

# Intro to Regression

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- 2 Linear Regression
- 3 Ridge Regression
- 4 LASSO
- 5 Probabilistic model for linear regression
- 6 Kernel Ridge Regression
- 7 Dessert for the most curious students: Elastic Net

## 1 Regression Problem

2 Linear Regression

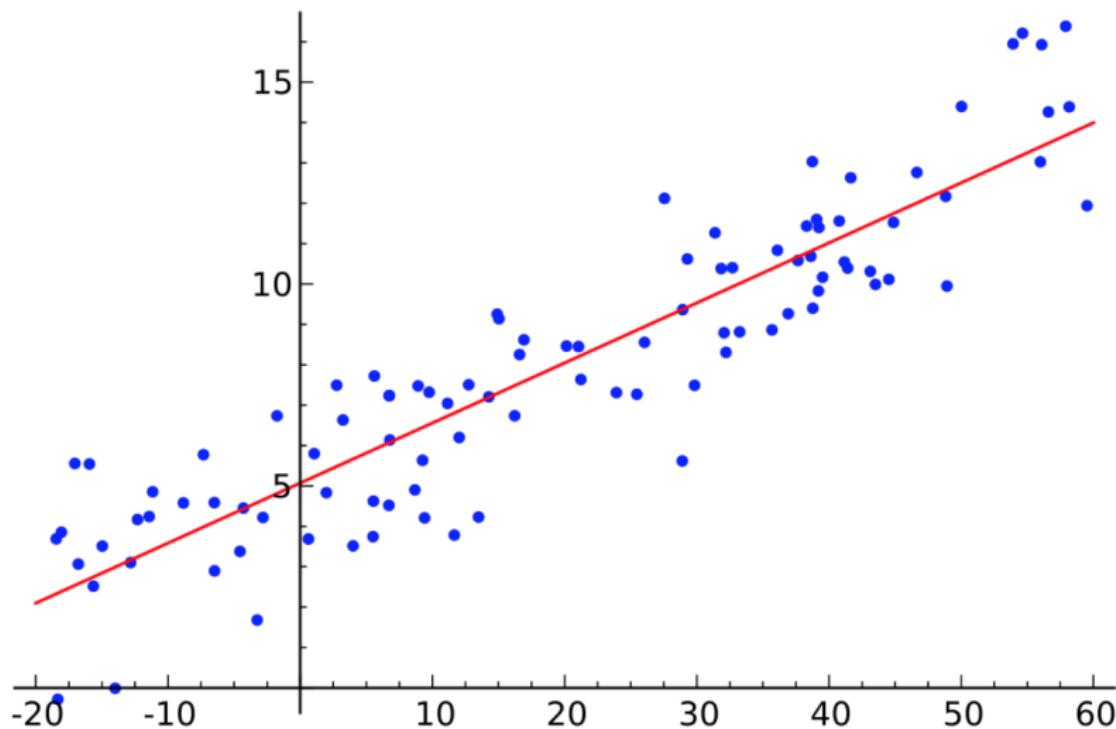
3 Ridge Regression

4 LASSO

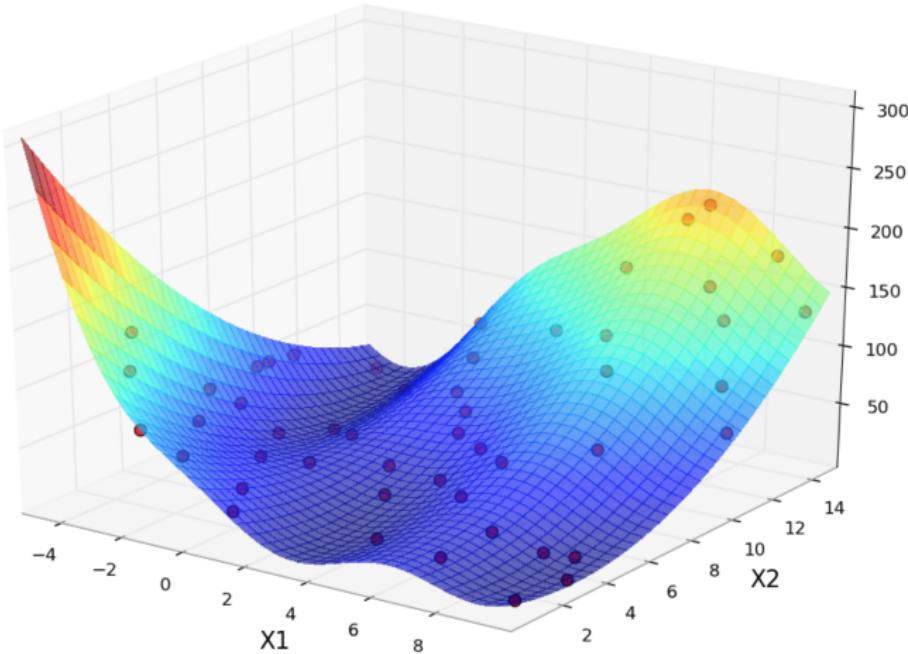
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## Branin function approximation: model prediction



