Elementary Mathematics II (Differential Equations and Dynamics) (MTH 102)

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Linear Equations

Definition:

A differential equation

$$\frac{dy}{dx} + p(x)y = q(x) \qquad (1)$$

is called a first order linear differential equation or **Linear equations.**

- The differential equation may not always come in the form (1).

Solution: Integrating factor (I.F.)

Linear equation are solved with help of an I.F. given by

$$I.F. = \mu = e^{\int p(x)dx}$$

Multiply the integrating factor by the Linear equation (1)

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y = e^{\int p(x)dx}\, q(x)$$

$$\frac{d}{dx}\left(e^{\int p(x)dx}y\right) = e^{\int p(x)dx}\, q(x)$$

Integrating both sides

$$\int \frac{d}{dx}\left(e^{\int p(x)dx} y\right) dx = \int e^{\int p(x)dx} q(x)\, dx$$

$$y \cdot e^{\int p(x)dx} = \int e^{\int p(x)dx} q(x)\, dx$$

The general solution of the Linear equation (1) is

$$\boxed{y(x) = e^{-\int p(x)dx} \int e^{\int p(x)dx} q(x)\, dx + c e^{-\int p(x)dx}}$$

Necessary steps to solve Linear Equation

1. Rearrange the equation in the form $\frac{dy}{dx} + p(x)y = q(x)$

2. Isolate $p(x)$ & $q(x)$, obtain the integrating factor I.F.

$$\mu = e^{\int p(x)dx}$$

3. The general solution is given by

$$y \cdot \mu = \int \mu \cdot q(x)\, dx$$

4. Integrate R.H.S. and add a constant of integration

5. Make y the subject formula.

Examples

1. Solve the differential equation

$$x\frac{dy}{dx} + y - x\sin x = 0$$

Solution:

Rearrange the equation in form of linear equation

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \sin x$$

$$\Rightarrow p(x) = \frac{1}{x}, \qquad q(x) = \sin x$$

$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Therefore, from step (3), the solution takes the form

$$yx = \int x \sin x \, dx$$

Using integration by parts for R.H.S

$u = x \Rightarrow du = dx$ $\qquad$ $dv = \sin x \Rightarrow v = -\cos x$

$$yx = -x\cos x + \int \cos x + c$$

$$yx = -x\cos x + \sin x + c$$

$$\boxed{y(x) = \frac{1}{x}(\sin x + c) - \cos x}$$

Examples

2. Solve the differential equation

$$\frac{dz}{dx} - \left(\frac{2}{x}\right) z = \frac{2}{3} x^4$$

Solution:

$$\Rightarrow p(x) = -\frac{2}{x}, \qquad q(x) = \frac{2}{3} x^4$$

$$\mu = e^{-2\int \frac{1}{x} dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\mu = \frac{1}{x^2}$$

Therefore, from step (3), the solution takes the form

$$\frac{y}{x^2} = \int \frac{1}{x^2} \cdot \frac{2}{3} x^4 dx$$

Simplify R.H.S. and integration using power rule

$$\frac{y}{x^2} = \frac{2}{3} \int x^2 dx$$

$$\frac{y}{x^2} = \frac{2}{3} \cdot \frac{1}{3} x^3 + c$$

$$\boxed{y(x) = \frac{2}{9} x^5 + cx^2}$$

Examples

3. Solve the differential equation

$$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x$$

Solution:

Rearrange the equation in form of linear equation

$$\frac{dy}{dx} - \left(\frac{3x^2 - 1}{x^3 - x}\right)y = \frac{x^5 - 2x^3 + x}{(x^3 - x)} = x^2 - 1$$

$$\Rightarrow p(x) = \frac{3x^2 - 1}{x^3 - x}, \qquad q(x) = x^2 - 1$$

$$\mu = e^{-\int \frac{3x^2-1}{x^3-x}dx} = e^{-\ln(x^3-x)} = \frac{1}{x^3-x}$$

Therefore, from step (3), the solution takes the form

$$\frac{y}{x^3-x} = \int \frac{1}{x^3-x} \cdot (x^2-1)dx$$

Simplify the R.H.S. and integrate

$$\frac{y}{x^3-x} = \int \frac{x^2-1}{x(x^2-1)}dx = \int \frac{1}{x}dx$$

$$\frac{y}{x^3-x} = \ln x + c$$

$$\boxed{y(x) = (x^3-x)[\ln x + c]}$$

Exercise / Class-work / Assignment

Solve the following Linear equations

1. $x\frac{dy}{dx} = y + x^3 + 3x^2 - 2x$

2. $\mathrm{y}\ln y\,dx + (x - \ln y)dy = 0$

3. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$

4. $\frac{dp}{d\theta} + 3p = 2$

Bernoulli Equations

Definition:

A differential equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n \qquad (2)$$

is called a Bernoulli differential equation**.**

Named after the german Mathematician and Physicist 01/1700 – 03/1782

Method of Solution

Divide (2) by y^n

$$y^{-n}\frac{dy}{dx}+p(x)y^{1-n}=q(x) \qquad (3)$$

Let $z=y^{1-n} \Rightarrow \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$. Multiply (3) by $(1-n)$, then (3) becomes

