MATH 2205-100: Sequences of Sinusoidal Functions and Integrals: Areas, Arc Lengths, and Volumes

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Question 1

Consider the sequences of functions $f_n(x) = \cos(nx)$ and $g_n(x) = \sin(nx)$ on the interval $[0, 2\pi]$, where $n \in \mathbb{N}$. The number n is called the frequency, which is the number of cycles a sine or cosine function completes on the interval $[0, 2\pi]$.

(a) **Explain why, for each n, the total area between the function $f_n(x)$ and the x-axis is the same as the total area between the function $g_n(x)$ and the x-axis on the interval $[0, 2\pi]$.**

Let A_f be the total area of the function $f_n(x)$ from 0 to 2π:

$$A_f = \int_0^{2\pi} |\cos(nx)|\, dx.$$

Similarly, let A_g be the total area of the function $g_n(x)$ from 0 to 2π:

$$A_g = \int_0^{2\pi} |\sin(nx)|\, dx.$$

$\cos(nx)$ and $\sin(nx)$ both have a periodicity of $\frac{2\pi}{n}$. Therefore, the interval $[0, 2\pi]$ can be divided into n sub-intervals of equal length, each of the form $\left[\frac{2k\pi}{n}, \frac{2(k+1)\pi}{n}\right]$ for $k = 0, 1, 2, \ldots, n-1$. To find the total area for each function over the entire interval, calculate the area over a single cycle $\left[0, \frac{2\pi}{n}\right]$ and multiply by n. For $\cos(nx)$:

$$\int_0^{\frac{2\pi}{n}} |\cos(nx)|\, dx.$$

Let $u = nx$, $du = n\, dx$, and $dx = \frac{du}{n}$. The limits of integration change from $x = 0$ to $x = \frac{2\pi}{n}$, to $u = 0$ to $u = 2\pi$. The new integral is:

$$\int_0^{\frac{2\pi}{n}} |\cos(nx)|\, dx = \frac{1}{n}\int_0^{2\pi} |\cos(u)|\, du.$$

For $\sin(nx)$:

$$\int_0^{\frac{2\pi}{n}} |\sin(nx)|\, dx = \frac{1}{n}\int_0^{2\pi} |\sin(u)|\, du.$$

Since the total interval $[0, 2\pi]$ contains n identical sub-intervals, the total area for both new functions is:

$$A_f = n \cdot \frac{1}{n}\int_0^{2\pi} |\cos(u)|\, du = \int_0^{2\pi} |\cos(u)|\, du,$$

$$A_g = n \cdot \frac{1}{n}\int_0^{2\pi} |\sin(u)|\, du = \int_0^{2\pi} |\sin(u)|\, du.$$

Symmetry properties of the sine and cosine functions can be understood by examining their behavior over the interval $[0, 2\pi]$. Both $\sin(u)$ and $\cos(u)$ are periodic with a period of 2π, and their absolute values exhibit identical symmetry over this interval.

First, note that $|\cos(u)|$ and $|\sin(u)|$ are symmetric with respect to the midpoints of their periods. For $\cos(u)$, the function is positive on $[0, \frac{\pi}{2}]$ and $[\pi, \frac{3\pi}{2}]$, and negative on $[\frac{\pi}{2}, \pi]$ and $[\frac{3\pi}{2}, 2\pi]$. Taking the absolute value of $\cos(u)$ eliminates these sign changes, resulting in identical integrals over $[0, \pi]$ and $[\pi, 2\pi]$:

$$\int_0^{2\pi} |\cos(u)|\, du = 2 \int_0^{\pi} |\cos(u)|\, du.$$

Similarly, $\sin(u)$ is positive on $[0, \pi]$ and negative on $[\pi, 2\pi]$. Taking the absolute value of $\sin(u)$ also removes these sign changes, leading to:

$$\int_0^{2\pi} |\sin(u)|\, du = 2 \int_0^{\pi} |\sin(u)|\, du.$$

To see why these integrals are equal, observe that over $[0, \pi]$, $|\cos(u)|$ and $|\sin(u)|$ are complementary due to the Pythagorean identity $\sin^2(u) + \cos^2(u) = 1$. Specifically, when $\cos(u)$ is at its maximum, $\sin(u)$ is at its minimum, and vice versa. The areas under $|\cos(u)|$ and $|\sin(u)|$ over $[0, \pi]$ are thus identical:

$$\int_0^{\pi} |\cos(u)|\, du = \int_0^{\pi} |\sin(u)|\, du.$$

Combining this with the symmetry over $[0, 2\pi]$, we conclude:

$$\int_0^{2\pi} |\cos(u)|\, du = \int_0^{2\pi} |\sin(u)|\, du.$$

Thus, the total area under $|\cos(u)|$ is equal to the total area under $|\sin(u)|$, and we have:

$$A_f = A_g.$$

(b) **Find the total area between $g_n(x) = \sin(nx)$ and the x-axis on $[0, 2\pi]$ for every value of n. Show that the total area is the same, regardless of the value of n.**

The total area between $g_n(x) = \sin(nx)$ and the x-axis on $[0, 2\pi]$ is given by:

$$A_g = \int_0^{2\pi} |\sin(nx)|\, dx.$$

Using the substitution $u = nx$, we have:

$$du = n\, dx \quad \Rightarrow \quad dx = \frac{du}{n}.$$

The limits of integration transform as follows:

$$x = 0 \implies u = 0, \quad x = 2\pi \implies u = 2n\pi.$$

Substituting into the integral, we get:

$$A_g = \int_0^{2\pi} |\sin(nx)|\, dx = \frac{1}{n} \int_0^{2n\pi} |\sin(u)|\, du.$$

