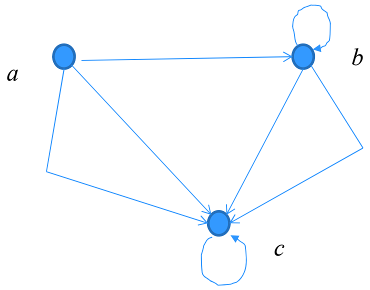
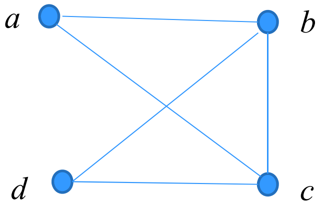
Graphs: basics

Basic types of graphs:

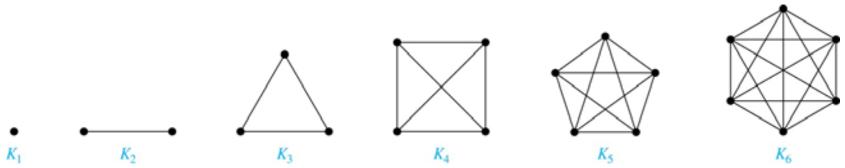
- **Directed graphs**
- **Undirected graphs**





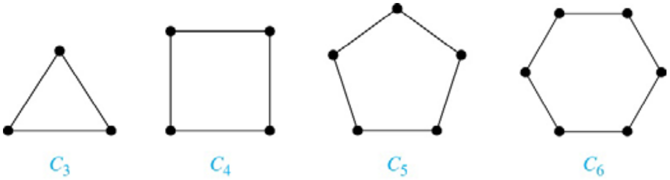
Complete graphs

A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



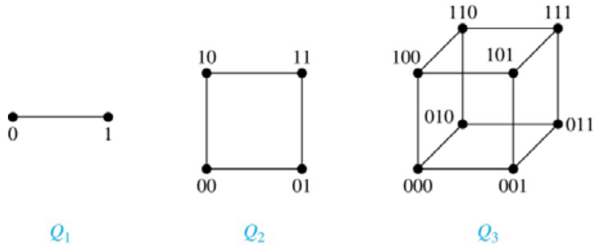
A cycle

A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



N-dimensional hypercube

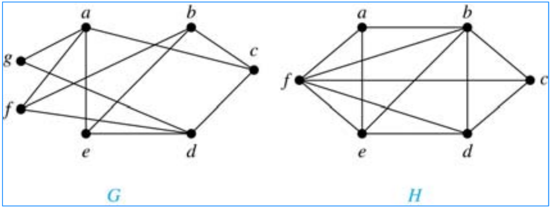
An *n-dimensional hypercube*, or *n-cube,* $\boldsymbol{Q_n}$, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



Bipartite graphs

Definition: A simple graph G is **bipartite** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2.

Note: An equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.



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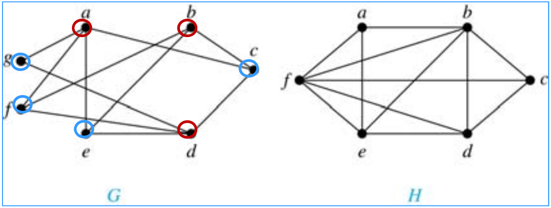
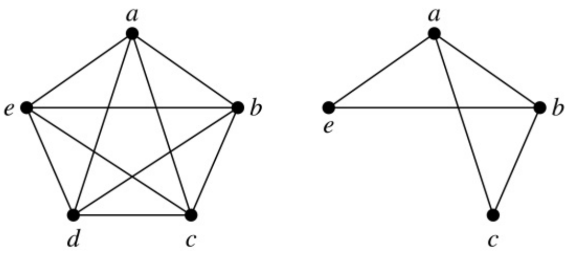


Subgraphs

Definition: A ***subgraph of a graph*** $G = (V,E)$ is a graph (W,F), where $W \subset V$ and $F \subset E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

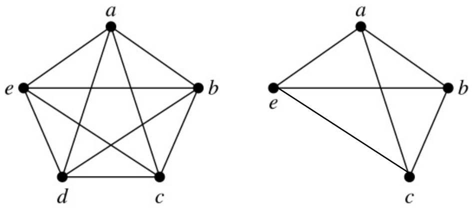
Example: K_5 and one of its subgraphs.



