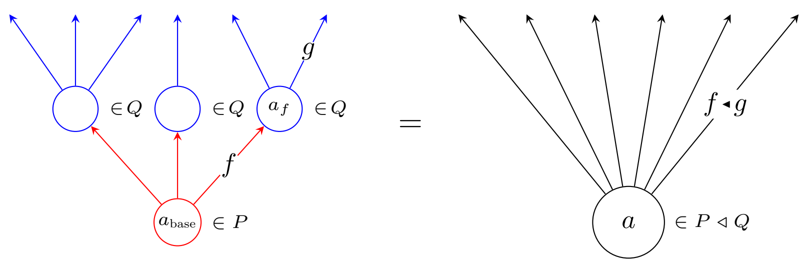
POLYNOMIAL COMONOIDS AND BICOMODULES

Recall the *substitution product* of polynomials P and Q, denoted $P \triangleleft Q$.

- A position a in $P \triangleleft Q$ consists of a position a_{base} in P and positions a_f in Q for each direction f from a_{base}.
- A direction from position a in $P \triangleleft Q$ consists of a direction f from a_{base} and a direction g from a_f.



We denote such a direction from such a position in a substitution product by $f \blacktriangleleft g$.[1] Accordingly, $\text{id}_{\blacktriangleleft}$ will denote the unique direction from the unique position in the unit for substitution $\text{id}_{\triangleleft}$ (a.k.a. the polynomial y).[2]

Note the following identity for transformations α and β between polynomials.

$$(\alpha \triangleleft \beta)^\sharp_a (f \blacktriangleleft g) = \underbrace{\alpha^\sharp_{(a_{\text{base}})}(f)}_{\star} \blacktriangleleft \beta^\sharp_{(a_\star)}(g).$$

Or in brief, "we have $(\alpha \triangleleft \beta)^\sharp (f \blacktriangleleft g) = \alpha^\sharp(f) \blacktriangleleft \beta^\sharp(g)$ whenever this makes sense."[3]

Proposition 1. *Polynomial comonoids are categories.*

Proof. Let C be a polynomial comonoid. Denote counit by ε and comultiplication by δ.

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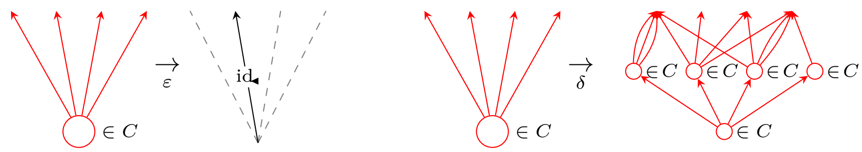
[1] Be aware there may be other directions named $f \blacktriangleleft g$ from other positions in $P \triangleleft Q$.

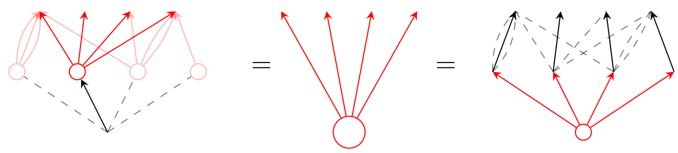
[2] Given directions f, g, and h respectively belonging to polynomials P, Q, and R, directions of the form $(f \blacktriangleleft g) \blacktriangleleft h$ belonging to $(P \triangleleft Q) \triangleleft R$ and directions of the form $f \blacktriangleleft (g \blacktriangleleft h)$ belonging to $P \triangleleft (Q \triangleleft R)$ are identified under the relevant monoidal coherence isomorphism. Hence brackets can be omitted.

Similarly, for any direction f belonging to a polynomial P, we have that $\text{id}_{\blacktriangleleft} \blacktriangleleft f$ and $f \blacktriangleleft \text{id}_{\blacktriangleleft}$ (respectively belonging to $\text{id}_{\triangleleft} \triangleleft P$ and $P \triangleleft \text{id}_{\triangleleft}$) are both canonically identified with f.

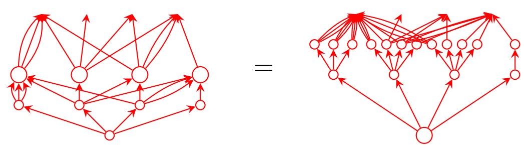
[3] Meaning, $(\alpha \triangleleft \beta)^\sharp_a (f \blacktriangleleft g)$ is defined $\iff$ there are u and v such that $\alpha^\sharp_u(f) \blacktriangleleft \beta^\sharp_v(g)$ actually describes a direction from a; in this case, the former and latter are equal.

