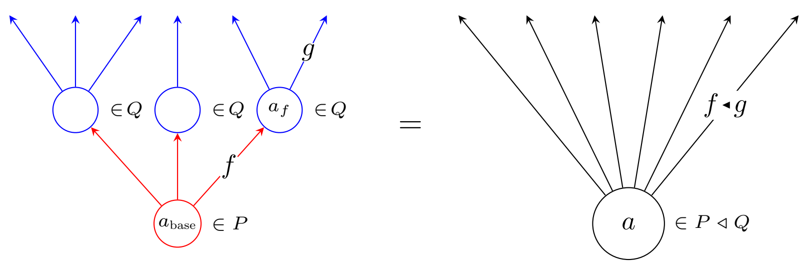
POLYNOMIAL COMONOIDS AND BICOMODULES

Recall the *substitution product* of polynomials P and Q, denoted $P \triangleleft Q$.

- A position a in $P \triangleleft Q$ consists of a position a_{base} in P and positions a_f in Q for each direction f from a_{base}.
- A direction from position a in $P \triangleleft Q$ consists of a direction f from a_{base} and a direction g from a_f.



We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$.[1] Accordingly, $\mathrm{id}_{\blacktriangleleft}$ will denote the unique direction from the unique position in the unit for substitution $\mathrm{id}_{\triangleleft}$ (a.k.a. the polynomial y).[2]

Note the following identity for transformations α and β between polynomials.

$$(\alpha \triangleleft \beta)^\sharp_a(f \blacktriangleleft g) = \underbrace{\alpha^\sharp_{(a_{\mathrm{base}})}(f)}_{\star} \blacktriangleleft \beta^\sharp_{(a_\star)}(g).$$

Or in brief, "we have $(\alpha \triangleleft \beta)^\sharp(f \blacktriangleleft g) = \alpha^\sharp(f) \blacktriangleleft \beta^\sharp(g)$ whenever this makes sense."[3]

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let C be a polynomial comonoid. Denote counit by ε and comultiplication by δ.

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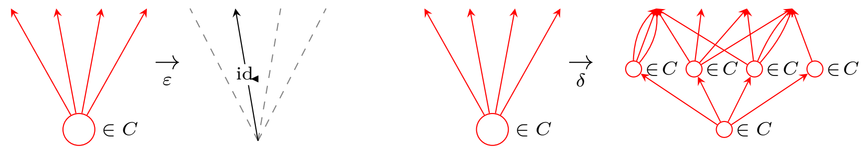
[1] Be aware there may be other directions named $f \blacktriangleleft g$ from other positions in $P \triangleleft Q$.

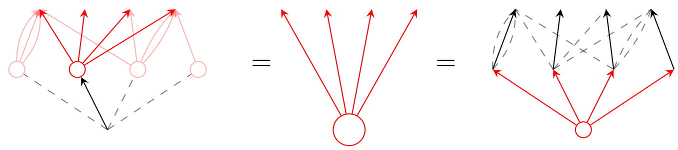
[2] Given directions f, g, and h respectively belonging to polynomials P, Q, and R, the directions of the form $(f \blacktriangleleft g) \blacktriangleleft h$ belonging to $(P \triangleleft Q) \triangleleft R$ and the directions of the form $f \blacktriangleleft (g \blacktriangleleft h)$ belonging to $P \triangleleft (Q \triangleleft R)$ are identified under the relevant monoidal coherence isomorphism. Hence brackets can be omitted.

Similarly, for any direction f belonging to a polynomial P, we have that $\mathrm{id}_{\blacktriangleleft} \blacktriangleleft f$ and $f \blacktriangleleft \mathrm{id}_{\blacktriangleleft}$ (respectively belonging to $\mathrm{id}_{\triangleleft} \triangleleft P$ and $P \triangleleft \mathrm{id}_{\triangleleft}$) are both canonically identified with f.

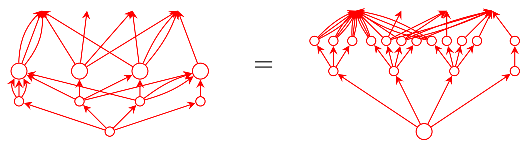
[3] Meaning, $(\alpha \triangleleft \beta)^\sharp_a(f \blacktriangleleft g)$ exists $\iff$ there are u and v such that $\alpha^\sharp_u(f) \blacktriangleleft \beta^\sharp_v(g)$ actually describes a direction from a; in this case the two sides are equal.

(We are assuming the analogue of this property holds for α and β themselves: if any $\alpha^\sharp_u(f)$ describes a direction from a, then $\alpha^\sharp_a(f)$ is defined with that value — and likewise for β. This is trivially so if across the domains of α and β we use different names for different directions.)