- **Training data:** sample drawn i.i.d. from set  $X$  according to some distribution  $D$

$$S = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_m, y_m)\} \in X \times Y,$$

with  $Y \subseteq \mathbb{R}$  is a measurable set,  $X \subseteq \mathbb{R}^d$ ,  $\mathbf{x}_i \in \mathbb{R}^d$

- **Loss function:**  $L : Y \times Y \rightarrow \mathbb{R}_+$  a measure of closeness, e.g.  
 $L(y, y') = (y - y')^2$  or  $L(y, y') = |y - y'|^p$  for some  $p \geq 1$
- **Problem:** find a hypothesis  $\hat{f} : X \rightarrow \mathbb{R}$  in  $\mathbb{H}$  with small generalization error w.r.t. target  $f$

$$R_D(\hat{f}) = \mathbb{E}_{\mathbf{x} \sim D}[L(\hat{f}(\mathbf{x}), f(\mathbf{x}))]$$

- Empirical error:

$$\hat{R}_S(f) = \frac{1}{m} \sum_{i=1}^m L(f(\mathbf{x}_i), y_i)$$

- In much of what follows:
  - $Y = \mathbb{R}$  or  $Y = [-M, M]$  for some  $M > 0$
  - $L(y, y') = (y - y')^2$  is a mean squared error

- **Object**  $x$ : place to open a new restaurant
- **Label**  $y$ : revenue after one year of operation
- **Features**: demographic properties of a considered city district, prices for real estate in a local neighborhood, availability of offices nearby, etc.
- **Challenges:**
  - small sample size
  - a lot of features ( $d \gg 1$ )
  - outliers/incorrect measurements
  - non-homogeneous data (big cities vs. local towns)

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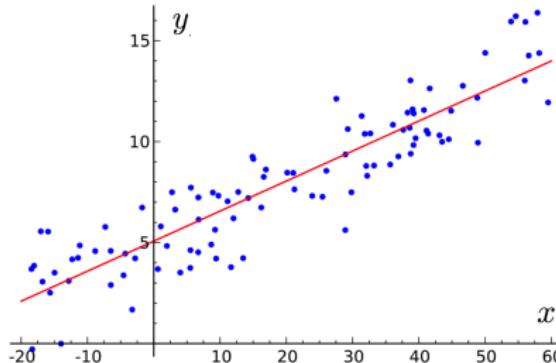
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- Hypothesis set: linear functions

$$\mathbb{H} = \{\mathbf{x} \rightarrow \mathbf{x}^\top \mathbf{w} + b : \mathbf{w} \in \mathbb{R}^d, b \in \mathbb{R}\}$$

- **Optimization problem:** empirical risk minimization

$$F(\mathbf{w}, b) = \frac{1}{m} \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 \rightarrow \min_{\mathbf{w}, b}$$



- Rewrite the objective function as  $F(\mathbf{w}) = \frac{1}{m} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$ , where

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & 1 \\ \vdots & \vdots \\ \mathbf{x}_m & 1 \end{bmatrix} \in \mathbb{R}^{m \times (d+1)}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_d \\ b \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

- Convex and differentiable function

$$\nabla F(\mathbf{w}) = \frac{2}{m} \mathbf{X}^\top (\mathbf{X}\mathbf{w} - \mathbf{y})$$

$$\nabla F(\mathbf{w}) = 0 \Leftrightarrow \mathbf{X}^\top (\mathbf{X}\mathbf{w} - \mathbf{y}) = 0 \Leftrightarrow \mathbf{X}^\top \mathbf{X}\mathbf{w} = \mathbf{X}^\top \mathbf{y}$$

- **Solution:**

$$\mathbf{w} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} \text{ if } \mathbf{X}^\top \mathbf{X} \text{ invertible}$$

- Computational complexity:  $O(md^2 + d^3)$  for  $O(d^3)$  matrix inversion
- Poor guarantees in general, no regularization
- For multivariate output labels in  $\mathbb{R}^{d_y}$ ,  $d_y > 1$ , solve  $d_y$  distinct linear regression problems

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- **Optimization problem:**

$$F(\mathbf{w}, b) = \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 + \lambda \|\mathbf{w}\|^2 \rightarrow \min_{\mathbf{w}, b}$$

where  $\lambda \geq 0$  is a regularization parameter

- **Benefits:**

- directly based on generalization bound (**strict result!**)
- generalization of linear regression
- closed-form solution
- can be used with kernels (**may be later**)

- Assume  $b = 0$ : often a constant feature is used
- Rewrite the objective function as

$$F(\mathbf{w}) = \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2 + \lambda\|\mathbf{w}\|^2.$$

- Convex and differentiable function

$$\begin{aligned}\nabla F(\mathbf{w}) &= 2\lambda\mathbf{W} + 2\mathbf{X}^\top(\mathbf{X}\mathbf{w} - \mathbf{y}) \\ \nabla F(\mathbf{w}) = 0 &\Leftrightarrow (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})\mathbf{w} = \mathbf{X}^\top\mathbf{y}\end{aligned}$$

- **Solution:**

$$\mathbf{w} = \underbrace{(\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}}_{\text{always invertible!}} \mathbf{X}^\top\mathbf{y}$$