(We are assuming the analogue of this property holds for α and β themselves: if any $\alpha^\sharp_u(f)$ has the same name as a direction from a, then $\alpha^\sharp_a(f)$ is defined with that value — and likewise for β. This is trivially so if across the domains of α and β we use different names for different directions.)





$\delta \mathbin{⨟} (\varepsilon \triangleleft \mathrm{id}_C)$ $\qquad$ id_C $\qquad$ $\delta \mathbin{⨟} (\mathrm{id}_C \triangleleft \varepsilon)$



$\delta \mathbin{⨟} (\delta \triangleleft \mathrm{id}_C)$ $\qquad$ $\delta \mathbin{⨟} (\mathrm{id}_C \triangleleft \delta)$

Observe first that the right identity law forces $\delta_1(a)_{\text{base}} = a$ for all $a \in C(1)$. Therefore the expression $\delta_1(a)_f$ for $f \in C[a]$ has a well-defined meaning.

We gather the data for a category $\mathcal{C}$.

- The set of objects $\mathrm{Ob}(\mathcal{C})$ is $C(1)$, the set of positions in C.
- The set of arrows $\mathrm{Arr}(\mathcal{C})$ is $\sum_{a \in C(1)} C[a]$, the set of all directions in C.
- The source map s sends each $f \in C[a]$ to a. (Hence the polynomial C is described by the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$.)
- The target map t sends each $f \in C[a]$ to $\delta_1(a)_f$.
- The identity map e sends each $a \in C(1)$ to $\varepsilon_a^\sharp(\mathrm{id}_\blacktriangleleft)$.
- The composition map m sends each pair of compatible arrows $f \in C[a], g \in C[t(f)]$ to $\delta_a^\sharp(f \blacktriangleleft g)$.

Next, observe that if we have such prerequisite data (not laws) for a category, subject to just the law $s(e(a)) = a$, then we find that further imposing the left identity law $m(e(s(f)), f) = f$ (and requiring that both sides are defined whenever one is) automatically forces the law $t(e(a)) = a$ to hold.[4] Similarly, if we have the law $s(m(f,g)) = s(f)$ as well as $t(e(a)) = a$, then the associativity law $m(m(f,g),h) = m(f, m(g,h))$ forces $t(m(f,g)) = t(g)$.[5]

We verify the data from above satisfy the laws of a category.

- The law $s(e(a)) = a$ is true by construction; $e(a)$ is a direction from the position a.
- The law $s(m(f,g)) = s(f)$ is true by construction; $m(f,g)$ is a direction from the position $s(f)$.
- The left identity law $m(e(s(f)), f) = f$ is directly expressed by the comonoid left identity law, which identifies $\delta^\sharp(\varepsilon^\sharp(\mathrm{id}_\blacktriangleleft) \blacktriangleleft f)$ with f whenever this makes sense.

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[4]We have $m(e(s(e(a))), e(a)) = e(a)$, since the right side is defined. The left side reduces to $m(e(a), e(a))$. This expression only makes sense if $t(e(a)) = s(e(a))$, which is a.

[5]Given that f and g are composable, we have $m(m(f,g), e(t(g))) = m(f, m(g, e(t(g))))$, since the right side is defined. The left side only makes sense if $t(f,g) = s(e(t(g)))$, which is $t(g)$.

- The right identity law $m(f, e(t(f))) = f$ is directly expressed by the comonoid right identity law, which identifies $\delta^\sharp(f \blacktriangleleft \varepsilon^\sharp(\mathrm{id}_{\blacktriangleleft}))$ with f whenever this makes sense.
- The associativity law $m(m(f,g),h) = m(f, m(g,h))$ is directly expressed by the comonoid associativity law, which identifies $\delta^\sharp(\delta^\sharp(f \blacktriangleleft g) \blacktriangleleft h)$ with $\delta^\sharp(f \blacktriangleleft \delta^\sharp(g \blacktriangleleft h))$ whenever this makes sense.
- The law $t(e(a)) = a$ is forced to hold (due to the comonoid left identity law).
- The law $t(m(f,g)) = t(g)$ is forced to hold (due to the comonoid associativity law).