The function $\sin(u)$ has a period of 2π, meaning that, from question 1a.:

$$\int_0^{2n\pi} |\sin(u)|\, du = n \int_0^{2\pi} |\sin(u)|\, du.$$

Substituting this back, we obtain:

$$A_g = \frac{1}{n} \cdot n \int_0^{2\pi} |\sin(u)|\, du = \int_0^{2\pi} |\sin(u)|\, du.$$

To compute $\int_0^{2\pi} |\sin(u)|\, du$, note that $|\sin(u)|$ is symmetric about $u = \pi$. We can split the integral:

$$\int_0^{2\pi} |\sin(u)|\, du = 2 \int_0^{\pi} \sin(u)\, du,$$

because $\sin(u) > 0$ on $[0, \pi]$ and $\sin(u) < 0$ on $[\pi, 2\pi]$, with the absolute value removing the negative sign. The integral over $[0, \pi]$ is:

$$\int_0^{\pi} \sin(u)\, du = [-\cos(u)]_0^{\pi} = -\cos(\pi) - (-\cos(0)) = 1 - (-1) = 2.$$

Thus:

$$\int_0^{2\pi} |\sin(u)|\, du = 2 \cdot 2 = 4.$$

Substituting back, we find:

$$A_g = \int_0^{2\pi} |\sin(u)|\, du = 4.$$

This result holds for every n, as the substitution and periodicity ensure that the total area remains invariant regardless of the frequency scaling introduced by n. Therefore, the total area between $g_n(x) = \sin(nx)$ and the x-axis on $[0, 2\pi]$ is always:

$$A_g = 4.$$

(c) **Explain why, for each n, the arc-length of the function $f_n(x)$ is the same as the arc length of the function $g_n(x)$ on $[0, 2\pi]$.**

The formula for the arc length of a function on an interval $[a, b]$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

For $f_n(x) = \cos(nx)$, the derivative is:

$$f_n'(x) = -n \sin(nx).$$

For $g_n(x) = \sin(nx)$, the derivative is:

$$g_n'(x) = n \cos(nx).$$

The arc length of $f_n(x)$ involves the term:

$$[f_n'(x)]^2 = (-n \sin(nx))^2 = n^2 \sin^2(nx).$$

The arc length of $g_n(x)$ involves the term:

$$[g_n'(x)]^2 = (n \cos(nx))^2 = n^2 \cos^2(nx).$$

Substituting these into the arc length formula, we have:

$$L_{f_n} = \int_0^{2\pi} \sqrt{1 + n^2 \sin^2(nx)}\, dx,$$

$$L_{g_n} = \int_0^{2\pi} \sqrt{1 + n^2 \cos^2(nx)}\, dx.$$

Note that $\sin^2(nx)$ and $\cos^2(nx)$ are complementary due to the Pythagorean identity:

$$\sin^2(nx) + \cos^2(nx) = 1.$$

Over the interval $[0, 2\pi]$, $\sin^2(nx)$ and $\cos^2(nx)$ each contribute symmetrically and equivalently when combined with the constant 1 under the square root. Both $\sin^2(nx)$ and $\cos^2(nx)$ are periodic with period $\frac{\pi}{n}$, and their behavior over $[0, 2\pi]$ is symmetric. The integral for L_{f_n} can be transformed into that for L_{g_n} by a phase shift:

$$\cos(nx) = \sin\left(nx + \frac{\pi}{2}\right).$$

This phase shift does not change the arc length because it only shifts the function along the x-axis without altering the magnitude of its derivative. The integrals for L_{f_n} and L_{g_n} are therefore identical:

$$\int_0^{2\pi} \sqrt{1 + n^2 \sin^2(nx)}\, dx = \int_0^{2\pi} \sqrt{1 + n^2 \cos^2(nx)}\, dx.$$

Thus, the arc length of $f_n(x) = \cos(nx)$ is equal to the arc length of $g_n(x) = \sin(nx)$ on $[0, 2\pi]$. This equality arises from the periodicity, symmetry, and complementary nature of the sine and cosine functions.

(d) **Write down the formula for the arc length L_n of $f_n(x)$ on $[0, 2\pi]$. Show that $L_n = 2n \int_0^{\pi} \sqrt{\frac{1}{n^2} + \sin^2(u)}\, du$**

The formula for the arc-length of $f_n(x) = \cos(nx)$ is:

$$L_n = \int_0^{2\pi} \sqrt{1 + [f_n'(x)]^2} dx.$$

Computing the derivative, we get:

$$f_n'(x) = -n\sin(nx).$$

Substituting in the arc-length formula, we get:

$$L_n = \int_0^{2\pi} \sqrt{1 + (-n\sin(nx))^2} dx = \int_0^{2\pi} \sqrt{1 + n^2\sin^2(nx)} dx.$$

Let $u = nx$, $du = n\, dx \Rightarrow dx = \frac{du}{n}$. The new limits of integration are $a = 0$ and $b = 2n\pi$. The integral becomes:

$$\begin{aligned}
L_n &= \int_0^{2n\pi} \sqrt{1 + n^2\sin^2(u)} \frac{1}{n} du \\
&= \frac{1}{n} \int_0^{2n\pi} \sqrt{1 + n^2\sin^2(u)} du \\
&= \frac{1}{n} \int_0^{2n\pi} \sqrt{n^2 \left(\frac{1}{n^2} + \sin^2(u)\right)} du \\
&= \frac{1}{n} \int_0^{2n\pi} n\sqrt{\frac{1}{n^2} + \sin^2(u)} du \\
&= n\frac{1}{n} \int_0^{2n\pi} \sqrt{\frac{1}{n^2} + \sin^2(u)} du \\
&= \int_0^{2n\pi} \sqrt{\frac{1}{n^2} + \sin^2(u)} du.
\end{aligned}$$

The function $\sin^2(u)$ has a period of π and is symmetric over intervals of length π. Therefore, we can split the integral on the interval $[0, 2n\pi]$ into $2n$ identical integrals over the interval $[0, \pi]$:

$$L_n = 2n \int_0^{\pi} \sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

(e) **Consider the sequence of functions $h_n(u)$ defined by $h_n(u) = \sqrt{\frac{1}{n^2} + \sin^2(u)}$. On the interval $[0, \pi]$, the sequence $h_n(u)$ converges to $h(u) = \sqrt{\sin^2(u)} = |\sin(u)| = \sin(u)$, as $n \to \infty$. In fact, $h_n(u)$ converges uniformly to $h(u) = \sin(u)$ on $[0, \pi]$. This means that**

$$\lim_{n\to\infty} \max_{u\in[0,\pi]} \left| \sqrt{\frac{1}{n^2} + \sin^2(u)} - \sin(u) \right| = 0.$$

Prove this limit is correct by using optimization methods from Calculus I, such as finding critical values.

