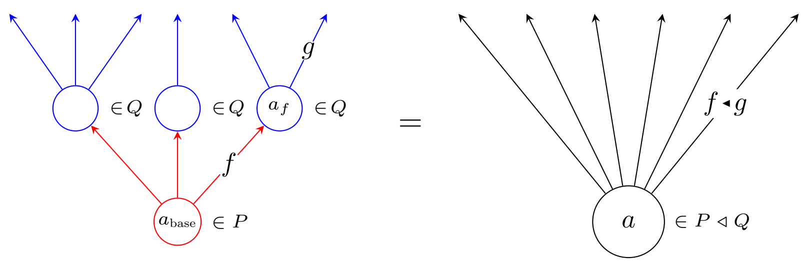
POLYNOMIAL COMONOIDS AND BICOMODULES

Recall the *substitution product* of polynomials P and Q, denoted $P \triangleleft Q$.

- A position a in $P \triangleleft Q$ consists of a position a_{base} in P and positions a_f in Q for each direction f from a_{base}.
- A direction from position a in $P \triangleleft Q$ consists of a direction f from a_{base} and a direction g from a_f.



We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$.[1] Accordingly, $\mathrm{id}_{\blacktriangleleft}$ will denote the unique direction from the unique position in the unit for substitution $\mathrm{id}_{\triangleleft}$ (a.k.a. the polynomial y).

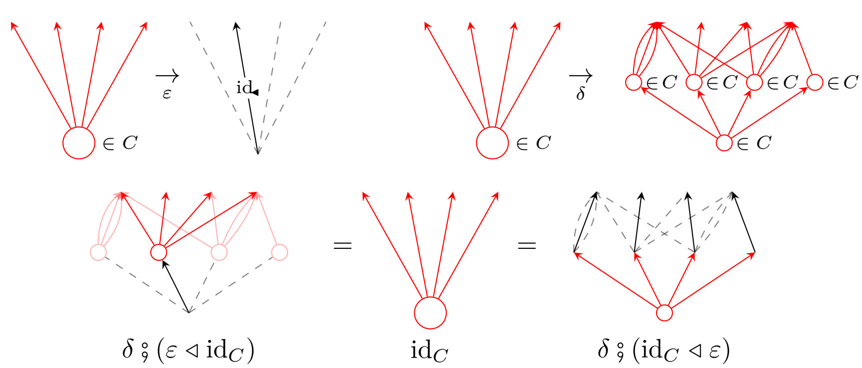
Note the following identity for transformations α and β between polynomials.

$$(\alpha \triangleleft \beta)^\sharp_a(f \blacktriangleleft g) = \underbrace{\alpha^\sharp_{(a_{\text{base}})}(f)}_{\star} \blacktriangleleft \beta^\sharp_{(a_\star)}(g).$$

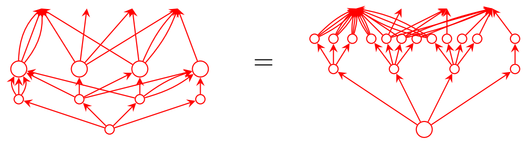
Or in brief, "we have $(\alpha \triangleleft \beta)^\sharp(f \blacktriangleleft g) = \alpha^\sharp(f) \blacktriangleleft \beta^\sharp(g)$ whenever this makes sense."

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let C be a polynomial comonoid. Denote counit by ε and comultiplication by δ.



[1]Be aware there may be other directions named $f \blacktriangleleft g$ from other positions in $P \triangleleft Q$.



$\delta \mathbin{⨟} (\delta \triangleleft \mathrm{id}_C)$ $\delta \mathbin{⨟} (\mathrm{id}_C \triangleleft \delta)$

Observe first that the right identity law forces $\delta_1(a)_{\text{base}} = a$ for all $a \in C(1)$. Therefore the expression $\delta_1(a)_f$ for $f \in C[a]$ has a well-defined meaning.

We gather the data for a category $\mathcal{C}$.

