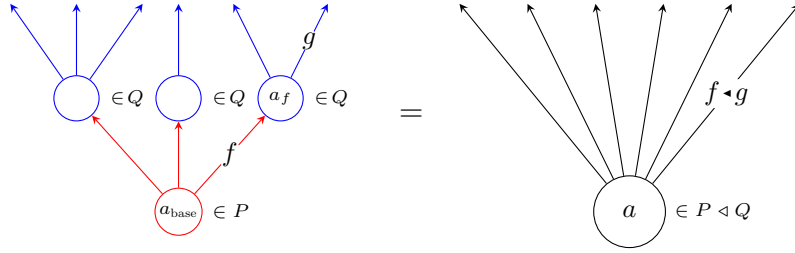


POLYNOMIAL COMONOIDS AND BICOMODULES

Recall the *substitution product* of polynomials P and Q , denoted $P \triangleleft Q$.

- A position a in $P \triangleleft Q$ consists of a position a_{base} in P and positions a_f in Q for each direction f from a_{base} .
- A direction from position a in $P \triangleleft Q$ consists of a direction f from a_{base} and a direction g from a_f .



We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$.¹ Accordingly, $\text{id}_\blacktriangleleft$ will denote the unique direction from the unique position in the unit for substitution id_\triangleleft (a.k.a. the polynomial y).

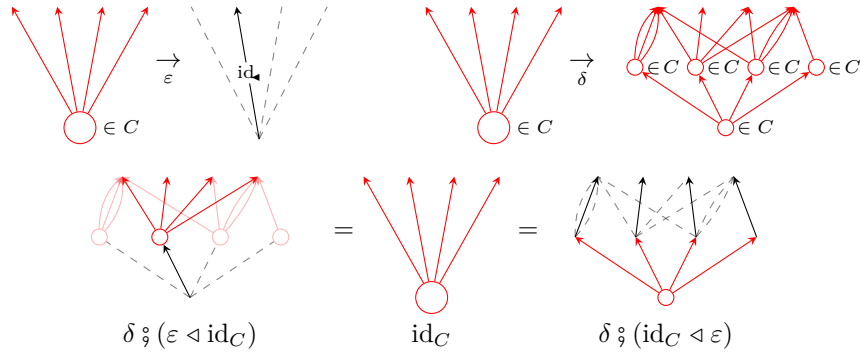
Note the following identity for transformations α and β between polynomials.

$$(\alpha \triangleleft \beta)_a^\#(f \blacktriangleleft g) = \underbrace{\alpha_{(a_{\text{base}})}^\#(f)}_{\star} \blacktriangleleft \beta_{(a_\star)}^\#(g).$$

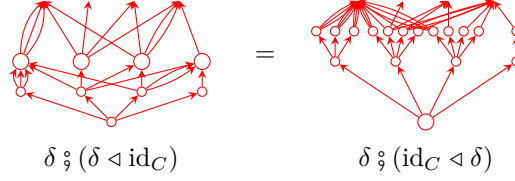
Or in brief, “we have $(\alpha \triangleleft \beta)^\#(f \blacktriangleleft g) = \alpha^\#(f) \blacktriangleleft \beta^\#(g)$ whenever this makes sense.”

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let C be a polynomial comonoid. Denote counit by ε and comultiplication by δ .



¹Be aware there may be other directions named $f \blacktriangleleft g$ from other positions in $P \triangleleft Q$.



Observe first that the right identity law forces $\delta_1(a)_{\text{base}} = a$ for all $a \in C(1)$. Therefore the expression $\delta_1(a)_f$ for $f \in C[a]$ has a well-defined meaning.

We gather the data for a category \mathcal{C} .

- The set of objects $\text{Ob}(\mathcal{C})$ is $C(1)$, the set of positions in C .
- The set of arrows $\text{Arr}(\mathcal{C})$ is $\sum_{a \in C(1)} C[a]$, the set of all directions in C .
- The source map s sends each $f \in C[a]$ to a . (Hence the polynomial C is described by the bundle $\text{Arr}(\mathcal{C}) \xrightarrow{s} \text{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in C[a]$ to $\delta_1(a)_f$.
- The identity map e sends each $a \in C(1)$ to $\varepsilon_a^\#(\text{id}_\bullet)$.
- The composition map m sends each pair of compatible arrows $f \in C[a], g \in C[t(f)]$ to $\delta_a^\#(f \blacktriangleleft g)$.