Subgraphs

Definition: Let $G = (V, E)$ be a simple graph. The *subgraph induced* by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E if and only if both endpoints are in W.

Example: K_5 and the subgraph induced by $W = \{a,b,c,e\}$.



Representation of graphs

Definition: An *adjacency list* can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

Example:

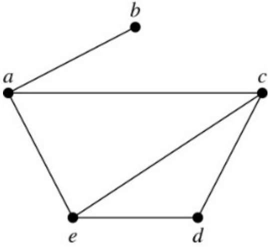


TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

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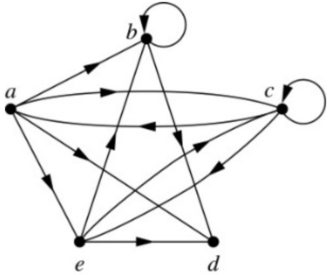


TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

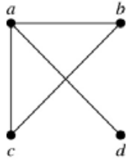
Adjacency matrices

Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, with respect to the listing of vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.

– In other words, if the graphs adjacency matrix is $\mathbf{A}_G = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Example:



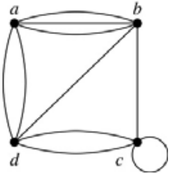
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.

Adjacency matrices

- Adjacency matrices can also be used to represent graphs with loops and multiple edges.
- A loop at the vertex v_i is represented by a 1 at the (i, i)th position of the matrix.
- When multiple edges connect the same pair of vertices v_i and v_j, (or if multiple loops are present at the same vertex), the (i, j)th entry equals the number of edges connecting the pair of vertices.

Example: The adjacency matrix of the pseudograph shown here using the ordering of vertices a, b, c, d.

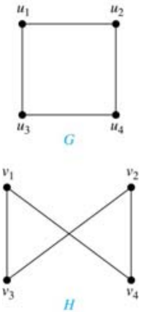


$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Graph isomorphism

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a **one-to-one and onto function** f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 , for all a and b in V_1 . Such a function f is called an *isomorphism*. Two simple graphs that are not isomorphic are called *nonisomorphic*.

Example:



Are the two graphs isomorphic?

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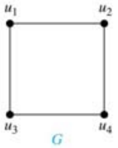

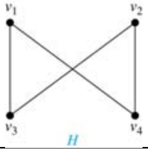

Are the two graphs isomorphic?

u1 → v1
u2 → v4
u3 → v2
u4 → v3

Connectivity in the graphs, paths

Informal Definition: A *path* is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these.

Applications: Numerous problems can be modeled with paths formed by traveling along edges of graphs such as:

- determining whether a message can be sent between two computers.
- efficiently planning routes for mail/message delivery.

Connectivity in the graphs

- We can use the adjacency matrix of a graph to find the **number of the different paths between two vertices in the graph.**

Theorem: Let G be a graph with adjacency matrix **A** with respect to the ordering $v_1, \ldots, v_n$ of vertices (with directed or undirected edges, multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where $r > 0$ is a positive integer, equals the (i,j)th entry of $\mathbf{A}^r$.

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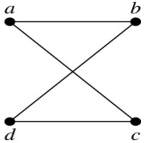

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Paths of length 4.

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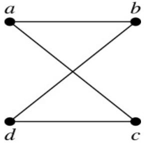


$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

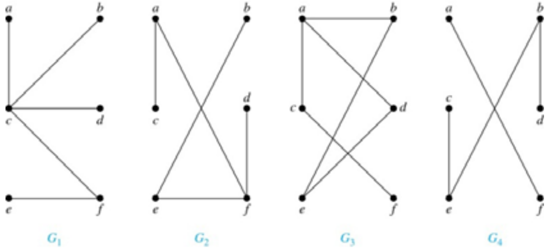
Paths of length 4: The adjacency matrix of G (ordering the vertices as a, b, c, d) is given above. Hence the number of paths of length four from a to d is the (1, 4)th entry of $\mathbf{A}^4$

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Trees

Definition: A *tree* is a **connected undirected** graph with **no simple circuits**.