- The set of objects $\mathrm{Ob}(\mathcal{C})$ is $C(1)$, the set of positions in C.
- The set of arrows $\mathrm{Arr}(\mathcal{C})$ is $\sum_{a \in C(1)} C[a]$, the set of all directions in C.
- The source map s sends each $f \in C[a]$ to a. (Hence the polynomial C is described by the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in C[a]$ to $\delta_1(a)_f$.
- The identity map e sends each $a \in C(1)$ to $\varepsilon^\sharp_a(\mathrm{id}_◂)$.
- The composition map m sends each pair of compatible arrows $f \in C[a], g \in C[t(f)]$ to $\delta^\sharp_a(f ◂ g)$.

Next, observe that if we have such prerequisite data (not laws) for a category, subject to just the law $s(e(a)) = a$, then we find that further imposing the left identity law $m(e(s(f)), f) = f$ (and requiring that both sides are defined whenever one is) automatically forces the law $t(e(a)) = a$ to hold.[2] Similarly, if we have the law $s(m(f,g)) = s(f)$ as well as $t(e(a)) = a$, then the associativity law $m(m(f,g),h) = m(f,m(g,h))$ forces $t(m(f,g)) = t(g)$.[3]

We verify the data from above satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a.
- The law $s(m(f,g)) = s(f)$ is true by construction; $m(f,g)$ is a direction from the position $s(f)$.
- The left identity law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left identity law, which identifies $\delta^\sharp(\varepsilon^\sharp(\mathrm{id}_◂) ◂ f)$ with f whenever this makes sense.
- The right identity law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right identity law, which identifies $\delta^\sharp(f ◂ \varepsilon^\sharp(\mathrm{id}_◂))$ with f whenever this makes sense.
- The associativity law $m(m(f,g),h) = m(f,m(g,h))$ is directly expressed by the comonoid associativity law, which identifies $\delta^\sharp(\delta^\sharp(f ◂ g) ◂ h)$ with $\delta^\sharp(f ◂ \delta^\sharp(g ◂ h))$ whenever this makes sense.
- The law $t(e(a)) = a$ is forced to hold (due to the comonoid left identity law).
- The law $t(m(f,g)) = t(g)$ is forced to hold (due to the comonoid associativity law).

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[2]We have $m(e(s(e(a))), e(a)) = e(a)$, since the right side is defined. The left side reduces to $m(e(a), e(a))$. This expression only makes sense if $t(e(a)) = s(e(a))$, which is a.

[3]Given that f and g are composable, we have $m(m(f,g), e(t(g))) = m(f, m(g, e(t(g))))$, since the right side is defined. The left side only makes sense if $t(f,g) = s(e(t(g)))$, which is $t(g)$.

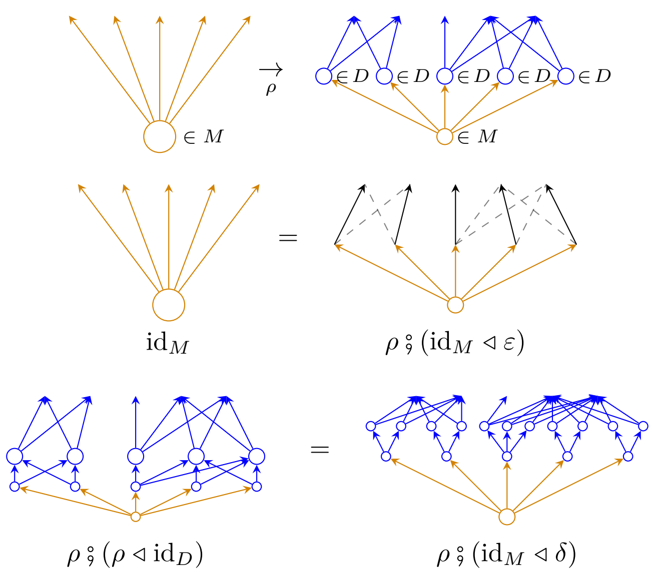
Conversely, let $\mathcal{C}$ be a category. We immediately obtain the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$. Let C denote the polynomial described by this bundle (the "outfacing polynomial" of $\mathcal{C}$). We will exhibit a comonoid struture on C.