Next, observe that if we have such prerequisite data (not laws) for a category, subject to just the law $s(e(a)) = a$, then we find that further imposing the left identity law $m(e(s(f)), f) = f$ (and requiring that both sides are defined whenever one is) automatically forces the law $t(e(a)) = a$ to hold.² Similarly, if we have the law $s(m(f, g)) = s(f)$ as well as $t(e(a)) = a$, then the associativity law $m(m(f, g), h) = m(f, m(g, h))$ forces $t(m(f, g)) = t(g)$.³

We verify the data from above satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a .
- The law $s(m(f, g)) = s(f)$ is true by construction; $m(f, g)$ is a direction from the position $s(f)$.
- The left identity law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left identity law, which identifies $\delta^\#(\varepsilon^\#(\text{id}_\bullet) \blacktriangleleft f)$ with f whenever this makes sense.
- The right identity law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right identity law, which identifies $\delta^\#(f \blacktriangleleft \varepsilon^\#(\text{id}_\bullet))$ with f whenever this makes sense.
- The associativity law $m(m(f, g), h) = m(f, m(g, h))$ is directly expressed by the comonoid associativity law, which identifies $\delta^\#(\delta^\#(f \blacktriangleleft g) \blacktriangleleft h)$ with $\delta^\#(f \blacktriangleleft \delta^\#(g \blacktriangleleft h))$ whenever this makes sense.
- The law $t(e(a)) = a$ is forced to hold (due to the comonoid left identity law).
- The law $t(m(f, g)) = t(g)$ is forced to hold (due to the comonoid associativity law).

²We have $m(e(s(e(a))), e(a)) = e(a)$, since the right side is defined. The left side reduces to $m(e(a), e(a))$. This expression only makes sense if $t(e(a)) = s(e(a))$, which is a .

³Given that f and g are composable, we have $m(m(f, g), e(t(g))) = m(f, m(g, e(t(g))))$, since the right side is defined. The left side only makes sense if $t(f, g) = s(e(t(g)))$, which is $t(g)$.

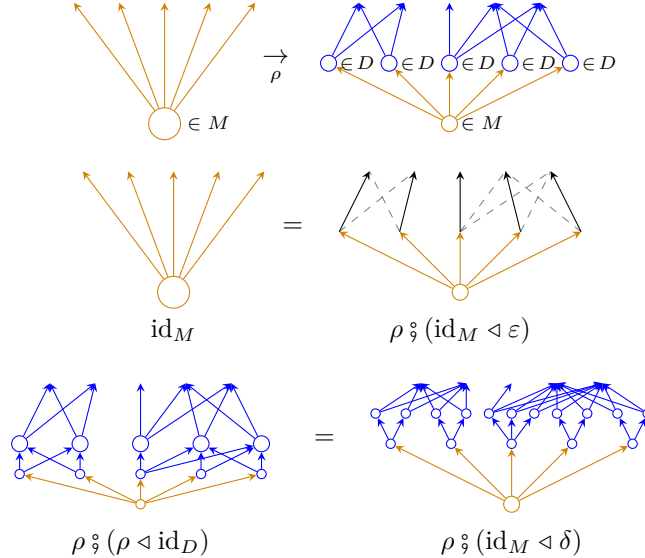
Conversely, let \mathcal{C} be a category. We immediately obtain the bundle $\text{Arr}(\mathcal{C}) \xrightarrow{s} \text{Ob}(\mathcal{C})$. Let C denote the polynomial described by this bundle (the “outfacing polynomial” of \mathcal{C}). We will exhibit a comonoid struture on C .

- The counit ε singles out the identity in each object’s set of outfacing maps.
- The comultiplication δ endows each object a with the map δ_a^\sharp sending $f \blacktriangleleft g$ to $m(f, g)$ for all arrows of the form $f : a \rightarrow b$, $g : b \rightarrow c$ (through which the map δ_1 is implicit).

