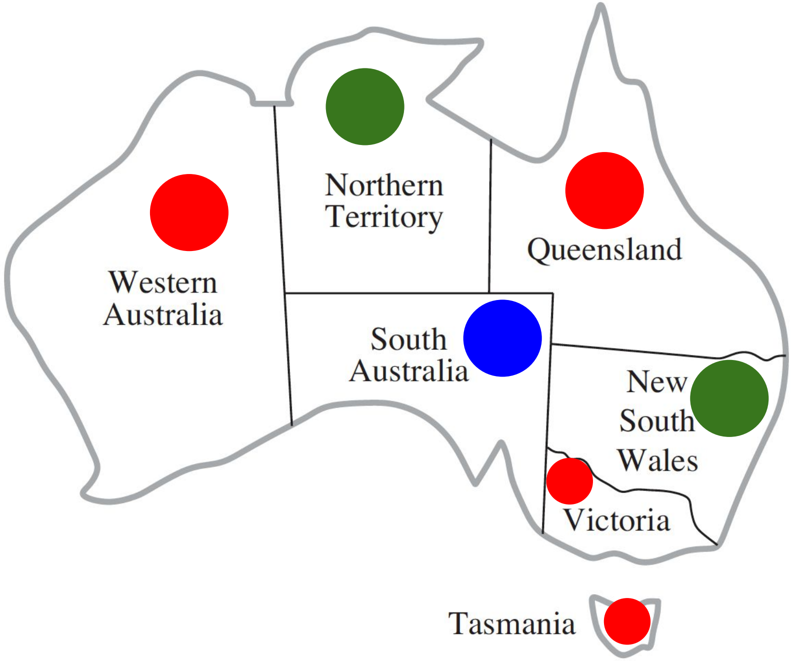
UMSL
Mathematics & Computer Science

6. Constraint Satisfaction Problems

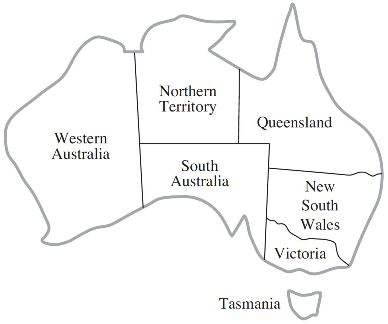
6.1.1 Australia Map Coloring Problem

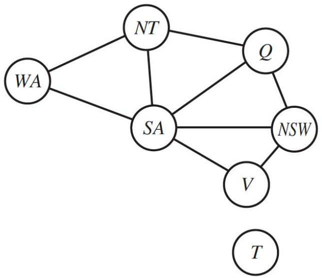
Problem: Color each region either red, green, or blue in such a way that no neighboring regions have the same color.



Example Solution - {WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T =red}

6.1.1 Formulating Map Coloring Problem as CSP





Variables are the regions:

X = {WA, NT, Q, NSW, V, SA, T}

The domain of each variable is the following set

D_i = {red, green, blue}

The nine constraints (for each regions border):

C = {SA≠WA, SA≠NT, SA≠Q, SA≠NSW, SA≠V, WA≠NT, NT≠Q, Q≠NSW, NSW≠V}

SA≠WA is a shortcut for ⟨(SA,WA), {(red , green), (red , blue), (green, red), (green, blue), (blue, red), (blue, green)}⟩

For example, (SA, WA) can take the values such as (red, green)

{ } - set
( ) - tuple / ordered sequence
⟨⟩- vector

State Space Approach vs CSP

State Space Approach

- Problems can be solved by searching in a space of states
- These states can be evaluated by domain-specific heuristics and tested to see whether they are goal states
- From the point of view of the search algorithm, however, each state is atomic, or indivisible—a black box with no internal structure

