, i.e., the prior was absorbed as a constant term along with the term representing the probability of the data. One interpretation of improper priors is that it can be thought of as a uniform distribution over the set of possible values that the parameters can take. The parameters to be estimated, $\boldsymbol{\theta}$, consist of m, c, ks, S_f, and S_e. The posterior distribution of $\boldsymbol{\theta}$ is given as:

$$p(\boldsymbol{\theta}/Y) = c_4 p(Y/\boldsymbol{\theta}) \tag{13}$$

where Y is the data used in the Bayesian estimation. In Equation 13, the dependence on the model class has not been shown. Scaled discrete Fourier transform (DFT) values of the noise-contaminated acceleration were used as data. The scaled DFT of acceleration, Y_k, at the k^{th}frequency is defined as follows:

$$Y_k = \sqrt{\frac{\Delta t}{N_d}} \sum_{j=0}^{N_d-1} y_j e^{\frac{-2\pi \mathbf{i} jk}{N_d}} \tag{14}$$

In Equation 14, Δt is the sampling time; y_j is the j^{th} discrete acceleration measurement; N_d is the total number of discrete acceleration data points used and $\mathbf{i}$ is the imaginary unit. The k^{th} frequency f_k in Hz is $k/N_d\Delta t$. The scaled DFT of acceleration, Y_k, can also be expressed as:

$$Y_k = H_k F_k + E_k \tag{15}$$

In the above equation, H_k is the frequency response function (FRF); F_k is the scaled DFT of input force and E_k is the scaled DFT of channel noise. The FRF, in this case is:

$$H_k = \frac{-(2\pi f_k)^2}{-m(2\pi f_k)^2 + ic(2\pi f_k) + k_s} \tag{16}$$

The scaled DFT values, Y_k, are complex numbers and are assumed to have a complex Gaussian distribution, i.e., their real and imaginary parts have a joint Gaussian distribution. The justification for assuming such a distribution is given in [13] and is primarily attributed to the central limit theorem. The probability distribution of Y_k under the assumptions of complex Gaussian and circular symmetry is given as:

$$p(Y_k/\boldsymbol{\theta}) = \frac{1}{\pi G_k(\boldsymbol{\theta})} exp\left[\frac{-Y_k Y_k^*}{G_k(\boldsymbol{\theta})}\right] \tag{17}$$

where,

$$G_k(\boldsymbol{\theta}) = \mathbb{E}[Y_k Y_k^*] = D_k S_f + S_e \tag{18}$$

In Equation 18, $\mathbb{E}$ is the expectation operator, and the expression of Y_k from Equation 15 is substituted. $Y_k{}^*$ is the complex conjugate of Y_k. Further, the input force and the channel noise are assumed to be independent of each other. The expanded form of D_k from the above equation is:

$$D_k = \frac{(2\pi f_k)^4}{\left(-m(2\pi f_k)^2 + k_s\right)^2 + \left(c(2\pi f_k)\right)^2} \tag{19}$$

If Yks are independent, the likelihood function can be formed as:

$$p(Y/\boldsymbol{\theta}) = \prod_{k=1}^{N_f} p(Y_k/\boldsymbol{\theta}) \tag{20}$$

In Equation 20, N_f is the number of the significant scaled DFT values used. These significant values can be selected by plotting the magnitude of the scaled DFTs. To get a more mathematically tractable form, we take the negative natural log of Equation 20:

$$L(\boldsymbol{\theta}) = N_f ln(\pi) + \sum_{k=1}^{N_f} ln(D_k S_f + S_e) + \sum_{k=1}^{N_f} \frac{Y_k Y_k^*}{D_k S_f + S_e} \tag{21}$$

$L(\boldsymbol{\theta})$ is called as the negative log-liklihood function (NLLF). From $L(\boldsymbol{\theta})$, the most probable value (MPV) of $\boldsymbol{\theta}$ is estimated as:

$$\boldsymbol{\theta}^* = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} L(\boldsymbol{\theta}) \tag{22}$$

After the formation of the NLLF, $\boldsymbol{\theta}^*$ can be estimated in MATLAB by using the built-in function *fminsearch*. This estimate of $\boldsymbol{\theta}^*$ is equivalent to the estimate from the Maximum likelihood estimation (MLE), a classical estimation technique. However, unlike classical estimation techniques, we will be quantifying the uncertainty too here, a trademark of Bayesian methods.

