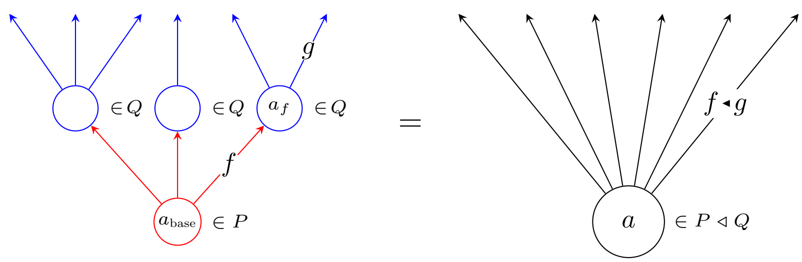
POLYNOMIAL COMONOIDS AND BICOMODULES

Recall the *substitution product* of polynomials P and Q, denoted $P \triangleleft Q$.

- A position a in $P \triangleleft Q$ consists of a position a_{base} in P and positions a_f in Q for each direction f from a_{base}.
- A direction from position a in $P \triangleleft Q$ consists of a direction f from a_{base} and a direction g from a_f.



We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$.[1] Accordingly, $\text{id}_{\blacktriangleleft}$ will denote the unique direction from the unique position in the unit for substitution $\text{id}_{\triangleleft}$ (a.k.a. the polynomial y).

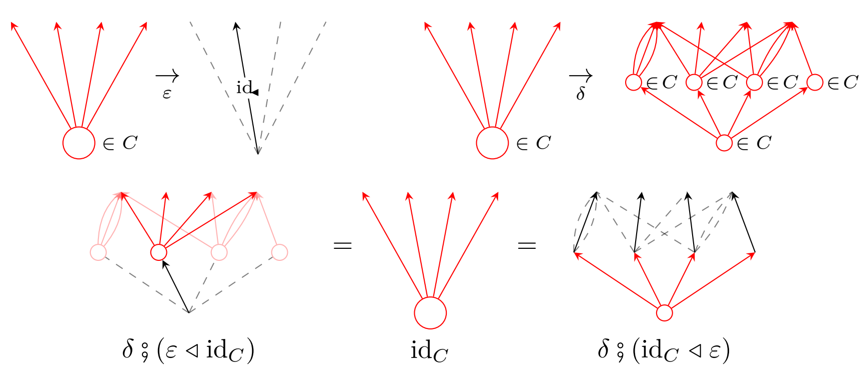
Note the following identity for transformations α and β between polynomials.

$$(\alpha \triangleleft \beta)^{\sharp}_{a}(f \blacktriangleleft g) = \underbrace{\alpha^{\sharp}_{(a_{\text{base}})}(f)}_{\star} \blacktriangleleft \beta^{\sharp}_{(a_{\star})}(g).$$

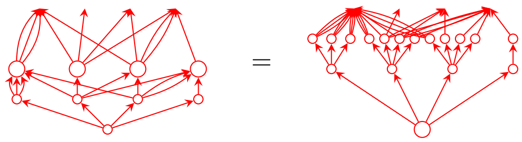
Or in brief, "we have $(\alpha \triangleleft \beta)^{\sharp}(f \blacktriangleleft g) = \alpha^{\sharp}(f) \blacktriangleleft \beta^{\sharp}(g)$ whenever this makes sense."

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let C be a polynomial comonoid. Denote counit by ε and comultiplication by δ.



[1] Be aware there may be other directions named $f \blacktriangleleft g$ from other positions in $P \triangleleft Q$.



$\delta \mathbin{⨟} (\delta \triangleleft \mathrm{id}_C)$ $\qquad$ $\delta \mathbin{⨟} (\mathrm{id}_C \triangleleft \delta)$

Observe first that the right identity law forces $\delta_1(a)_{\text{base}} = a$ for all $a \in C(1)$. Therefore the expression $\delta_1(a)_f$ for $f \in C[a]$ has a well-defined meaning.

We gather the data for a category $\mathcal{C}$.

- The set of objects $\mathrm{Ob}(\mathcal{C})$ is $C(1)$, the set of positions in C.
- The set of arrows $\mathrm{Arr}(\mathcal{C})$ is $\sum_{a \in C(1)} C[a]$, the total set of directions in C.
- The source map s sends each $f \in C[a]$ to a. (Hence the polynomial C is described by the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in C[a]$ to $\delta_1(a)_f$.
- The identity map e sends each $a \in C(1)$ to $\varepsilon_a^\sharp(\mathrm{id}_{\blacktriangleleft})$.
- The composition map m sends each pair of compatible arrows $f \in C[a], g \in C[t(f)]$ to $\delta_a^\sharp(f \mathbin{\blacktriangleleft} g)$.