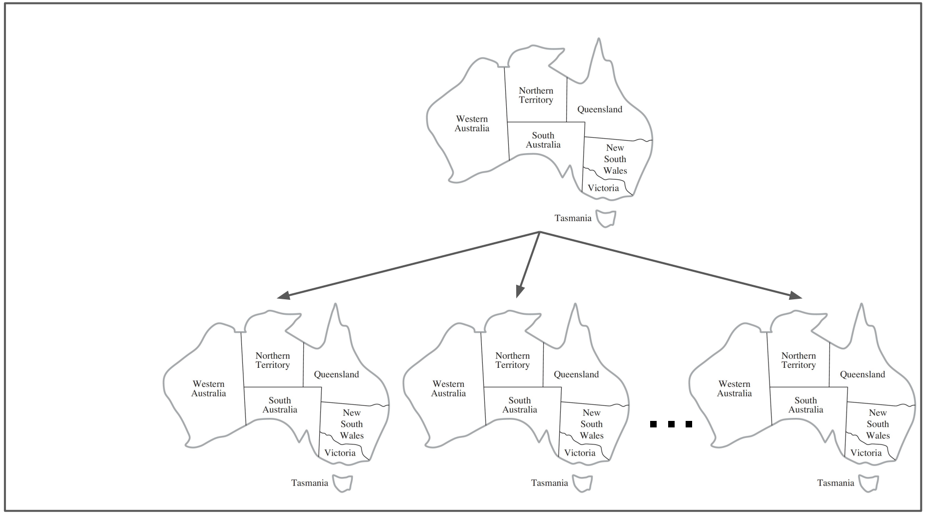
Constraint Satisfaction Approach

(more efficient for many problems)

- Use a **factored representation for each state**: a set of variables, each of which has a value
- Problem is solved when each variable has a value that satisfies all the constraints on the variable

CSP search algorithms take advantage of the **structure of states** and use **general-purpose** rather than problem-specific heuristics to enable the solution of complex problems.
Examples: Shortest Path (global perspective is important) vs Sudoku (local solutions lead to global)

State space search approach



6.1 Defining CSPs

A constraint satisfaction problem consists of three components - X, D, and C:

X is a set of variables, $\{X_1, . . . ,X_n\}$
D is a set of domains, $\{D_1, . . . ,D_n\}$, one for each variable
C is a set of constraints that specify allowable combinations of values

- Each domain D_i consists of a set of allowable values, $\{v_1, . . . , v_k\}$ for variable X_i
- Each constraint C_i consists of a pair **⟨scope, rel⟩**
    - **scope** is a tuple of variables that participate in the constraint
    - **rel** is a relation that defines the values that those variables can take on.

    Example: SA≠WA is a shortcut for ⟨(SA,WA), {(red , green), (red , blue), (green, red), (green, blue), (blue, red), (blue, green)}⟩

- Each state in a CSP is defined by an assignment of values to some or all of the variables, $\{X_i = v_i, X_j = v_j , . . .\}$

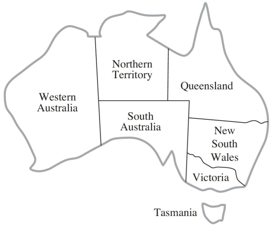
Why formulate a problem as CSP?

1. CSPs yield a natural representation for a wide variety of problems
   - if you already have a CSP-solving system, it is often easier to solve a problem using it than to design a custom solution using another search technique
2. CSP solvers can be faster than state-space searchers
   - because CSP solvers can quickly eliminate large swatches of the search space

Example:

- Once we have chosen {SA=blue} in the Australia problem, we can conclude that none of the five neighboring variables can take on the value blue.
- Without taking advantage of constraint propagation, a search procedure would have to consider **$3^5 = 243$** assignments for the five neighboring variables
- With constraint propagation we never have to consider blue as a value, so we have only $2^5 = 32$ assignments
- a reduction of 87%

Provide a simpler example!



6.1.2 Scheduling the Assembly of a Car

(Sub-)problem: Scheduling consists of 15 tasks

- Install axles (front and back), affix all four wheels (right and left, front and back), tighten nuts for each wheel, affix hubcaps, and inspect the final assembly.

Variables

- Each task can be modeled as a variable, e.g. install wheel
- Value of the variable = **time that the task starts**
- Example, **$Axle_F$ = 0 indicates front axle installation begins at time 0**
- All 15 variables, X = {$Axle_F$, $Axle_B$, $Wheel_{RF}$, $Wheel_{LF}$, $Wheel_{RB}$, $Wheel_{LB}$, $Nuts_{RF}$, $Nuts_{LF}$, $Nuts_{RB}$, $Nuts_{LB}$, Cap_{RF}, Cap_{LF}, Cap_{RB}, Cap_{LB}, Inspect}

Constraints

- Assert that one task must occur before another
  - for example, **a wheel must be installed before the hubcap is put on**
- Assert that only so many tasks can go on at once (for example, when sharing tools)
- Specify that a task takes a certain **amount of time to complete**

6.1.2 Constraints for the Car Assembly Problem

It takes 10 minutes to install an axle:

