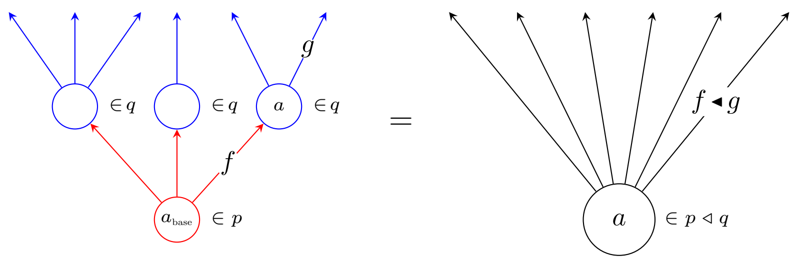
POLYNOMIAL BICOMODULES ARE PARAMETRIC RIGHT ADJOINTS

Recall that the substitution product of polynomials p and q, denoted $p \triangleleft q$, is characterized as follows.

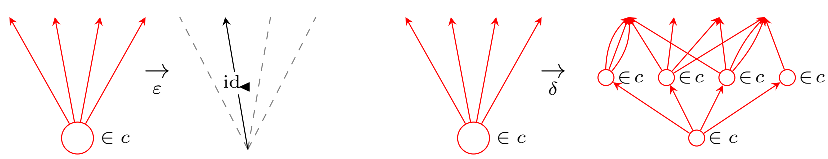
- A position a in $p \triangleleft q$ consists of a position a_{base} in p and positions a_f in q for each direction f from a_{base}.
- A direction from position a in $p \triangleleft q$ consists of a direction f from a_{base} and a direction g from a_f.



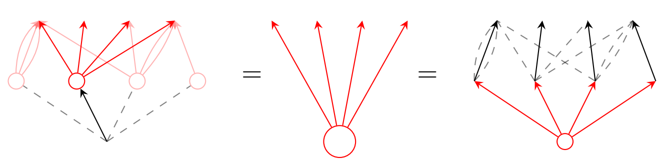
We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$. Accordingly, $\text{id}_{\blacktriangleleft}$ will denote the unique direction from the unique position in the unit for substitution $\text{id}_{\triangleleft}$ (a.k.a. the polynomial y).[1]

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let c be a polynomial comonoid. Denote counit by ε and comultiplication by δ.



Observe first that the right unit law forces $(\delta_1(a))_{\text{base}} = a$ for all $a \in c(1)$.



$(\varepsilon \triangleleft \text{id}_c) \circ \delta$ $\qquad$ id_c $\qquad$ $(\text{id}_c \triangleleft \varepsilon) \circ \delta$

---

[1] Given directions f, g, and h respectively belonging to polynomials p, q, and r, the directions of the form $(f \blacktriangleleft g) \blacktriangleleft h$ belonging to $(p \triangleleft q) \triangleleft r$ and the directions of the form $f \blacktriangleleft (g \blacktriangleleft h)$ belonging to $p \triangleleft (q \triangleleft r)$ are identified under the relevant monoidal coherence isomorphism. Hence brackets can be omitted.

Similarly, for any direction f belonging to a polynomial p, we have that $\text{id}_{\blacktriangleleft} \blacktriangleleft f$ and $f \blacktriangleleft \text{id}_{\blacktriangleleft}$ (respectively belonging to $\text{id}_{\triangleleft} \triangleleft p$ and $p \triangleleft \text{id}_{\triangleleft}$) are both canonically identified with f.

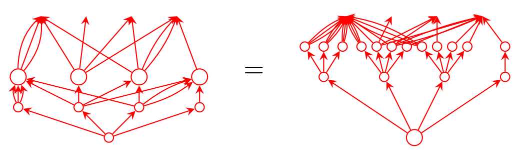
Therefore the expression $(\delta_1(a))_f$ for $f \in c[a]$ has a well-defined meaning.

We gather the data of a category $\mathcal{C}$.

- The set of objects $\mathrm{Ob}(\mathcal{C})$ is $c(1)$, i.e., the set of positions in c.
- The set of arrows $\mathrm{Arr}(\mathcal{C})$ is $\sum_{a \in c(1)} c[a]$, i.e., the set of all directions in c.
- The source map s sends each $f \in c[a]$ to a. (Hence the polynomial c is described by the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in c[a]$ to $(\delta_1(a))_f$.
- The identity map e sends each $a \in c(1)$ to $\varepsilon^\sharp(a, \mathrm{id}_\blacktriangleleft)$.
- The composition map m sends each pair of compatible arrows $f \in c[a], g \in c[t(f)]$ to $\delta^\sharp(a, f \blacktriangleleft g)$.

Now we verify these data satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a.
- The law $t(e(a)) = a$ is forced to hold by the comonoid left unit law, which identifies $\delta^\sharp(a, f \blacktriangleleft g)$ with f.
- The law $s(m(f, g)) = s(f)$ is true by construction; $m(f, g)$ is a direction from the position $s(f)$.
- The law $t(m(f, g)) = t(g)$.
- The left unit law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left unit law.
- The right unit law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right unit law.
- The associativity law $m(m(f, g)h) = m(f, m(g, h))$ is directly expressed by the comonoid associativity law.



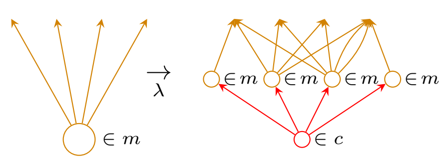
$(\delta \triangleleft \mathrm{id}_c) \circ \delta$ $\quad$ $(\mathrm{id}_c \triangleleft \delta) \circ \delta$

Conversely, let $\mathcal{C}$ be a category. We immediately obtain the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$. Let c denote the polynomial described by this bundle (the "outfacing polynomial" of $\mathcal{C}$); we exhibit a comonoid struture on c.c

Lastly, these translation processes between polynomial comonoids and categories are inverse by construction. □

Proposition 2. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

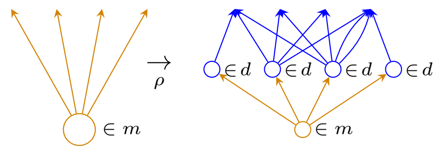
Proof. Let c be a polynomial comonoid and let m be a left comodule on c. Denote left comodule comultiplication by λ.



□

Proposition 3. *A polynomial right comodule amounts to a set of copresheaves.*

Proof. Let d be a polynomial comonoid and let m be a right comodule on d. Denote right comodule comultiplication by ρ.



$\square$

Proposition 4. *Polynomial bicomodules are prafunctors.*

Proof. $\square$

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof. $\square$

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

Proof. Recall bicomodules from d to 0 are copresheaves on d (and maps between such bicomodules are copresheaf maps). Hence each bicomodule m from c to d induces a functor F_m from d-copresheaves to c-copresheaves by precomposition. Accordingly, we have $F_{m \triangleleft_d n} \cong F_m \circ F_n$ (for bicomodules m from c to d and n from d to e).

We show that the prafunctor corresponding to the bimodule m is F_m. $\square$