Next, observe that if we have such prerequisite data (not laws) for a category, subject to just the law $s(e(a)) = a$, then we find that further imposing the left identity law $m(e(s(f)), f) = f$ (and requiring that both sides are defined whenever one is) automatically forces the law $t(e(a)) = a$ to hold.[2] Similarly, if we have the law $s(m(f,g)) = s(f)$ as well as $t(e(a)) = a$, then the associativity law $m(m(f,g),h) = m(f, m(g,h))$ forces $t(m(f,g)) = t(g)$.[3]

With this in mind, we verify the data from above satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a.
- The law $s(m(f,g)) = s(f)$ is true by construction; $m(f,g)$ is a direction from the position $s(f)$.
- The left identity law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left identity law, which identifies $\delta^\sharp(\varepsilon^\sharp(\mathrm{id}_{\blacktriangleleft}) \mathbin{\blacktriangleleft} f)$ with f whenever this makes sense.
- The right identity law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right identity law, which identifies $\delta^\sharp(f \mathbin{\blacktriangleleft} \varepsilon^\sharp(\mathrm{id}_{\blacktriangleleft}))$ with f whenever this makes sense.
- The associativity law $m(m(f,g),h) = m(f, m(g,h))$ is directly expressed by the comonoid associativity law, which identifies $\delta^\sharp(\delta^\sharp(f \mathbin{\blacktriangleleft} g) \mathbin{\blacktriangleleft} h)$ with $\delta^\sharp(f \mathbin{\blacktriangleleft} \delta^\sharp(g \mathbin{\blacktriangleleft} h))$ whenever this makes sense.
- The law $t(e(a)) = a$ is forced to hold (due to the comonoid left identity law).
- The law $t(m(f,g)) = t(g)$ is forced to hold (due to the comonoid associativity law).

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[2] We have $m(e(s(e(a))), e(a)) = e(a)$, since the right side is defined. The left side reduces to $m(e(a), e(a))$. This expression only makes sense if $t(e(a)) = s(e(a))$, which is a.

[3] Given that f and g are composable, we have $m(m(f,g), e(t(g))) = m(f, m(g, e(t(g))))$, since the right side is defined. The left side only makes sense if $t(f,g) = s(e(t(g)))$, which is $t(g)$.

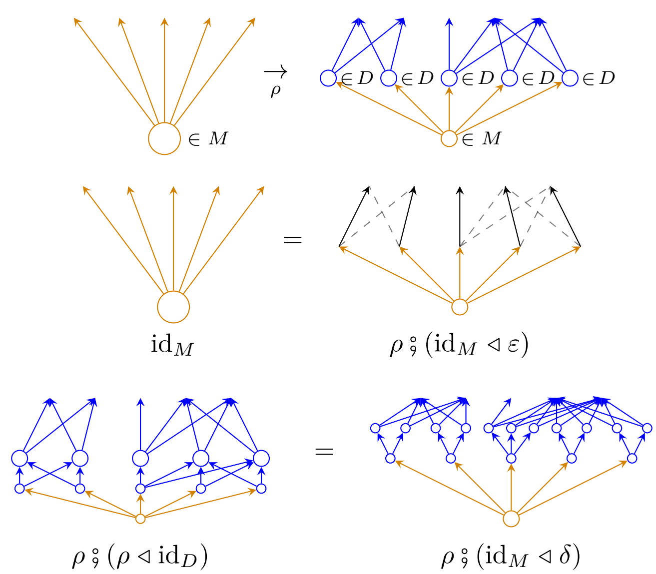
Conversely, let $\mathcal{C}$ be a category. We immediately obtain the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$. Let C denote the polynomial described by this bundle (the "outfacing polynomial" of $\mathcal{C}$). We will exhibit a comonoid struture on C.

- The counit ε singles out the identity in each object's set of outfacing maps.
- The comultiplication δ endows each object a with the map $\delta_a^\sharp$ sending $f \blacktriangleleft g$ to $m(f, g)$ for all arrows of the form $f : a \to b$, $g : b \to c$ (through which the map δ_1 is implicit).

The above processes of translation between the prerequisite data (not laws) for a category subject to just $s(e(a)) = a$ and $s(m(f, g)) = g$, and the prerequisite data (not laws) for a polynomial comonoid subject to just $\delta_1(a)_{\text{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity comonoid laws, in this context, directly translate to the identity and associativity category laws. $\square$

Proposition 2. *A polynomial right comodule amounts to a family of copresheaves.*

Proof. Let D be a polynomial comonoid with counit ε and comultiplication δ, and let M be a right comodule on D. Denote right comodule comultiplication by ρ.



Observe first that the identity law forces $\rho_1(a)_{\text{base}} = a$ for all $a \in M(1)$. Therefore the expression $\rho_1(a)_z$ for $z \in M[a]$ has a well-defined meaning.

Let $\mathcal{D}$ be the category underlying D. We gather the data for a family of copresheaves $\{Z_a\}_{a \in A}$ on $\mathcal{D}$.

- The family's indexing set A is $M(1)$, the set of positions in M.
- The total set of elements $\sum_{d \in \mathrm{Ob}(\mathcal{D})} Z_a(d)$ in Z_a is $M[a]$, the set of directions from a.
- The bundle map t assigning each element z in Z_a its indexing object in $\mathrm{Ob}(\mathcal{D}) = D(1)$ is given by $\rho_1(a)_z$.
- The multiplication map m sends each element $z \in Z_a(d)$ and compatible arrow $f \in D[d]$ to $\rho_a^\sharp(z \blacktriangleleft f)$.