$\delta \mathbin{⨟} (\varepsilon \triangleleft \mathrm{id}_C)$ id_C $\delta \mathbin{⨟} (\mathrm{id}_C \triangleleft \varepsilon)$



$\delta \mathbin{⨟} (\delta \triangleleft \mathrm{id}_C)$ $\delta \mathbin{⨟} (\mathrm{id}_C \triangleleft \delta)$

Observe first that the right identity law forces $(\delta_1(a))_{\text{base}} = a$ for all $a \in C(1)$. Therefore the expression $(\delta_1(a))_f$ for $f \in C[a]$ has a well-defined meaning.

We gather the data for a category $\mathcal{C}$.

- The set of objects $\mathrm{Ob}(\mathcal{C})$ is $C(1)$, i.e., the set of positions in C.
- The set of arrows $\mathrm{Arr}(\mathcal{C})$ is $\sum_{a \in C(1)} C[a]$, i.e., the set of all directions in C.
- The source map s sends each $f \in C[a]$ to a. (Hence the polynomial C is described by the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in C[a]$ to $(\delta_1(a))_f$.
- The identity map e sends each $a \in C(1)$ to $\varepsilon^\sharp_a(\mathrm{id}_\lrcorner)$.
- The composition map m sends each pair of compatible arrows $f \in C[a], g \in C[t(f)]$ to $\delta^\sharp_a(f \blacktriangleleft g)$.

Next, observe that if we have such prerequisite data (not laws) for a category, subject to just the law $s(e(a)) = a$, then we find that further imposing the left identity law $m(e(s(f)), f) = f$ (requiring that both sides are defined whenever one is) automatically forces the law $t(e(a)) = a$ to hold.[4] Similarly, if we have the law $s(m(f,g)) = s(f)$ as well as $t(e(a)) = a$, then the associativity law $m(m(f,g),h) = m(f, m(g,h))$ forces $t(m(f,g)) = t(g)$.[5]

We verify the data from above satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a.
- The law $s(m(f,g)) = s(f)$ is true by construction; $m(f,g)$ is a direction from the position $s(f)$.
- The left identity law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left identity law, which identifies $\delta^\sharp(\varepsilon^\sharp(\mathrm{id}_\lrcorner) \blacktriangleleft f)$ with f.
- The right identity law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right identity law, which identifies $\delta^\sharp(f \blacktriangleleft \varepsilon^\sharp(\mathrm{id}_\lrcorner))$ with f.

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[4] We have $m(e(s(e(a))), e(a)) = e(a)$, since the right side is defined. The left side reduces to $m(e(a)), e(a))$. This expression only makes sense if $t(e(a)) = s(e(a))$, which is a.

[5] Given that f and g are composable, we have $m(m(f,g), e(t(g))) = m(f, m(g, e(t(g))))$, since the right side is defined. The left side only makes sense if $t(f,g) = s(e(t(g)))$, which is $t(g)$.

- The associativity law $m(m(f,g)h) = m(f, m(g,h))$ is directly expressed by the comonoid associativity law, which identifies $\delta^\sharp(\delta^\sharp(f \blacktriangleleft g) \blacktriangleleft h)$ with $\delta^\sharp(f \blacktriangleleft \delta^\sharp(g \blacktriangleleft h))$.
- The law $t(e(a)) = a$ is forced to hold (due to the comonoid left identity law).
- The law $t(m(f,g)) = t(g)$ is forced to hold (due to the comonoid associativity law).

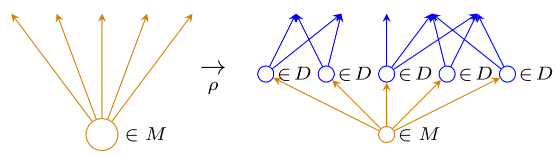
Conversely, let $\mathcal{C}$ be a category. We immediately obtain the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$. Let C denote the polynomial described by this bundle (the "outfacing polynomial" of $\mathcal{C}$); we exhibit a comonoid struture on C.

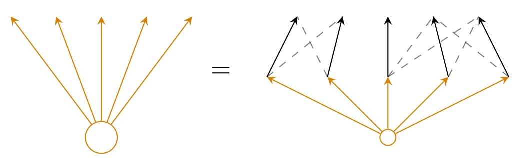
- The counit ε singles out the identity in each object's set of outfacing maps.
- The comultiplication δ endows each object a with the map $\delta^\sharp_a$ sending $f \blacktriangleleft g$ to $f \mathbin{⨟} g$ for all arrows of the form $f : a \to b$, $g : b \to c$ (through which the map δ_0 is implicit).

