Elementary Mathematics II (Differential Equations and Dynamics) (MTH 102)

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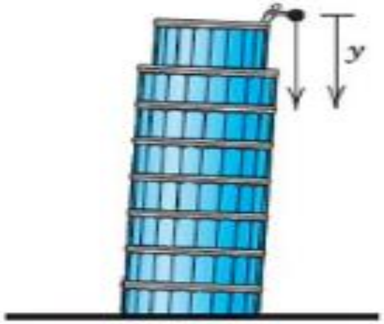
Differential Equations

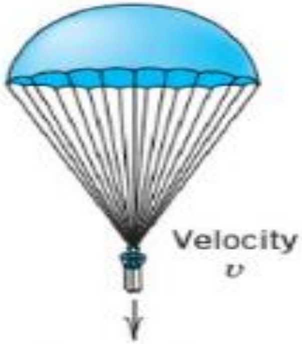
- Differential equations play a very important role in understanding physical sciences
- Differential equations are a class of equations that helps us understand the dynamics in physics, engineering, chemistry and other disciplines
- Equations known as mathematical models represent certain problems with interactions of components as variables.
- Differential equations occurred since 1690s- Newton, Leibnitz, Bernoulli, etc

Applications of Differential Equations



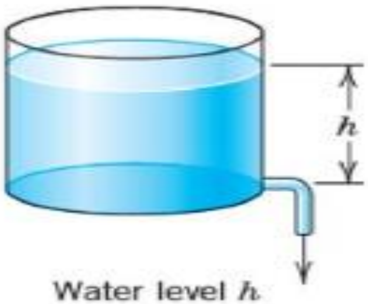
Falling stone

$y'' = g = const.$



Parachutist

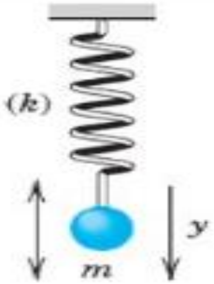
$mv' = mg - bv^2$



Water level h

Outflowing water

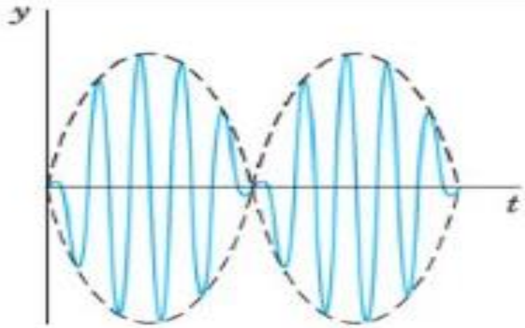
$h' = -k\sqrt{h}$



Displacement y

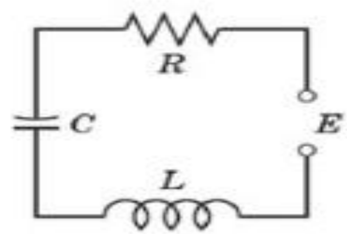
Vibrating mass on a spring

$my'' + ky = 0$



Beats of a vibrating system