Examples:

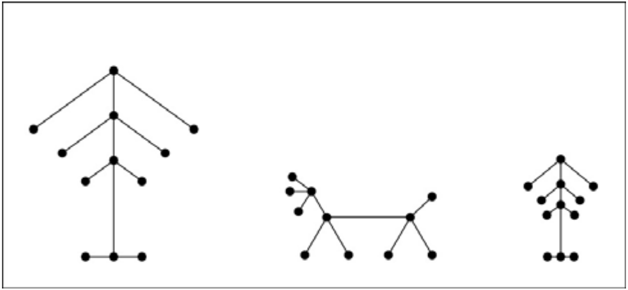


Tree: yes **Tree: yes** **Tree: no** **Tree: no**

Connectivity in the graphs

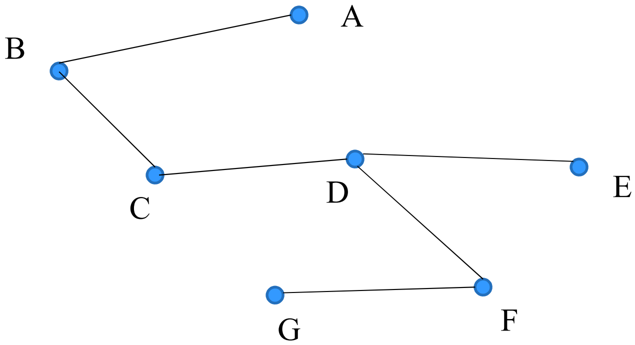
Definition: A *forest* is a graph that has no simple circuit, but is not connected. Each of the connected components in a forest is a tree.

Example:



Trees

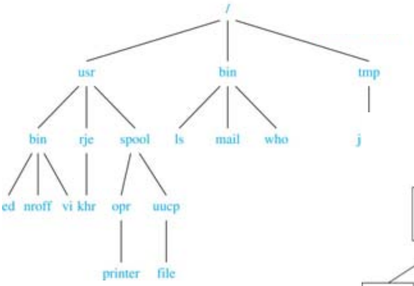
Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

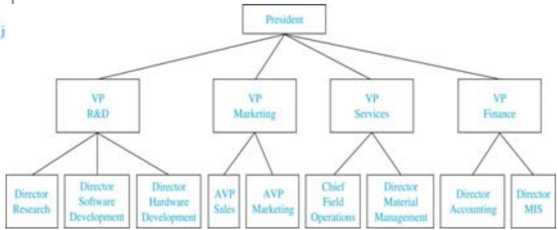


Application of trees

Examples:

- The organization of a computer file system into directories, subdirectories, and files is naturally represented as a tree.
- structure of organizations.

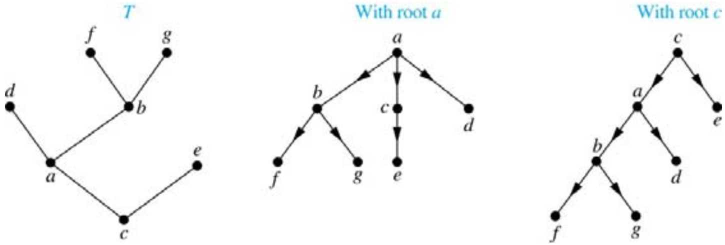




Rooted trees

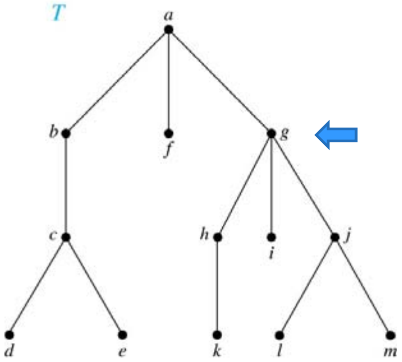
Definition: A *rooted tree* is a tree in which one vertex has been designated as the *root* and every edge is directed away from the root.