For estimating the posterior covariance matrix of $\boldsymbol{\theta}$, Laplace's method is used. In the Laplace's method, if $p(\boldsymbol{\theta}/Y)$ has a unique maximum, one can use a Gaussian approximation around $\boldsymbol{\theta}^*$ to find the covariance matrix of $\boldsymbol{\theta}$. Generally, when one considers many Y_ks, $p(\boldsymbol{\theta}/Y)$ tends to have a unique maximum. In Laplace's method, the posterior covariance matrix, $\boldsymbol{C_\theta}$, is estimated as the inverse of the Hessian matrix, $\boldsymbol{H}$, of the NLLF evaluated at $\boldsymbol{\theta}^*$.

$$\boldsymbol{C_\theta} = \boldsymbol{H}_{\boldsymbol{\theta}^*}^{-1} \tag{23}$$

The Hessian matrix, $\boldsymbol{H}_{\boldsymbol{\theta}^*}$, from Equation 23 can be evaluated by using finite difference approximations [4]. The estimated $\boldsymbol{\theta}^*$ and $C_{\boldsymbol{\theta}}$ can be considered as the mean and covariance matrix of an equivalent multivariable Gaussian distribution, representing $p(\boldsymbol{\theta}/Y)$.

4.1.1 Initial guess of parameters

Before we can solve for the MPVs, we need to set an initial guess of $\boldsymbol{\theta}$ to start the optimization algorithm needed for Equation 22. In a real-life scenario, the mass of the SDOF system can be measured and used as the initial guess for the mass, m^g. An estimate of the natural frequency, f_n^g, of the system can be roughly identified from the peak of the magnitude plot of the scaled DFTs. From m^g and f_n^g, we can calculate the initial guess for stiffness, k_s^g, as:

$$k_s^g = (2\pi f_n^g)^2 m^g \tag{24}$$

For the damping parameter, the initial guess, c^g, is evaluated as:

$$c^g = 2m^g(2\pi f_n^g)\xi^g \tag{25}$$

In Equation 25, ξ^g can be chosen to be 1%. The initial guess for the spectral density of the input force can be calculated as:

$$S_F^g = \sum_{k=1}^{N_f} \frac{Y_k Y_k^*}{N_f D_k} \tag{26}$$

Equation 26 is derived for the case of high signal-to-noise ratio. This equation can be derived by taking the partial derivative of Equation 21 with respect to S_f and then equating it to zero after neglecting Se. Unfortunately, a similar mathematically tractable result cannot be derived for the initial guess of channel noise, S_e^g. The value of S_e^g can be calculated by forming a separate MLE or Bayesian estimation problem.

4.2 Implementation on simulated data

For the formation of the NLLF in the numerical SDOF case, 208 Y_ks are selected around the peak of the scaled DFT values of acceleration data. The magnitudes of these Y_ks are shown in Fig. 2. The initial guesses of m^g, f_n^g, and S_e^g are taken as 1.9kg, 0.63Hz, and $1.1(\mu g)^2$/Hz, respectively. The initial guesses of the remaining parameters are calculated from m^g and f_n^g by using the expressions from Section 4.1.2. The values of k_s^g, c_g, and S_f^g were 29.77N/m, 0.15Ns/m, and 3.38N/Hz, respectively. The MPVs and the covariance matrix are evaluated according to the method discussed in Section 4.1. Table 1 lists the mean values (MPVs) and coefficient of variances (COV) that are identified along with the actual values.

The COV is the ratio of standard deviation to the mean. The identified marginal PDFs of the system's physical parameters (m, c, and k_s) are shown in Fig. 3. From 1, the damping parameter indicated the maximum uncertainty among the various physical parameters of the system. This is because the NLLF formed from Equations 11 and 12 is more sensitive to damping, which explains, in part, why damping estimates always show high variability. The channel noise parameter, S_e, showed an extremely high uncertainty. This high uncertainty value could be due to the numerical issues that arise while calculating the covariance matrix by Laplace's method using the finite difference method. MCMC methods can eliminate this problem. It is generally advised to formulate the Bayesian estimation problem by using dimensionless parameters, like in Section 5, to mitigate neumerical issues. Bayesian methods can also be used to identify a multi-degree-of-freedom systems' mass, damping, and stiffness matrices.