Next, observe that if we have such prerequisite data (not laws) for a copresheaf on $\mathcal{D}$, then we find that imposing the copresheaf associativity law $m(m(z,f),g) = m(z,m(f,g))$ forces the law $t(m(z,f)) = t(f)$ to hold.[4] (The argument works the same for copresheaves as it does for categories.)

With this in mind, we verify each Z_a satisfies the laws of a copresheaf on $\mathcal{D}$.

- The identity law $m(z,e(t(z))) = z$ is directly expressed by the right comodule identity law, which identifies $\rho^\sharp(z \blacktriangleleft \varepsilon^\sharp(\mathrm{id}_\bullet))$ with z whenever this makes sense.
- The associativity law $m(m(z,f),g) = m(z,m(f,g))$ is directly expressed by the right comodule associativity law, which identifies $\rho^\sharp(\rho^\sharp(z \blacktriangleleft f) \blacktriangleleft g)$ with $\rho^\sharp(z \blacktriangleleft \delta^\sharp(f \blacktriangleleft g))$ whenever this makes sense.
- The law $t(m(z,f)) = t(f)$ is forced to hold (due to the right comodule associativity law).

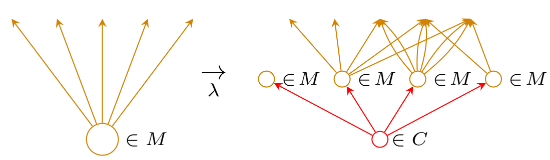
Conversely, let $\{Z_a\}_{a\in A}$ be a family of copresheaves on $\mathcal{D}$. Let M denote the polynomial described by the family of total sets of elements $\{\sum_{d\in \mathrm{Ob}(\mathcal{D})} Z_a(d)\}_{a\in A}$. We will exhibit a right D-comodule struture on M.

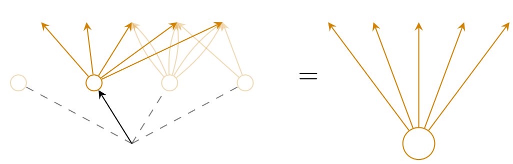
- The right comodule comultiplication ρ endows each position $a \in A$ with the map $\rho_a^\sharp$ sending each $z \blacktriangleleft f$ to $m(z,f)$ for all $z \in Z_a(d), f : d \to d' \in \mathcal{D}$. This implicitly determines $\rho_1(a)$ as long as the domain of $\rho_a^\sharp$ is nonempty; otherwise let $\rho_1(a)$ be the unique position in $M \triangleleft D$ with $\rho_1(a)_{\mathrm{base}} = a$.

The above processes of translation between the prerequisite data (not laws) for a family of presheaves on $\mathcal{D}$, and the prerequisite data (not laws) for a right D-comodule subject to just $\rho_1(a)_{\mathrm{base}} = a$, are inverse by construction. Moreover, the identity and associativity right comodule laws, in this context, directly translate to the identity and associativity copresheaf laws. □

Proposition 3. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

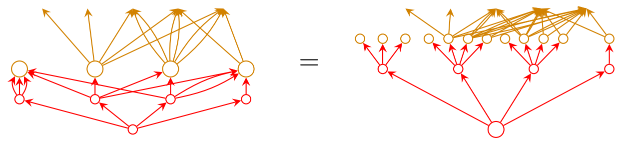
Proof. Let C be a polynomial comonoid with counit ε and comultiplication δ, and let M be a left comodule on C. Denote left comodule comultiplication by λ.





$\lambda \mathbin{⨾} (\varepsilon \triangleleft \mathrm{id}_M)$ id_M

[4] Given that z and f are composable, we have $m(m(z,f),e(t(f))) = m(z,m(f,e(t(f))))$, since the right side is defined. The left side only makes sense if $t(z,f) = s(e(t(f)))$, which is $t(f)$.



$\lambda \mathbin{⨾} (\delta \triangleleft \mathrm{id}_M)$ $\qquad\qquad$ $\lambda \mathbin{⨾} (\mathrm{id}_C \triangleleft \lambda)$

Let $\mathcal{C}$ be the category underlying C. We gather the data for a copresheaf X on $\mathcal{C}$.

- The total set of elements $\sum_{c \in \mathrm{Ob}(\mathcal{C})} X(c)$ in X is $M(1)$, the set of positions in M.
- The bundle map t assigning each element x in X its indexing object in $\mathrm{Ob}(\mathcal{C}) = C(1)$ is given by $\lambda_1(x)_{\mathrm{base}}$.
- The multiplication map m sends each element $x \in M(1)$ and compatible arrow $f \in C[t(x)]$ to $\lambda_1(x)_f$.

Now we gather the remaining data of a presheaf Z on $\int_{\mathcal{C}} X$, the category of elements of X. (We will have accumulated the data sans laws of a copresheaf on $\mathcal{C}$ and presheaf on its category of elements; we are still yet to verify X satisfies the laws of a copresheaf on $\mathcal{C}$.)