- We can easily prove that

$$(\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^\top = \mathbf{X}^\top (\mathbf{X} \mathbf{X}^\top + \lambda \mathbf{I})^{-1}$$

- **Dual solution:** thus we get that

$$\mathbf{w} = (\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^\top \mathbf{y} = \mathbf{X}^\top \underbrace{(\mathbf{X} \mathbf{X}^\top + \lambda \mathbf{I})^{-1} \mathbf{y}}_{\text{new variable } \alpha}$$

- With

$$\alpha = (\mathbf{X} \mathbf{X}^\top + \lambda \mathbf{I})^{-1} \mathbf{y},$$

we can represent  $\mathbf{w}$  as

$$\mathbf{w} = \mathbf{X}^\top \alpha = \sum_{i=1}^m \alpha_i \mathbf{x}_i^\top,$$

- We can use the dual representation of the solution

$$\hat{f}(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} = \sum_{i=1}^m \alpha_i (\mathbf{x}_i^\top \mathbf{x}).$$

Type	Solution	Prediction
Primal	$O(md^2 + d^3)$	$O(d)$
Dual	$O(\kappa m^2 + m^3)$	$O(\kappa m)$

Here  $\kappa$  denotes the time complexity of computing a dot product;  
Euclidian dot product  $\kappa = O(d)$ .

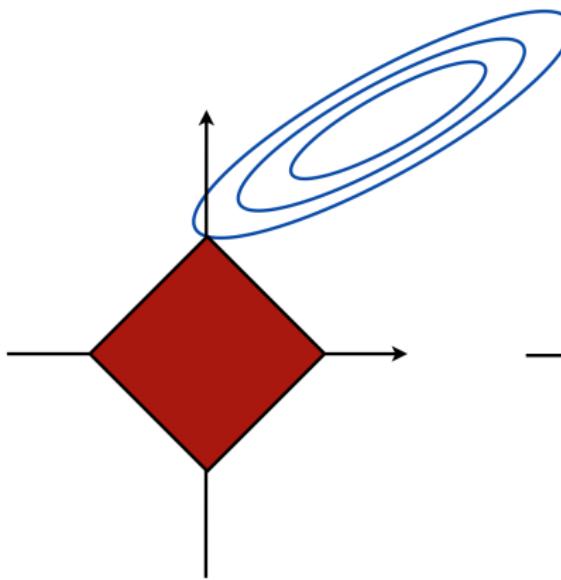
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- **Optimization problem:** “Least Absolute Shrinkage and Selection Operator”

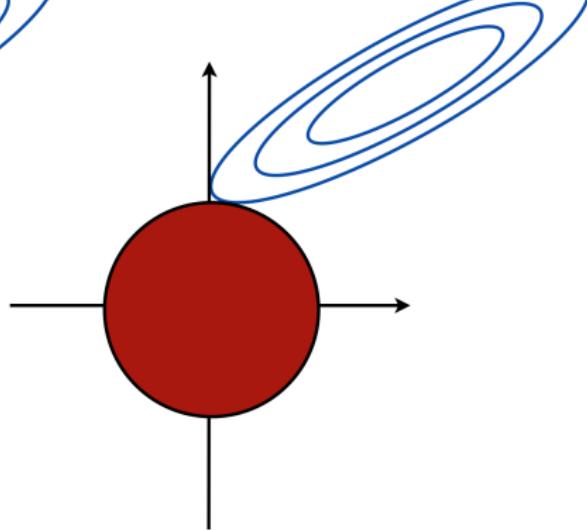
$$F(\mathbf{w}, b) = \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 + \lambda \|\mathbf{w}\|_1 \rightarrow \min_{\mathbf{w}, b},$$

where  $\lambda \geq 0$  is a regularization parameter

- **Solution:** equivalent convex quadratic programming (QP)
  - general: standard QP solvers
  - specific algorithms: LARS (least angle regression procedure), entire path of solution



$L_1$  regularization



$L_2$  regularization

- Advantages
  - strong theoretical guarantees
  - sparse solution
  - feature selection
- Disadvantages
  - no natural use of kernels (next slides!)
  - no closed-form solution (not necessary, but can be convenient for theoretical analysis)
- **Empirical recipe** to provide better prediction performance:
  - First, perform variable selection using LASSO
  - Second, re-estimate model parameters with selected variables using Ridge Regression

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- Our hypothesis is wrong. We'll always have errors:

$$y \neq \mathbf{x}^\top \mathbf{w}.$$

- Our hypothesis with an error:

$$y = \mathbf{x}^\top \mathbf{w} + \varepsilon, \varepsilon \sim \mathcal{N}(0, \sigma^2).$$

- Then the conditional density for  $y \sim \mathcal{N}(\mathbf{x}^\top \mathbf{w}, \sigma^2)$  is:

$$p(y|\mathbf{w}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - \mathbf{x}^\top \mathbf{w})^2\right).$$

- The logarithm of the conditional density has a familiar form:

$$\log p(y|\mathbf{x}, \mathbf{w}) = \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sigma^2}(y - \mathbf{x}^\top \mathbf{w})^2$$

