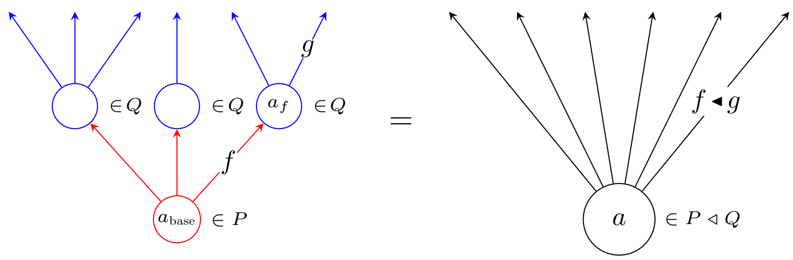
POLYNOMIAL BICOMODULES ARE PARAMETRIC RIGHT ADJOINTS

Recall that the substitution product of polynomials P and Q, denoted $P \triangleleft Q$, is characterized as follows.

- A position a in $P \triangleleft Q$ consists of a position a_{base} in P and positions a_f in Q for each direction f from a_{base}.
- A direction from position a in $P \triangleleft Q$ consists of a direction f from a_{base} and a direction g from a_f.



We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$. Accordingly, $\mathrm{id}_{\blacktriangleleft}$ will denote the unique direction from the unique position in the unit for substitution $\mathrm{id}_{\triangleleft}$ (a.k.a. the polynomial y).[1]

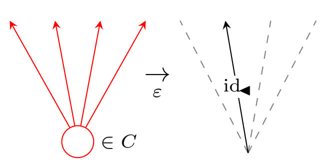
Let α and β be maps between polynomials. As we will frequently encounter maps composed in parallel via $\triangleleft$, we will repeatedly use the following identity.

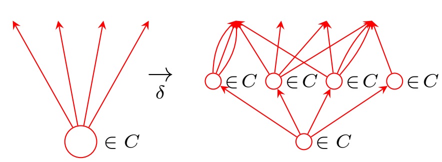
$$(\alpha \triangleleft \beta)^\sharp_x (f \blacktriangleleft g) = \underbrace{\alpha^\sharp_{(x_{\mathrm{base}})}(f)}_{\star} \blacktriangleleft \beta^\sharp_{(x_\star)}(g).$$

Or in brief, "$(\alpha \triangleleft \beta)^\sharp (f \blacktriangleleft g) = \alpha^\sharp(f) \blacktriangleleft \beta^\sharp(g)$ wherever this makes sense."

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let C be a polynomial comonoid. Denote counit by ε and comultiplication by δ.

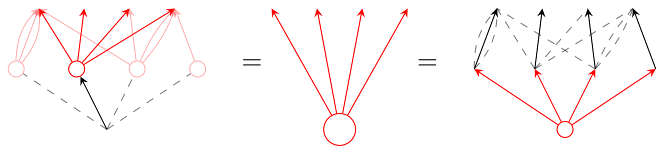




Observe first that the right unit law forces $(\delta_1(a))_{\mathrm{base}} = a$ for all $a \in C(1)$.

[1]Given directions f, g, and h respectively belonging to polynomials P, Q, and R, the directions of the form $(f \blacktriangleleft g) \blacktriangleleft h$ belonging to $(P \triangleleft Q) \triangleleft R$ and the directions of the form $f \blacktriangleleft (g \blacktriangleleft h)$ belonging to $P \triangleleft (Q \triangleleft R)$ are identified under the relevant monoidal coherence isomorphism. Hence brackets can be omitted.

Similarly, for any direction f belonging to a polynomial P, we have that $\mathrm{id}_{\blacktriangleleft} \blacktriangleleft f$ and $f \blacktriangleleft \mathrm{id}_{\blacktriangleleft}$ (respectively belonging to $\mathrm{id}_{\triangleleft} \triangleleft P$ and $P \triangleleft \mathrm{id}_{\triangleleft}$) are both canonically identified with f.



$(\varepsilon \triangleleft \mathrm{id}_C) \circ \delta$ $\qquad$ id_C $\qquad$ $(\mathrm{id}_C \triangleleft \varepsilon) \circ \delta$

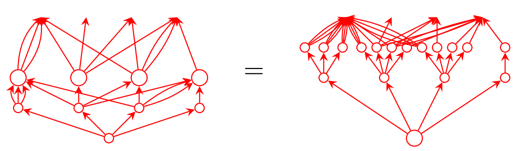
Therefore the expression $(\delta_1(a))_f$ for $f \in C[a]$ has a well-defined meaning.

We gather the data of a category $\mathcal{C}$.

- The set of objects $\mathrm{Ob}(\mathcal{C})$ is $C(1)$, i.e., the set of positions in C.
- The set of arrows $\mathrm{Arr}(\mathcal{C})$ is $\sum_{a \in C(1)} C[a]$, i.e., the set of all directions in C.
- The source map s sends each $f \in C[a]$ to a. (Hence the polynomial C is described by the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in C[a]$ to $(\delta_1(a))_f$.
- The identity map e sends each $a \in C(1)$ to $\varepsilon_a^\sharp(\mathrm{id}_\blacktriangleleft)$.
- The composition map m sends each pair of compatible arrows $f \in C[a], g \in C[t(f)]$ to $\delta_a^\sharp(f \blacktriangleleft g)$.