- The set $Z(x)$ for $x \in \mathrm{Ob}(\int_{\mathcal{C}} X) = M(1)$ is $M[x]$, the set of directions from x. Hence we obtain the bundle map s from the total set of elements $\sum_{x \in \mathrm{Ob}(\int_{\mathcal{C}} X)} Z(x)$ to $\mathrm{Ob}(\int_{\mathcal{C}} X)$ sending each $z \in Z(x)$ to x.
- The multiplication map m sends each arrow $f|_x : x \to w$ (in $\int_{\mathcal{C}} X$, lying over $f : t(x) \to t(w)$ in $\mathcal{C}$) and w-indexed element $z \in Z(w) = M[w]$ to $\lambda_x^\sharp(f \blacktriangleleft z)$.

  (To be clear, the domain of this map is the set of tuples (x, f, z) such that $t(x) = s(f)$ and $m(x, f) = s(z)$. This indeed coincides with the set of pairs $(f|_x, z)$ that should belong in the domain of multiplication for our presheaf on $\int_{\mathcal{C}} X$, since an arrow $f|_x$ in $\int_{\mathcal{C}} X$ is a pair (x, f) such that $t(x) = s(f)$, and the target of this arrow is $m(x, f)$.)

We will also use the following notation for identities and composition in $\int_{\mathcal{C}} X$.

- If x is an element of X, then $e(x)$ will refer to $e(t(x))|_x$.
- If $f|_x$ and $g|_{m(x,f)}$ are composable arrows in $\int_{\mathcal{C}} X$, then $m(f|_x, g|_{m(x,f)})$ will refer to $m(f, g)|_x$.

We verify X satisfies the laws of a copresheaf on $\mathcal{C}$.

- The identity law $m(x, e(t(x))) = x$ is the content of the left comodule identity law as regards positions, which says that $\lambda_1(x)_{\left(\varepsilon^\sharp_{\lambda_1(x)_{\mathrm{base}}}(\mathrm{id}_{\blacktriangleleft})\right)} = x$.
- The associativity law $m(m(x, f), g) = m(x, m(f, g))$ is the content of the left comodule associativity law as regards positions, which says that $\lambda_1(\lambda_1(x)_f)_g = \lambda_1(x)_{\left(\delta^\sharp_{\lambda_1(x)_{\mathrm{base}}}(f \blacktriangleleft g)\right)}$.
- The law $t(m(x, f)) = t(f)$ is forced to hold by the associativity law, as we have seen previously.

Now we verify Z satisfies the laws of a presheaf on $\int_{\mathcal{C}} X$.

- The identity law $m(e(s(z)), z) = z$ is the content of the left comodule identity law as regards directions, which identifies $\lambda^\sharp(\varepsilon^\sharp(\mathrm{id}_{\blacktriangleleft}) \blacktriangleleft z)$ with z whenever this makes sense.

- The associativity law $m(m(f|_x, g|_{m(x,f)}), z) = m(f|_x, m(g|_{m(x,f)}, z))$ is the content of the left comodule associativity law as regards directions, which identifies $\lambda^\sharp(\delta^\sharp(f \blacktriangleleft g) \blacktriangleleft z)$ with $\lambda^\sharp(f \blacktriangleleft \lambda^\sharp(g \blacktriangleleft z))$ whenever this makes sense.
- The law $s(m(f|_x, z)) = s(f|_x)$ is forced to hold by the associativity law, as we have seen previously (in the dual scenario).

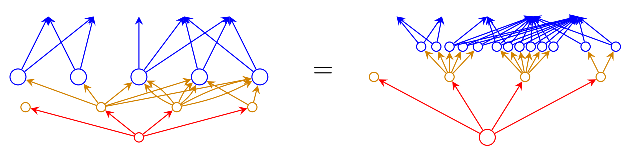
Conversely, let X be a copresheaf on $\mathcal{C}$ and Z be a presheaf on $\int_\mathcal{C} X$. We immediately obtain the bundle $\sum_{x \in \mathrm{Ob}(\int_\mathcal{C} X)} Z(x) \xrightarrow{s} \mathrm{Ob}(\int_\mathcal{C} X)$. Let M denote the polynomial described by this bundle. We will exhibit a left C-comodule struture on M.

- The left comodule comultiplication λ is defined on positions by $\lambda_1(x)_{\text{base}} = t(x)$ and $\lambda_1(x)_f = m(x, f)$ for all $f \in C[t(x)]$.
- For each position x, the map $\lambda^\sharp_x$ sends each $f \blacktriangleleft z$ to $m(f|_x, z)$ for all $f \in C[t(x)], z \in Z(m(x, f))$.

The above processes of translation between the prerequisite data (not laws) for a presheaf on the category of elements of a copresheaf on $\mathcal{C}$, and the prerequisite data (not laws) for a left C-comodule, are inverse by construction. Moreover, we the identity and associativity left comodule laws, in this context, directly translate to the identity and associativity copresheaf and presheaf laws. □

Proposition 4. *Polynomial bicomodules are prafunctors between presheaf categories.*

Proof. Let C and D be polynomial comonoids and let M a bicomodule from C to D with left module comultiplication λ and right module comultiplication ρ.