- The counit ε singles out the identity in each object's set of outfacing maps.
- The comultiplication δ endows each object a with the map $\delta_a^\sharp$ sending $f \blacktriangleleft g$ to $m(f,g)$ for all arrows of the form $f : a \to b$, $g : b \to c$ (through which the map δ_1 is implicit).

The above processes of translation between the prerequisite data (not laws) for a category subject to just $s(e(a)) = a$ and $s(m(f,g)) = g$, and the prerequisite data (not laws) of a polynomial comonoid subject to just $\delta_1(a)_{\text{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity comonoid laws, in this context, directly translate to the identity and associativity category laws. $\square$

Proposition 2. *A polynomial right comodule amounts to a family of copresheaves.*

Proof. Let D be a polynomial comonoid with counit ε and comultiplication δ, and let M be a right comodule on D. Denote right comodule comultiplication by ρ.



Observe first that the identity law forces $\rho_1(a)_{\text{base}} = a$ for all $a \in M(1)$. Therefore the expression $\rho_1(a)_z$ for $z \in M[a]$ has a well-defined meaning.

Let $\mathcal{D}$ be the category corresponding to D. We gather the data for a family of copresheaves $\{Z_a\}_{a \in A}$ on $\mathcal{D}$.

- The family's indexing set A is $M(1)$, the set of positions in M.
- The total set of elements $\sum_{d \in \mathrm{Ob}(\mathcal{D})} Z_a(d)$ in Z_a is $M[a]$, the set of directions from a.
- The bundle map t assigning each element z in Z_a its indexing object in $\mathrm{Ob}(\mathcal{D}) = D(1)$ is given by $\rho_1(a)_z$.
- The multiplication map m sends each element $z \in Z_a(d)$ and compatible arrow $f \in D[d]$ to $\rho_a^\sharp(z \blacktriangleleft f)$.

Next, observe that if we have such prerequisite data (not laws) for a copresheaf on $\mathcal{D}$, then we find that imposing the copresheaf associativity law $m(m(z,f),g) = m(z,m(f,g))$ forces the law $t(m(z,f)) = t(f)$ to hold.[4] (The argument works the same for copresheaves as it does for categories.)

We verify each Z_a satisfies the laws of a copresheaf on $\mathcal{D}$.

- The identity law $m(z,e(t(z))) = z$ is directly expressed by the right comodule identity law, which identifies $\rho^\sharp(z \blacktriangleleft \varepsilon^\sharp(\mathrm{id}_{\blacktriangleleft}))$ whenever this makes sense.
- The associativity law $m(m(z,f),g) = m(z,m(f,g))$ is directly expressed by the right comodule associativity law, which identifies $\rho^\sharp(\rho^\sharp(z \blacktriangleleft f) \blacktriangleleft g)$ with $\rho^\sharp(z \blacktriangleleft \delta^\sharp(f \blacktriangleleft g))$ whenever this makes sense.
- The law $t(m(z,f)) = t(f)$ is forced to hold (due to the right comodule associativity law).

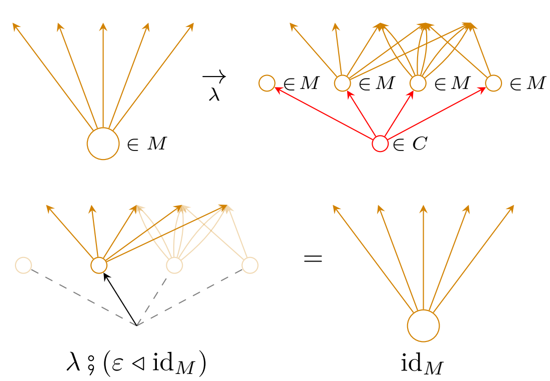
Conversely, let $\{Z_a\}_{a\in A}$ be a family of copresheaves on $\mathcal{D}$. Let M denote the polynomial described by the family of total sets of elements $\{\sum_{d\in \mathrm{Ob}(\mathcal{D})} Z_a(d)\}_{a\in A}$. We will exhibit a right D-comodule struture on M.