- The logarithm of the conditional density has the form:

$$\log p(y|\mathbf{x}, \mathbf{w}) = \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sigma^2}(y - \mathbf{x}^\top \mathbf{w})^2$$

- The conditional density for a sample  $S$  has the form:

$$p(\mathbf{y}|\mathbf{X}, \mathbf{w}) = \prod_{i=1}^m p(y_i|\mathbf{x}_i, \mathbf{w})$$

- The conditional density for a sample  $S$  has the form:

$$\log p(\mathbf{y}|\mathbf{X}, \mathbf{w}) = \sum_{i=1}^m \log p(y_i|\mathbf{x}_i, \mathbf{w}) = m \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sigma^2} \sum_{i=1}^m (y_i - \mathbf{x}_i^\top \mathbf{w})^2$$

- Our optimization problem is the following (the likelihood maximization)

$$\log p(\mathbf{y}|\mathbf{X}, \mathbf{w}) \rightarrow \max_{\mathbf{w}} \Leftrightarrow \frac{1}{m} \sum_{i=1}^m (y_i - \mathbf{x}_i^\top \mathbf{w})^2 \rightarrow \min_{\mathbf{w}}$$

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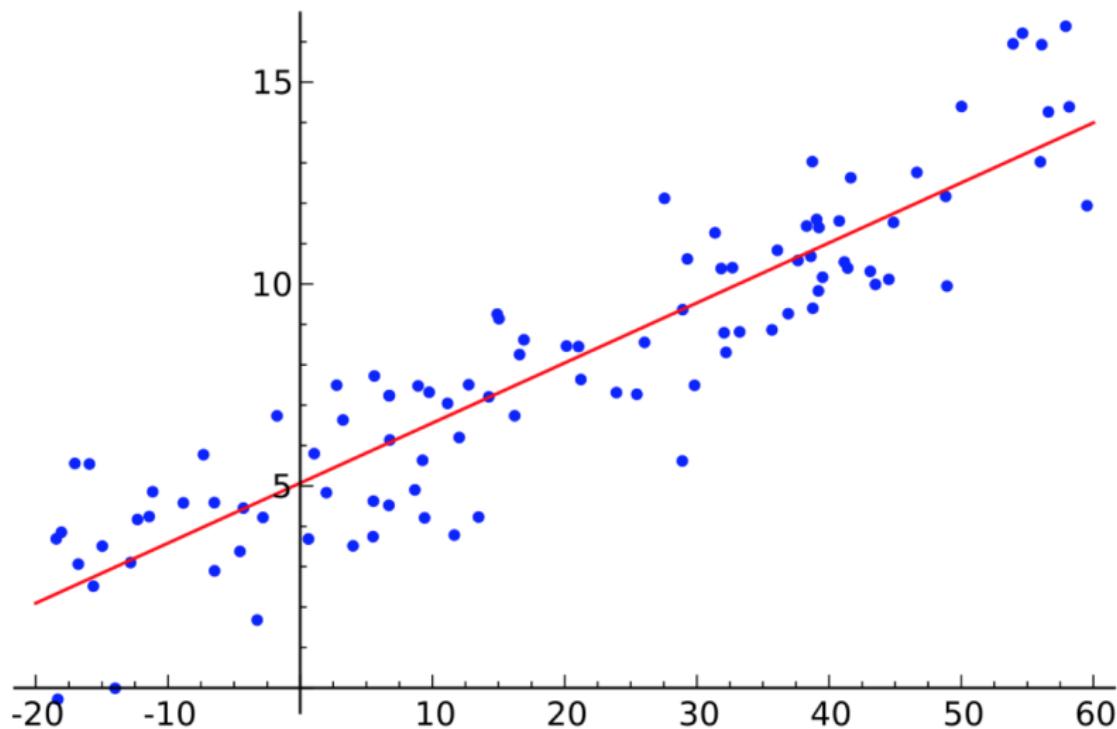
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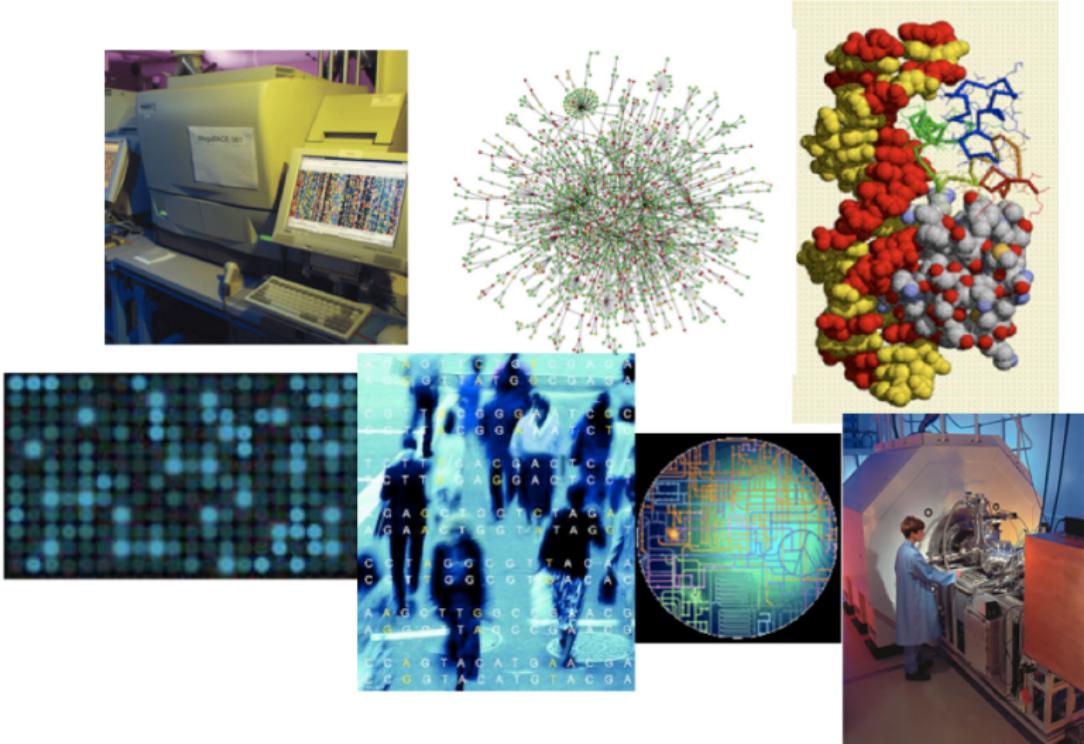
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## What we know how to solve

