The **Verdict** Library Reference Manual

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Abstract

Verdict is a collection of subroutines for evaluating the geometric qualities of triangles, quadrilaterals, tetrahedra, and hexahedra using a variety of metrics. A metric is a real number assigned to one of these shapes depending on its particular vertex coordinates. These metrics are used to evaluate the input to finite element, finite volume, boundary element, and other types of solvers that approximate the solution to partial differential equations defined over regions of space. The geometric qualities of these regions is usually strongly tied to the accuracy these solvers are able to obtain in their approximations.

The subroutines are written in C++ and have a simple C interface. Each metric may be evaluated individually or in combination. When multiple metrics are evaluated at once, they share common calculations to lower the cost of the evaluation.

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Chapter 1

Introduction

This is Verdict, a library for evaluating the geometric qualities of regions of space. These regions of space are typically those used as problem domains for partial differential equations, and are typically partitioned into subregions known as finite elements, finite volumes, or boundary elements. Subregions are typically simple shapes defined by a few vertices. The subregion shapes Verdict currently supports include triangles, quadrilaterals, tetrahedra, and hexahedra.

1.1 Measuring Quality with Metrics

Verdict evaluates geometric qualities on a subregion of the partition with metrics. A metric is a real number that may be assigned to the subregion. Some metrics, which we will call **proper metrics**, are normalized so that their values are 1 for ideally-shaped subregions and their values tend to ∞ for ill-defined, poor quality, or degenerate subregions. Examples where a proper metric should tend to ∞ include non-planar quadrilaterals, triangles with edges of vastly different lengths, and subregions with coincident vertices.

1.2 History of **Verdict**

Verdict has its first roots in the VERDE[1] project. VERDE is a simple program to read Exodus meshes, and analyze them for possible problems. Quality was one of the areas of analysis that VERDE covers. It was realized that VERDE and CUBIT[2] did not yield the same results when analyzing the geometric

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[1] http://www.cs.sandia.gov/capabilities/VerdeMeshVerificationSuite/index.html
[2] http://cubit.sandia.gov/

qualities of meshes. As a result, Verdict was created so that both VERDE and CUBIT could share the same code and produce the same results. Verdict also has roots in the CUBIT project and many other contributors, both theoretical and practical. Verdict was initially licensed under the LGPL. Meanwhile, the Visualization Tool Kit (VTK)[3] did not have any support for general purpose mesh quality assessment (with the exception of a method to calculate the tetrahedral radius ratio, written by Leila Baghdadi, Hanif Ladak, and David Steinman at the Imaging Research Labs, Robarts Research Institute). Due to the need for such a tool, and because Kitware was unwilling at that time to include LGPL libraries in the VTK repository, Philippe Pébay and David Thompson generalized the `vtkMeshQuality`[4] filter in 2004 to compute one or more measures of geometric quality for each 2-D and 3-D cell (triangle, quadrilateral, tetrahedron, or hexahedron) of a mesh for a variety of metrics. In addition to computing per-element qualities, methods to compute their average, minimum, maximum, and variance over the entire mesh where also added. These descriptive statistics are stored in the output mesh's FIELDDATA. This filter allows forx further processing and/or visualization of the per-element quality, for instance using ParaView[5] as illustrated in Figure 1.1
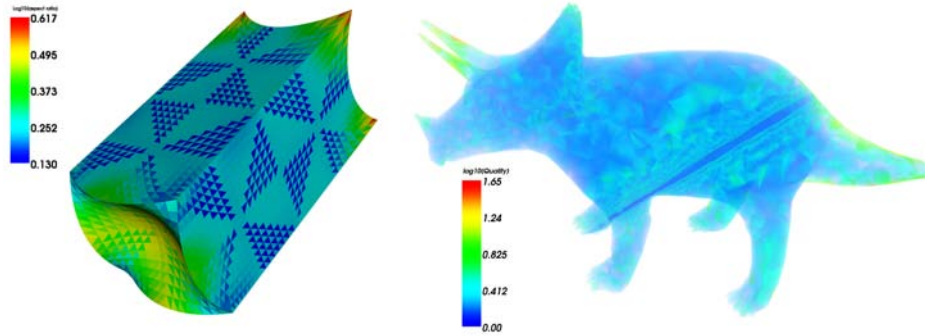
Figure 1.1: Surface (left) and volume (right) renderings with ParaView of the per-element base-10 logarithms of the aspect ratios for two tetrahedral meshes.