The above processes of translation between the prerequisite data (not laws) for a category subject to just $s(e(a)) = a$ and $s(m(f, g)) = g$, and the prerequisite data (not laws) of a polynomial comonoid subject to just $\delta_1(a)_{\text{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity comonoid laws, in this context, directly translate to the identity and associativity category laws. \square

Proposition 2. *A polynomial right comodule amounts to a family of copresheaves.*

Proof. Let D be a polynomial comonoid with counit ε and comultiplication δ , and let M be a right comodule on D . Denote right comodule comultiplication by ρ .



Observe first that the identity law forces $\rho_1(a)_{\text{base}} = a$ for all $a \in M(1)$. Therefore the expression $\rho_1(a)_x$ for $x \in M[a]$ has a well-defined meaning.

Let \mathcal{D} be the category corresponding to D . We gather the data for a family of copresheaves $\{X_a\}_{a \in A}$ on \mathcal{D} .

- The family’s indexing set A is $M(1)$, the set of positions in M .
- The total set of elements $\sum_{d \in \text{Ob}(\mathcal{D})} X_a(d)$ in X_a is $M[a]$, the set of directions from a .
- The bundle map t assigning each element x in X_a its indexing object in $\text{Ob}(\mathcal{D}) = D(1)$ is given by $\rho_1(a)_x$.
- The multiplication map m sends each element $x \in X_a(d)$ and compatible arrow $f \in D[d]$ to $\rho_a^\sharp(x \blacktriangleleft f)$.

Next, observe that if we have such prerequisite data (not laws) for a copresheaf on \mathcal{D} , then we find that imposing the copresheaf associativity law $m(m(x, f), g) = m(x, m(f, g))$ forces the law $t(m(x, f)) = t(f)$ to hold.⁴ (The argument works the same for copresheaves as it does for categories.)

We verify each X_a satisfies the laws of a copresheaf on \mathcal{D} .

- The identity law $m(x, e(t(x))) = x$ is directly expressed by the right comodule identity law, which identifies $\rho^\sharp(x \blacktriangleleft \varepsilon^\sharp(\text{id}_\bullet))$ whenever this makes sense.
- The associativity law $m(m(x, f), g) = m(x, m(f, g))$ is directly expressed by the right comodule associativity law, which identifies $\rho^\sharp(\rho^\sharp(x \blacktriangleleft f) \blacktriangleleft g)$ with $\rho^\sharp(x \blacktriangleleft \delta^\sharp(f \blacktriangleleft g))$ whenever this makes sense.
- The law $t(m(x, f)) = t(f)$ is forced to hold (due to the right comodule associativity law).

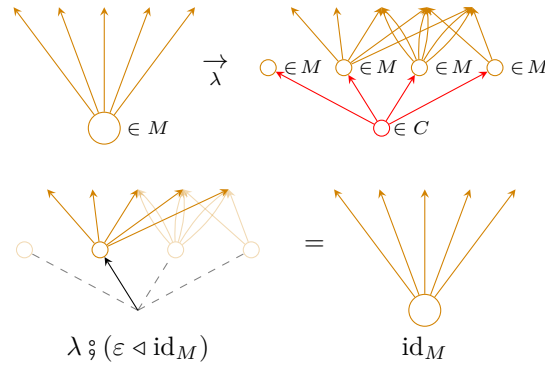
Conversely, let $\{X_a\}_{a \in A}$ be a family of copresheaves on \mathcal{D} . Let M denote the polynomial described by the family of total sets of elements $\{\sum_{d \in \text{Ob}(\mathcal{D})} X_a(d)\}_{a \in A}$. We will exhibit a right D -comodule structure on M .

- The right comodule comultiplication ρ endows each position $a \in A$ with the map ρ_a^\sharp sending each $x \blacktriangleleft f$ to $m(x, f)$ for all $x \in X_a(d)$, $f : d \rightarrow d' \in \mathcal{D}$. This implicitly determines $\rho_1(a)$ as long as the domain of ρ_a^\sharp is nonempty; otherwise let $\rho_1(a)$ be the unique position in $M \blacktriangleleft D$ with $\rho_1(a)_{\text{base}} = a$.