$Axle_F + 10 \leq Wheel_{RF}$; $Axle_F + 10 \leq Wheel_{LF}$; $Axle_B + 10 \leq Wheel_{RB}$; $Axle_B + 10 \leq Wheel_{LB}$

It takes 1 minute to affix the wheels:

$Wheel_{RF} + 1 \leq Nuts_{RF}$; $Wheel_{LF} + 1 \leq Nuts_{LF}$; $Wheel_{RB} + 1 \leq Nuts_{RB}$; $Wheel_{LB} + 1 \leq Nuts_{LB}$;

It takes 2 minutes to tighten nuts (before attaching the hubcaps):

$Nuts_{RF} + 2 \leq Cap_{RF}$; $Nuts_{LF} + 2 \leq Cap_{LF}$; $Nuts_{RB} + 2 \leq Cap_{RB}$; $Nuts_{LB} + 2 \leq Cap_{LB}$

Have to share one tool that helps put the axle in place:

We need a **disjunctive constraint** to say that $Axle_F$ and $Axle_B$ must not overlap in time

$(Axle_F + 10 \leq Axle_B)$ **or** $(Axle_B + 10 \leq Axle_F)$

Inspection takes 3 minutes:

$X + 3 \leq Inspect$ (for each X)

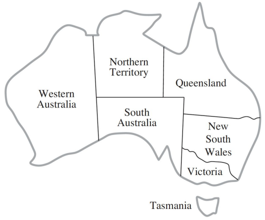
What will be the constraint/s for - “Get the whole assembly done in 30 min”?

Hint: Can this constraint be used to define domains?

Approaches for Solving CSPs (List of topics we cover)

- Constraint Propagation / Inference
  - Requires the concepts of node consistency, arc consistency, path consistency, k-consistency
  - Arc consistency can solve an ‘easy’ sudoku problem
- Backtracking search
  - Variable and value ordering
  - Intelligent backtracking
- Local search
- Using the structure of the problem to find quick solutions
  - The Tree-CSP-Search algorithm
  - How to reduce constraint graphs to tree?
    - The cutset conditioning algorithm
    - Tree decomposition approach

6.2.2 What is Arc Consistency?



- Example 1 (works):
    - X and Y are both 'set of digits'
    - Consider a constraint $X = Y^2$ OR $\langle(X,Y), \{(0,0), (1,1), (2,4), (3,9)\}\rangle$
    - To **make X arc-consistent with respect to Y**, we reduce X's domain to {0, 1, 2, 3}
    - To **make Y arc-consistent with respect to X**, we reduce Y's domain to {0, 1, 4, 9}
- Arc consistency is reducing the domain of the variables (X) to consistent values.
- **X_i is arc-consistent with respect to X_j** if for every value in the current domain D_i there is some value in domain D_j that satisfies the binary constraint on the arc (X_i, X_j).
- A network is arc-consistent is every variable is arc-consistent with every other variable.
- Example 2 (does nothing):
    - SA and WA are in the set {red, green, blue}
    - The inequality constraint on (SA, WA) is {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}
    - To make SA arc-consistent with respect to WA, we will reduce SA's domain to {red, green, blue}.

6.2.6 The Sudoku Puzzle

The Sudoku Puzzle

- The Sudoku board consists of 81 squares
- Some of the squares are already filled with digits from 1 to 9
- **Goal:** Fill in the remaining squares such that no digit appears twice in any row, column, or 3x3 box
- There is exactly one solution
- Even the hardest Sudoku problems yield to a CSP solver in less than 0.1 second

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

6.2.6 Variables, Domains, and Constraints for Sudoku

- Variables:
  - A1 through A9 for the top row
  - I1 through I9 for the bottom row
- Variable Domain:
  - The empty squares have the domain {1, 2, 3, 4, 5, 6, 7, 8, 9}
  - The prefilled squares have a domain consisting of a single value
- Constraints:
  - 27 "Alldiff" constraints: one for each **row**, **column**, and **box of 9 squares**

$$
\begin{aligned}
&Alldiff(A1, A2, A3, A4, A5, A6, A7, A8, A9)\\
&Alldiff(B1, B2, B3, B4, B5, B6, B7, B8, B9)\\
&\ldots\\
&Alldiff(A1, B1, C1, D1, E1, F1, G1, H1, I1)\\
&Alldiff(A2, B2, C2, D2, E2, F2, G2, H2, I2)\\
&\ldots\\
&Alldiff(A1, A2, A3, B1, B2, B3, C1, C2, C3)\\
&Alldiff(A4, A5, A6, B4, B5, B6, C4, C5, C6)\\
&\ldots
\end{aligned}
$$