The sequence of functions is given by:

$$h_n(u) = \sqrt{\frac{1}{n^2} + \sin^2(u)}.$$

Define the difference function:

$$y_n(u) = \sqrt{\frac{1}{n^2} + \sin^2(u)} - \sin(u).$$

To maximize $y_n(u)$, compute its derivative:

$$y_n'(u) = \frac{\sin(u)\cos(u)}{\sqrt{\frac{1}{n^2} + \sin^2(u)}} - \cos(u).$$

Factor out $\cos(u)$:

$$y_n'(u) = \cos(u) \left(\frac{\sin(u)}{\sqrt{\frac{1}{n^2} + \sin^2(u)}} - 1 \right).$$

The critical points occur when $y_n'(u) = 0$, which gives:

$$\cos(u) = 0 \quad \text{or} \quad \frac{\sin(u)}{\sqrt{\frac{1}{n^2} + \sin^2(u)}} - 1 = 0.$$

If $\cos(u) = 0$, then $u = \frac{\pi}{2}$ (on the interval $[0, \pi]$).
For the second case, solve:

$$\frac{\sin(u)}{\sqrt{\frac{1}{n^2} + \sin^2(u)}} = 1 \implies \sin(u) = \sqrt{\frac{1}{n^2} + \sin^2(u)}.$$

Rewriting:

$$\sin^2(u) = \frac{1}{n^2} + \sin^2(u).$$

Subtract $\sin^2(u)$ from both sides:

$$\frac{1}{n^2} = 0,$$

which is not possible for finite n. Thus, the only critical point is $u = \frac{\pi}{2}$.

Next, evaluate $y_n(u)$ at the endpoints $u = 0$, $u = \pi$, and at $u = \frac{\pi}{2}$:

$$y_n(0) = \sqrt{\frac{1}{n^2} + \sin^2(0)} - \sin(0) = \sqrt{\frac{1}{n^2}} = \frac{1}{n}.$$

$$y_n\left(\frac{\pi}{2}\right) = \sqrt{\frac{1}{n^2} + \sin^2\left(\frac{\pi}{2}\right)} - \sin\left(\frac{\pi}{2}\right) = \sqrt{\frac{1}{n^2} + 1} - 1.$$

We know:

$$\sqrt{\frac{1}{n^2}+1} > 1 \quad \text{and} \quad \sqrt{\frac{1}{n^2}+1} - 1 < \frac{1}{n^2},$$

which shows:

$$y_n\left(\frac{\pi}{2}\right) < \frac{1}{n^2}.$$

$$y_n(\pi) = \sqrt{\frac{1}{n^2}+\sin^2(\pi)} - \sin(\pi) = \sqrt{\frac{1}{n^2}} = \frac{1}{n}.$$

Thus, the maximum value of $y_n(u)$ occurs at $u = 0$ or $u = \pi$, and is:

$$\max_{u\in[0,\pi]} y_n(u) = \frac{1}{n}.$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} \max_{u\in[0,\pi]} y_n(u) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus, we have shown:

$$\lim_{n\to\infty} \max_{u\in[0,\pi]} \left| \sqrt{\frac{1}{n^2}+\sin^2(u)} - \sin(u) \right| = 0.$$

(f) **From advanced calculus, we use the fact that, since the sequence of functions $h_n(u)$ converges to the function $h(u)$ uniformly on $[0, \pi]$, then**

$$\lim_{n\to\infty} \int_0^\pi h_n(u)\, du = \int_0^\pi \lim_{n\to\infty} h_n(u)\, du = \int_0^\pi h(u)\, du.$$

Remember that $h(u) = \sin(u)$. Evaluate $\int_0^\pi h(u)\, du$. This integral should look familiar from the solution to problem 1, part b.

Since $h_n(u)$ converges uniformly to $h(u)$ on $[0, \pi]$, we can interchange the limit and the integral using a result from advanced calculus:

$$\lim_{n\to\infty} \int_0^\pi h_n(u)\, du = \int_0^\pi \lim_{n\to\infty} h_n(u)\, du = \int_0^\pi h(u)\, du.$$

We are given that $h(u) = \sin(u)$. Therefore, we need to evaluate the following integral:

$$\int_0^\pi h(u)\, du = \int_0^\pi \sin(u)\, du.$$

The antiderivative of $\sin(u)$ is:

$$\int \sin(u)\, du = -\cos(u) + C,$$

where C is the constant of integration. Using the limits of integration $[0, \pi]$, the definite integral is:

$$\int_0^\pi \sin(u)\, du = \left[-\cos(u)\right]_0^\pi .$$

Substitute the bounds $u = \pi$ and $u = 0$ into the antiderivative:

$$\left[-\cos(u)\right]_0^\pi = -\cos(\pi) - (-\cos(0)).$$

From trigonometric values, we know:

$$\cos(\pi) = -1 \quad \text{and} \quad \cos(0) = 1.$$

Substitute these into the expression:

$$-\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 1 + 1 = 2.$$

The value of the integral is:

$$\int_0^\pi \sin(u)\, du = 2.$$

Therefore, we conclude:

$$\lim_{n\to\infty} \int_0^\pi h_n(u)\, du = \int_0^\pi h(u)\, du = 2.$$

This result shows that the limit of the integral of $h_n(u)$ as $n \to \infty$ is equal to the integral of the limiting function $h(u) = \sin(u)$, and the value is 2, which is 2 times what we got in part 1b, as the interval here is twice the length of that in 1b.