$y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 = \omega$

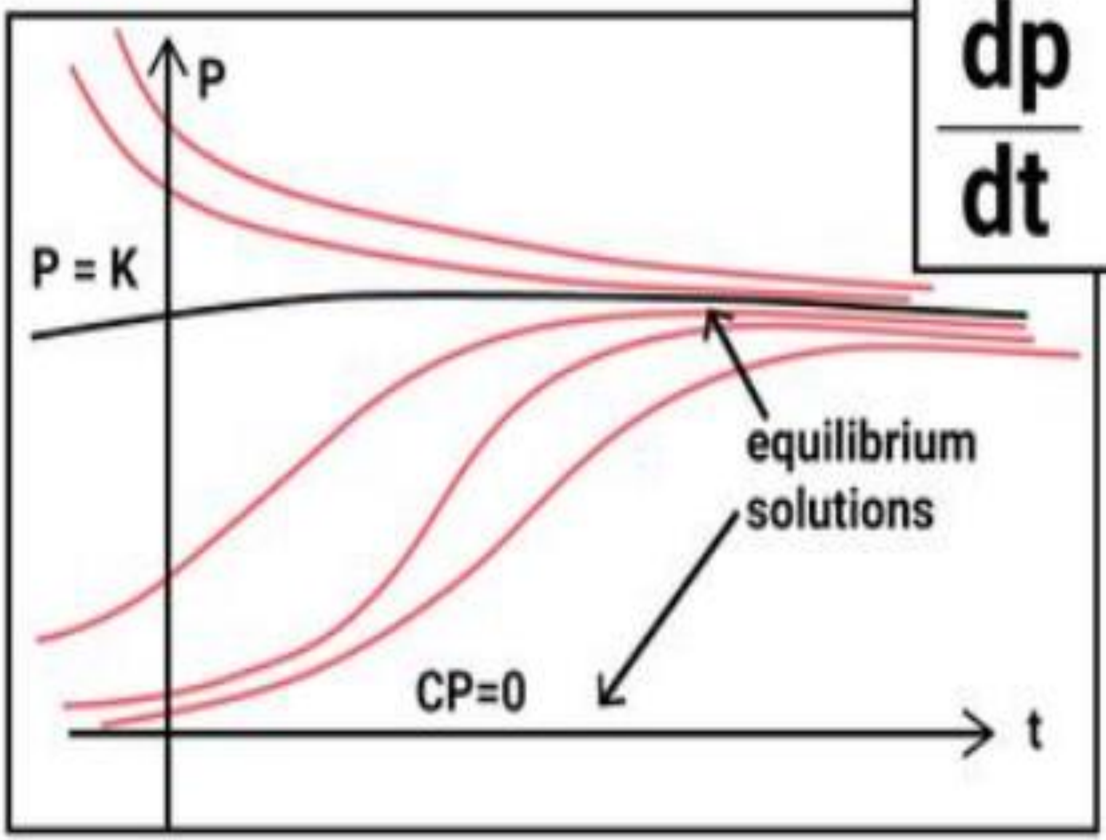


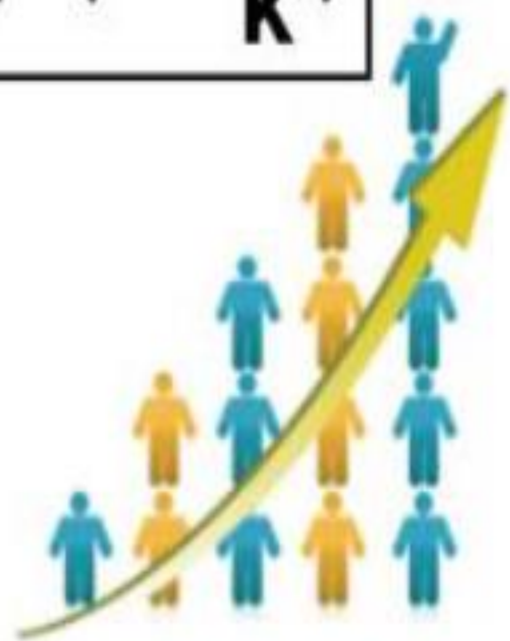
Current I in an RLC circuit

$LI'' + RI' + \frac{1}{C}I = E'$

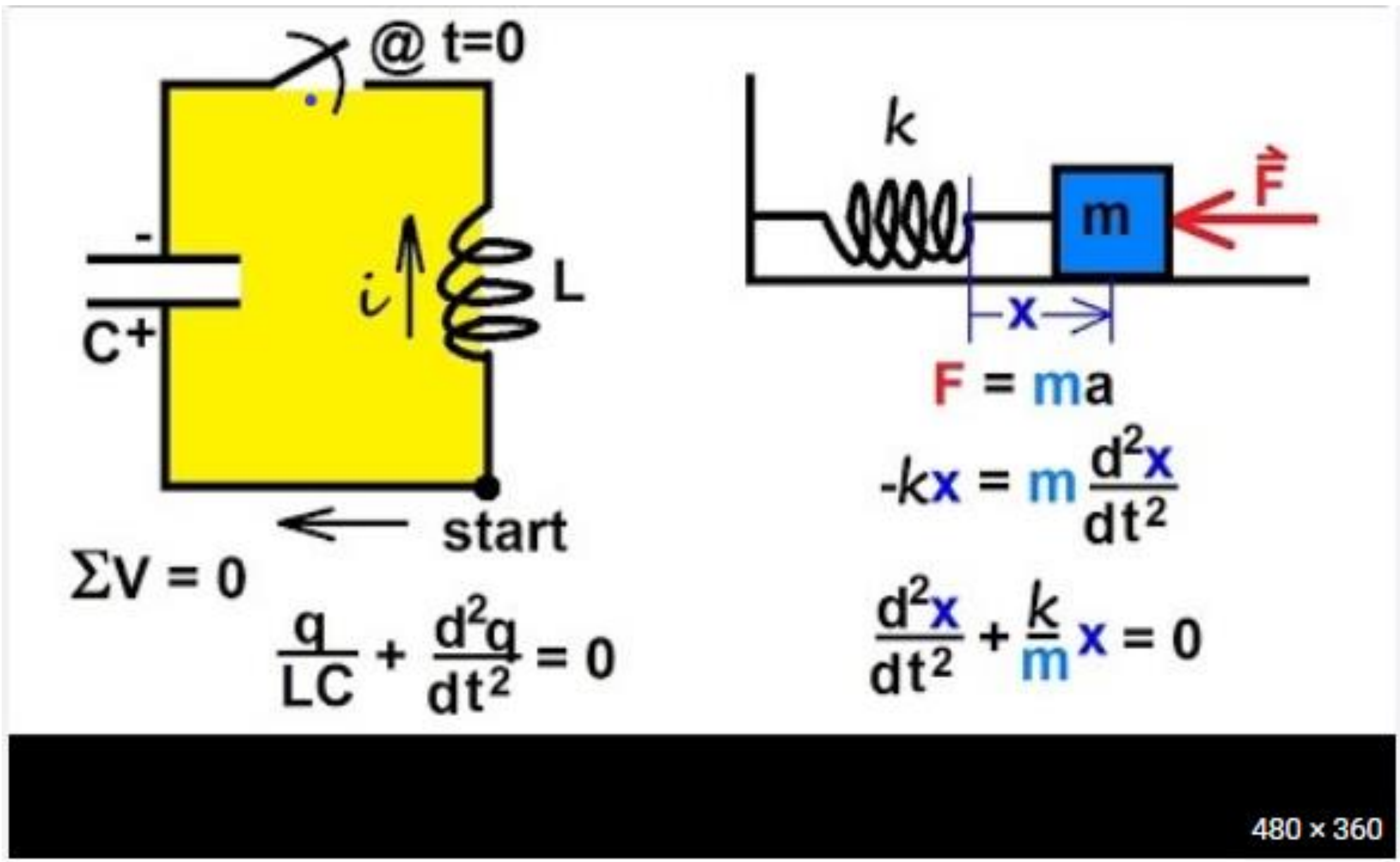
Human Population, Malthus

$$\frac{dp}{dt} = kp\left(1-\frac{p}{k}\right)$$





Electricity (Kirchhoff) and Resistive Force (Newton)



What is a Differential Equation?

Definition:

Any equation containing differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \left(\frac{dy}{dx}\right)^2, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}$, etc is called a Differential Equation (DE).

Examples:

1. $\frac{dy}{dx} - 5y = 0$
2. $5\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$
3. $\frac{d^2x}{dt^2} + 5\frac{d^2x}{dt^2} - 3t = \cos x$
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Ordinary Differential Equation (ODE)

A differential equation involving only ordinary derivatives w.r.t. to a single independent variable is called an ODE.

A general form of such equation is

$$g\left(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0 \qquad (1)$$

- The order of such equation is n.
- It is expected that the solution to the equation is of the form $y = f(x)$ which is expected to satisfy (1).

Formation of ODE

- Consider the function $y = \sin 2x$

$$\frac{d^2y}{dx^2} + 4y = 0$$

- The rate of flow of electricity by the voltage V w.r.t. t is transmitted between resistance R used and other parameters such that $V - \frac{g}{R} = ce^{-Rt}$. Obtain a first order differential equation generated from this physical phenomenon given that R, c, g are constants.

$$\frac{dV}{dt} + RV = g$$

Order of an ODE

Definition:

The highest derivative involved in a DE denotes the order of the differential equation.

- Note: $\frac{d^n y}{dx^n}$ is a differential coefficient of order n

Examples:

1. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 1$ Order 2

2. $\frac{d^4 y}{dx^4} - 6xy = 0$ Order 4

Degree of an ODE

Definition:

The power to which the highest ordered derivative is raised after removing the radicals in a DE is called the degree of the differential equation.