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

6.2.6 Arc Consistency for Solving Sudoku

Consider the variable E6:

- Initially the domain of E6 is {1, 2, …, 9}
- To make E6 arc-consistent with respect to the eight other variables in the box (D4 to F6) we will reduce the domain of E6 to {3, 4, 5, 6, 9}
  - i.e. we remove 1, 2, 7, and 8
- Next, to make E6 arc-consistent will the eight variables in the row E, we will reduce the domain of E6
  - In this case, the domain remains same
- Further, to make E6 arc-consistent with eight variables in the column 6, we will reduce the domain of E6 down to {4}
  - because 3, 5, 6, and 9 are all in the column
- This solves E6

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Arc consistency is reducing the domain of the variables (X) to consistent values (by satisfying constraints).

6.2.6 Arc Consistency for Solving Sudoku

Consider the variable I6:

- I6 = {1, 2, 3, …, 9}
- Applying arc consistency in its column, we eliminate 5, 6, 2, 4, 8, 9, and 3
  - I6 = {1, 7}
- We eliminate 1 by arc consistency with I5
  - I6 = {7}

Consider the variable A6:

- A6 = {1, 2, 3, …, 9}
- Applying arc consistency in its column, we eliminate 5, 6, 2, 4, 8, 9, 3, and 7
  - A6 = {1}

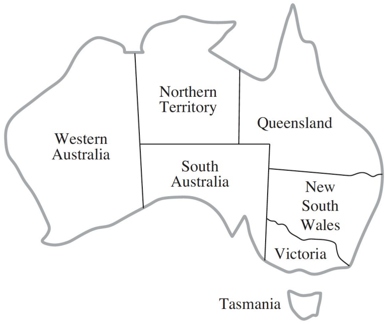
AC algorithms work only for easiest Sudoku puzzles!

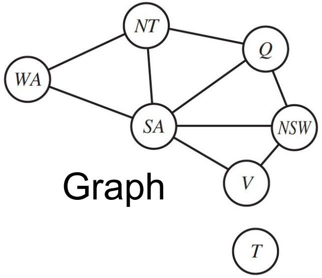
	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7					4			8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

How do ‘we’ know that we have to solve I6 after E6?

Can Arc-consistency solve the car assembly problem?

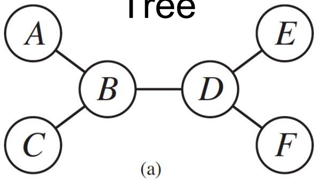
6.5 A Reduced Australia Map Coloring Problem





Graph

Tree



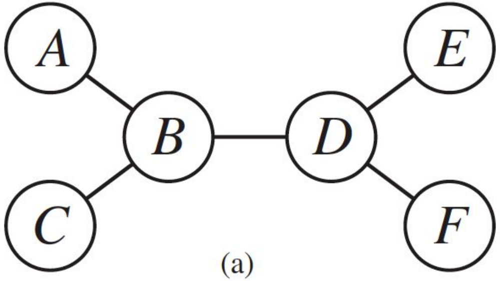
(a)

- We would like to color the states A, B, C, D, E, and F with {green, blue, red} colors
- A **constraint graph is a tree when any two variables are connected by only one path**

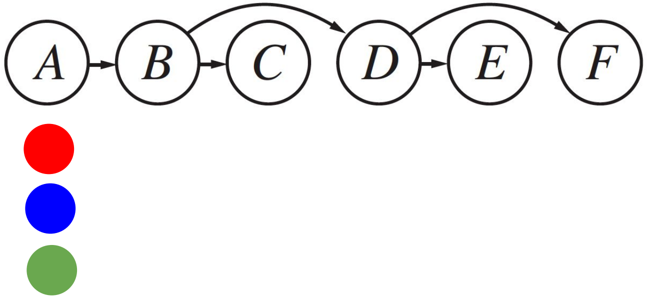
6.5 The TREE-CSP-SOLVER Algorithm

function TREE-CSP-SOLVER(*csp*) **returns** a solution, or failure
  inputs: *csp*, a CSP with components X, D, C

  $n \leftarrow$ number of variables in X
  $assignment \leftarrow$ an empty assignment
  $root \leftarrow$ any variable in X
  $X \leftarrow$ TOPOLOGICALSORT(X, $root$)
  for $j = n$ **down to** 2 **do**
    MAKE-ARC-CONSISTENT(PARENT(X_j), X_j)
    if it cannot be made consistent **then return** *failure*
  for $i = 1$ **to** n **do**
    $assignment[X_i] \leftarrow$ any consistent value from D_i
    if there is no consistent value **then return** *failure*
  return *assignment*