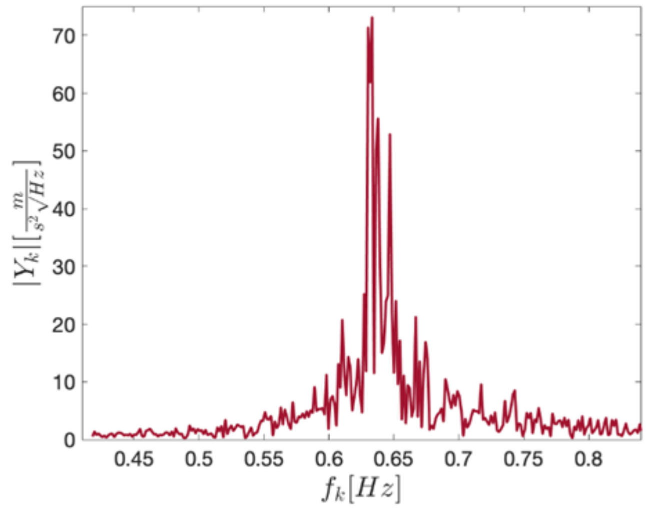

Figure 3. The magnitudes of the scaled DFTs, i.e., Y_ks used for forming the NLLF

5. Experimental implementation of Bayesian FEMU for damage detection

Bayesian FEMU is used to detect damage in a four-story shear building experimental model (Fig. 5). The columns of the model are fabricated from mild steel plates of thickness 2mm and 60mm width. The model had a total height of 1286mm, and the floor masses were fabricated from 20mm thick mild steel plates.

Before performing any dynamic tests, the stiffness' at the various floor levels are estimated using a static-pullback test in the undamaged state. The estimated floor stiffness' is shown in Table 2. In Table 2, k_{fi} corresponds to the floor stiffness estimated for the i^{th} floor. The measured stiffness values may be used to form the prior; however, here, these values are used to form the analytical stiffness matrix and the likelihood function instead. The first objective is to find the PDFs of the stiffness parameters in both undamaged and damaged states using the estimated modal parameters and Bayesian FEMU. Then, using the statistical properties of these PDFs, one can attempt to detect damage. The damaged state here corresponds to the state in which two bolts were loosened at each side of the second-floor columns (see Fig. 5). The modal identification is performed separately for each state using the stochastic subspace identification [14] on the recorded ambient acceleration data. The data were recorded by four piezo-electric accelerometers (PCB 393B04) placed on each floor, as illustatrated in Fig. 6 for around eight minutes at 2000Hz sampling (later down sampled to 200Hz). A pedestal fan is used to simulate ambient input force. Only the identified natural frequencies and the full displacement mode shapes of the four dominant modes are used as data for Bayesian updating. The results from the modal identification of the undamaged state, i.e., natural frequencies, damping ratios and mode shapes are shown in Fig 7. The results from the damaged states have not been shown for brevity.

5.1 Details of the formulation

The analytical relationship used here for updating is the undamped eigen value equation. The analytical global stiffness matrix, $\boldsymbol{K}(\boldsymbol{\theta})$, is parametrized as:

$$\boldsymbol{K}(\boldsymbol{\theta}) = \sum_{i=1}^{4} \theta_i \boldsymbol{K}_i \tag{27}$$

where $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3, \theta_4\}^T$ are the dimensionless stiffness parameters and the matrix, $\boldsymbol{K}_i$, is the contribution of i^{th} floor stiffness to the global stiffness matrix , formed conforming to shear building assumptions. The data from Table 2 are used

Table 1. Identified means and COVs with actual values

	m	c	k_s	S_f	S_e
Actual	2 kg	0.16Ns/m	32N/m	$4N^2$/Hz	$1(\mu g)^2$/Hz
Mean	1.92 kg	0.13Ns/m	30.8N/m	$3.32N^2$/Hz	$1.17(\mu g)^2$/Hz
COV %	0.31	7.23	0.03	0.26	1.29×10^5