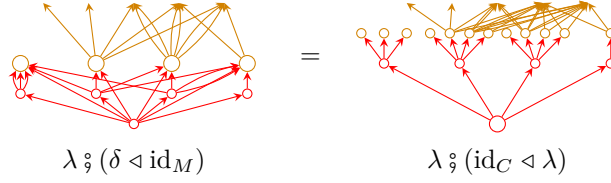
The above processes of translation between the prerequisite data (not laws) for a family of presheaves on \mathcal{D} , and the prerequisite data (not laws) of a right D -comodule subject to just $\rho_1(a)_{\text{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity right comodule laws, in this context, directly translate to the identity and associativity copresheaf laws. \square

Proposition 3. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

Proof. Let C be a polynomial comonoid with counit ε and comultiplication δ , and let M be a left comodule on C . Denote left comodule comultiplication by λ .



⁴Given that x and f are composable, we have $m(m(x, f), e(t(f))) = m(x, m(f, e(t(f))))$, since the right side is defined. The left side only makes sense if $t(x, f) = s(e(t(f)))$, which is $t(f)$.



Let \mathcal{C} be the category corresponding to C . We gather the data for a copresheaf X on \mathcal{C} .

- The total set of elements $\sum_{c \in \text{Ob}(\mathcal{C})} X(c)$ in X is $M(1)$, the set of positions in M .
- The bundle map t assigning each element x in X its indexing object in $\text{Ob}(\mathcal{C}) = C(1)$ is given by $\lambda_1(x)_{\text{base}}$.
- The multiplication map m sends each element $x \in M(1)$ and compatible arrow $f \in C[t(x)]$ to $\lambda_1(x)_f$.

Now we gather the remaining data of a presheaf Z on $\int_{\mathcal{C}} X$, the category of elements of X . (We will have accumulated the data sans laws of a copresheaf on \mathcal{C} and presheaf on its category of elements; we are still yet to verify X satisfies the laws of a copresheaf on \mathcal{C} .)

- The set $Z(x)$ for $x \in \text{Ob}(\int_{\mathcal{C}} X) = M(1)$ is $M[x]$, the set of directions from x . Hence we obtain the bundle map s from the total set of elements $\sum_{x \in \text{Ob}(\int_{\mathcal{C}} X)} Z(x)$ to $\text{Ob}(\int_{\mathcal{C}} X)$ sending each $z \in Z(x)$ to x .
- The multiplication map m sends each arrow $f|_w : w \rightarrow x$ (in $\int_{\mathcal{C}} X$, lying over $f : t(w) \rightarrow t(x)$ in \mathcal{C}) and x -indexed element $z \in Z(x) = M[x]$ to $\lambda_w^\#(f \bullet z)$.

(To be clear, the domain of this map is the set of tuples (w, f, z) such that $t(w) = s(f)$ and $m(w, f) = s(z)$. This is indeed the set of pairs $(f|_w, z)$ that should belong in the domain of multiplication for our presheaf on $\int_{\mathcal{C}} X$, since an arrow $f|_w$ in $\int_{\mathcal{C}} X$ is a pair (w, f) such that $t(w) = s(f)$, and the target of this arrow is $m(w, f)$.)

We will also use the following notation for identities and composition in $\int_{\mathcal{C}} X$.

- If x is an element of X , then $e(x)$ will refer to $e(t(x))|_x$.
- If $f|_w$ and $g|_{m(w, f)}$ are composable arrows in $\int_{\mathcal{C}} X$, then $m(f|_w, g|_{m(w, f)})$ will refer to $m(f, g)|_w$.

We verify X satisfies the laws of a copresheaf on \mathcal{C} .

- The identity law $m(x, e(t(x))) = x$ is the content of the left comodule identity law for positions, which says that $\lambda_1(x)_{\varepsilon_{\lambda_1(x)_{\text{base}}}^\#(\text{id}_\bullet)} = x$.
- The associativity law $m(m(x, f), g) = m(x, m(f, g))$ is the content of the left comodule associativity law for positions, which says that $\lambda_1(\lambda_1(x)_f)_g = \lambda_1(x)_{\delta_{\lambda_1(x)_{\text{base}}}^\#(f \blacktriangleright)}^\#$.
- The law $t(m(x, f)) = t(f)$ is forced to hold by the associativity law, as we have seen previously.