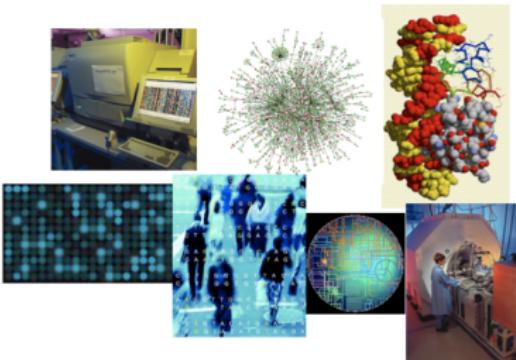
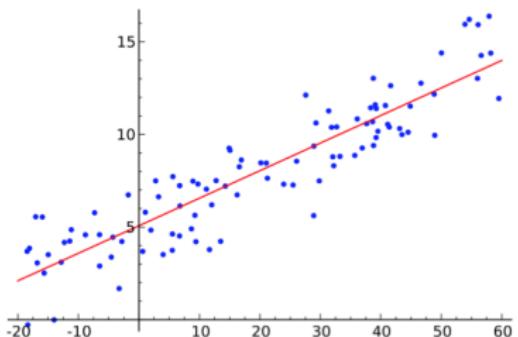


But real data are often more complicated ...



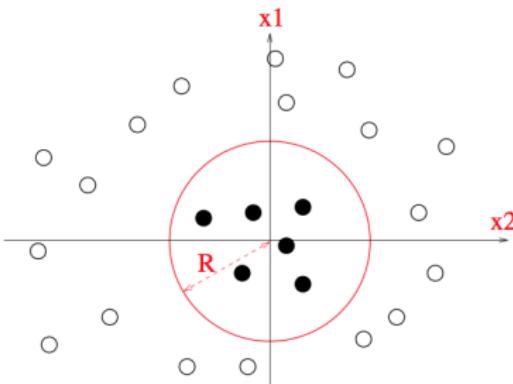
# Main goal of using kernels

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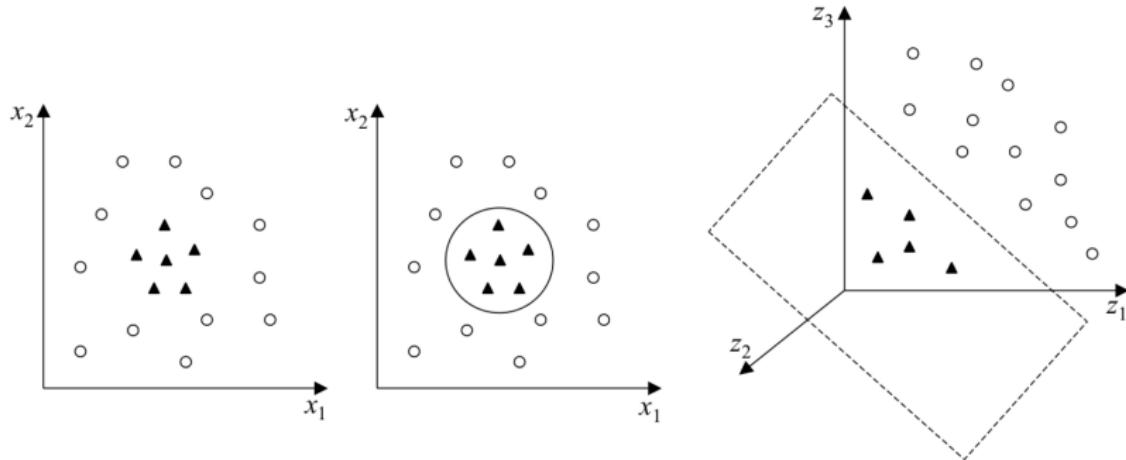
Show some classic examples of how to extend well-understood, linear statistical learning techniques to real-world, complicated, structured, high-dimensional data (texts, time series, graphs, distributions, permutations, ...)