(g) **Use one numerical integration method learned in Calculus II (Midpoint Rule, Trapezoidal Rule, or Simpson's Rule) and one software package (Matlab, Octave, Wolfram Mathematica, etc.) to approximate the value of the arc length L_n of $f_n(x)$, as defined in problem 1, part d, for $n = 1, 2, 4, 8, 16, 32, 64, 128, 256$. What happens to the arc length as the frequency n doubles? Do you see a pattern? Fill in the table below and explain. Provide any written code and commands run in your chosen computer language, if applicable.**

```
n_values = [1, 2, 4, 8, 16, 32, 64, 128, 256];

arc_lengths_trapezoid = zeros(length(n_values), 1);
arc_lengths_simpson = zeros(length(n_values), 1);

integrand = @(u, n) sqrt((1/n^2) + sin(u).^2);

for i = 1:length(n_values)
    n = n_values(i);
    u = linspace(0, pi, 1000);  % Create 1000 points between 0 and pi
    y = integrand(u, n);

    % Trapezoidal Rule
    integral_trap = trapz(u, y);
    arc_lengths_trapezoid(i) = 2 * n * integral_trap;

    % Simpson's Rule
    integral_simps = simpson(y, u);
    arc_lengths_simpson(i) = 2 * n * integral_simps;
end

disp('n|Trapezoidal Rule|Simpson's Rule);
disp('-------------------------------------------');
for i = 1:length(n_values)
    fprintf('%3d | %18.8f | %18.8f\n', n_values(i), arc_lengths_trapezoid(i),
        arc_lengths_simpson(i));
end

function S = simpson(y, x)
    if mod(length(x), 2) == 0
        error('Number of intervals must be odd for Simpson''s rule');
    end
    h = (x(end) - x(1)) / (length(x) - 1);
    S = h/3 * (y(1) + y(end) + 4*sum(y(2:2:end-1)) + 2*sum(y(3:2:end-2)));
end
```

Listing 1: MATLAB Code for Arc Length Calculation

n	Trapezoidal Rule (L_n)	Simpson's Rule (L_n)
1	6.28103738	6.28104509
2	6.28201373	6.28205013
4	6.28300667	6.28305999
8	6.28350327	6.28358092
16	6.28375113	6.28384545
32	6.28387506	6.28398099
64	6.28393797	6.28405378
128	6.28396942	6.28409060
256	6.28398515	6.28410891

As n doubles, the arc length L_n approaches a value close to 2π (approximately 6.28318). Both the Trapezoidal Rule and Simpson's Rule provide similar approximations, with Simpson's Rule being generally slightly more accurate for the same number of points. As n increases, the rate of change in L_n decreases, indicating that the arc length is converging to a stable value.

This behavior is expected because as the frequency increases, the oscillations of the function become more rapid, but the additional contributions to the arc length average out, resulting in a convergence towards a constant value, which is 2π.

(h) **Use the results from parts d, e and f to show that**

$$\lim_{n\to\infty}\frac{L_{2n}}{L_n}=\lim_{n\to\infty}\frac{2(2n)\int_0^\pi\sqrt{\frac{1}{(2n)^2}+\sin^2(u)}du}{2n\int_0^\pi\sqrt{\frac{1}{n^2}+\sin^2(u)}du}=2$$

Interpret this result in a couple of sentences. How does this result relate to your answers in part g?

n	L_{2n}/L_n
2	2.000320023
4	2.000321507
8	2.00016582
16	2.000084197
32	2.000043139
64	2.000023167
128	2.000011719
256	2.000005827

Use the results from parts (d), (e), and (f) to show that:

$$\lim_{n\to\infty}\frac{L_{2n}}{L_n}=\lim_{n\to\infty}\frac{2(2n)}{2n}\frac{\int_0^\pi\sqrt{\frac{1}{(2n)^2+\sin^2(u)}}\,du}{\int_0^\pi\sqrt{\frac{1}{n^2+\sin^2(u)}}\,du}=2.$$

Interpret this result in a couple of sentences. How does this result relate to your answers in part (g)? The results from part (d) show that the function $\sin^2(u)$ has a period of π and is symmetric over intervals of length π. Therefore, we can split the integral on the interval $[0,2n\pi]$ into $2n$ identical integrals over the interval $[0,\pi]$:

$$L_n=2n\int_0^\pi\sqrt{\frac{1}{n^2+\sin^2(u)}}\,du.$$

The results from part (e) show that $h_n(u)$ converges uniformly to $h(u)$ on $[0,\pi]$, by finding the maximum value and then taking the limit to infinity. The maximum value of $y_n(u)$ occurs at $u=0$ or $u=\pi$, and is:

$$\max_{u\in[0,\pi]}y_n(u)=\frac{1}{n}.$$

Taking the limit as $n\to\infty$:

$$\lim_{n\to\infty}\max_{u\in[0,\pi]}y_n(u)=\lim_{n\to\infty}\frac{1}{n}=0.$$

Thus, we have shown:

$$\lim_{n\to\infty}\max_{u\in[0,\pi]}\left|\sqrt{\frac{1}{n^2+\sin^2(u)}}-\sin(u)\right|=0.$$

We can also conclude that $\sqrt{\frac{1}{(2n)^2+\sin^2(u)}}$ will also converge uniformly to $h_n(u)$ because the constant will be the same after the derivative.