- The right comodule comultiplication ρ endows each position $a \in A$ with the map $\rho^\sharp_a$ sending each $z \blacktriangleleft f$ to $m(z,f)$ for all $z \in Z_a(d)$, $f : d \to d' \in \mathcal{D}$. This implicitly determines $\rho_1(a)$ as long as the domain of $\rho^\sharp_a$ is nonempty; otherwise let $\rho_1(a)$ be the unique position in $M \triangleleft D$ with $\rho_1(a)_{\mathrm{base}} = a$.

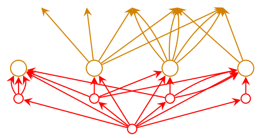
The above processes of translation between the prerequisite data (not laws) for a family of presheaves on $\mathcal{D}$, and the prerequisite data (not laws) of a right D-comodule subject to just $\rho_1(a)_{\mathrm{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity right comodule laws, in this context, directly translate to the identity and associativity copresheaf laws. □

Proposition 3. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

Proof. Let C be a polynomial comonoid with counit ε and comultiplication δ, and let M be a left comodule on C. Denote left comodule comultiplication by λ.



[4]Given that z and f are composable, we have $m(m(z,f),e(t(f))) = m(z,m(f,e(t(f))))$, since the right side is defined. The left side only makes sense if $t(z,f) = s(e(t(f)))$, which is $t(f)$.



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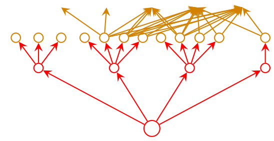


$\lambda \fatsemi (\delta \triangleleft \mathrm{id}_M)$ $\lambda \fatsemi (\mathrm{id}_C \triangleleft \lambda)$

Let $\mathcal{C}$ be the category corresponding to C. We gather the data for a copresheaf X on $\mathcal{C}$.

- The total set of elements $\sum_{c \in \mathrm{Ob}(\mathcal{C})} X(c)$ in X is $M(1)$, the set of positions in M.
- The bundle map t assigning each element x in X its indexing object in $\mathrm{Ob}(\mathcal{C}) = C(1)$ is given by $\lambda_1(x)_{\mathrm{base}}$.
- The multiplication map m sends each element $x \in M(1)$ and compatible arrow $f \in C[t(x)]$ to $\lambda_1(x)_f$.

Now we gather the remaining data of a presheaf Z on $\int_{\mathcal{C}} X$, the category of elements of X. (We will have accumulated the data sans laws of a copresheaf on $\mathcal{C}$ and presheaf on its category of elements; we are still yet to verify X satisfies the laws of a copresheaf on $\mathcal{C}$.)

- The set $Z(x)$ for $x \in \mathrm{Ob}(\int_{\mathcal{C}} X) = M(1)$ is $M[x]$, the set of directions from x. Hence we obtain the bundle map s from the total set of elements $\sum_{x \in \mathrm{Ob}(\int_{\mathcal{C}} X)} Z(x)$ to $\mathrm{Ob}(\int_{\mathcal{C}} X)$ sending each $z \in Z(x)$ to x.
- The multiplication map m sends each arrow $f|_w : w \to x$ (in $\int_{\mathcal{C}} X$, lying over $f : t(w) \to t(x)$ in $\mathcal{C}$) and x-indexed element $z \in Z(x) = M[x]$ to $\lambda^\sharp_w(f \blacktriangleleft z)$.

  (To be clear, the domain of this map is the set of tuples (w, f, z) such that $t(w) = s(f)$ and $m(w, f) = s(z)$. This is indeed the set of pairs $(f|_w, z)$ that should belong in the domain of multiplication for our presheaf on $\int_{\mathcal{C}} X$, since an arrow $f|_w$ in $\int_{\mathcal{C}} X$ is a pair (w, f) such that $t(w) = s(f)$, and the target of this arrow is $m(w, f)$.)