- Efficient computation of inner products in high dimension
- Non-linear decision boundary
- Learning with non-vectorial inputs
- More informative features
- Kernels allow to perform pairwise comparisons



- Linear separation is impossible in most problems
- Non-linear mapping  $\Phi : X \rightarrow \mathbb{H}$  from input space to high-dimensional feature space
- Generalization ability: independent of  $\dim(\mathbb{H})$ , depends only on  $d$  and  $m$

## Example: polynomial kernel



For  $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ , let  $\Phi(\mathbf{x}) = (x_1^2, \sqrt{2}x_1x_2, x_2^2) \in \mathbb{R}^3$ . Then

$$\begin{aligned} K(\mathbf{x}', \mathbf{x}) &= \Phi(\mathbf{x}') \cdot \Phi(\mathbf{x})^\top \quad [\text{dot product of features}] \\ &= x_1^2(x'_1)^2 + 2x_1x_2x'_1x'_2 + x_2^2(x'_2)^2 \\ &= (x_1x'_1 + x_2x'_2)^2 = (\mathbf{x}' \cdot \mathbf{x}^\top)^2 \end{aligned}$$

- **Idea:**

- Define  $K : X \times X \rightarrow \mathbb{R}$  called kernel, such that

$$\Phi(\mathbf{x}) \cdot \Phi(\mathbf{x}')^\top = K(\mathbf{x}, \mathbf{x}')$$

- $K$  is often interpreted as a similarity measure

- **Benefits:**

- Efficiency:  $K$  is often more efficient to compute than  $\Phi$  and the dot product
- Flexibility:  $K$  can be chosen arbitrarily so long as the existence of  $\Phi$  is guaranteed (PDS condition or Mercer's condition)

- **Definition:**

$$\forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^d, K(\mathbf{x}, \mathbf{x}') = (\mathbf{x}' \cdot \mathbf{x}^\top + c)^p, c > 0$$

- **Example:** for  $p = 2$  and  $d = 2$ ,

$$K(\mathbf{x}, \mathbf{x}') = (x_1 x'_1 + x_2 x'_2 + c)^2$$

$$= [x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2}cx_1, \sqrt{2}cx_2, c] \cdot \begin{bmatrix} (x'_1)^2 \\ (x'_2)^2 \\ \sqrt{2}x'_1x'_2 \\ \sqrt{2}cx'_1 \\ \sqrt{2}cx'_2 \\ c \end{bmatrix}$$

- **Gaussian kernels:**

$$K(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{2\sigma^2}\right), \sigma \neq 0$$

- **Sigmoid kernels:**

$$K(\mathbf{x}, \mathbf{x}') = \tanh(a(\mathbf{x} \cdot \mathbf{x}') + b), a, b > 0$$

- **Definition:** a kernel  $K : X \times X \rightarrow \mathbb{R}$  is *positive definite symmetric* (PDS) if for any  $\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq X$  the matrix  $\mathbf{K} = [K(\mathbf{x}_i, \mathbf{x}_j)]_{ij} \in \mathbb{R}^{m \times m}$  is symmetric positive semi-definite (SPSD)
- Matrix  $\mathbf{K}$  SPSD if symmetric and one of the 2 equivalent conditions:
  - its eigenvalues are non-negative
  - for any  $\mathbf{c} \in \mathbb{R}^{m \times 1}$ ,  $\mathbf{c}^\top \mathbf{K} \mathbf{c} = \sum_{i,j=1}^m c_i c_j K(\mathbf{x}_i, \mathbf{x}_j) \geq 0$
- **Terminology:** PDS for kernels, SPSD for kernel matrices

- Usual linear ridge regression in dual representation

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^m \alpha_i (\mathbf{x} \cdot \mathbf{x}_i^\top)$$

with

$$\boldsymbol{\alpha} = (\mathbf{X}\mathbf{X}^\top + \lambda\mathbf{I})^{-1}\mathbf{y}$$

- Kernel ridge regression

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^m \alpha_i (\Phi(\mathbf{x}) \cdot \Phi(\mathbf{x}_i)^\top) = \sum_{i=1}^m \alpha_i K(\mathbf{x}_i, \mathbf{x})$$

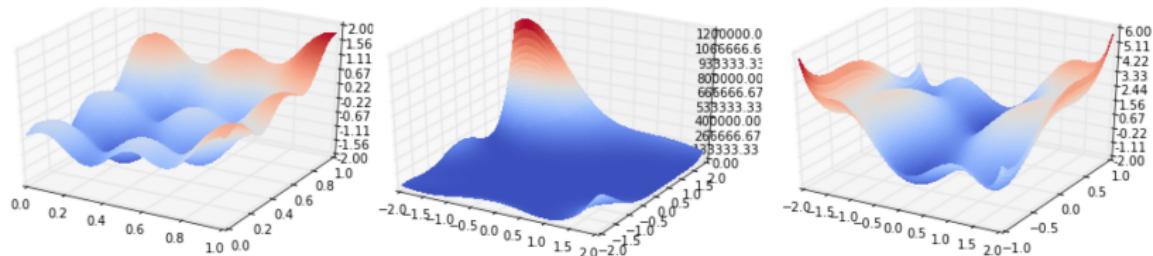
with

$$\boldsymbol{\alpha} = (\Phi(\mathbf{X}) \cdot \Phi(\mathbf{X})^\top + \lambda\mathbf{I})^{-1}\mathbf{y} = (\mathbf{K} + \lambda\mathbf{I})^{-1}\mathbf{y},$$

where

$$\mathbf{K} = \{\Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{x}_j)^\top\}_{i,j=1}^m = \{K(\mathbf{x}_i, \mathbf{x}_j)\}_{i,j=1}^m$$

