Laboratory 5
Applications of First-Order Differential Equations

(§3.2 of the Nagle/Saff/Snider text)

First-order differential equations can be used to model real world phenomena that vary with respect to time. This section looks at some examples of this modeling. Our objectives are as follows.

1. Use differential equations in modeling changes in population by means of the exponential, logistic, and modified logistic population models.

2. Use differential equations in modeling various other growth and decay problems.

3. Use differential equations to model mixing problems.

Population Growth

Exponential growth

A simple way of modeling the growth (or decline) of a population N is to assume that the rate of change in the population at any time t is proportional to the quantity $N(t)$ that is present (or remaining). Formally, this rate is change is

$$\frac{dN}{dt} = rN. \tag{L5.1}$$

Separating variables and solving for $N(t)$ with initial value $N(t_0) = N_0$ yields the *exponential* function $N(t) = N(t_0)e^{r(t-t_0)}$. The proportionality constant r is the difference in the per capita birth and death rates ($r = b - d$).

- If $r > 0$ the population is growing, if $r < 0$ the population is declining (what happens if $r = 0$?)

- Slowly reproducing species such as elephants, killer whales, and certain plants have a small r while rapidly reproducing organisms such as bacteria, lake trout, and small insects have a large r.

Logistic growth

Equation (L5.1) exhibits unbounded population growth at an exponential rate assuming unlimited resources. In comparison, *logistic* growth limits the size of the population. Let $N_{\max}$ denote the maximum sustainable population size (also called *carrying capacity*) of the habitat of study. Then $N_{\max} - N$ is the number of new individuals that the habitat can accept and $\frac{N_{\max}-N}{N_{\max}}$ the percentage of $N_{\max}$ available for population growth. As the population approaches the carrying capacity $N_{\max}$, the percentage of $N_{\max}$ available for growth is small as a result of overcrowding and more competition for limited resources. On the other hand the percentage $\frac{N_{\max}-N}{N_{\max}} \to 1$ for population sizes well below the carrying capacity. Formally the logistic population growth is then modeled by the ODE

$$\frac{dN}{dt} = rN\frac{N_{\max} - N}{N_{\max}} = rN\left(1 - \frac{N}{N_{\max}}\right) = rN - cN^2 \tag{L5.2}$$

with $c = r/N_{\max}$. Equation (L5.2) succinctly shows that logistic population growth is the combination of the process of inhibited growth with the process of competition among pairs of individuals at a rate c.

Modified logistic growth

We can modify the logistic equation (L5.2) by including the possibility that when the population size is too small, say less than N_{min}, the inability to find a suitable mate leads to the extinction of the species. The *modified logistic* equation (also called Nagumo equation) is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{N_{\text{max}}}\right)\left(\frac{N}{N_{\text{min}}} - 1\right). \tag{L5.3}$$

Note that $dN/dt < 0$ for $0 < N < N_{\text{min}}$.

Effect of stocking/harvesting

We can expand upon either model by adding (to model stocking) or subtracting (to model harvesting or hunting) an amount $\alpha > 0$ for each time period. For example a logistic model with harvesting reads

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{N_{\text{max}}}\right) - \alpha.$$

Problems

1. Generate the direction fields of the exponential, logistic, and modified logistic models for population growth and record any observable differences. Use $N_{\text{max}} = 20$, $N_{\text{min}} = 10$, $r = 0.8$. What are the long term trends in the population in each case? Use commands like those below.

```
f = @(t,N) .8.*N; %Employ entry-wise multiplication .* for use with dirfiield7
dirfield7(f,[0,10],[0,30],'arrow')
```

2. Two bacteria are placed in a petri dish with unlimited resources and growth unchecked according to exponential law. If they divide every 20 minutes, how many bacteria are present after one day? Write an equation that models this problem and use MATLAB to find a numerical answer.

3. Suppose that the population of a species of fish in a certain lake is growing according to a logistic model with $r = 0.3$ and $N_{\text{max}} = 3000$. Assume that initially there are 2500 fishes of that species in the lake. Determine the correct IVP for each of the scenarios below and in each case determine from the direction field of the differential equation the long term behavior of the fish population.