We will also use the following notation for identities and composition in $\int_{\mathcal{C}} X$.

- If x is an element of X, then $e(x)$ will refer to $e(t(x))|_x$.
- If $f|_w$ and $g|_{m(w,f)}$ are composable arrows in $\int_{\mathcal{C}} X$, then $m(f|_w, g|_{m(w,f)})$ will refer to $m(f, g)|_w$.

We verify X satisfies the laws of a copresheaf on $\mathcal{C}$.

- The identity law $m(x, e(t(x))) = x$ is the content of the left comodule identity law as regards positions, which says that $\lambda_1(x)_{\varepsilon^\sharp_{\lambda_1(x)_{\mathrm{base}}}(\mathrm{id}_\blacktriangleleft)} = x$.
- The associativity law $m(m(x, f), g) = m(x, m(f, g))$ is the content of the left comodule associativity law as regards positions, which says that $\lambda_1(\lambda_1(x)_f)_g = \lambda_1(x)_{\delta^\sharp_{\lambda_1(x)_{\mathrm{base}}}(f \blacktriangleleft g)}$.
- The law $t(m(x, f)) = t(f)$ is forced to hold by the associativity law, as we have seen previously.

Now we verify Z satisfies the laws of a presheaf on $\int_{\mathcal{C}} X$.

- The identity law $m(e(s(z)), z) = z$ is the content of the left comodule identity law as regards directions, which identifies $\lambda^\sharp(\varepsilon^\sharp(\mathrm{id}_\blacktriangleleft) \blacktriangleleft z)$ with z whenever this makes sense.

- The associativity law $m(m(f|_w, g|_{m(w,f)}), z) = m(f|_w, m(g|_{m(w,f)}, z))$ is the content of the left comodule associativity law as regards directions, which identifies $\lambda^\sharp(\delta^\sharp(f \triangleleft g) \triangleleft z)$ with $\lambda^\sharp(f \triangleleft \lambda^\sharp(g \triangleleft z))$ whenever this makes sense.
- The law $s(m(f|_w, z)) = s(f|_w)$ is forced to hold by the associativity law, as we have seen previously (in the dual scenario).

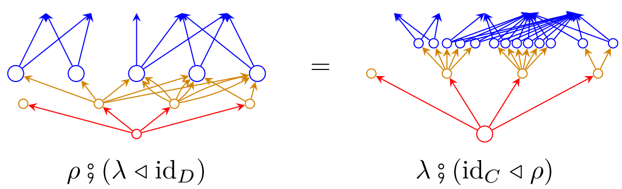
Conversely, let X be a copresheaf on $\mathcal{C}$ and Z be a presheaf on $\int_{\mathcal{C}} X$. We immediately obtain the bundle $\sum_{x \in \mathrm{Ob}(\int_{\mathcal{C}} X)} Z(x) \xrightarrow{s} \mathrm{Ob}(\int_{\mathcal{C}} X)$. Let M denote the polynomial described by this bundle. We will exhibit a left C-comodule struture on M.

- The left comodule comultiplication λ is defined on positions by $\lambda_1(x)_{\mathrm{base}} = t(x)$ and $\lambda_1(x)_f = m(x, f)$ for all $f \in C[t(x)]$.
- For each position x, the map $\lambda^\sharp_x$ sends each $f \triangleleft z$ to $m(f|_x, z)$ for all $f \in C[t(x)], z \in Z(m(x, f))$.

The above processes of translation between the prerequisite data (not laws) for a presheaf on the category of elements of a copresheaf on $\mathcal{C}$, and the prerequisite data (not laws) of a left C-comodule, are inverse by construction. Moreover, we saw earlier that the identity and associativity left comodule laws, in this context, directly translate to the identity and associativity copresheaf and presheaf laws. □

Proposition 4. *Polynomial bicomodules are prafunctors between presheaf categories.*

Proof. Let C and D be polynomial comonoids and let M a bimodule from C to D with left module comultiplication λ and right module comultiplication ρ.