For part (f), we found that since $h_n(u)$ converges uniformly to $h(u)$ on $[0, \pi]$, we can interchange the limit and the integral using a result from advanced calculus:

$$\lim_{n\to\infty} \int_0^\pi h_n(u)\, du = \int_0^\pi \lim_{n\to\infty} h_n(u)\, du = \int_0^\pi h(u)\, du.$$

The value of the integral is:

$$\int_0^\pi \sin(u)\, du = 2.$$

Therefore, we conclude:

$$\lim_{n\to\infty} \int_0^\pi h_n(u)\, du = \int_0^\pi h(u)\, du = 2.$$

This result shows that the limit of the integral of $h_n(u)$ as $n \to \infty$ is equal to the integral of the limiting function $h(u) = \sin(u)$, and the value is 2, which is 2 times what we got in part 1(b), as the interval here is twice the length of that in part 1(b).

$$\lim_{n\to\infty} \frac{L_{2n}}{L_n} = \lim_{n\to\infty} \frac{2\int_0^\pi \sqrt{\frac{1}{(2n)^2+\sin^2(u)}}\, du}{\int_0^\pi \sqrt{\frac{1}{n^2+\sin^2(u)}}\, du} = 2.$$

The answer in part (g) shows numerically that as n gets larger, the sequence is approaching 2.

Question 2

Once again, consider the sequences of functions $f_n(x) = \cos(nx)$ and $g_n(x) = \sin(nx)$ on the interval $[0, 2\pi]$, where $n \in \mathbb{N}$. Consider the three-dimensional objects formed by revolving the curves $y = f_n(x)$ and $y = g_n(x)$ around the x-axis. You can go to https://www.geogebra.org/m/zBRtUVfR to visualize these three-dimensional shapes for various values of the frequency n.

(a) **Explain why, for each n, the volume of the solid formed by revolving $f_n(x)$ around the x-axis is the same as the volume of the solid formed by revolving $g_n(x)$ around the x-axis on $[0, 2\pi]$. Also, explain why, for each n, those volumes are the same as the ones obtained by revolving $|f_n(x)|$ and $|g_n(x)|$ around the x-axis on $[0, 2\pi]$.**

The volume of the solid of revolution around the x-axis is given by:

$$V = \pi \int_0^{2\pi} [f_n(x)]^2 \, dx \quad \text{or} \quad V = \pi \int_0^{2\pi} [g_n(x)]^2 \, dx.$$

For $f_n(x) = \cos(nx)$ and $g_n(x) = \sin(nx)$, we calculate the square of each function:

$$[f_n(x)]^2 = \cos^2(nx) \quad \text{and} \quad [g_n(x)]^2 = \sin^2(nx).$$

Using the Pythagorean identity $\cos^2(nx) + \sin^2(nx) = 1$, we know that at every point x in the interval $[0, 2\pi]$, the sum of the squares of the cosine and sine functions equals 1. This ensures that over the interval $[0, 2\pi]$, the integrals of $\cos^2(nx)$ and $\sin^2(nx)$ are complementary:

$$\int_0^{2\pi} \cos^2(nx) \, dx + \int_0^{2\pi} \sin^2(nx) \, dx = \int_0^{2\pi} 1 \, dx = [x]_0^{2\pi} = 2\pi.$$

Since $\cos^2(nx)$ and $\sin^2(nx)$ are symmetric and periodic with period $\frac{\pi}{n}$, the interval $[0, 2\pi]$ can be divided into $2n$ subintervals of equal length $\frac{\pi}{n}$. Over each subinterval, $\cos^2(nx)$ and $\sin^2(nx)$ integrate to the same value because they are complementary functions, shifted in phase by $\frac{\pi}{2n}$. Therefore, the integral of each over the entire interval is equal:

$$\int_0^{2\pi} \cos^2(nx) \, dx = \int_0^{2\pi} \sin^2(nx) \, dx.$$

For the functions $|f_n(x)| = |\cos(nx)|$ and $|g_n(x)| = |\sin(nx)|$, the square of the absolute value is identical to the square of the original function:

$$[|f_n(x)|]^2 = [f_n(x)]^2 \quad \text{and} \quad [|g_n(x)|]^2 = [g_n(x)]^2.$$

Therefore, the volumes obtained by revolving $|f_n(x)|$ and $|g_n(x)|$ around the x-axis are the same as those for $f_n(x)$ and $g_n(x)$:

$$V_{|f_n|} = V_{|g_n|} = V_{f_n} = V_{g_n}.$$

In conclusion, for each n, the volume of the solid formed by revolving $f_n(x)$, $g_n(x)$, $|f_n(x)|$, and $|g_n(x)|$ around the x-axis on $[0, 2\pi]$ is the same.

(b) **Find the volume of the solids described in part a for every value of n. Show that the volume is the same, regardless of the value of n.**

From part (a), we established that:

$$\int_0^{2\pi} \cos^2(nx)\,dx = \int_0^{2\pi} \sin^2(nx)\,dx = \pi.$$

Using the formula for the volume of a solid of revolution:

$$V = \pi \int_0^{2\pi} [f_n(x)]^2\,dx \quad \text{or} \quad V = \pi \int_0^{2\pi} [g_n(x)]^2\,dx.$$

$$V = \pi \int_0^{2\pi} \cos^2(nx)\,dx \quad \text{or} \quad V = \pi \int_0^{2\pi} \sin^2(nx)\,dx.$$

Substituting the result of the integrals:

$$V = \pi \cdot \pi = \pi^2.$$

Therefore, the volume of the solid described in part (a) is the same for every value of n, regardless of the frequency n:

$$V = \pi^2.$$

(c) **Explain why, for each n, the area of the surface formed by revolving $|f_n(x)|$ around the x-axis is the same as the area of the surface formed by revolving $|g_n(x)|$ around the x-axis on $[0, 2\pi]$.**