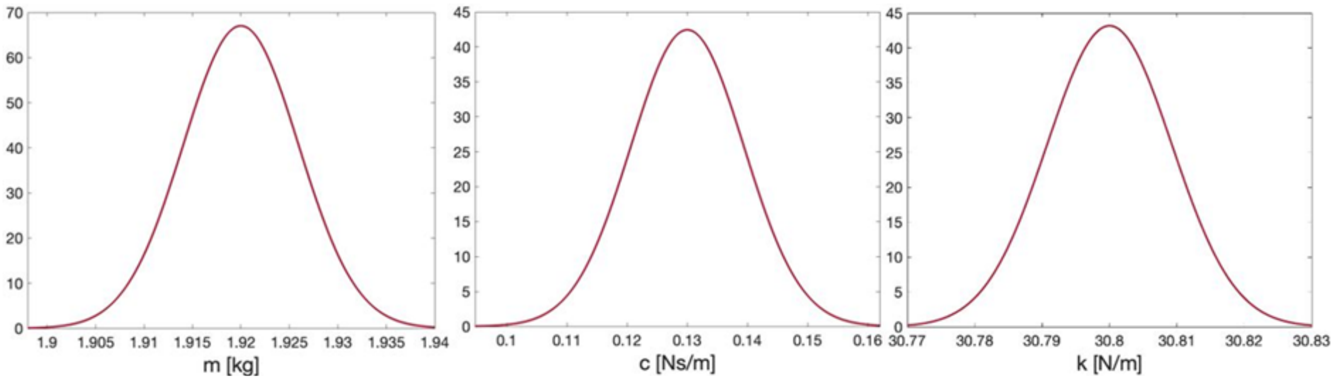

Figure 4. Identified marginal PDFs of m, c, and k_s

Table 2. Estimated floor stiffness values of the undamaged state from static pull test

	k_{f1}	k_{f2}	k_{f3}	k_{f4}
Estimated floor stiffness values (kN/m)	10.33	10.40	6.73	7.71

for forming the $\boldsymbol{K}_i$s. For example, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$are:

$$\boldsymbol{K}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.33 \end{bmatrix} kN/m, \tag{28}$$

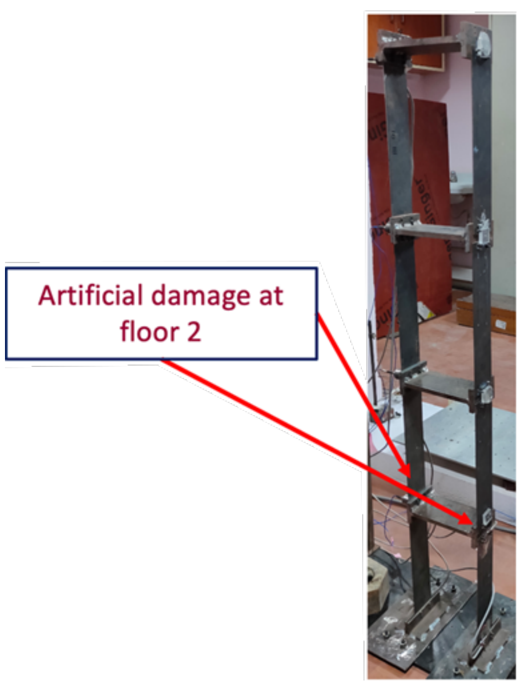

Figure 5. The experimental model used

$$\boldsymbol{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -10.40 & -10.40 \\ 0 & 0 & -10.40 & 10.44 \end{bmatrix} kN/m \tag{29}$$

The analytical global mass matrix used is a consistent mass matrix formed by weighing the individual components of the experimental model. A consistent mass matrix is used instead