Now we verify Z satisfies the laws of a presheaf on $\int_{\mathcal{C}} X$.

- The identity law $m(e(s(z)), z) = z$ is the content of the left comodule identity law for directions, which identifies $\lambda^\#(\varepsilon^\#(\text{id}_\bullet) \bullet z)$ with z whenever this makes sense.

- The associativity law $m(m(f|_w, g|_{m(w,f)}), z) = m(f|_w, m(g|_{m(w,f)}, z))$ is the content of the left comodule associativity law for directions, which identifies $\lambda^\#(\delta^\#(f \blacktriangleleft g) \blacktriangleleft z)$ with $\lambda^\#(f \blacktriangleleft \lambda^\#(g \blacktriangleleft z))$ whenever this makes sense.
- The law $s(m(f|_w, z)) = s(f|_w)$ is forced to hold by the associativity law, as we have seen previously (in the dual scenario).

Conversely, let X be a copresheaf on \mathcal{C} and Z be a presheaf on $\int_{\mathcal{C}} X$. We immediately obtain the bundle $\sum_{x \in \text{Ob}(\int_{\mathcal{C}} X)} Z(x) \xrightarrow{s} \text{Ob}(\int_{\mathcal{C}} X)$. Let M denote the polynomial described by this bundle. We will exhibit a left C -comodule structure on M .

- The left comodule comultiplication λ is defined on positions by $\lambda_1(x)_{\text{base}} = t(x)$ and $\lambda_1(x)_f = m(x, f)$ for all $f \in C[t(x)]$.
- For each position x , the map $\lambda_x^\#$ sends each $f \blacktriangleleft z$ to $m(f|_x, z)$ for all $f \in C[t(x)]$, $z \in Z(m(x, f))$.

The above processes of translation between the prerequisite data (not laws) for a presheaf on the category of elements of a copresheaf on \mathcal{C} , and the prerequisite data (not laws) of a left C -comodule, are inverse by construction. Moreover, we saw earlier that the identity and associativity left comodule laws, in this context, directly translate to the identity and associativity copresheaf and presheaf laws. \square

Proposition 4. *Polynomial bicomodules are prafunctors between presheaf categories.*

Proof. Let C and D be polynomial comonoids and let M a bimodule from C to D with left module comultiplication λ and right module comultiplication ρ .

$$\rho \circ (\lambda \triangleleft \text{id}_D) = \lambda \circ (\text{id}_C \triangleleft \rho)$$

We will show that M amounts to a profunctor⁵ $(\int_{\mathcal{C}} X) \circ \bullet \mathcal{D}$, where \mathcal{C} is the category corresponding to C , X is a copresheaf on \mathcal{C} , and \mathcal{D} is the category corresponding to D .

Such a profunctor is the same as a prafunctor $\mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}^{\mathcal{C}}$. Indeed,

$$\begin{array}{c}
 \text{prafunctors } \mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}^{\mathcal{C}} \\
 \updownarrow \\
 \text{right adjoint functors } \mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}^{\mathcal{C}}/X \text{ for any } \mathcal{C}\text{-copresheaf } X \\
 \updownarrow \\
 \text{left adjoint functors } \mathbf{Set}^{\mathcal{C}}/X \rightarrow \mathbf{Set}^{\mathcal{D}} \\
 \updownarrow \\
 \text{left adjoint functors } \mathbf{Set}^{\int_{\mathcal{C}} X} \rightarrow \mathbf{Set}^{\mathcal{D}} \\
 \updownarrow \\
 \text{functors } (\int_{\mathcal{C}} X)^{\text{op}} \rightarrow \mathbf{Set}^{\mathcal{D}} \\
 \updownarrow \\
 \text{profunctors } (\int_{\mathcal{C}} X) \circ \bullet \mathcal{D}.
 \end{array}$$

⁵The notation $\mathcal{A} \circ \bullet \mathcal{B}$ for a profunctor $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \mathbf{Set}$ is due to Michael Shulman.