We will show that M amounts to a profunctor[5] $(\int_{\mathcal{C}} X) \mathrel{\circ\!\!-\!\!\bullet} \mathcal{D}$, where $\mathcal{C}$ is the category corresponding to C, X is a copresheaf on $\mathcal{C}$, and $\mathcal{D}$ is the category corresponding to D.

Such a profunctor is the same as a prafunctor $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$. Indeed,

$$\begin{gathered}
\text{prafunctors } \mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}} \\
\updownarrow \\
\text{right adjoint functors } \mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}/X \text{ for any } \mathcal{C}\text{-copresheaf } X \\
\updownarrow \\
\text{left adjoint functors } \mathbf{Set}^{\mathcal{C}}/X \to \mathbf{Set}^{\mathcal{D}} \\
\updownarrow \\
\text{left adjoint functors } \mathbf{Set}^{\int_{\mathcal{C}} X} \to \mathbf{Set}^{\mathcal{D}} \\
\updownarrow \\
\text{functors } (\textstyle\int_{\mathcal{C}} X)^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{D}} \\
\updownarrow \\
\text{profunctors } (\textstyle\int_{\mathcal{C}} X) \mathrel{\circ\!\!-\!\!\bullet} \mathcal{D}.
\end{gathered}$$

[5]The notation $\mathcal{A} \mathrel{\circ\!\!-\!\!\bullet} \mathcal{B}$ for a profunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$ is due to Michael Shulman.

We have already seen that M as a left $\mathcal{C}$-comodule amounts to a $\mathcal{C}$-copresheaf X and a presheaf Z on $\int_{\mathcal{C}} X$, and that M as a right $\mathcal{D}$-comodule amounts to a family of $\mathcal{D}$-copresheaves $\{Z_i\}_{i\in I}$.

The set $M(1)$ of positions in M serves as both the set of elements in the $\mathcal{C}$-copresheaf X, that is, $\mathrm{Ob}(\int_{\mathcal{C}} X)$, as well as the indexing set I for $\{Z_i\}_{i\in I}$. For any position $x \in M(1)$, the set $M[x]$ of directions from x serves as both $Z(x)$, the set of x-indexed elements in the $\int_{\mathcal{C}} X$-presheaf Z, as well as the set of elements in the $\mathcal{D}$-copresheaf Z_x.

Moreover, M and its left and right comodule structures are recovered from such information. That is, the data of a $\mathcal{C}$-copresheaf X, a $\int_{\mathcal{C}} X$-presheaf Z, and a $\mathcal{D}$-copresheaf structure on $Z(x)$ for each element x in X is the same as a polynomial M equipped with the structure of a left $\mathcal{C}$-module and right $\mathcal{D}$-module, assuming no compatibility.

The law $m(m(f|_w, z), g) = m(f|_w, m(z, g))$, encoding naturality of the maps $Z(f|_w) : Z(x) \to Z(w)$ (with respect to the $\mathcal{D}$-copresheaf structure on $Z(x)$ and $Z(w)$), is directly expressed by the bicomodule law, which identifies $\rho^\sharp(\lambda^\sharp(f \blacktriangleleft z) \blacktriangleleft g)$ with $\lambda^\sharp(f \blacktriangleleft \rho^\sharp(z \blacktriangleleft g))$ whenever this makes sense.

Thus, a bicomodule M from $\mathcal{C}$ to $\mathcal{D}$ is the same as a functor $(\int_{\mathcal{C}} X)^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{D}}$, i.e., a profunctor $(\int_{\mathcal{C}} X) \multimapdotboth \mathcal{D}$ (i.e., a prafunctor from $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$). $\square$

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof. First, recall[6] that under the correspondence between prafunctors and profunctors, a natural transformation between prafunctors $F, G : \mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$ respectively corresponding to profunctors $M : (\int_{\mathcal{C}} X_M) \multimapdotboth \mathcal{D}, N : (\int_{\mathcal{C}} X_N) \multimapdotboth \mathcal{D}$ (where X_M and X_N are $\mathcal{C}$-copresheaves) is identified with a copresheaf map α_1 from X_M to X_N and maps $\alpha^\sharp_x|_d$ from $N(\alpha_1(x), d)$ to $M(x, d)$, natural in x and d.