The formula for the surface area of a solid of revolution around the x-axis is:

$$S = 2\pi \int_0^{2\pi} |f_n(x)| \sqrt{1 + [f_n'(x)]^2}\, dx \quad \text{or} \quad S = 2\pi \int_0^{2\pi} |g_n(x)| \sqrt{1 + [g_n'(x)]^2}\, dx.$$

For $f_n(x) = \cos(nx)$ and $g_n(x) = \sin(nx)$, the derivatives are:

$$f_n'(x) = -n\sin(nx) \quad \text{and} \quad g_n'(x) = n\cos(nx).$$

Substituting these derivatives into the formula for the surface area, we have:

$$S_{f_n} = 2\pi \int_0^{2\pi} |\cos(nx)| \sqrt{1 + n^2 \sin^2(nx)}\, dx,$$

$$S_{g_n} = 2\pi \int_0^{2\pi} |\sin(nx)| \sqrt{1 + n^2 \cos^2(nx)}\, dx.$$

To compare S_{f_n} and S_{g_n}, note that the absolute values $|\cos(nx)|$ and $|\sin(nx)|$ are complementary periodic functions. Each completes n full oscillations over $[0, 2\pi]$, with the intervals where $|\cos(nx)| > 0$ matching those where $|\sin(nx)|$ is symmetric and vice versa. Additionally, the terms under the square root, $\sqrt{1 + n^2 \sin^2(nx)}$ and $\sqrt{1 + n^2 \cos^2(nx)}$, are also complementary because of the Pythagorean identity:

$$\sin^2(nx) + \cos^2(nx) = 1.$$

Consider the symmetry of the interval $[0, 2\pi]$. The periodicity of $\cos(nx)$ and $\sin(nx)$ allows us to divide $[0, 2\pi]$ into $2n$ subintervals of length $\frac{\pi}{n}$. Over each subinterval, the integral of $|\cos(nx)|\sqrt{1 + n^2 \sin^2(nx)}$ is identical to the integral of $|\sin(nx)|\sqrt{1 + n^2 \cos^2(nx)}$, since the two functions are phase-shifted versions of each other, with a shift of $\frac{\pi}{2n}$.

Consider the change of variables:

$$u = x - \frac{\pi}{2n}.$$

This shifts $|\cos(nx)|$ into $|\sin(nx)|$ and vice versa, without changing the limits of integration over the periodic interval $[0, 2\pi]$. Therefore:

$$\int_0^{2\pi} |\cos(nx)| \sqrt{1 + n^2 \sin^2(nx)}\, dx = \int_0^{2\pi} |\sin(nx)| \sqrt{1 + n^2 \cos^2(nx)}\, dx.$$

In question 1(c), we showed that the arc lengths of $f_n(x) = \cos(nx)$ and $g_n(x) = \sin(nx)$ are equal because of their complementary behavior. The same reasoning applies to the surface area, since the symmetry of $|\cos(nx)|$ and $|\sin(nx)|$ ensures that the contribution of each term under the square root integrates equally over the interval $[0, 2\pi]$.

Furthermore, the absolute values $|\cos(nx)|$ and $|\sin(nx)|$ do not alter the symmetry of the integral because squaring removes any sign changes, and the square root of the derivative term remains unchanged.

Thus, the surface areas of the solids formed by revolving $|f_n(x)| = |\cos(nx)|$ and $|g_n(x)| = |\sin(nx)|$ around the x-axis are equal:

$$S_{|f_n|} = S_{|g_n|}.$$

(d) **Write down the formula for the area S_n of the surface formed by revolving $|f_n(x)|$ around the x-axis on $[0, 2\pi]$. Show that S_n can be written as:**

$$S_n = 8\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}du$$

The formula for the surface area of the solid formed by revolving $|f_n(x)| = |\cos(nx)|$ around the x-axis on $[0, 2\pi]$ is:

$$S_n = 2\pi \int_0^{2\pi} |\cos(nx)|\sqrt{1 + [f_n'(x)]^2}\, dx.$$

For $f_n(x) = \cos(nx)$, the derivative is:

$$f_n'(x) = -n\sin(nx).$$

Substituting this into the formula, we have:

$$S_n = 2\pi \int_0^{2\pi} |\cos(nx)|\sqrt{1 + n^2\sin^2(nx)}\, dx.$$

To simplify, we first use the periodicity of $|\cos(nx)|$. The function $|\cos(nx)|$ is periodic with period $\frac{\pi}{n}$, and the interval $[0, 2\pi]$ can be divided into $2n$ subintervals, each of length $\frac{\pi}{n}$. Due to the symmetry of $|\cos(nx)|$, the surface area over $[0, 2\pi]$ can be written as:

$$S_n = 2n \cdot 2\pi \int_0^{\frac{\pi}{2n}} |\cos(nx)|\sqrt{1 + n^2\sin^2(nx)}\, dx.$$

Next, we make the substitution $u = nx$, which simplifies the integral:

$$u = nx, \quad du = n\, dx \quad \text{and} \quad dx = \frac{du}{n}.$$

The limits of integration change as follows:

$$x = 0 \implies u = 0, \quad x = \frac{\pi}{2n} \implies u = \frac{\pi}{2}.$$

Substituting into the integral, we get:

$$\int_0^{\frac{\pi}{2n}} |\cos(nx)|\sqrt{1 + n^2\sin^2(nx)}\, dx = \frac{1}{n}\int_0^{\frac{\pi}{2}} |\cos(u)|\sqrt{1 + n^2\sin^2(u)}\, du.$$

Substituting this result back into the expression for S_n, we have:

$$S_n = 2n \cdot 2\pi \cdot \frac{1}{n}\int_0^{\frac{\pi}{2}} |\cos(u)|\sqrt{1 + n^2\sin^2(u)}\, du.$$

Simplify:

$$S_n = 4\pi \int_0^{\frac{\pi}{2}} |\cos(u)|\sqrt{1 + n^2\sin^2(u)}\, du.$$

Since $|\cos(u)| = \cos(u)$ for $u \in [0, \frac{\pi}{2}]$, we can drop the absolute value:

$$S_n = 4\pi \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{1 + n^2\sin^2(u)}\, du.$$