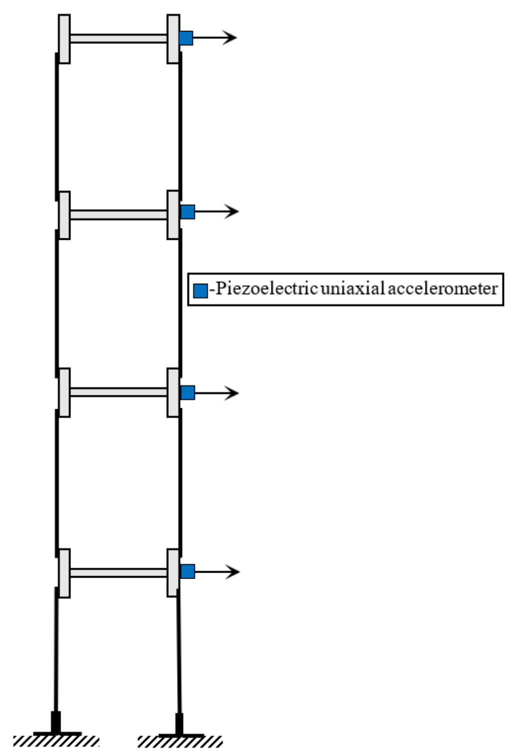

Figure 6. The location of the accelerometers

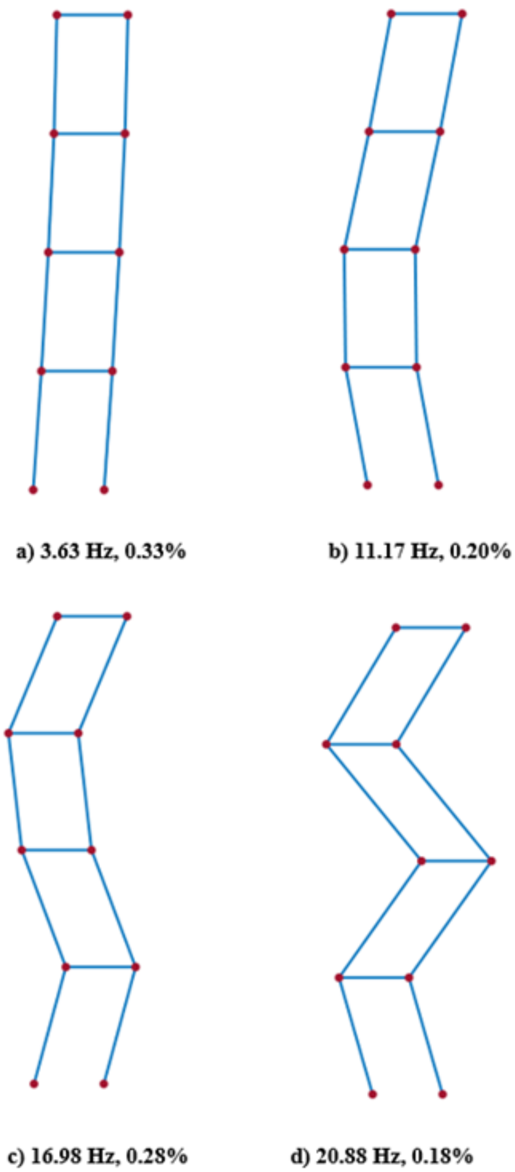

Figure 7. The identified modal parameters of the undamaged state

of a traditional lumped mass matrix because it better agrees with the results obtained from the modal analysis. This could be because the columns' mass is not much smaller compared to the floor masses.

Like in the numerical example from Section 3, here also, an improper prior is used. The likelihood function is formed in the same manner discussed in Section 3.3. The likelihood function has a Gaussian form, and the identified natural frequencies and displacement mode shapes are used as f_i^m and ϕ_i^m, respectively. σ_is and the diagonal matrix $\boldsymbol{C_i}$ are formed as:

$$\sigma_i = \frac{1}{100} f_i^m \tag{30}$$

$$diag(\boldsymbol{C}_i) = \left\{ \left(\frac{1}{100}\phi_{1i}^m\right)^2, \left(\frac{1}{100}\phi_{2i}^m\right)^2, \left(\frac{1}{100}\phi_{3i}^m\right)^2, \left(\frac{1}{100}\phi_{4i}^m\right)^2 \right\}^T \tag{31}$$

Equations 30 and 31 were formed by assuming COVs of 1%, a reasonable assumption for natural frequencies and mode shapes. In Equation 31, ϕ_{ji}^m stands for the estimated displacement mode shape amplitude at the j^{th} degree-of-freedom corresponding to the i^{th} mode. The mode shapes were normalized such that the amplitude at the fourth floor is unity.

