DATA 605 HW Assignment 3

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September 9, 2022

1 Problem Set 1

1.1 What is the rank of the matrix A?

If:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

Then the reduced row echelon form of A is:

```
library(pracma)
A = matrix(c(1,2,3,4,
             -1,0,1,3,
             0,1,-2,1,
             5,4,-2,-3),
             4, byrow=TRUE)
rref(A)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

The reduced row echelon form of A being equal to I_n is equivalent to A having rank n. Therefore rank(A) = 4

1.2 Given an $m \times n$ matrix where $m > n$, what can be the maximum rank? The mini-mum rank, assuming that the matrix is non-zero?

Since the rank(A) is equal to the common dim of the column and row spaces, the maximum rank can be at most n. Assuming the matrix is non-zero, the minimum rank(A) is 1.

1.3 What is the rank of matrix B?

If:

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

Then the reduced row echelon form of B is:

```
library(pracma)
B = matrix(c(1,2,1,
             3,6,3,
             2,4,2),
             3, byrow=TRUE)
rref(B)
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    1
## [2,]    0    0    0
## [3,]    0    0    0
```

The vectors with leading 1s form the basis for their respective row and column spaces of B, since B is in reduced row echelon form. This implies the row space of B has dim(1) and the column space has dim(3). Therefore, rank(B) = 1 due to the common dim of both spaces. Upon inspection, it is apparent that all rows of B are linearly dependent. R2 = 3 * R1 and R3 = 2 * R1.

2 Problem Set 2

Compute the eigenvalues and eigenvectors of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

The eigenvalues of matrix A are computed by solving the characteristic equation $\det(\lambda I - A) = 0$. Since A is a triangular matrix the determinate is equal to the product of the diagonal.

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 4)(\lambda - 6)$$

Setting each component of the factored polynomial equal to 0 yields our eigenvalues of $\lambda = 1, 4, 6$

The eigenvectors of A are the vectors $\vec{x}$ that satisfy $A\vec{x} = \lambda\vec{x}$ or alternatively satisfy $(\lambda I - A)\vec{x} = 0$
For $\lambda = 1$:

$$(1I - A) = \begin{bmatrix} 0 & -2 & -3 \\ 0 & -3 & -5 \\ 0 & 0 & -5 \end{bmatrix}$$

The null space of a matrix is equal to the null space of the reduced row echelon form of that matrix. Applying the following row operations yields our RREF. R3 = R3 * -1/5, R2 = R2 + 5R3, R2 = R2 * -1/3, R1 = R1 -2R2 - 3R3.

$$RREF(1I - A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $(\lambda I - A)\vec{x} = 0 \rightarrow 0x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ yields the eigenspace for $\lambda = 1:$

$$E_1 = \text{Span}\left(s \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \forall s \in \mathbf{R} \right)$$

For $\lambda = 4:$

$$(4I - A) = \begin{bmatrix} 3 & -2 & -3 \\ 0 & 0 & -5 \\ 0 & 0 & -2 \end{bmatrix}$$

Applying the following row operations yields our RREF. R3 = R3 * -1/2, R2 = R2 + 5R3, R1 = R1 + 3R3, R1 = R1 * 1/3.

$$RREF(4I - A) = \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $(\lambda I - A)\vec{x} = 0 \rightarrow x_3 = 0$ and $x_1 = 2/3x_2$. Setting $x_2 = s$ and $x_1 = 2/3s$ yields the eigenspace for $\lambda = 4$:

$$E_4 = \text{Span}\left(s \times \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} \forall s \in \mathbf{R} \right)$$

For $\lambda = 6$:

$$(6I - A) = \begin{bmatrix} 5 & -2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying the following row operations yields our RREF. R2 = R2 * 1/2, R1 = R1 + 2R2, R1 = R1 * 1/5.

$$RREF(6I - A) = \begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Setting $x_3 = s$ and solving $(\lambda I - A)\vec{x} = 0 \rightarrow x_1 = 8/5\text{s}$ and $x_2 = 5/2\text{s}$. This yields the eigenspace for $\lambda = 6$:

$$E_6 = \text{Span}\left(s \times \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} \forall s \in \mathbf{R} \right)$$

Verifying solutions with R

```
A = matrix(c(1,2,3,
             0,4,5,
             0,0,6),
             3, byrow=TRUE)
ev <- eigen(A)
values <- ev$values
print(values)

## [1] 6 4 1

vectors <- ev$vectors
print(vectors)

##           [,1]      [,2] [,3]
## [1,] 0.5108407 0.5547002    1
## [2,] 0.7981886 0.8320503    0
## [3,] 0.3192754 0.0000000    0
```