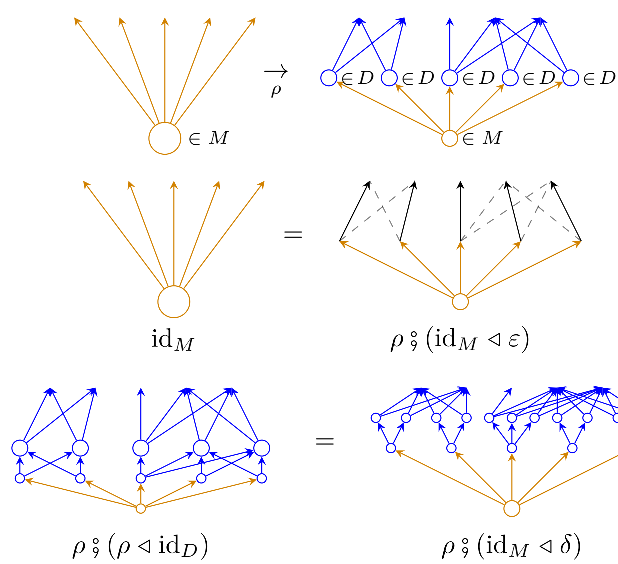
Conversely, let $\mathcal{C}$ be a category. We immediately obtain the bundle $\mathrm{Arr}(\mathcal{C}) \xrightarrow{s} \mathrm{Ob}(\mathcal{C})$. Let C denote the polynomial described by this bundle (the "outfacing polynomial" of $\mathcal{C}$). We exhibit a comonoid struture on C.

- The counit ε singles out the identity in each object's set of outfacing maps.
- The comultiplication δ endows each object a with the map $\delta^\sharp_a$ sending $f \blacktriangleleft g$ to $m(f,g)$ for all arrows of the form $f : a \to b$, $g : b \to c$ (through which the map δ_0 is implicit).

The above processes of translation between the prerequisite data (not laws) for a category subject to just $s(e(a)) = a$ and $s(m(f,g)) = g$, and the prerequisite data (not laws) of a polynomial comonoid subject to just $\delta_1(a)_{\mathrm{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity category laws, in this context, directly translate to the identity and associativity comonoid laws. □

Proposition 2. *A polynomial right comodule amounts to a family of copresheaves.*

Proof. Let (D, ε, δ) be a polynomial comonoid and let M be a right comodule on D. Denote right comodule comultiplication by ρ.



Observe first that the identity law forces $\rho_1(a)_{\mathrm{base}} = a$ for all $a \in M(1)$. Therefore the expression $\rho_1(a)_x$ for $x \in M[a]$ has a well-defined meaning.

Let $\mathcal{D}$ be the category corresponding to D. We gather the data for a family of copresheaves $\{X_a\}_{a\in A}$ on $\mathcal{D}$.

- The family's indexing set A is $M(1)$, the set of positions in M.
- The total set of elements $\sum_{d\in\mathrm{Ob}(\mathcal{D})} X_a(d)$ in X_a is $M[a]$, the set of directions a.
- The bundle map t assigning each element x in X_a its indexing object in $\mathrm{Ob}(\mathcal{D}) = D(1)$ is given by $\rho_1(a)_x$.
- The multiplication map m sends each element $x \in X_a(d)$ and compatible arrow $f \in D[d]$ to $\rho^\sharp_a(x \blacktriangleleft f)$.

Next, observe that if we have such prerequisite data (not laws) for a copresheaf on $\mathcal{D}$, then we find that imposing the copresheaf associativity law $m(m(x,f),g) = m(x, m(f,g))$ forces the law $t(m(x,f)) = t(f)$ to hold.[6] (The argument works the same for copresheaves as it does for categories.)

We verify each X_a satisfies the laws of a copresheaf on $\mathcal{D}$.

- The identity law $m(x, e(t(x))) = x$ is directly expressed by the right comodule identity law, which identifies $\rho^\sharp(x \blacktriangleleft \varepsilon^\sharp(\mathrm{id}_\blacktriangleleft))$ whenever this makes sense.
- The associativity law $m(m(x,f),g) = m(x,m(f,g))$ is directly expressed by the right comodule associativity law, which identifies $\rho^\sharp(\rho^\sharp(x \blacktriangleleft f) \blacktriangleleft g)$ with $\rho^\sharp(x \blacktriangleleft \delta^\sharp(f \blacktriangleleft g))$ whenever this makes sense.
- The law $t(m(x,f)) = t(f)$ is forced to hold (due to the right comodule associativity law).

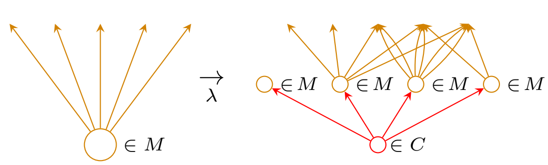
Conversely, let $\{X_a\}_{a\in A}$ be a family of copresheaves on $\mathcal{D}$. Let M denote the polynomial described by the family of total sets of elements $\{\sum_{d\in\mathrm{Ob}(\mathcal{D})} X_a(d)\}_{a\in A}$. We exhibit a right D-comodule struture on M.