   (a) each year 150 fishes are harvested from the lake.

   ```
   f = @(t,N) ??? %fill in correct ODE using entry-wise operations .* and ./
   dirfield7(f,0:5:100,0:65:3000,'line'); % Or use 'line_scaled'
   hold on
   [t,N] = ode45(f,[0,100],2500);
   plot(t,N);
   hold off
   ```

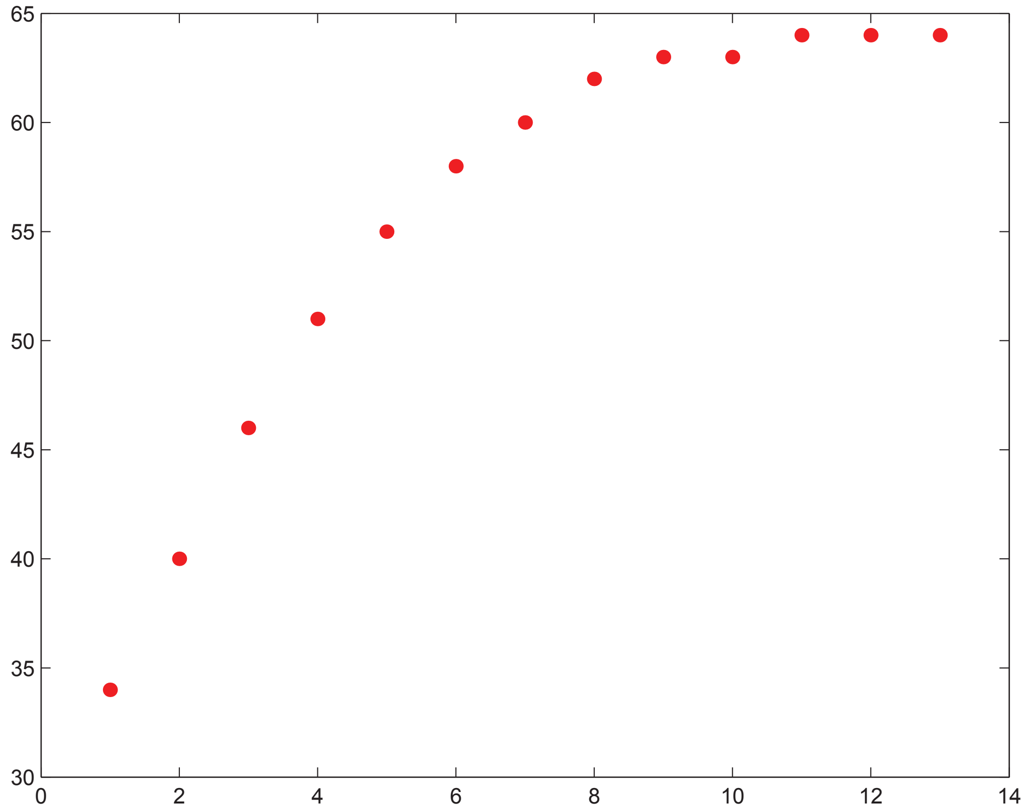
   (b) each year 25% of the fishes are harvested from the lake.

   (c) what is the maximum safe fixed amount to harvest each year in order to assure that there will always be some fish in the lake?

4. The population of a species of birds in a natural preserve has been recorded each year for the past thirteen years. We wish to determine the model which represents the behavior of this population best:

year	1	2	3	4	5	6	7	8	9	10	11	12	13
population	34	40	46	51	55	58	60	62	63	63	64	64	64

```
>> years = 1:13; population = [34,40,46,51,55,58,60,62,63,63,64,64,64];
>> plot(years,population,'.r','MarkerSize',20)
```



(a) Which of the three population models (exponential, logistic, modified logistic) would you use?

(b) Make a guess for the values of the parameter(s) in your model.

(c) Solve the resulting IVP using `ode45` and plot the solution together on top of the given population point plot above. Use an IC at $t = 1$.