We have already seen that M as a left \mathcal{C} -comodule amounts to a \mathcal{C} -copresheaf X and a presheaf Z on $\int_{\mathcal{C}} X$, and that M as a right \mathcal{D} -comodule amounts to a family of \mathcal{D} -copresheaves $\{X_i\}_{i \in I}$.

The set $M(1)$ of positions in M serves as both the set of elements in the \mathcal{C} -copresheaf X , that is, $\text{Ob}(\int_{\mathcal{C}} X)$, as well as the indexing set I for $\{X_i\}_{i \in I}$. For any position $x \in M(1)$, the set $M[x]$ of directions from x serves as both $Z(x)$, the set of x -indexed elements in the $\int_{\mathcal{C}} X$ -presheaf Z , as well as the set of elements in the \mathcal{D} -copresheaf X_x .

Moreover, M and its left and right comodule structures are recovered from such information. That is, the data of a \mathcal{C} -copresheaf X , a $\int_{\mathcal{C}} X$ -presheaf Z , and a \mathcal{D} -copresheaf structure on $Z(x)$ for each element x in X is the same as a polynomial M equipped with the structure of a left \mathcal{C} -module and right \mathcal{D} -module, assuming no compatibility.

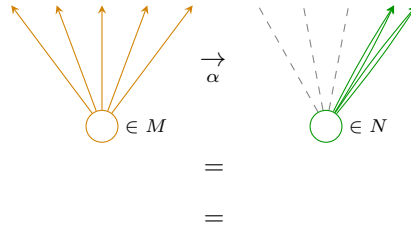
The law $m(m(f|_w, z), g) = m(f|_w, m(z, g))$, encoding naturality of the maps $Z(f|_w) : Z(x) \rightarrow Z(w)$ (with respect to the \mathcal{D} -copresheaf structure on $Z(x)$ and $Z(w)$), is directly expressed by the bicomodule law, which identifies $\rho^\#(\lambda^\#(f \blacktriangleleft z) \blacktriangleright g)$ with $\lambda^\#(f \blacktriangleleft \rho^\#(z \blacktriangleright g))$ whenever this makes sense.

Thus, a bicomodule M from \mathcal{C} to \mathcal{D} is the same as a functor $(\int_{\mathcal{C}} X)^{\text{op}} \rightarrow \mathbf{Set}^{\mathcal{D}}$, i.e., a profunctor $(\int_{\mathcal{C}} X) \circ \bullet \mathcal{D}$ (i.e., a prafunctor from $\mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}^{\mathcal{C}}$). \square

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof. Under the correspondence between prafunctors and profunctors, a natural transformation between prafunctors $F, G : \mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}^{\mathcal{C}}$ respectively corresponding to profunctors $M : (\int_{\mathcal{C}} X_M) \circ \bullet \mathcal{D}, N : (\int_{\mathcal{C}} X_N) \circ \bullet \mathcal{D}$ (where X_M and X_N are \mathcal{C} -copresheaves) is identified with a copresheaf map α_1 from X_M to X_N and maps $(\alpha_x^\#)_d$ from $N(\alpha_1(x), d)$ to $M(x, d)$, natural in x and d .

Let C and D be polynomial comonoids. Let M and N be bimodules from C to D , and let α be a bimodule map from M to N .



As usual, α consists of maps $\alpha_1 : C(1) \rightarrow D(1)$ and $\alpha_x^\# : D[\alpha_1(x)] \rightarrow C[x]$ for all positions $x \in C(1)$.

Observe first that the \square

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

Proof. Bicomodules from D to 0 specialize to copresheaves on D (and maps between such bicomodules are copresheaf maps). Hence each bicomodule M from C to D induces a functor F_M from D -copresheaves to C -copresheaves by precomposition. Accordingly, we have $F_{M \triangleleft_D N} \cong F_N \circ F_M$ (for bicomodules M from C to D and N from D to E).

We show that the prafunctor corresponding to the bimodule M is F_M .

□