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)p(x)y^{1-n}=(1-n)q(x)$$

Which yields a Linear equation

$$\frac{dz}{dx}+P(x)z=Q(x)$$

$$\Rightarrow P(x) = (1-n)p(x), \qquad Q(x) = (1-n)q(x)$$

Therefore the new I.F. $\mu = e^{(1-n)\int p(x)dx}$

The solution of the Bernoulli equations, takes the form

$$z \cdot e^{(1-n)\int p(x)dx} = \int e^{(1-n)\int p(x)dx} Q(x)\,dx$$

$$z = e^{-(1-n)\int p(x)dx} \int e^{(1-n)\int p(x)dx} Q(x)\,dx + ce^{-(1-n)\int p(x)dx}$$

$$\boxed{y^{1-n} = e^{-(1-n)\int p(x)dx} \int e^{(1-n)\int p(x)dx} Q(x)\,dx + ce^{-(1-n)\int p(x)dx}}$$

Necessary steps to solve Bernoulli Equation (Method 1)

1. Rearrange in the form $\frac{dy}{dx}+p(x)y=q(x)y^n$

2. Divide both sides by y^n, to yield

$$y^{-n}\frac{dy}{dx}+p(x)y^{1-n}=q(x) \qquad (2)$$

3. Use the substitution $z=y^{1-n} \Rightarrow \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$

4. Multiply $(1-n)$ to both sides of (2) to obtain

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)p(x)y^{1-n}=(1-n)q(x)$$

Bernoulli Equation (Method 1 Cont'd)

5. Form a new linear equation

$$\frac{dz}{dx} + P(x)z = Q(x)$$

6. Obtain $P(x) = (1-n)p(x)$, $Q(x) = (1-n)q(x)$

7. Obtain Integrating Factor I.F.

$$\mu = e^{\int P(x)dx}$$

8. The general solution is given by

$$z \cdot \mu = \int \mu \cdot Q(x)\, dx \xRightarrow[z=y^{1-n}]{} y^{1-n} \cdot \mu = \int \mu \cdot Q(x)\, dx$$

9. Integrate R.H.S and add a constant of integration

Necessary steps to solve Bernoulli Equation (Method 2)

1. Rearrange in the form $\frac{dy}{dx} + p(x)y = q(x)y^n$
2. Isolate $p(x)$, $q(x)$, & n obtain the integrating factor I.F.
3. Obtain $P(x)$, $Q(x)$ with $z = y^{1-n}$, we have that

$$P(x) = (1-n)p(x),\ \mathrm{Q}(x) = (1-n)q(x)$$

$$\mu = e^{\int P(x)dx}$$

4. The general solution is given by

$$z \cdot \mu = \int \mu \cdot Q(x)\, dx \xRightarrow[z=y^{1-n}]{} y^{1-n} \cdot \mu = \int \mu \cdot Q(x)\, dx$$

5. Integrate R.H.S and add a constant of integration

1. Solve the differential equation $\frac{dy}{dx} - y = xy^5$

Divide both sides by y^5

$$y^{-5}\frac{dy}{dx} - y^{-4} = x$$

Let $z = y^{-4} \Rightarrow \frac{dz}{dx} = -4y^{-5}\frac{dy}{dx}$. Multiply both sides by -4

$$-4y^{-5}\frac{dy}{dx} + 4 \cdot (y^{-4}) = -4x$$

$$\frac{dz}{dx} + 4z = -4x$$

$$\Rightarrow P(x) = 4, \qquad Q(x) = -4x$$

$$\mu = e^{4\int dx} = e^{4x}$$

Therefore, the solution takes the form

$$ze^{4x} = -\int 4xe^{4x}dx$$

Using integration by parts for R.H.S

$$u = 4x \Rightarrow du = 4dx \qquad dv = e^{4x} \Rightarrow v = \frac{1}{4}e^{4x}$$

$$ze^{4x} = -\left(4x \cdot \frac{1}{4}e^{4x} + \frac{4}{4}\int e^{4x}dx\right)$$

$$ze^{4x} = -xe^{4x} + \frac{1}{4}e^{4x} - c$$

$$\boxed{y^{-4} = -x + \frac{1}{4} - ce^{-4x}}$$

2. Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} + \frac{y^2}{x^2} = 0$

Divide both sides by y^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{x} \cdot y^{-1} = -\frac{1}{x^2}$$

Let $z = y^{-1} \Rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$. Multiply both sides by -1

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x} \cdot y^{-1} = \frac{1}{x^2}$$

$$\frac{dz}{dx} - \left(\frac{1}{x}\right) \cdot z = \frac{1}{x^2}$$

$$\Rightarrow P(x) = -\left(\frac{1}{x}\right), \qquad Q(x) = \frac{1}{x^2}$$

$$\mu = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

Therefore, the solution takes the form

$$\frac{z}{x} = -\int \frac{1}{x} \cdot \frac{1}{x^2} dx = \int x^{-3} dx$$

Integrating R.H.S using power rule

$$\frac{z}{x} = \frac{x^{-2}}{-2} + c$$

$$\frac{z}{x} = -\frac{x^{-2}}{2} + c$$

$$\boxed{y^{-1} = -\frac{1}{2x} + cx}$$

Exercise / Class-work / Assignment

Solve the following Bernoulli equations

1. $\frac{dy}{dx} - y = xy^2$

2. $\mathrm{x}dy + ydx = x^3y^6dx$

3. $yy' - xy^2 + x = 0$

4. $2\frac{dx}{dy} - xy + \cos y = 0$