(d) If the plots do not reasonably coincide, adjust your model.

(e) At each time $t_i = 1, 2, \ldots$ we thus have two values for the population: the data N_i from the above table and the value $y(t_i)$ computed use `ode45` for a specific guess of the parameter(s). Evaluate the quantity $\displaystyle\sum_{i=1}^{13}(y_i - y(t_i))^2$. If your result is much larger than 0.89 adjust your parameter(s) (say to 3 decimal digits).

(f) Does it make sense to have a model giving a non-integer population size?

Other Models, Parameter Estimation

Differential equations similar to population models can be used to model other phenomena such as heating/cooling, spread of diseases, and mixing problems. Here are three examples.

5. **Newton's law of cooling** states that the rate $\frac{dT}{dt}$ at which the temperature T of an object changes with time is proportional to the difference of temperature between the object and the ambient medium. Mathematically we write

$$\frac{dT}{dt} = k\left(T_{\text{air}} - T\right).$$

Milk is brought out of a refrigerator compartment kept at 40 degrees into a warm room of unknown temperature. After 4 minutes the temperature of the milk is 54 degrees and after 6 minutes the temperature of the milk is 59 degrees.

(a) What is the temperature of the room? For this you may want to use the following program:

```
function LAB05pb5
clc
tdata = [4;6]; Tdata = [54;59];
k = ??; Tair = ??;              % FILL-IN initial guess
param = [k;Tair];
[param,residual] = fminsearch(@F,param,[],tdata,Tdata);
k = param(1); Tair = param(2);
[t,T] = ode45(@f,[0,40],40,[],k,Tair);
figure(1)
set(gcf,'DefaultAxesFontsize',16)
plot(t,T,'r-',tdata,Tdata,'b.','LineWidth',2,'MarkerSize',30);
xlabel('Time [hours]'); ylabel('Temperature [degrees]')
grid on
disp(['k = ' num2str(k) '  Tair = ' num2str(Tair)  ...
        ' residual = ' num2str(residual)])
%---------------------------------------------------------
function Fv = F(param,tdata,Tdata)
k = param(1); Tair = param(2); T0 = ??;      % FILL-IN IC
[t,T] = ode45(@f,[0;tdata],T0,[],k,Tair);
Fv = norm(T(2:end)-Tdata,2)^2;
%---------------------------------------------------------
function dTdt = f(t,T,k,Tair)
dTdt = ??;                              % FILL-IN ODE
```

(b) When will the milk be within 1 degree of room temperature? Answer graphically or use the commands

```
[t,T] = ode45(@f,[0,40],40,[],k,Tair);
i = find(T>Tair-1); i = i(1); t = round(t(i));
disp(['The temperature of milk is within one degree of air '...
        'temperature after about ' num2str(t) ' minutes.'])
```

(c) Does the choice of initial guess for `k` and `Tair` matter?

6. **Spread of Disease**. Suppose that an infectious disease is spreading among a population of N individuals. For simplicity we will assume that the incubation period is zero and that the changes in the progress of the decease are continuous. The population is comprised of 3 types of individuals: infected (i individuals), susceptible (s individuals), and removed (r individuals, either immune, dead, or removed into isolation). Susceptible individuals can become infected at a rate proportional to both susceptible and infected population sizes si (a second-order rate of change). Infected individuals become removed at a rate proportional to the infected population size only (a first-order rate of change).

$$\boxed{s} \xrightarrow{a} \boxed{i} \xrightarrow{b} \boxed{r}$$

Accordingly, the following equations hold:

$$\frac{ds}{dt} = -asi, \quad \frac{di}{dt} = asi - bi, \quad \frac{dr}{dt} = bi. \tag{L5.4}$$

Note that $\frac{d}{dt}(s+i+r) = 0$, i.e., $s+i+r = cst = N$. From (L5.4) one easily obtain

$$\frac{di}{ds} = \frac{asi - bi}{-asi} = \frac{b/a}{s} - 1, \tag{L5.5}$$

giving the rate of change of infected individuals with respect to those susceptible.