Note: An *unrooted tree* can be converted into different rooted trees when one of the vertices is chosen as the root.



Rooted trees - terminology

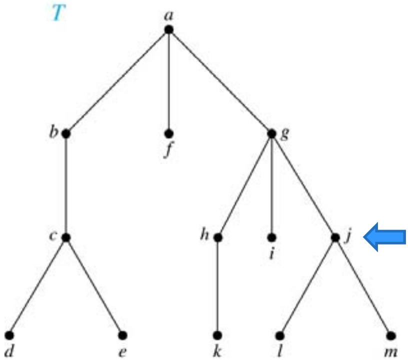
- If *v* is a vertex of a rooted tree other than the root, the *parent* of *v* is the unique vertex *u* such that there is a directed edge from *u* to *v*. When *u* is a parent of *v*, *v* is called a *child* of *u*. Vertices with the same parent are called *siblings*.



Parent of g: *a*
Children of g: *h,j,k*
Siblings of g: *b,f*

Rooted trees - terminology

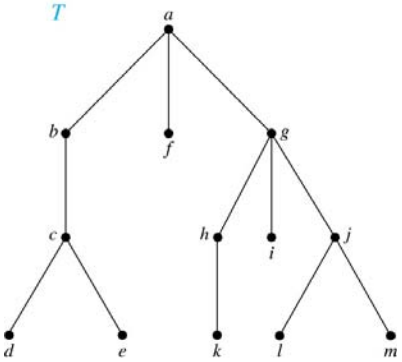
- The *ancestors* of a vertex are the vertices on the path from the root to this vertex, excluding the vertex itself and including the root. The *descendants* of a vertex v are those vertices that have v as an ancestor.



Ancestors of j: *g,a*
Descendants of j: *l,m*

Rooted trees - terminology

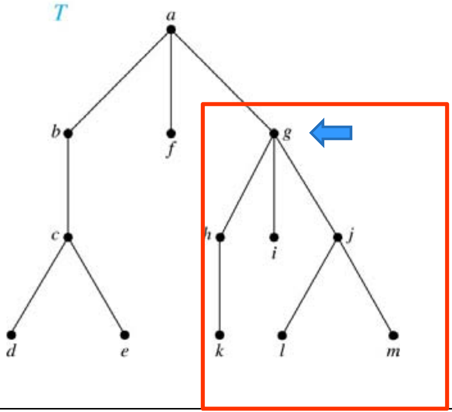
- A vertex of a rooted tree with no children is called a *leaf*. Vertices that have children are called *internal vertices*.



Leafs: *d,e,k, l,m*
Examples of internal nodes: *b,g,h*

Rooted trees - terminology

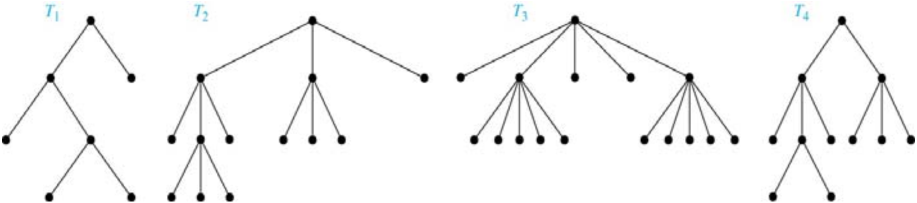
- If *a* is a vertex in a tree, the *subtree* with *a* as its root is the subgraph of the tree consisting of *a* and its descendants and all edges incident to these descendants.



M-ary tree

Definition: A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. An m-ary tree with $m = 2$ is called a *binary* tree.

Example: Are the following rooted trees full m-ary trees for some positive integer m?



Binary trees

Definition: A *binary tree* is an ordered rooted where each internal vertex has at most two children. If an internal vertex of a binary tree has two children, the first is called the *left child* and the second the *right child*. The tree rooted at the left child of a vertex is called the *left subtree* of this vertex, and the tree rooted at the right child of a vertex is called the *right subtree* of this vertex.