Now we verify these data satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a.
- The law $t(e(a)) = a$ is forced to hold by the comonoid left unit law, which identifies with .
- The law $s(m(f,g)) = s(f)$ is true by construction; $m(f,g)$ is a direction from the position $s(f)$.
- The law $t(m(f,g)) = t(g)$.
- The left unit law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left unit law.
- The right unit law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right unit law.
- The associativity law $m(m(f,g)h) = m(f, m(g,h))$ is directly expressed by the comonoid associativity law.



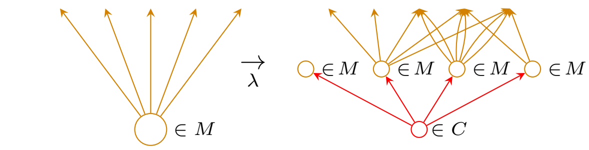
$(\delta \triangleleft \mathrm{id}_C) \circ \delta$ $\qquad$ $(\mathrm{id}_C \triangleleft \delta) \circ \delta$

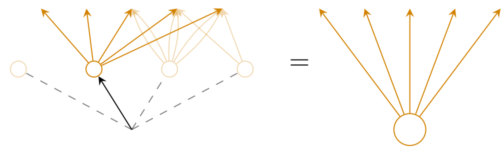
Conversely, let $\mathcal{C}$ be a category. We immediately obtain the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$. Let C denote the polynomial described by this bundle (the "outfacing polynomial" of $\mathcal{C}$); we exhibit a comonoid struture on C.

Lastly, these translation processes between polynomial comonoids and categories are inverse by construction. □

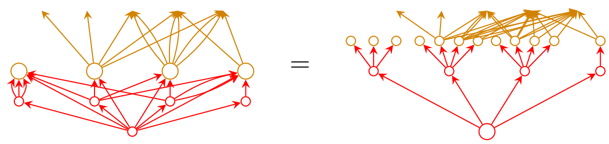
Proposition 2. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

Proof. Let C be a polynomial comonoid and let M be a left comodule on C. Denote left comodule comultiplication by λ.





$(\varepsilon \triangleleft \mathrm{id}_M) \circ \lambda$ $\qquad$ id_M

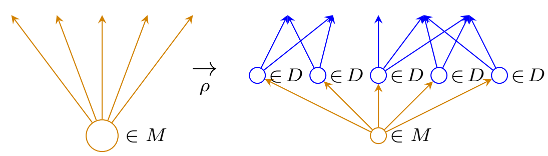


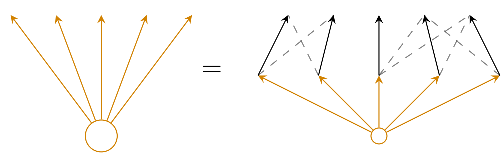
$(\delta \triangleleft \mathrm{id}_M) \circ \lambda$ $\qquad$ $(\mathrm{id}_C \triangleleft \lambda) \circ \lambda$

□

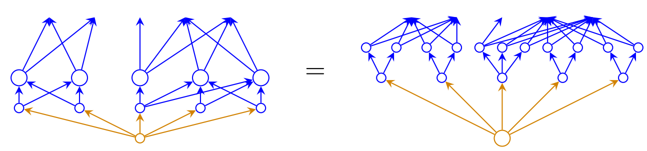
Proposition 3. *A polynomial right comodule amounts to a set of copresheaves.*

Proof. Let D be a polynomial comonoid and let M be a right comodule on D. Denote right comodule comultiplication by ρ.





id_M $\qquad$ $(\mathrm{id}_M \triangleleft \varepsilon) \circ \rho$



$(\rho \triangleleft \mathrm{id}_D) \circ \rho$ $\qquad$ $(\mathrm{id}_M \triangleleft \delta) \circ \rho$

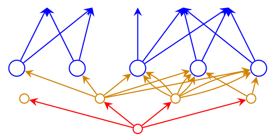
□

Proposition 4. *Polynomial bicomodules are prafunctors.*

Proof. □

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof.



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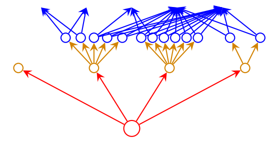

$(\lambda \triangleleft \mathrm{id}_D) \circ \rho$ $(\mathrm{id}_C \triangleleft \rho) \circ \lambda$

□

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

Proof. Recall bicomodules from D to 0 are copresheaves on D (and maps between such bicomodules are copresheaf maps). Hence each bicomodule M from D to D induces a functor F_M from D-copresheaves to C-copresheaves by precomposition. Accordingly, we have $F_{M \triangleleft_D N} \cong F_M \circ F_N$ (for bicomodules M from C to D and N from D to E).

We show that the prafunctor corresponding to the bimodule M is F_M. □