For finding the MPVs (mean) of the dimensionless stiffness parameters ($\boldsymbol{\theta}$), the NLLF function is minimized. While the covariance matrix of the $\boldsymbol{\theta}$ is evaluated by using Laplace's method, which was explained in Section 4.1.

5.2 Damage detection

The mean and covariance matrix of, $\boldsymbol{\theta}$, has to be estimated for both the damaged and undamaged states. The estimated natural frequencies and displacement mode shapes corresponding to each state are used as data while forming the respective likelihood functions. For minimizing the NLLF of the undamaged state, the initial values of all parameters are chosen to be unity. While for the damaged case, the mean values estimated after updating in the undamaged case were used as the initial values. The means and COVs estimated for both the states are listed in Table 3. The estimated means and standard deviations were used for calculating the probabilistic damage measures for each floor, as shown in Fig 8. The probabilistic damage measure is calculated by using the following equation [4, 11]:

$$p^{dam}(d) = \boldsymbol{\Phi}\left[\frac{(1-d)\theta_{und}^* - \theta_{dam}^*}{\sqrt{(1-d)^2(\sigma_{und})^2 + (\sigma_{dam})^2}}\right] \tag{32}$$

Table 3. Identified mean and COV of the non-dimensional stiffness parameters

		θ_1	θ_2	θ_3	θ_4
Undamaged state	Mean	1.04	0.88	1.36	1.27
	COV%	0.71	0.72	0.73	0.72
Damaged state	Mean	0.99	0.74	1.34	1.27
	COV%	0.73	0.75	0.75	0.74

$p^{dam}(d)$ is the probabilistic damage measure which is the probability of exceedance of a damage level, d (reduction of the stiffness parameter). The curves in Fig. 6 should be interpreted as follows. For floor 2, the probability that damage is 18% or more is around 0.5. While the prbability that damage is 15% or more is 1. Equation 32 approximates the various integrals involved. In Equation 32, $\boldsymbol{\Phi}$ represents the cumulative distribution function of the standard Gaussian PDF; θ_{und}^* and θ_{dam}^* are the means of the stiffness parameters of the undamaged and damaged states, respectively; and σ_{und} and σ_{dam} are the standard deviations of the undamaged and damaged states, respectively. Fig. 8 demonstrated clear-cut evidence of damage on the second floor, and the first three objectives of SHM are achieved through this figure.

6. Conclusions

The Bayesian probability provides a rigorous framework for SHM by identifying the uncertainties in the model parameters. By employing

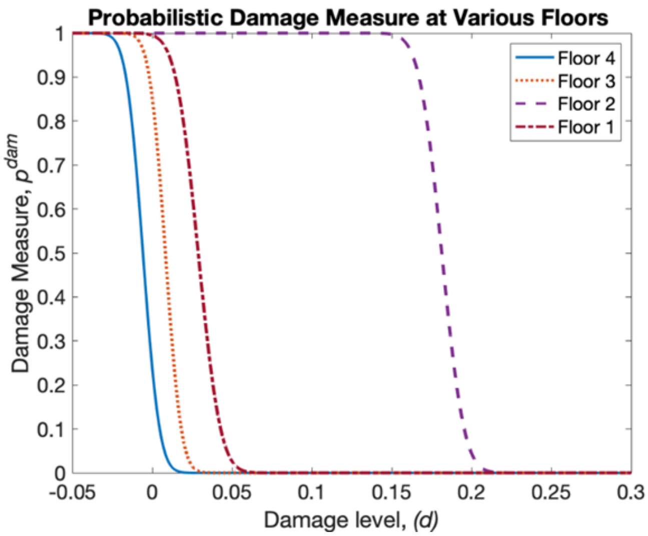

Figure 8. Plot of damage measure

Bayesian framework, the SHM scheme is made more robust as this helps understand whether the changes in the values of the parameters are due to uncertainty or any actual damage. The various advantages of Bayesian updating are discussed. Numerical and experimental examples are provided to help the reader to appreciate the capabilities of Bayesian methods. It is envisioned that this article, as an excellent starting point, would encourage practicing engineers to delve into SHM and probabilistic methods.

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