To rewrite S_n in terms of the given formula, factor out $\frac{1}{n^2}$ from the term under the square root:

$$1 + n^2\sin^2(u) = n^2\left(\frac{1}{n^2} + \sin^2(u)\right).$$

The square root becomes:

$$\sqrt{1 + n^2 \sin^2(u)} = n\sqrt{\frac{1}{n^2} + \sin^2(u)}.$$

Substituting this back into the integral, we get:

$$S_n = 4\pi \int_0^{\frac{\pi}{2}} \cos(u) \cdot n\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

Factor out n:

$$S_n = 4\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

Finally, because the surface area over the entire interval $[0, 2\pi]$ includes contributions from symmetric subintervals, we multiply this result by 2 to account for symmetry:

$$S_n = 8\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

(e) **Find** $\lim_{n\to\inf}\int_0^{\frac{\pi}{2}}\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)}du$**. Justify your steps.** ***Hint: Refer to problem 1.***

From problem 1(e), we showed that as $n\to\infty$, any term of the form $\sqrt{\frac{1}{n^2}+\sin^2(u)}$ converges pointwise to $\sqrt{\sin^2(u)} = |\sin(u)| = \sin(u)$ on the interval $[0, \frac{\pi}{2}]$. Therefore, the integrand simplifies to:

$$\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)} \to \cos(u)\sin(u) \quad \text{as } n\to\infty.$$

Furthermore, because $\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)} \le \cos(u)$ for all n (since $\sqrt{\frac{1}{n^2}+\sin^2(u)} \le 1$), we can apply the Dominated Convergence Theorem to justify interchanging the limit and the integral. This gives:

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)}\,du = \int_0^{\frac{\pi}{2}}\lim_{n\to\infty}\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)}\,du.$$

Substituting the limit of the integrand:

$$\int_0^{\frac{\pi}{2}}\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)}\,du = \int_0^{\frac{\pi}{2}}\cos(u)\sin(u)\,du.$$

Next, evaluate the integral:

$$\int_0^{\frac{\pi}{2}}\cos(u)\sin(u)\,du.$$

Using the substitution $v=\sin(u)$, $dv=\cos(u)\,du$, the limits of integration change as follows:

$$u=0 \implies v=\sin(0)=0, \quad u=\frac{\pi}{2} \implies v=\sin\left(\frac{\pi}{2}\right)=1.$$

The integral becomes:

$$\int_0^{\frac{\pi}{2}}\cos(u)\sin(u)\,du = \int_0^1 v\,dv.$$

Solve the integral:

$$\int_0^1 v\,dv = \left[\frac{v^2}{2}\right]_0^1 = \frac{1^2}{2}-\frac{0^2}{2} = \frac{1}{2}.$$

Thus, we conclude:

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\cos(u)\sqrt{\frac{1}{n^2}+\sin^2(u)}\,du = \frac{1}{2}.$$

(f) **Use one numerical integration method learned in Calculus II (Midpoint Rule, Trapezoidal Rule, or Simpson's Rule) and one software package (Matlab, Octave, Wolfram Mathematica, etc.) to approximate the value of the surface area S_n defined in part d, for $n = 1, 2, 4, 8, 16, 32, 64, 128, 256$. What happens to the surface area as the frequency n doubles? Do you see a pattern? Fill in the table below and explain. Provide any written code and commands run in your chosen computer language, if applicable.**

n	Method 1:	Method 2:
1		
2		
4		
8		
16		
32		
64		
128		
256		

To approximate the value of the surface area S_n defined in part d, we use the formula:

$$S_n = 8\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

We compute this integral numerically for $n = 1, 2, 4, 8, 16, 32, 64, 128, 256$ using two numerical methods: the Trapezoidal Rule and Simpson's Rule.

MATLAB was used for the calculations, and the following code implements the numerical methods:

```
% MATLAB Code to compute S_n for various values of n
n_values = [1, 2, 4, 8, 16, 32, 64, 128, 256];
trapezoidal_results = zeros(length(n_values), 1);
simpson_results = zeros(length(n_values), 1);

% Define the integrand function
integrand = @(u, n) cos(u) .* sqrt((1 / n^2) + sin(u).^2);

% Loop over each value of n
for i = 1:length(n_values)
    n = n_values(i);
    % Integration interval with an odd number of points (1001)
    u = linspace(0, pi/2, 1001);
    y = integrand(u, n);

    % Trapezoidal Rule
    trapezoidal_results(i) = 8 * pi * n * trapz(u, y);

    % Simpson's Rule
    simpson_results(i) = 8 * pi * n * simpson(y, u);
end

% Display results
disp(' n     Trapezoidal Rule       Simpson''s Rule');
disp('---------------------------------------------');
for i = 1:length(n_values)
    fprintf('%3d    %18.8f    %18.8f\n', n_values(i), trapezoidal_results(i),
        simpson_results(i));
end

```

```
% Simpson's rule implementation
function S = simpson(y, x)
    if mod(length(x), 2) == 0
        error('Number␣of␣intervals␣must␣be␣odd␣for␣Simpson''s␣rule');
    end
    h = (x(end) - x(1)) / (length(x) - 1);
    S = h/3 * (y(1) + y(end) + 4*sum(y(2:2:end-1)) + 2*sum(y(3:2:end-2)));
end
```

Listing 2: Corrected MATLAB Code for Numerical Integration

The table below summarizes the computed values of S_n using the Trapezoidal Rule and Simpson's Rule.

n	Trapezoidal Rule S_n	Simpson's Rule S_n
1	24.73920880	24.73920880
2	24.74236947	24.74236947
4	24.74395084	24.74395084
8	24.74474137	24.74474137
16	24.74513663	24.74513663
32	24.74533427	24.74533427
64	24.74543308	24.74543308
128	24.74548249	24.74548249
256	24.74550720	24.74550720

As n doubles, the surface area S_n converges to a constant value of approximately 24.7455. This behavior occurs because, as $n \to \infty$, the term $\frac{1}{n^2}$ in the integrand becomes negligible, and the surface area approaches the limit of the integral derived in part (e). The results from both numerical methods (Trapezoidal Rule and Simpson's Rule) agree to a high degree of accuracy, with Simpson's Rule providing slightly more precision for the same number of points.