$\rho \mathbin{⨟} (\lambda \triangleleft \mathrm{id}_D) \qquad \lambda \mathbin{⨟} (\mathrm{id}_C \triangleleft \rho)$

We will show that M amounts to a profunctor[5] $(\int_\mathcal{C} X) \multimap\!\bullet\ \mathcal{D}$, where $\mathcal{C}$ is the category underlying C, X is a copresheaf on $\mathcal{C}$, and $\mathcal{D}$ is the category underlying D.

Such a profunctor is the same as a prafunctor $\mathbf{Set}^\mathcal{D} \to \mathbf{Set}^\mathcal{C}$. Indeed,

$$\begin{array}{c}
\text{prafunctors } \mathbf{Set}^\mathcal{D} \to \mathbf{Set}^\mathcal{C} \\
\updownarrow \\
\text{right adjoint functors } \mathbf{Set}^\mathcal{D} \to \mathbf{Set}^\mathcal{C}/X \text{ for any } \mathcal{C}\text{-copresheaf } X \\
\updownarrow \\
\text{left adjoint functors } \mathbf{Set}^\mathcal{C}/X \to \mathbf{Set}^\mathcal{D} \\
\updownarrow \\
\text{left adjoint functors } \mathbf{Set}^{\int_\mathcal{C} X} \to \mathbf{Set}^\mathcal{D} \\
\updownarrow \\
\text{functors } (\int_\mathcal{C} X)^{\mathrm{op}} \to \mathbf{Set}^\mathcal{D} \\
\updownarrow \\
\text{profunctors } (\int_\mathcal{C} X) \multimap\!\bullet\ \mathcal{D}.
\end{array}$$

[5]The notation $\mathcal{A} \multimap\!\bullet\ \mathcal{B}$ for a profunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$ is due to Michael Shulman.

We have already seen that M as a left $\mathcal{C}$-comodule amounts to a $\mathcal{C}$-copresheaf X and a presheaf Z on $\int_{\mathcal{C}} X$, and that M as a right $\mathcal{D}$-comodule amounts to a family of $\mathcal{D}$-copresheaves $\{Z_i\}_{i\in I}$.

The set $M(1)$ of positions in M serves as both the set of elements in the $\mathcal{C}$-copresheaf X, that is, $\mathrm{Ob}(\int_{\mathcal{C}} X)$, as well as the indexing set I for $\{Z_i\}_{i\in I}$. For any position $x \in M(1)$, the set $M[x]$ of directions from x serves as both $Z(x)$, the set of x-indexed elements in the $\int_{\mathcal{C}} X$-presheaf Z, as well as the set of elements in the $\mathcal{D}$-copresheaf Z_x.

Moreover, M and its left and right comodule structures are recovered from such information. That is, the data of a $\mathcal{C}$-copresheaf X, a $\int_{\mathcal{C}} X$-presheaf Z, and a $\mathcal{D}$-copresheaf structure on $Z(x)$ for each element x in X is the same as a polynomial M equipped with the structure of a left $\mathcal{C}$-module and right $\mathcal{D}$-module, assuming no compatibility.

The law $m(m(f|_x, z), g) = m(f|_x, m(z, g))$, encoding naturality of the maps $Z(f|_x) : Z(w) \to Z(x)$ (with respect to the $\mathcal{D}$-copresheaf structure on $Z(w)$ and $Z(x)$), is directly expressed by the bicomodule law, which identifies $\rho^\sharp(\lambda^\sharp(f \blacktriangleleft z) \blacktriangleleft g)$ with $\lambda^\sharp(f \blacktriangleleft \rho^\sharp(z \blacktriangleleft g))$ whenever this makes sense.

Thus, a bicomodule M from $\mathcal{C}$ to $\mathcal{D}$ is the same as a functor $(\int_{\mathcal{C}} X)^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{D}}$, i.e., a profunctor $(\int_{\mathcal{C}} X) \circ\!\!\to\!\!\bullet\ \mathcal{D}$ (i.e., a prafunctor from $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$). □

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof. First, we see what a natural transformation between prafunctors translates to under the correspondence between prafunctors and profunctors.

Recall[6] that a prafunctor $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$ is the same as a functor from $\mathcal{C}$ into the category of prafunctors $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}$, and that prafunctors $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}$ are coproducts of representables. Hence a prafunctor decomposes into a $\mathcal{C}$-shaped diagram of coproducts of representable functors $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}$, and maps of prafunctors are maps of such diagrams. A natural transformation between such coproducts of representables

$$\sum_{x\in X(c)} \mathrm{Hom}(Z_x, -) \quad \to \quad \sum_{x'\in X'(c)} \mathrm{Hom}(Z'_{x'}, -)$$

is the same as a function α_1 from $X(c)$ to $X'(c)$ and a map of $\mathcal{D}$-presheaves $\alpha^\sharp_x : Z_{\alpha_1(x)} \to Z_x$ for each $x \in X(c)$ (by the Yoneda lemma). On that account, we can see how a $\mathcal{C}$-shaped diagram of natural transformations is a $\mathcal{D}$-copresheaf-valued presheaf on the category of elements of a $\mathcal{C}$-copresheaf: the arrows of $\mathcal{C}$ act as functions between the indexing sets of $\mathcal{D}$-copresheaf families $\{Z_x\}_{x\in X(c)}$, and along each application of an arrow is a $\mathcal{D}$-copresheaf map in the opposite direction.

