Elementary Mathematics II (Differential Equations and Dynamics) (MTH 102)

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Exact Equations

Definition:

A differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

A first order differential equation is said to be an exact differential equation in domain D if there exist a function F of two variables $(x,y) \in D$ such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \qquad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.

Exact Equations

Theorem:

If M and N have continuous first partial derivatives at all Points $(x, y) \in D$, then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is exact iff

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

Examples: How to show that an ODE is Exact?

1. Show that the DE $2xy\,dx + (1+x^2)dy = 0$?

$$M(x,y) = 2xy \qquad\qquad N(x,y) = (1+x^2)$$

$$\frac{\partial M(x,y)}{\partial y} = 2x \qquad\qquad \frac{\partial N(x,y)}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

- Hence the differential Equation is exact

Examples: How to show that an ODE is Exact?

2. Is the DE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$ exact?

$$M(x,y) = 3x^2 + 4xy \qquad N(x,y) = 2x^2 + 2y$$

$$\frac{\partial M(x,y)}{\partial y} = 4x \qquad \frac{\partial N(x,y)}{\partial x} = 4x$$

$$\Rightarrow \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

- Hence the differential Equation is exact

Examples: How to show that an ODE is Exact?

3. Is the DE $(x^2 + y^2 + x)dx + xydy = 0$ exact?

$$M(x, y) = x^2 + y^2 + x \qquad N(x, y) = xy$$

$$\frac{\partial M(x, y)}{\partial y} = 2y \qquad \frac{\partial N(x, y)}{\partial x} = y$$

$$\Rightarrow \frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$$

- Hence the differential Equation is NOT exact

Examples: How to show that an ODE is Exact?

4. Is the DE $2xydx + (4y + 3x^2)dy = 0$ exact?

$$M(x,y) = 2xy \qquad N(x,y) = 4y + 3x^2$$

$$\frac{\partial M(x,y)}{\partial y} = 2x \qquad \frac{\partial N(x,y)}{\partial x} = 6x$$

$$\Rightarrow \frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$

- Hence the differential Equation is NOT exact

Method of Solution

The Idea is to seek for the function $F(x,y) = c$ such that

If $\frac{\partial F(x,y)}{\partial x} = M(x,y) \Rightarrow F(x,y) = \int M(x,y)dx + \varphi(y)$

But $\frac{\partial F(x,y)}{\partial y} = N(x,y) \Rightarrow \frac{\partial}{\partial y}\int M(x,y)dx + \frac{\partial \varphi(y)}{\partial y} = N(x,y)$

Hence $\frac{\partial \varphi(y)}{\partial y} = N(x,y) - \int M(x,y)dx$

Therefore $\varphi(y) = \int \left[N(x,y) - \int M(x,y)dx\right]dy$

$$\boxed{.F(x,y) = \int M(x,y)\,dx + \int \left[N(x,y) - \int M(x,y)dx\right]dy = c}$$

Method of Solution (Comparison)

The Idea is to seek for the function $F(x, y) = c$ such that

If $\frac{\partial F(x,y)}{\partial x} = M(x, y) \Rightarrow F(x, y) = \int M(x, y)dx + \varphi(y)$

But $\frac{\partial F(x,y)}{\partial y} = N(x, y) \Rightarrow \frac{\partial}{\partial y}\int M(x, y)dx + \frac{\partial \varphi(y)}{\partial y} = N(x, y)$

Hence $\frac{\partial \varphi(y)}{\partial y} = N(x, y) - \int M(x, y)dx$

Therefore $\varphi(y) = \int\left[N(x, y) - \int M(x, y)dx\right]dy$

$$\boxed{.F(x, y) = \int M(x, y)\,dx + \int\left[N(x, y) - \int M(x, y)dx\right]dy = c}$$

Solve the DE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$

- Show that it exact $\Rightarrow$ $M_y = N_x$. Then seek $F(x, y)$ such that $\frac{\partial F(x,y)}{\partial x} = M(x, y)$ & $\frac{\partial F(x,y)}{\partial y} = N(x, y)$

If $\frac{\partial F(x,y)}{\partial x} = 3x^2 + 4xy \Rightarrow F(x, y) = \int(3x^2 + 4xy)\, dx + \varphi(y)$

$$\Rightarrow F(x, y) = x^3 + 2x^2y + \varphi(y)$$

.$\frac{\partial F(x,y)}{\partial y} = N(x, y) \Rightarrow \frac{\partial}{\partial y}[x^3 + 2x^2y + \varphi(y)] = 2x^2 + \frac{\partial\varphi(y)}{\partial y}$

$$.\Rightarrow N(x,y) = 2x^2 + \frac{\partial \varphi(y)}{\partial y}$$

But from the question $N(x,y) = 2x^2 + 2y$

Therefore

$$\frac{\partial \varphi(y)}{\partial y} = 2y$$

Hence

$$\varphi(y) = \int 2y\, dy = y^2$$

$$\boxed{\Rightarrow F(x,y) = x^3 + 2x^2y + y^2 = c}$$

Method of Solution (Grouping)