- The comodule map ρ endows each position $a \in A$ with the map $\delta^\sharp_a$ sending each $x \blacktriangleleft f$ to $x \fatsemi f$ for all $x \in X_a(d)$, $f : d \to d' \in \mathcal{D}$ (through which the map δ_0 is implicit).

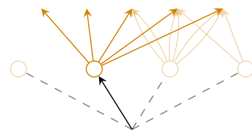
The above processes of translation between the prerequisite data (not laws) for a family of presheaves on $\mathcal{D}$, and the prerequisite data (not laws) of a right D-comodule subject to just $\rho_1(a)_{\mathrm{base}} = a$, are inverse by construction. Moreover, we saw earlier that the identity and associativity copresheaf laws, in this context, directly translate to the identity and associativity right comodule laws. $\square$

Proposition 3. *A polynomial left comodule amounts to a copresheaf and a presheaf on that copresheaf's category of elements.*

Proof. Let (C, ε, δ) be a polynomial comonoid and let M be a left comodule on C. Denote left comodule comultiplication by λ.

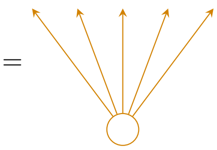


[6]Given that x and f are composable, we have $m(m(x,f), e(t(f))) = m(x, m(f, e(t(f))))$, since the right side is defined. The left side only makes sense if $t(x,f) = s(e(t(f)))$, which is $t(f)$.

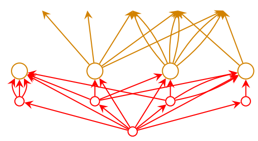


$\lambda \mathbin{⨾} (\varepsilon \triangleleft \mathrm{id}_M)$

$=$



id_M



$\lambda \mathbin{⨾} (\delta \triangleleft \mathrm{id}_M)$

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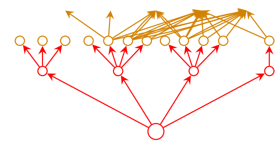


$\lambda \mathbin{⨾} (\mathrm{id}_C \triangleleft \lambda)$

Let $\mathcal{C}$ be the category corresponding to C. We gather the data for a copresheaf X on $\mathcal{C}$.

- The total set of elements $\sum_{c \in \mathrm{Ob}(\mathcal{C})} X(c)$ in X is $M(1)$, the set of positions in M.
- The bundle map t assigning each element x in X its indexing object in $\mathrm{Ob}(\mathcal{C}) = C(1)$ is given by $\lambda_1(x)_{\mathrm{base}}$.
- The multiplication map m sends each element $x \in M(1)$ and compatible arrow $f \in C[t(x)]$ to $\lambda_1(x)_f$.

Now we gather the remaining data of a presheaf Z on $\int_{\mathcal{C}} X$, the category of elements of X. (We will have accumulated the data sans laws of a copresheaf on $\mathcal{C}$ and presheaf on its category of elements; we are still yet to verify X satisfies the laws of a copresheaf on $\mathcal{C}$.)

- The set $Z(x)$ for $x \in \mathrm{Ob}(\int_{\mathcal{C}} X) = M(1)$ is $M[x]$, the set of directions from x. Hence we obtain the bundle map s from the total set of elements $\sum_{x \in \mathrm{Ob}(\int_{\mathcal{C}} X)} Z(x)$ to $\mathrm{Ob}(\int_{\mathcal{C}} X)$ sending each $z \in Z(x)$ to x.
- The multiplication map m sends each arrow $f|_w : w \to x$ (in $\int_{\mathcal{C}} X$, lying over $f : t(w) \to t(x)$ in $\mathcal{C}$) and x-indexed element $z \in Z(x) = M[x]$ to $\lambda^\sharp_w(f \blacktriangleleft z)$.

  (To be clear, the domain of this map is the set of tuples (w, f, z) such that $t(w) = s(f)$ and $m(w, f) = s(z)$. This is indeed the set of pairs $(f|_w, z)$ belonging in the domain of multiplication for our presheaf on $\int_{\mathcal{C}} X$, since an arrow $f|_w$ in $\int_{\mathcal{C}} X$ is a pair (w, f) such that $t(w) = s(f)$, and the target of this arrow is $m(w, f)$.)