Putting it all together, a natural transformation between the prafunctors corresponding to profunctors $Z : (\int_{\mathcal{C}} X)^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{D}}$ and $Z' : (\int_{\mathcal{C}} X')^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{D}}$ (where X and X' are $\mathcal{C}$-copresheaves) amounts to a copresheaf map α_1 from X to X' and maps $\alpha^\sharp_x$ from $Z'(\alpha_1(x))(d)$ to $Z(x)(d)$, natural in x and d. (Moreover, vertical composition of natural transformations between prafunctors corresponds to the evident composition of such α maps.)

[6] We have not shown this in this paper.

Let C and D be polynomial comonoids (with underlying categories $\mathcal{C}$ and $\mathcal{D}$). Let M and N be bicomodules from C to D, and let α be a bicomodule map from M to N.

As usual, α consists of maps $\alpha_1 : C(1) \to D(1)$ and $\alpha_x^\sharp : D[\alpha_1(x)] \to C[x]$ for all positions $x \in C(1)$. Translated into language about the corresponding profunctors, α_1 is a map from the set of elements of X_M to the set of elements of X_N (where X_M and X_N are the $\mathcal{C}$-copresheaves induced by M and N, respectively), and $\alpha_x^\sharp$ is a map from $Z_N(\alpha_1(x))$ to $Z_M(x)$ (where Z_M and Z_N are the $\int_\mathcal{C} X_M$-presheaf and $\int_\mathcal{C} X_N$-presheaf induced by M and N, respectively).

We verify this α corresponds to a natural transformation between prafunctors $\mathbf{Set}^\mathcal{D} \to \mathbf{Set}^\mathcal{C}$ in the sense described above.

- The naturality in c law $\alpha_1(m(x,f)) = m(\alpha_1(x), f)$ for the $\mathcal{C}$-copresheaf map is the content of the left comodule map law $\alpha \mathbin{⨾} \lambda_N = \lambda_M \mathbin{⨾} (\mathrm{id}_C \triangleleft \alpha)$ as regards positions.

  (This also forces the law $t(\alpha_1(x)) = t(x)$, i.e., α_1 preserves indexing objects.)
- The law $t(\alpha_x^\sharp(z)) = t(z)$ (i.e., $\alpha_x^\sharp$ preserves indexing objects of elements) is the content of the right comodule map law $\alpha \mathbin{⨾} \rho_N = \rho_M \mathbin{⨾} (\alpha \triangleleft \mathrm{id}_D)$ as regards positions.
- The naturality in x law $\alpha_x^\sharp(m(f|_{\alpha_1(x)}, z)) = m(f|_x, \alpha^\sharp_{m(x,f)}(z))$ is the content of the left comodule map as regards directions, which identifies $\alpha^\sharp({\lambda_N}^\sharp(f \blacktriangleleft z))$ with ${\lambda_M}^\sharp(f \blacktriangleleft \alpha^\sharp(z))$ whenever this makes sense.
- The naturality in d law $\alpha_x^\sharp(m(z,f)) = m(\alpha_x^\sharp(z), f)$ for the maps $\alpha_x^\sharp$ as $\mathcal{D}$-copresheaf maps is the content of the right comodule map law as regards directions, which identifies $\alpha^\sharp({\rho_N}^\sharp(z \blacktriangleleft f))$ with ${\rho_M}^\sharp(\alpha^\sharp(z) \blacktriangleleft f)$ whenever this makes sense.

Conversely, any natural transformation between prafunctors $\mathbf{Set}^\mathcal{D} \to \mathbf{Set}^\mathcal{C}$ gives rise to such an α, satisfying such laws.

(Moreover, note that vertical compositions are preserved under the correspondence between prafunctor natural transformations and bicomodule maps, since vertical composition of such natural transformations is achieved by composing the α maps relating the underlying profunctors.) □

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

Proof. Bicomodules from D to 0 specialize to copresheaves on $\mathcal{D}$ (and maps between such bicomodules are copresheaf maps). Hence each bicomodule M from C to D induces a functor M^* from $\mathcal{D}$-copresheaves to $\mathcal{C}$-copresheaves by precomposition. Accordingly, we have $(M \triangleleft_D N)^* \cong N^* \mathbin{⨾} M^*$ (where $M : C \nrightarrow D$ and $N : D \nrightarrow E$ are bicomodules).

We show that the prafunctor underlying the bicomodule M is the same functor M^* induced by horizontal precomposition of M.