Examples:

1. $\left(\frac{d^2y}{dx^2}\right)^2 + 3\left(\frac{dy}{dx}\right)^3 - 3x\frac{dy}{dx} = 2y$ — Degree 2, Order 2

2. $\left(\frac{d^3y}{dx^3}\right)^4 + 3\left(\frac{dy}{dx}\right)^3 - 3\frac{dy}{dx} = \cos x$ — Degree 4, Order 3

3. $\frac{d^3y}{dx^3} + 2xy\left(\frac{dy}{dx}\right)^2 - 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ — Degree 1, Order 3

Linear and Nonlinear ODEs

A linear ODE of order n is given by

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

- The dependent variable y and its various derivatives occur to the first degree only
- There is no product of dependent variable y and its derivatives in the DE

$$3x\frac{d^2 y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$$

Nonlinear

$$\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} - y = 0$$

Linear

Initial and Boundary Conditions

Initial Value Problems (IVP) are differential equations together with initial conditions satisfied by the solution of the ordinary differential equation

- $\frac{dy}{dx} + 3y = x, \quad y(0) = 6$

Boundary Value Problems (BVP) are differential equations together with boundary conditions satisfied by the solution of the ordinary differential equation

- $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = \sin x, \qquad y(0) = 0, y(\pi) = 1$

First Order ODE

First order ODE are equations of the following form:

- $\frac{dy}{dx} = f(x,y) = p(x)q(y)$ (Variable Separable)
- $\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right)$ (Homogeneous)
- $M(x,y)dx + N(x,y)dy = 0$ (Exact)
- $\frac{dy}{dx} + p(x)y = q(x)$ (Linear)
- $\frac{dy}{dx} + p(x)y = q(x)y^n$ (Bernoulli)

- Sometimes the form of these equations determine the method of solution.

Variable Separable

Suppose an ODE

- $\frac{dy}{dx} = f(x, y) = p(x)q(y)$

The equation can be resolved such that we have

$$\frac{dy}{q(y)} = p(x)dx \longrightarrow \text{Separation of Variables}$$

Then we can Integrate both sides to have a solution

$$\int \frac{dy}{q(y)} = \int p(x)dx + c$$

Example 1. Solve the ODE $\frac{dy}{dx} = x^2 y$

Observe that the RHS is variable separable. Hence

$$\frac{dy}{y} = x^2 dx$$

Integrating both sides

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{x^3}{3} + c$$

$$y(x) = Ae^{\frac{x^3}{3}}$$

$$\boxed{A = e^c}$$

Example 2. Solve the ODE $\frac{dy}{dx} = x(1 + y^2)$

Observe that the RHS is variable separable. Hence

$$\frac{dy}{1 + y^2} = xdx$$

Integrating both sides

$$\int \frac{dy}{1 + y^2} = \int xdx$$

$$\Rightarrow \qquad \tan^{-1} y = \frac{x^2}{2} + c$$

$$\Rightarrow \qquad y = \tan\left(\frac{x^2}{2} + c\right)$$

Example 3. Solve the ODE $\frac{dy}{dx} = \frac{1}{2}x(1 - y^2)$

Observe that the RHS is variable separable. Hence

$$\frac{2dy}{1 - y^2} = xdx$$

$$\int \frac{2dy}{1 - y^2} = \int xdx$$

Partial Fraction

$$\frac{2}{1 - y^2} = \frac{A}{1 - y} + \frac{B}{1 + y}$$

$$2 = A(1 + y) + B(1 - y)$$

- $y = 1 \Rightarrow 2 = 2A \Rightarrow A = 1$
- $y = -1 \Rightarrow 2 = 2B \Rightarrow B = 1$

$$\frac{2}{1 - y^2} = \left[\frac{1}{1 + y} + \frac{1}{1 - y}\right]$$

$$\int \frac{2dy}{1-y^2} = \int xdx$$

$$\int \left[\frac{1}{1+y} + \frac{1}{1-y}\right] dy = \int xdx$$

$$\ln|1+y| \; - \ln|1-y| = \frac{x^2}{2} + c$$

$$\Rightarrow \qquad \ln\left[\frac{1+y}{1-y}\right] = \frac{x^2}{2} + c$$

$$\Rightarrow \qquad \left[\frac{1+y}{1-y}\right] = e^{\frac{x^2}{2}+c} = Ae^{\frac{x^2}{2}}$$

$$\boxed{A = e^c}$$

$$\Rightarrow \qquad y = \left[\frac{Ae^{\frac{x^2}{2}} - 1}{Ae^{\frac{x^2}{2}} + 1}\right]$$

Solve the following variable separable ODEs

1. $\frac{dy}{dx} = \sin(x+2)e^{y}$

2. $\frac{dy}{dx} = \sec y \tan x$

3. $x \sin y\, dx + (x^2+1) \cos y\, dy = 0 \qquad y(1) = \frac{\pi}{2}$

4. $(1+x^3)dy - x^2 y dx = 0$