## Example: Kernel ridge regression (I)



## Example: Kernel ridge regression (II)

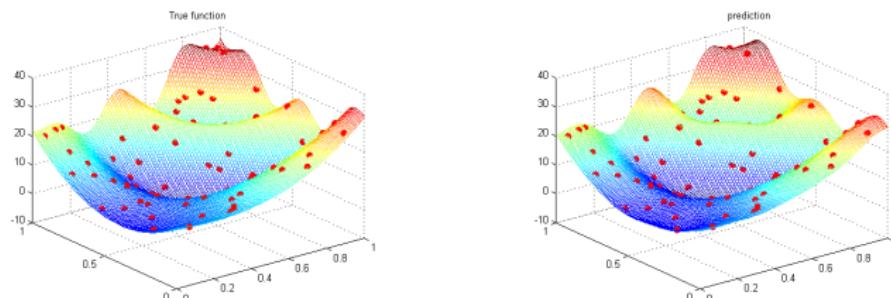


Figure – Mystery function (left) and an approximation (right) for the training sample of size 80

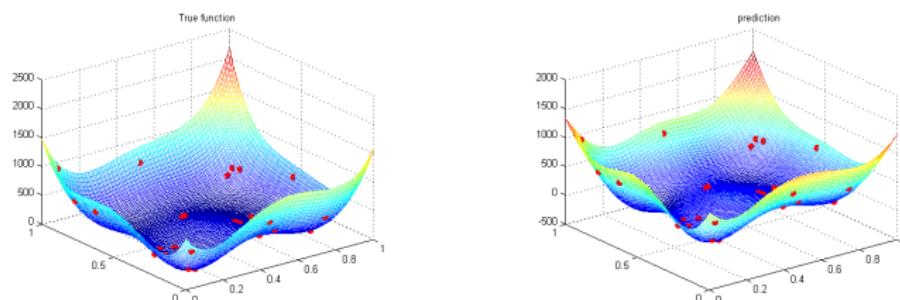


Figure – Himmelblau function (left) and an approximation (right) for the training sample of size 40

- Advantages
  - strong theoretical guarantees
  - generalization to outputs in  $\mathbb{R}^p$ : single matrix inversion
  - use of kernels
- Disadvantages
  - solution is not sparse
  - training time for large matrices: low-rank approximations of kernel matrix, e.g., Nyström approximation, partial Cholesky decomposition
- In practice in many cases regression based on decision trees and gradient boosting is used (next lectures!)

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- Ordinary Least Squares (OLS):

$$F(\mathbf{w}, b) = \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 \rightarrow \min_{\mathbf{w}, b}$$

- Ridge Regression:

$$F(\mathbf{w}, b) = \lambda \|\mathbf{w}\|_2^2 + \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 \rightarrow \min_{\mathbf{w}, b}$$

where  $\lambda \geq 0$  is a regularization parameter

- LASSO:

$$F(\mathbf{w}, b) = \lambda \|\mathbf{w}\|_1 + \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 \rightarrow \min_{\mathbf{w}, b}$$

- Elastic Net:

$$F(\mathbf{w}, b) = \lambda ((1 - \alpha) \|\mathbf{w}\|_1 + \alpha \|\mathbf{w}\|_2^2) + \sum_{i=1}^m (\mathbf{x}_i^\top \mathbf{w} + b - y_i)^2 \rightarrow \min_{\mathbf{w}, b}$$

where  $\lambda \geq 0$  is a regularization parameter,  $\alpha \in [0, 1]$  is a parameter of a convex combination

- **Prediction accuracy** and **model interpretation** are two important aspects of regression models
- LASSO is a penalized regression method to improve OLS and Ridge regression
- LASSO enforces shrinkage and variable selection simultaneously for better prediction and model interpretation
- Disadvantages of LASSO:
  - In the  $d > m$  case, the LASSO selects at most  $m$  variables before it saturates because of the nature of the convex optimization problem. However, e.g. in gene selection problems  $d > m$
  - LASSO can not do **group selection**, which is important e.g. in gene selection problems

- If there is a group of variables among which the pairwise correlations are very high, then the LASSO tends to select only one variable from the group and does not care which one is selected
- It has been empirically observed that the prediction performance of the LASSO is dominated by ridge regression
- **Elastic Net:** Regression, variable selection, with the capacity of selecting groups of correlated variables.
- *Elastic Net is like a stretchable fishing net that retains 'all the big fish'*