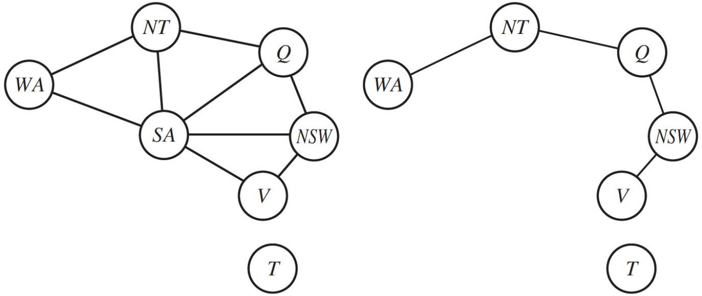


(a)



6.5 Reducing Graphs to Trees

- We have Tree-CSP-Solver algorithm. How to reduce a graph to a tree?
  - Assign values to some variables, so that the remaining graph becomes a tree.
- For example, if we delete "South Australia" the remaining constraint graph becomes a tree.
  - For example, we can fix the color of SA to "blue" and remove "blue" from the domain of all adjacent states (equivalent to deleting).
  - {SA} is **cycle cutset** because after removing {SA} the constraint graph becomes a tree

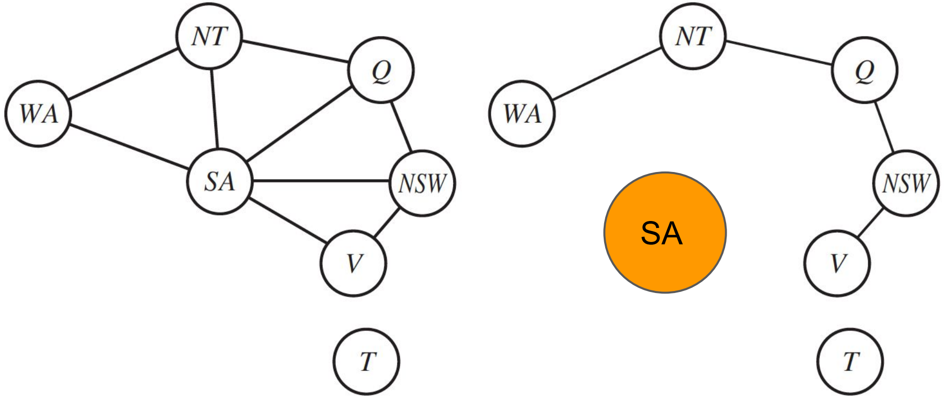


Tree is special form of graph i.e. minimally connected graph and having only one path between any two vertices.

- **Cycle cutset** is a subset of variables in a constraint graph such that after removing the cycle cutset, the graph becomes a tree.

6.5 Cutset Conditioning Algorithm

1. Step 1: Choose a cycle cutset, i.e. a subset S of the CSP's variables such that the constraint graph becomes a tree after removal of S.
2. Step 2: For each possible assignment to the variables in S that satisfies all constraints on S
   a. remove from the domains of the remaining variables any values that are inconsistent with the assignment for S, and
   b. If the remaining CSP has a solution, return it together with the assignment for S.



Example Classwork Problem

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Sample Problem 1: Provide the steps (with numbering) how you will apply the Cutset Conditioning Algorithm and the Tree-CSP-Solver algorithm to color the following map with 4 colors - blue, white, red or green - such that no two neighboring countries have the same color.

Sample Problem 2: Color the following map with 4 colors - blue, white, red or green - such that no two neighbouring countries have the same color.

1. Draw the constraint graph
2. Apply Cutset Conditioning algorithm
    a. Choose a cycle cutset S (remaining variables for a tree T)
    b. (Loop) Assign constraint ‘satisfying’ values to S
        i. Update the domain of the variables in T (by deleting all values of the variables in S)
        ii. Apply Tree-CSP-Solver to solve T
            1. Choose root
            2. Make graph directed (assume that we apply BFS)
            3. Build a topological order
            4. Assign colors
        iii. Return solution if T can be solved, i.e. all variables have consistent values

Summary

- Arc consistency algorithm can solve problems such as ‘easy’ Sudoku.
- Cutset conditioning can reduce a general CSP to a tree-structured one and is quite efficient if a small cutset can be found.