Following a change in Verdict's licensing scheme, from the LGPL to a modified BSD-style license, it was decided in late 2006 to use Verdict in VTK for

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[3]http://www.vtk.org/

[4]http://www.vtk.org/doc/nightly/html/classvtkMeshQuality.html

[5]http://www.paraview.org/

the same reasons that Verdict was initially created, by:

1. moving all metric implementations from `vtkMeshQuality` to Verdict while retaining the best implementation when the same metric was implemented in both software packages;

2. using `vtkMeshQuality` as a wrapper around Verdict; and

3. resolving naming inconsistencies and redundancies.

It is important to former Verdict users to note that item 3 has resulted in changes to Verdicts API, although efforts have been made to preserve backwards-compatibility as often as possible; this document underscores these modifications.

1.3   Organization

After this introduction, the document can be broadly divided into sections covering first the practical aspects and second the theoretical aspects of geometric quality evaluation. In these sections:

- Instructions on the practical aspects of obtaining, building, and installing the library are described.

- Notes on the application programming interface (API) are provided.

- A summary of each metric is provided, sorted first by the shape of the subregion they deal with and then by name.

The summaries in the last section form the bulk of the document and each contains a mathematical description of its quality metric q. In addition to a formula for each metric, information on the typical, acceptable, and total range of values taken on are presented in a tabular form according to the conventions outlined below in §1.4. Each summary table also contains a note on the dimension of its metric – where we use dimension in terms of the units associated with each metric value. Proper metrics have no dimension (which is denoted with a 1) but some metrics such as area, volume, or maximum angle do have dimension. We use L to denote dimensions of length and A to denote dimensions of angle. When a metric has a dimension repeated, an exponent is used to show the count. For example, volume has 3 length dimensions and would be denoted L^3. While the precise units of length depend on the input coordinates, angles are always reported in

degrees. The summary table also contains an entry for the value that the metric takes on for some ideally-shaped subregion, when the metric is shape-invariant (unlike, *e.g.* volume metrics which are not preserved by scaling). For triangular, quadrilateral, tetrahedral, and hexahedral shapes, this is respectively an equilateral triangle, a square, a regular tetrahedron, and a cube. For proper metrics, this value will be $q = 1$. Finally, each summary table has a reference to a book or paper where its metric is defined and discussed. If no reference is listed, the metric is one that is traditionally used but not present in the literature we are aware of.

Where possible, notes on the intended use of the metric are included. Verdict provides a variety of metrics for each subregion shape it supports. Since a metric is a single real number, it cannot completely describe the shape of its corresponding subregion. Thus, most metrics are used to identify a single type of problem with a subregion's shape. Because many numerical techniques are used to solve partial differential equations, the best metric to characterize the geometric quality of a region will vary.

1.4   Metric Ranges

Each metric may take on any value on the real number line, but typically subsets of this range are of interest since values for misbehaved or geometrically degenerate elements are often used to segregate or eliminate elements. The summary tables provided for each metric include three intervals on the real number line:

Acceptable Range :	Well-behaved elements will have metrics in this range.
Normal Range :	All elements except those with degeneracies will have metrics in this range.
Full Range :	All elements including degenerate ones will have metrics in this range.

1.5   Metric Behavior

The metrics in this report are all checked for overflow like so:

1: Given a double-precision quality metric value q,
2: **if** $q > 0$ **then**
3:    $q \leftarrow \min(q, DBL\_MAX)$
4: **else**
5:    $q \leftarrow \max(q, -DBL\_MAX)$

6: **end if**

Where applicable, the metrics in Verdict were verified against theory for:

- Node order invariance.

- Continous solutions within the normal range.

Chapter 2

Obtain, Configure, Compile, Install

The Verdict repository now resides at Kitware, Inc. and is publicly available. A formal release has not yet been made since the repository has been moved and so you will need to obtain Verdict source code from CVS. If you intend to build VTK, you need not obtain or compile Verdict separately since it is included with VTK.

