Elementary Mathematics II (Differential Equations and Dynamics) (MTH 102)

Dr. Julius Ehigie
Dr. Joseph Aroloye

SCHOOL OF SCIENCE AND TECHNOLOGY
PAN-ATLANTIC UNIVERSITY

Homogeneous Equations

Definition:

A function $f(x,y)$ is said to be Homogeneous function if there exists if $f(tx,ty) = f(x,y)$ for all x, y.

Definition:

A first order differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

Which may be sometimes given as $\frac{dy}{dx} = f(x,y)$ is said to be homogeneous, if it can be expressed as

$$\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right)$$

explicitly.

Degree of Homogeneity

Definition:

A first order differential equation written in the form

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} := f(x,y)$$

is said to be homogenous of degree n, if

$$f(tx,ty) = \frac{M(tx,ty)}{N(tx,ty)} = \frac{t^n M(x,y)}{t^n N(x,y)}$$

$$\Rightarrow \frac{t^n M(x,y)}{t^n N(x,y)} = \frac{M(x,y)}{N(x,y)} = f(x,y)$$

- n is the degree of Homogeneity

Examples: How to show that an ODE is Homogeneous?

1. Show that the ODE $\left(y+\sqrt{x^2+y^2}\right)dx - xdy = 0$?

Observe that the ODE $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x} := f(x,y)$

$$f(tx,ty) = \frac{ty+\sqrt{t^2x^2+t^2y^2}}{tx} = \frac{ty+\sqrt{t^2}\cdot\sqrt{x^2+y^2}}{tx}$$

$$f(tx,ty) = \frac{ty+t\sqrt{x^2+y^2}}{tx} = \frac{t^1}{t^1}\cdot\frac{y+\sqrt{x^2+y^2}}{x}$$

$$\Rightarrow f(tx,ty) = f(x,y)$$

- The degree of Homogeneity is 1

2. Verify that the ODE $\frac{dy}{dx} = \frac{xy^3}{2y^4+x^4}$ is Homogeneous?

To Show that the ODE is Homogeneous $x \to tx$, $y \to ty$

Therefore $f(tx, ty) = \frac{tx \cdot (ty)^3}{2(ty)^4+(tx)^4}$

$$f(tx, ty) = \frac{t^4 \cdot xy^3}{2t^4y^4 + t^4x^4} = \frac{t^4 \cdot xy^3}{t^4(2y^4 + x^4)}$$

$$f(tx, ty) = \frac{t^4}{t^4} \cdot \frac{xy^3}{(2y^4 + x^4)} = \frac{xy^3}{2y^4 + x^4}$$

$$\Rightarrow f(tx, ty) = f(x, y)$$

- The degree of Homogeneity is 4

Method of Solution

We now consider the Homogeneous equation $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$

Suppose we take the substitution $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \Rightarrow v + x\frac{dv}{dx} = g(v)$$

We now have a variable separable equation $\frac{dv}{g(v)-v} = \frac{dx}{x}$

Integrating both sides $\int \frac{dv}{g(v)-v} = \int \frac{dx}{x}$

$v = \frac{y}{x} \Rightarrow$ yields $F(v) = \ln x + c \Rightarrow \boxed{F\left(\frac{y}{x}\right) = \ln x + c}$

Example 1. Solve the ODE $\frac{dy}{dx} = \frac{y+xe^{-y/x}}{x}$

Observe that $f(tx,ty) = f(x,y) \Leftrightarrow f(x,y) = g\left(\frac{y}{x}\right)$

Use the substitution $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Therefore $\frac{dy}{dx} = \frac{y+xe^{-y/x}}{x} \Rightarrow v + x\frac{dv}{dx} = v + e^{-v}$

$$\Rightarrow x\frac{dv}{dx} = e^{-v}$$

$$\Rightarrow e^{v}dv = \frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \int e^v dv = \int \frac{dx}{x}$$

$$\Rightarrow e^v = \ln x + c = \ln kx$$

$$\boxed{c = \ln k}$$

$$\Rightarrow v = \ln(\ln kx)$$

Substitute $v = \frac{y}{x}$, Therefore the solution becomes

$$\boxed{\Rightarrow y(x) = x \ln(\ln kx)}$$

Example 2. Solve the ODE $\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$

Observe that $f(tx, ty) = f(x, y) \Leftrightarrow f(x, y) = g\left(\frac{y}{x}\right)$

Use the substitution $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Therefore $\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = v + x\sin v$

$$\Rightarrow x\frac{dv}{dx} = x\sin v$$

$$\Rightarrow \csc v\, dv = dx$$

Integrating both sides

$$\Rightarrow \int \csc v \, dv = \int dx$$

$$\Rightarrow \ln \tan \frac{v}{2} = x + c$$

$$\Rightarrow \tan \frac{v}{2} = Ae^{x}$$

$$\boxed{A = e^{c}}$$

Substitute $v = \frac{y}{x}$, Therefore the solution becomes

$$\boxed{\Rightarrow y(x) = 2x \tan^{-1}\left(Ae^{x}\right)}$$

Example 3. Solve the ODE $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$, $y(1) = -2$

Observe that $f(tx, ty) = f(x, y) \Leftrightarrow f(x, y) = g\left(\frac{y}{x}\right)$

Use the substitution $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Therefore $\frac{dy}{dx} = \frac{x^2+y^2}{xy} \Rightarrow v + x\frac{dv}{dx} = \frac{1}{v} + v$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v}$$

$$\Rightarrow v dv = \frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \int v dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{v^2}{2} = \ln x + c = \ln kx$$

$$\boxed{c = \ln k}$$

$$\Rightarrow v^2 = 2 \ln kx$$

Substitute $v = \frac{y}{x}$, Therefore the solution becomes

$$\Rightarrow y^2 = x^2 (\ln K x^2)$$

$$\boxed{K = k^2}$$

$$\boxed{\Rightarrow y = x\sqrt{\ln K x^2}}$$

We now use the Initial values to obtain c

$y(1) = -2$, means when $x = 1$, $y = -2$

Therefore from the solution, $\boxed{\Rightarrow y = x\sqrt{\ln Kx^2}}$

$$\Rightarrow y(1) = 1 \cdot \left(\sqrt{\ln K}\right) = -2$$

$$\Rightarrow \ln K = 4$$

$$\Rightarrow K = e^4$$

$$\Rightarrow y = x\sqrt{\ln e^4 x^2} = x\sqrt{\ln x^2 + \ln e^4}$$

$$\boxed{\Rightarrow y(x) = x\sqrt{\ln x^2 + 4}}$$

Class Work / Assignment: Solve the following

1. $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

2. $\frac{dy}{dx} = \frac{y^2 - xy - x^2}{x^2}$

3. $(x + y)dx - xdy = 0$

4. $x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}$

5. $x\frac{dy}{dx} = y(\log y - \log x)$