We will also use the following notation for identities and composition in $\int_{\mathcal{C}} X$.

- If x is an element of X, then $e(x)$ will refer to $e(t(x))|_x$.
- If $f|_w$ and $g|_x$ are composable arrows in $\int_{\mathcal{C}} X$ (i.e., $x = m(w, f)$ — and $t(f) = s(g)$?), then $m(f|_w, g|_x)$ will refer to $m(f, g)|_w$.

Next, observe that if we have such prerequisite data (not laws) for a copresheaf on $\mathcal{C}$ and presheaf on its category of elements, then we find that imposing the presheaf identity law $m(e(s(z)), z) = z$ forces the copresheaf identity law $m(x, e(t(x))) = x$ to hold for $x = s(z)$.[7] Similarly, imposing the presheaf associativity law $m(m(f|_w, g|_x), z) =$

[7]The expression $m(e(s(z)), z)$ is short for $m(e(t(s(z)))|_{s(z)}, z)$, and if this is defined, then $m(s(z), e(t(s(z)))) = s(z)$.

$m(f|_w, m(g|_x, z))$ forces the copresheaf associativity law $m(m(w,f),g) = m(w, m(f,g))$ to hold for f and g such that one of these expressions is defined for some w, x, z.[8]

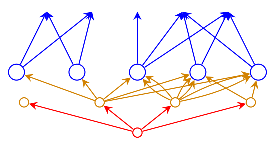
We verify Z satisfies the laws of a presheaf on $\int_{\mathcal{C}} X$.

- The identity law $m(e(s(z)), z) = z$ is expressed by the left comodule identity law, which identifies $\lambda^\sharp(\varepsilon^\sharp(\mathrm{id}_\cdot) \blacktriangleleft z)$ with z whenever this makes sense.
- The associativity law $m(m(f|_w, g|_x), z) = m(f|_w, m(g|_x, z))$ is expressed by the left comodule associativity law, which identifies $\lambda^\sharp(\delta^\sharp(f \blacktriangleleft g) \blacktriangleleft z)$ with $\lambda^\sharp(f \blacktriangleleft \lambda^\sharp(g \blacktriangleleft z))$ whenever this makes sense.

□

Proposition 4. *Polynomial bicomodules are prafunctors between presheaf categories.*

Proof. Let $(C, \varepsilon_C, \delta_C)$ and $(D, \varepsilon_D, \delta_D)$ be polynomial comonoids and let (M, λ, ρ) be a bimodule from C to D.



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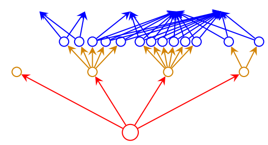

$\rho \mathbin{⨾} (\lambda \triangleleft \mathrm{id}_D)$ $\lambda \mathbin{⨾} (\mathrm{id}_C \triangleleft \rho)$

We will show that M amounts to a profunctor[9] $(\int_{\mathcal{C}} X) \mathrel{\circ\!\!-\!\!\bullet} \mathcal{D}$, where $\mathcal{C}$ is the category corresponding to C, X is the copresheaf on $\mathcal{C}$ induced by M as a left comodule, and $\mathcal{D}$ is the category corresponding to D.

Such a profunctor is equivalent to a prafunctor $\mathbf{Set}^{\mathcal{D}} \to \mathbf{Set}^{\mathcal{C}}$. Indeed,

$$\cong$$

□

Proposition 5. *Maps between bicomodules are natural transformations between prafunctors.*

Proof. □

Proposition 6. *Composition of bicomodules is composition of prafunctors.*

Proof. Bicomodules from D to 0 specialize copresheaves on D (and maps between such bicomodules are copresheaf maps). Hence each bicomodule M from D to D induces a functor F_M from D-copresheaves to C-copresheaves by precomposition. Accordingly, we have $F_{M \triangleleft_D N} \cong F_N \mathbin{⨾} F_M$ (for bicomodules M from C to D and N from D to E).

We show that the prafunctor corresponding to the bimodule M is F_M.

□

[8]The expression $m(m(f|_w, g|_{m(w,f)}), z)$ is "short" for $m(m(f,g)_w, z)$, and if this is defined, then $m(w, m(f,g)) = s(z)$. Meanwhile, if $m(f|_w, m(g|_{m(w,f)}, z))$ is defined, then $m(m(w,f),g) = s(z)$.

[9]The notation $\mathcal{A} \mathrel{\circ\!\!-\!\!\bullet} \mathcal{B}$ for a profunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$ is due to Michael Shulman.