Solve the DE $(x - xy^2)dx + (8y - x^2y)dy = 0$

- Expand all the terms

$$xdx - xy^2\,dx + 8ydy - x^2ydy = 0$$

- Integrate both sides and for any product x variable multiplying y variable divide by 2

$$.\int xdx - \frac{1}{2}\int xy^2\,dx + \int 8ydy - \frac{1}{2}\int x^2ydy = \int 0$$

$$.\frac{x^2}{2} - \frac{1}{4}x^2y^2 + 4y^2 - \frac{1}{4}x^2y^2 = c$$

$$\boxed{.\ F(x,y) = \frac{x^2}{2} - \frac{x^2y^2}{2} + 4y^2 = c}$$

Exercise / Class-work / Assignment

Show that the following DE are exact and solve

1. $(y^2 + x^2)\dfrac{dy}{dx} + 2xy = 0$

2. $e^y \sin y \dfrac{dy}{dx} + (1 + e^y) = \cos x$

3. $(2xy + 1)dx + (x^2 + 4y)dy = 0$

Integrating Factor (I.F.)

Definition:

If the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is not an exact equation in a domain D for all $(x,y) \in D$, but a function $\mu(x,y)$ can be found such that the DE

$$\mu(x,y)\, M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact in D, then $\mu(x,y)$ is called an **Integrating factor**

$$\boxed{\Rightarrow \frac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \frac{\partial}{\partial x}[\mu(x,y)N(x,y)]}$$

Some rules for obtaining Integrating Factor (I.F.)

1. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = f(x)$,

   then $\mu(x, y) = e^{\int f(x)dx}$ is an I.F.

2. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M} = g(y)$,

   then $\mu(x, y) = e^{-\int g(y)dy}$ is an I.F.

1. Show that the DE $(x^2+y^2+x)dx+xydy=0$ is not exact. Obtain an I.F. and solve completely

$$M(x,y)=x^2+y^2+x \qquad N(x,y)=xy$$

$$M_y=2y \qquad N_x=y$$

$$\Rightarrow M_y\neq N_x$$

But $\frac{M_y-N_x}{N}=\frac{2y-y}{xy}=\frac{1}{x}=f(x)\Rightarrow\mu=e^{\int\frac{dx}{x}}=e^{\ln x}=x$

Therefore the new exact DE becomes

$$(x^3+xy^2+x^2)dx+x^2ydy=0$$

We now solve by grouping method

Expand $\Rightarrow$

$$x^3dx + xy^2dx + x^2\,dx + x^2ydy = 0$$

Integrate both sides and multiply $(x * y)$ by $\frac{1}{2} \Rightarrow$

$$.\int x^3dx + \frac{1}{2}\int xy^2dx + \int x^2\,dx + \frac{1}{2}\int x^2ydy = \int 0$$

$$\frac{x^4}{4} + \frac{1}{4}x^2y^2 + \frac{x^3}{3} + \frac{1}{4}x^2y^2 = c$$

$$\boxed{F(x,y) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 = c}$$

2. Show that the DE $2xydx + (4y + 3x^2)dy = 0$ is not exact. Obtain an I.F. and solve completely

$$M(x, y) = 2xy \qquad N(x, y) = (4y + 3x^2)$$

$$M_y = 2x \qquad N_x = 6x$$

$$\Rightarrow M_y \neq N_x$$

But $\frac{M_y - N_x}{M} = \frac{2x-6x}{2xy} = -\frac{2}{y} = g(y) \Rightarrow \mu = e^{2\int\frac{dy}{y}} = e^{2\ln y} = y^2$

Therefore the new exact DE becomes

$$2xy^3\,dx + (4y^3 + 3x^2y^2)dy = 0$$

We now solve by grouping method

Expand $\Rightarrow$

$$2xy^3\,dx + 4y^3dy + 3x^2y^2dy = 0$$

Integrate both sides and multiply $(x * y)$ by $\frac{1}{2} \Rightarrow$

$$.\frac{1}{2}\int 2xy^3dx + \int 4y^3dy + \frac{1}{2}\int 3x^2y^2dy = \int 0$$

$$\frac{1}{2}x^2y^3 + y^4 + \frac{1}{2}x^2y^3 = c$$

$$\boxed{F(x,y) = x^2y^3 + y^4 = c}$$

Exercise / Class-work / Assignment

Find the I.F. to make the following DE to be exact and solve completely.

1. $(2y - 3x)dx + xdy = 0$

2. $ydx + x(x^2y - 1)dy = 0$

3. $(2y - x^3)dx + xdy = 0$