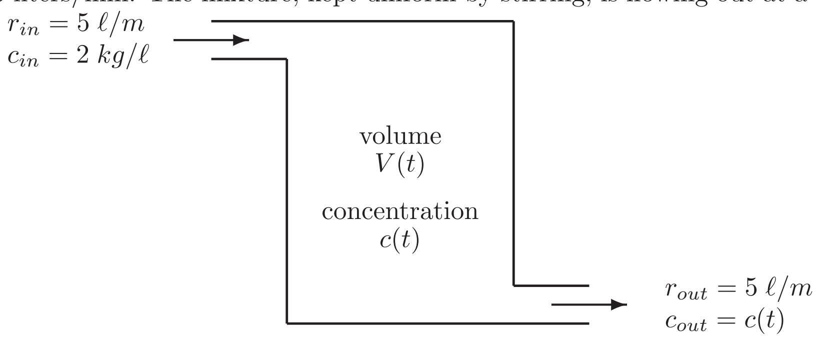
(a) Solve analytically (L5.5) for i in terms of s. Use an IC $i(s_0) = i_0$.

(b) Graph the solution for particular choices of b/a, s_0, and i_0. Use the following commands:

```
f = @(s,i) ????/s-1;          % choose constant b/a
i0 = ???; s0 = ???;           % select IC
[s,i] = ode45(f,[i0,2],s0);   % forward solution
plot(s,i); hold on;
[s,i] = ode45(f,[i0,1e-3],s0);   % backward solution
plot(s,i); hold off;
```

What do the graphs tell us about the relationship between the population size of infected individuals versus the population size of susceptible individuals?

7. **Mixing Problem** (§3.2). Suppose a brine solution containing 2 kg of salt per liter runs into a tank initially filled with 500 liters of water containing 50 kg of salt. The brine runs into the tank at a rate of 5 liters/min. The mixture, kept uniform by stirring, is flowing out at a rate of 5 liters/min.



(a) Find the concentration $c(10)$, in kg/ℓ, of salt in the tank after 10 minutes. Hints: note that the volume is constant; identify the rate of change $\frac{ds}{dt}$ of the amount of salt $s(t)$ in terms of r_{in}, r_{out}, c_{in}, and $c(t)$ and express $c(t)$ in terms of $s(t)$ to obtain an ODE in $s(t)$. Use the following MATLAB commands.

```
ode1 = @(t,s) ???;      % fill-in ODE
[t1,s1] = ode45(ode1,[0,10],s0);
concentration1 = s1(end)/500
```

(b) After 10 minutes a leak develops and an additional 1 ℓ/min of mixture flows out of the tank. What will be the concentration, in kg/ℓ, of salt in the tank 20 minutes after the leak develops? Hints: the initial amount of salt in this question is the amount after 10 minutes calculated in (a). Moreover the volume of the tank is now decreasing. Use the following MATLAB commands:

```
ode2 = @(t,s) 10-6*s/(510-t);     % explain the ODE
[t2,s2] = ode45(ode2,[10,??],s1(end));       % fill-in final time
concentration2 = s2(end)/(510-??)
```

What happens at $t = 510\ min$? What is the maximal amount of salt that was in the tank at any time before? What is the corresponding time? Hint: use the `max` and `find` commands.

(c) Assume that the tank has an horizontal cross-section $A = 50\ m^2$, the leak is at the bottom of the tank, and the rate of flow due to the leak is proportional to the square root of the height of water in the tank, with an initial rate of 1 ℓ/min. Write an IVP giving the rate of change $\frac{dV}{dt}$ of the volume $V(t)$ of water in the tank in terms of r_{in}, r_{out}, and $V(t)$. Repeat (b) with the corrected volume and rate out. Do you expect a higher or lower concentration of salt in the tank 20 minutes after the leak develops compared to the situation in (b)? Why?

(d) Plot the concentration $c(t)$ and volume $V(t)$ for $0 \le t \le 510$ for all scenarios.