The above processes of translation between the prerequisite data (not laws) for a category subject to just $s(e(a)) = a$ and $s(m(f,g)) = g$, and the prerequisite data (not laws) of a polynomial comonoid subject to just $(\delta_1(a))_{\mathrm{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity category laws, in this context, directly translate to the identity and associativity comonoid laws. □

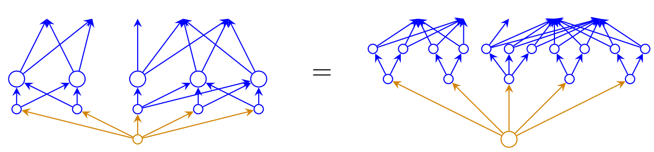
Proposition 2. *A polynomial right comodule amounts to a family of copresheaves.*

Proof. Let D be a polynomial comonoid and let M be a right comodule on D. Denote right comodule comultiplication by ρ.





id_M $\qquad$ $\rho \mathbin{⨟} (\mathrm{id}_M \triangleleft \varepsilon)$



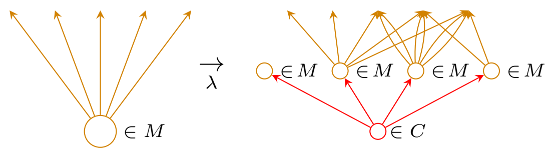
$\rho \mathbin{⨟} (\rho \triangleleft \mathrm{id}_D)$ $\qquad$ $\rho \mathbin{⨟} (\mathrm{id}_M \triangleleft \delta)$

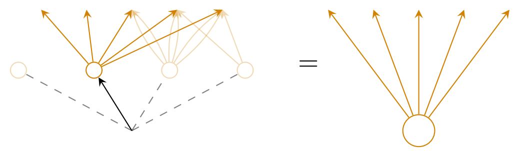
Let $\mathcal{D}$ be the category corresponding to D.

□

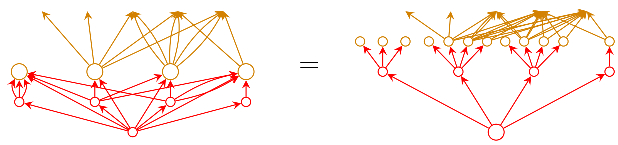
Proposition 3. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

Proof. Let C be a polynomial comonoid and let M be a left comodule on C. Denote left comodule comultiplication by λ.





$\lambda \mathbin{⨟} (\varepsilon \triangleleft \mathrm{id}_M)$ $\qquad$ id_M



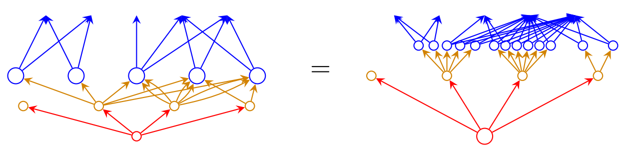
$\lambda \mathbin{⨟} (\delta \triangleleft \mathrm{id}_M)$ $\qquad$ $\lambda \mathbin{⨟} (\mathrm{id}_C \triangleleft \lambda)$

Let $\mathcal{D}$ be the category corresponding to D.

$\square$

Proposition 4. *Polynomial bicomodules are prafunctors between presheaf categories.*

Proof. Let C and D be polynomial comonoids and let M be a bimodule from C to D.



$\rho \mathbin{⨟} (\lambda \triangleleft \mathrm{id}_D)$ $\qquad$ $\lambda \mathbin{⨟} (\mathrm{id}_C \triangleleft \rho)$

We will show that M amounts to a profunctor[6] $(\int_{\mathcal{C}} R) \mathrel{\circ\!\!-\!\!\bullet} \mathcal{D}$, where $\mathcal{C}$ is the category corresponding to C, $\int_{\mathcal{C}} R$ is the category of elements of the copresheaf R on $\mathcal{C}$ induced by M as a left comodule, and $\mathcal{D}$ is the category corresponding to D.

Such a profunctor is equivalent to a prafunctor $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$. Indeed,

$$\cong$$

$\square$

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof. $\square$

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

[6]The notation $\mathcal{A} \mathrel{\circ\!\!-\!\!\bullet} \mathcal{B}$ for a profunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$ is due to Michael Shulman.

Proof. Recall bicomodules from D to 0 are copresheaves on D (and maps between such bicomodules are copresheaf maps). Hence each bicomodule M from D to D induces a functor F_M from D-copresheaves to C-copresheaves by precomposition. Accordingly, we have $F_{M \triangleleft_D N} \cong F_N \mathbin{⨟} F_M$ (for bicomodules M from C to D and N from D to E).

We show that the prafunctor corresponding to the bimodule M is F_M. $\square$