Let C and D be polynomial comonoids (with corresponding categories $\mathcal{C}$ and $\mathcal{D}$). Let M and N be bimodules from C to D (with respective left comodule comultiplication transformations λ_M and λ_N and respective right comodule comultiplication transformations ρ_M and ρ_N), and let α be a bimodule map from M to N.

As usual, α consists of maps $\alpha_1 : C(1) \to D(1)$ and $\alpha^\sharp_x : D[\alpha_1(x)] \to C[x]$ for all positions $x \in C(1)$. Translated into language about the corresponding profunctors, α_1 is a map from the set of elements of X_M to the set of elements of X_N (where X_M and X_N are the $\mathcal{C}$-copresheaves on induced by M and N, respectively), and $\alpha^\sharp_x$ is a map from $Z_N(\alpha_1(x))$ to $Z_M(x)$ (where Z_M and Z_N are the $\int_{\mathcal{C}} X_M$-presheaf and $\int_{\mathcal{C}} X_N$-presheaf induced by M and N, respectively). We define the map $\alpha^\sharp_x|_d$ for $d \in \mathcal{D}$ to be the restriction of $\alpha^\sharp_x$ to just the elements in the $\mathcal{D}$-copresheaf $Z_N(\alpha_1 x)$ with indexing object d.

We verify this α corresponds to a natural transformation between prafunctors $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$ in the sense described above.

- The naturality law $\alpha_1(m(x, f)) = m(\alpha_1(x), f)$ for the $\mathcal{C}$-copresheaf map is the content of the left comodule map law $\alpha \mathbin{⨾} \lambda_N = \lambda_M \mathbin{⨾} (\mathrm{id}_C \triangleleft \alpha)$ as regards positions.

  (This also forces the law $t(\alpha_1(x)) = t(x)$, i.e., α_1 preserves indexing objects.)

[6]We have not shown this in this paper.

- The law $t(\alpha^\sharp_x(z)) = t(z)$ (i.e., each $\alpha^\sharp_x|_d$ preserves the indexing object $d \in \mathcal{D}$) is the content of the right comodule map law $\alpha \mathbin{⨟} \rho_N = \rho_M \mathbin{⨟} (\alpha \triangleleft \mathrm{id}_D)$ as regards positions.
- The naturality in x law $\alpha^\sharp_x(m(f|_{\alpha_1(x)}, z)) = m(f|_x, \alpha^\sharp_{m(x,f)}(z))$ is the content of the left comodule map as regards directions, which identifies $\alpha^\sharp(\lambda_N{}^\sharp(f \blacktriangleleft z))$ with $\lambda_M{}^\sharp(f \blacktriangleleft \alpha^\sharp(z))$ whenever this makes sense.
- The naturality in d law $\alpha^\sharp_x(m(z, f)) = m(\alpha^\sharp_x(z), f)$ for the maps $alpha^\sharp_x$ as $\mathcal{D}$-copresheaf maps is the content of the right comodule map law as regards directions, which identifies $\alpha^\sharp(\rho_N{}^\sharp(z \blacktriangleleft f))$ with $\rho_M{}^\sharp(\alpha^\sharp(z) \blacktriangleleft f)$ whenever this makes sense.

Conversely, any natural transformation between prafunctors $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$ gives rise to such an α, satisfying such laws. □

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

Proof. Bicomodules from D to 0 specialize to copresheaves on D (and maps between such bicomodules are copresheaf maps). Hence each bicomodule M from C to D induces a functor F_M from D-copresheaves to C-copresheaves by precomposition. Accordingly, we have $F_{M \triangleleft_D N} \cong F_N \mathbin{⨟} F_M$ (for bicomodules M from C to D and N from D to E).

We show that the prafunctor corresponding to the bimodule M is F_M.

□