Recall[7] that the composite $M \triangleleft_D N$ of bicomodules $M : C \nrightarrow D$ and $N : D \nrightarrow E$ is the equalizer of $\mathrm{id}_M \triangleleft \lambda_N$ and $\rho_M \triangleleft \mathrm{id}_N$. Its bicomodule structure is given by the unique transformations

$$\lambda_{M \triangleleft_D N} : M \triangleleft_D N \to C \triangleleft M \triangleleft_D N \quad \text{and} \quad \rho_{M \triangleleft_D N} : M \triangleleft_D N \to M \triangleleft_D N \triangleleft E$$

[7]We have not shown this in this paper.

such that

$$\lambda_{M\triangleleft_D N} \mathbin{⨾} (\mathrm{id}_C \triangleleft \iota) = \iota \mathbin{⨾} (\lambda_M \triangleleft \mathrm{id}_N) \quad \text{and} \quad \rho_{M\triangleleft_D N} \mathbin{⨾} (\iota \triangleleft \mathrm{id}_E) = \iota \mathbin{⨾} (\mathrm{id}_M \triangleleft \rho_N),$$

where ι is the "inclusion" $M \triangleleft_D N \hookrightarrow M \triangleleft N$.

Concretely, the positions in $(M \triangleleft_D N)(1)$ are the positions a in $(M \triangleleft N)(1)$ such that $(\mathrm{id}_M \triangleleft \lambda_N)_1(a) = (\rho_M \triangleleft \mathrm{id}_N)_1(a)$, and the directions in $(M \triangleleft_D N)[a]$ are the equivalence classes of directions in $(M \triangleleft N)[a]$ under the equivalence relation generated by

$$(\mathrm{id}_M \triangleleft \lambda_N)^\sharp_a(z_M \blacktriangleleft d \blacktriangleleft z_N) \sim (\rho_M \triangleleft \mathrm{id}_N)^\sharp_a(z_M \blacktriangleleft d \blacktriangleleft z_N)$$

for all directions $z_M \blacktriangleleft d \blacktriangleleft z_N$ from $(\mathrm{id}_M \triangleleft \lambda_N)_1(a) = (\rho_M \triangleleft \mathrm{id}_N)_1(a)$, i.e., we have

$$z_M \blacktriangleleft {\lambda_N}^\sharp(d \blacktriangleleft z_N) \sim {\rho_M}^\sharp(z_M \blacktriangleleft d) \blacktriangleleft z_N$$

whenever this makes sense. The transformations $\lambda_{M\triangleleft_D N}$ and $\rho_{M\triangleleft_D N}$ respectively agree with $\lambda_M \triangleleft \mathrm{id}_N$ and $\mathrm{id}_M \triangleleft \rho_N$ on positions, and similarly they send directions $c \blacktriangleleft [z_M \blacktriangleleft z_N]$ to $[{\lambda_M}^\sharp(c \blacktriangleleft z_M) \blacktriangleleft z_N]$ and $[z_M \blacktriangleleft z_N] \blacktriangleleft e$ to $[z_M \blacktriangleleft {\rho_M}^\sharp(z_M \blacktriangleleft e)]$ whenever this makes sense. (As we will see, we will not end up needing to think about directions in $M \triangleleft_D N$ at all.)

We are interested in the case where $E = 0$. Here ρ_N is trivial and N is constant (just a set $N(1)$ of empty positions) with a $\mathcal{D}$-copresheaf structure induced by λ_N. As a right D-comodule, M describes a family of $\mathcal{D}$-copresheaves. Hence $(- \triangleleft_D -)$ may be viewed as an operation that takes as input a family of $\mathcal{D}$-copresheaves $\{Z_x\}_{x\in X}$, plus another $\mathcal{D}$-copresheaf U, and returns a constant polynomial P (i.e., a set). The positions in this P are the $p \in (M \triangleleft N)(1)$ such that $m(p_z, f) = p_{m(z,f)}$ for all $z \in M[p_{\mathrm{base}}]$. Such a p is the same as a natural transformation from any one of the Z_x to U.

This set P, whose elements are identified with natural transformations $Z_x \to U$, is endowed with a $\mathcal{C}$-copresheaf (left C-comodule) structure $\lambda_{M\triangleleft_D N}$, induced by λ_M. Indeed, λ_M supplies a covariant action by $\mathcal{C}$ on the indexing set X of $\{Z_x\}_{x\in X}$ and supplies a natural transformation $Z(f|_x) : Z_w \to Z_x$ whenever x is sent to w by $f \in \mathcal{C}$. So we obtain a covariant $\mathcal{C}$-action on the set of natural transformations from members of $\{Z_x\}_{x\in X}$ to U by precomposing such $Z(f|_x)$ maps.

In summary,

$$P(c) = (M^*(U))(c) = \sum_{x\in X(c)} \mathrm{Hom}_{\mathbf{Set}^{\mathcal{D}}}(Z_x, U).$$

This characterizes the behavior of M^* on objects. We see the expected form of the prafunctor underlying M. In particular, when U is the terminal $\mathcal{D}$-copresheaf, P is X (the $\mathcal{C}$-copresheaf induced by λ_M).