The optimization problem for **Elastic Net** is

$$F(\mathbf{w}, b) = \lambda ((1 - \alpha) \|\mathbf{w}\|_1 + \alpha \|\mathbf{w}\|_2^2) + \sum_{i=1}^m (\mathbf{w} \cdot \mathbf{x}_i^\top + b - y_i)^2 \rightarrow \min_{\mathbf{w}, b}$$

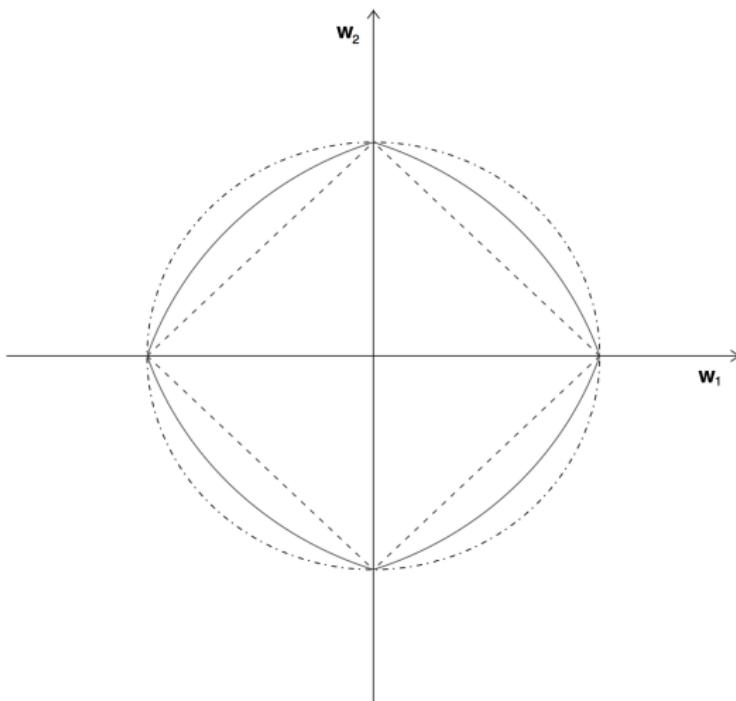
- **Elastic Net** combines  $l_1$  and  $l_2$  penalties
- Ridge regression:  $\alpha \rightarrow 1$ ; LASSO:  $\alpha \rightarrow 0$
- It can be proved that in case of orthogonal design matrix ( $\mathbf{X}^\top \mathbf{X} = \mathbf{I}$ ) analytical solution exists

$$\hat{w}_i(\text{Ridge}) = \hat{w}_i(\text{OLS}) / (1 + \lambda_2),$$

$$\hat{w}_i(\text{LASSO}) = (|\hat{w}_i(\text{OLS})| - \lambda_1/2)_+ \cdot \text{sign}(\hat{w}_i(\text{OLS})),$$

$$\hat{w}_i(\text{EN}) = \frac{(|\hat{w}_i(\text{OLS})| - \lambda_1/2)_+}{1 + \lambda_2} \cdot \text{sign}(\hat{w}_i(\text{OLS}))$$

where  $(x)_+ = \max(x, 0)$



**Figure –** Two-dimensional contour plots of the penalty (· · · · ·, shape of the ridge penalty; · · ·, contour of the lasso penalty; —, contour of the elastic net penalty with  $\alpha = 0.5$ ): we see that singularities at the vertices and the edges are strictly convex; the strength of convexity varies with  $\alpha$  [Hui Zou, Trevor Hastie]

- It can be proved that in Elastic Net, highly correlated predictors will have similar regression coefficients
- Given data  $\{\mathbf{X}, \mathbf{y}\}$  and parameters  $(\lambda, \alpha)$ , the response  $\mathbf{y}$  is centred and the predictors  $\mathbf{X}$  are standardized
- Let  $\hat{\mathbf{w}}(\lambda, \alpha)$  be the elastic net estimate. Suppose that  $\hat{w}_i(\lambda, \alpha)\hat{w}_j(\lambda, \alpha) > 0$
- Define

$$D_{(\lambda, \alpha)}(i, j) = \frac{1}{\|\mathbf{y}\|_1} |\hat{w}_i(\lambda, \alpha) - \hat{w}_j(\lambda, \alpha)|,$$

then

$$D_{(\lambda, \alpha)}(i, j) \leq \frac{1}{\lambda\alpha} \sqrt{2(1 - \rho)},$$

where  $\rho$  is the sample correlation between the  $i$ -th and  $j$ -th columns of the matrix  $\mathbf{X}$

- This theorem provides a quantitative description for the grouping effect of Elastic Net

- Elastic Net produces a sparse model with good prediction accuracy, while encouraging a grouping effect
- Efficient computation algorithm for Elastic Net is derived based on LARS
- Empirical results and simulations demonstrate its superiority over LASSO (LASSO can be viewed as a special case of Elastic Net)
- For Elastic Net, two parameters should be tuned/selected on training and validation data set. For LASSO, there is only one tuning parameter
- Although Elastic Net is proposed with the regression model, it can also be extend to classification problems (such as gene selection)