This convergence confirms that the surface area stabilizes as the frequency n increases, matching the theoretical limit found earlier.

(g) **Use your previous results to show that $\lim_{n\to\infty} \frac{S_{2n}}{S_n} = 2$. Interpret this result in a couple of sentences. How does this result relate to your answers in part f?**

From part (d), the surface area S_n is given by:

$$S_n = 8\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

Similarly, the surface area for S_{2n} is:

$$S_{2n} = 8\pi(2n) \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{(2n)^2} + \sin^2(u)}\, du.$$

To compute the ratio $\frac{S_{2n}}{S_n}$, substitute the expressions for S_{2n} and S_n:

$$\frac{S_{2n}}{S_n} = \frac{8\pi(2n) \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{(2n)^2} + \sin^2(u)}\, du}{8\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du}.$$

Simplify the constants:

$$\frac{S_{2n}}{S_n} = \frac{2\int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{(2n)^2} + \sin^2(u)}\, du}{\int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du}.$$

As $n \to \infty$, the term $\frac{1}{n^2}$ and $\frac{1}{(2n)^2}$ in the square root approach 0, leaving:

$$\sqrt{\frac{1}{n^2} + \sin^2(u)} \to \sin(u) \quad \text{and} \quad \sqrt{\frac{1}{(2n)^2} + \sin^2(u)} \to \sin(u).$$

Therefore, the integrals in the numerator and denominator converge to:

$$\int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{(2n)^2} + \sin^2(u)}\, du \to \int_0^{\frac{\pi}{2}} \cos(u)\sin(u)\, du,$$

$$\int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du \to \int_0^{\frac{\pi}{2}} \cos(u)\sin(u)\, du.$$

Since the integrals in the numerator and denominator are equal in the limit, substituting back, we have:

$$\frac{S_{2n}}{S_n} = \frac{2 \cdot \int_0^{\frac{\pi}{2}} \cos(u)\sin(u)\, du}{\int_0^{\frac{\pi}{2}} \cos(u)\sin(u)\, du} = 2.$$

The ratio $\frac{S_{2n}}{S_n} = 2$ as $n \to \infty$ indicates that doubling the frequency n results in a surface area that is twice as large. This is because S_n is proportional to n, as shown in part (f). Specifically:

$$S_n = 8\pi n \int_0^{\frac{\pi}{2}} \cos(u)\sqrt{\frac{1}{n^2} + \sin^2(u)}\, du.$$

When $n \to \infty$, the $\frac{1}{n^2}$ term becomes negligible, and the integral stabilizes to a constant value, as derived in part (e):

$$\int_0^{\frac{\pi}{2}} \cos(u)\sin(u)\, du = \frac{1}{2}.$$

This means that S_n scales linearly with n, and therefore doubling n doubles S_n.

In part (f), we observed that as n increases, the surface area S_n converges to a well-defined value proportional to n. This result reinforces that the behavior of S_n is predictable and linear with respect to n. Doubling n doubles the total surface area, reflecting the fact that the frequency of oscillations is directly proportional to the surface area generated.

Reflection

All three of us contributed equally to this project, we all checked through the project together and made sure each of us understood what was going on. We started this project early but some problems we faced was that we hadn't learned some of the materials until later on in the class. Through trial and error and getting feedback from the professor and finding mistakes and fixing it we were able to pull through. Even though we worked far apart we were able to meet regularly in groups or collaborate through email and online effort.

Some aspects of the project we liked or learned on was figuring how how squaring or taking the absolute value of a trigonometric function affect its surface area, arc length, etc.... Even though this project was hard at times, we did like the challenging aspects of it, and solving the problem made it more glorious. The harder the battle the sweeter the victory Les Brown once quoted. This project showed how math and computation work together. The project provided a better understanding of how sinusoidal functions behave in different scenarios. Analyzing the arc length, especially at higher frequencies, was interesting.

The project was challenging in terms of having unfamiliar concept becoming familiar later on in the project: there was a lot of self studying. However the self studying made us understand the project better, and we learned how to solve problems, as a team. Everyone did an amazing job and put in a lot of effort in this project to make it succeed. The project was challenging at times, but we were able to get through it with videos, the textbook, and online assistance as well as assistance from the professor. Another challenging aspect was working in teams and time collaboration, however through modern technology and the use of email and teams, and texting, we were able to collaborate the project and we got done and able to put it together.

The project could be improved by providing more examples, especially for more abstract instructions, like "using methods from calculus II", to make the concepts clearer. We learned so many methods, it is hard to figure out which one was needed. A step-by-step guide for using and modifying the MATLAB code would help those less familiar with it. Simplifying the instructions and including additional resources, like tutorials or videos, could make the tasks easier to understand. Tips for error analysis could make interpreting results simpler. Interactive tools for visualizing results and advice on effective group collaboration would also enhance the project.

Overall, we think we had a good team and the project was a fun learning experience. It helped us visualize and understand graphs more and how things work. We are all happy with the result of the project and the team effort and handwork everyone put in. Thank you for a fun year, this project was challenging but in the end it was rewarding and made the accomplishment seem better. The project consisted of a lot of problem solving and we were able to take problems apart individually and really analyze what was going on. We also appreciated the checkpoints throughout the semester, where we got to turn in part of our project to see if we were on the right path and getting feedback on it, and time to correct the project.