Next, we seek the behavior of M^* on maps. Recall[8] that the horizontal composite $\alpha \triangleleft_D \beta$ of bicomodule maps $\alpha : M \to M'$ and $\beta : N \to N'$ is the unique transformation $M \triangleleft N \to M' \triangleleft N'$ such that

$$(\alpha \triangleleft_D \beta) \mathbin{⨾} \iota_{M'\triangleleft_D N'} = \iota_{M\triangleleft_D N} \mathbin{⨾} (\alpha \triangleleft \beta),$$

where $\iota_{M\triangleleft_D N}$ and $\iota_{M'\triangleleft_D N'}$ are respectively the "inclusions" $M \triangleleft_D N \hookrightarrow M \triangleleft N$ and $M' \triangleleft_D N' \hookrightarrow M' \triangleleft N'$.

[8]We have not shown this in this paper.

Concretely, the transformation $\alpha \triangleleft_D \beta$ agrees with $\alpha \triangleleft \beta$ on positions, and similarly it sends directions $[z_{M'} \blacktriangleleft z_{N'}]$ to $[\alpha^\sharp(z_{M'}) \blacktriangleleft \beta^\sharp(z_{N'})]$ whenever this makes sense. (But again, we will not need to look at directions.)

We will be taking $\alpha = \mathrm{id}_M$, i.e., horizontally precomposing the bicomodule M with the map β. Since our $N, N' : D \nrightarrow 0$ are constant, β is entirely detemined by the map of positions β_1, which amounts to a natural transformation between $\mathcal{D}$-copresheaves. Hence $M \triangleleft_D \beta$ sends each position $a \in (M \triangleleft_D N)(1) \subseteq (M \triangleleft N)(1)$ to the position $a' \in (M \triangleleft_D N')(1) \subseteq (M \triangleleft N')(1)$ such that $a'_{\text{base}} = a_{\text{base}}$ and $a'_z = \beta_1(a_z)$ for all directions $z \in M[a_{\text{base}}]$. In other words, it is the map that sends a natural transformation $p \in P = (M \triangleleft_D N)(1)$ to $p \mathbin{⨟} \beta_1$. This characterizes the behavior of M^* on maps (which could already have been anticipated based on the formula for $M^*(U)$ given above).

The result, $M^*(U) = \sum_{x \in X(-)} \mathrm{Hom}(Z(x), U)$, is precisely the prafunctor between presheaf categories (or equivalently, $\mathcal{C}$-shaped diagram valued in coproducts of representables) corresponding to the profunctor $Z : (\mathbf{Set}^{\int_{\mathcal{C}} X})^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{D}}$. □

Proposition 7. *Horizontal composition of bicomodule maps is horizontal composition of natural transformations between prafunctors.*

Proof. Each bicomodule map $\alpha : M \to M' : C \nrightarrow D$ induces a natural transformation $\alpha^* : M^* \to M'^*$ by horizontal ($\triangleleft_D$) precomposition. Accordingly, we have that the natural transformation $(\alpha \triangleleft_D \beta)^* : (M \triangleleft_D N)^* \to (M' \triangleleft_D N')^*$ is equal to the horizontal composite (as natural transformations) of β^* and α^*, from $N^* \mathbin{⨟} M^*$ to $N'^* \mathbin{⨟} M'^*$ (where both $\alpha : M \to M' : C \nrightarrow D$ and $\beta : N \to N' : D \nrightarrow E$ are bicomodule maps).

Let N be a bicomodule $D \nrightarrow 0$ (i.e., a $\mathcal{D}$-copresheaf). The component of $\alpha^* : M^* \to M'^*$ at N is $\alpha \triangleleft_D \mathrm{id}_N : M^*(N) \to M'^*(N)$. The map on positions $(\alpha \triangleleft_D \mathrm{id}_N)_1$ agrees with $(\alpha \triangleleft \mathrm{id}_N)_1$, and since $M' \triangleleft_D N$ is constant, there are no directions to account for. This map $(\alpha \triangleleft_D \mathrm{id}_N)_1$ sends each position $a \in M \triangleleft_D N \subseteq M \triangleleft N$ to the position $a' \in M' \triangleleft_D N \subseteq M' \triangleleft N$ such that $a'_{\text{base}} = \alpha_1(a_{\text{base}})$ and $a'_z = a_{(\alpha^\sharp_{a_{\text{base}}}(z))}$ for all directions $z \in M'[\alpha_1(a'_{\text{base}})]$.

Recall $M \triangleleft_D N$ is the $\mathcal{C}$-copresheaf $\sum_{x \in X(-)} \mathrm{Hom}(Z(x), U)$, where Z is the functor $\int_{\mathcal{C}} X \to \mathbf{Set}^{\mathcal{D}}$ underlying M and U is the $\mathcal{D}$-copresheaf underlying N. We see that $(\alpha \triangleleft_D \mathrm{id}_N)_1$ sends each $p \in \mathrm{Hom}(Z(x), U)$ to $\alpha^\sharp_x \mathbin{⨟} p \in \mathrm{Hom}(Z(\alpha_1(x)), U)$. The relevant Yoneda embedding identifies this map between representable functors with the map of $\mathcal{D}$-copresheaves $\alpha^\sharp_x$.

Hence α^* is the same transformation of prafunctors derived from the data in α (where indexes of $\mathcal{D}$-copresheaves are mapped according to α_1 and elements are mapped backwards according to $\alpha^\sharp$). □