2.1 Prerequisites

To build Verdict you will need a C++ compiler. GNU's gcc 3 or above, Intel's icc 8 or above, Apple's Xcode 2.4 or above, Borland's bcc 3.2 or 5.5, and Microsoft's VC 6 or above are all known to work. Other compilers, including Sun's CC, DEC's cxx, IBM's xlC, HP's aCC, and SGI's CC, are untested but should work. Unless you are using Microsoft's Visual Studio compilers, you will also need "make".

It is not required, but CMake[1] version 2.4 or above is highly recommended. At some point in the future, this will be the only supported configuration system.

2.2 Obtaining **Verdict**

Fetch the source code from Kitware's CVS server at `www.vtk.org`. Kitware provides anonymous access using CVS pserver. Before you can retrieve the

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[1] http://www.cmake.org

source code, you must authenticate yourself to the CVS server. From a terminal window, run

```
cvs -d :pserver:anonymous@www.vtk.org:/cvsroot/VTK login
```

You will be prompted for a password. Use "vtk" (without the quotes). You may then retrieve Verdict by running

```
cvs -d :pserver:anonymous@www.vtk.org:/cvsroot/VTK -z3 \
    co -d Verdict VTK/Utilities/verdict
```

This will place the source code in a subdirectory named `Verdict`.
If the machine you will use to compile Verdict is behind a firewall, you will probably not be able to use the commands above to obtain the source code. If you have SSH access to a computer that is not behind a firewall and SSH port forwarding is not forbidden, you may port forward CVS requests using a pair of terminal windows. For the purposes of our description, say that you will build Verdict on a computer named `inside.thefirewall.com` (behind the firewall) and have SSH access to a computer named `outside.thefirewall.com` (which is not behind the firewall).
In the first terminal window on `inside.thefirewall.com`, run

```
ssh -L 2401:www.vtk.org:2401 outside.thefirewall.com
```

and enter your password as required. Then, while you are still logged into `outside.thefirewall.com`, type the following into a second terminal window on `inside.thefirewall.com`:

```
cvs -d :pserver:anonymous@localhost:/cvsroot/VTK login
cvs -d :pserver:anonymous@localhost:/cvsroot/VTK -z3 \
    co -d Verdict VTK/Utilities/verdict
```

After the first command, you'll have to enter the repository password "vtk".

2.3   Configuring **Verdict**

Now that you have the Verdict source in a directory named `Verdict`, you are ready to configure it. The recommended way to configure Verdict is to use CMake and perform an "out-of-source" build (where the object files are not stored in the same directory tree as the source code). To follow the recommended practice, create a directory named `Verdict/Build`. On Mac OS X, Linux, and other Unix-like systems, do the following[2]:

---

[2]On Mac OS X, you may wish to use "`ccmake -G Xcode ..`" in order to create Xcode project files instead of makefiles.

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```
cd Verdict/Build
ccmake ..
```

You will be presented with a text interface for changing configuration parameters. If the defaults are acceptable, press the 'c' key until an option to generate project files appears and then press the 'g' key. In practice, there are a few configuration parameters you may wish to change:

`BUILD_SHARED_LIBRARIES`	Should a shared or static Verdict library be created?
`CMAKE_BUILD_TYPE`	This should be `Release` unless you are developing Verdict, in which case it should be set to `Debug`.
`CMAKE_INSTALL_PREFIX`	By default, Verdict will be installed in `/usr/local/`
`VERDICT_ENABLE_TESTING`	Should tests of the quality metrics be compiled?

You may change them after you have run `ccmake`'s configuration stage the first time (by pressing the 'c' key).

On Windows machines, run the `CMakeSetup.exe` program that comes with CMake. Set the source directory to the full path to the `Verdict` directory containing the source code and the build directory to the full path to the `Verdict/Build` directory you just created. Click the configure button until the OK button is enabled and then click OK. As with other systems, you may wish to change some of the configuration parameters in the table above. If you choose not to use CMake, there is no configuration required or available.

2.4   Building **Verdict**

On systems where you have used CMake with the "Unix Makefiles" generator (the default for everything except Windows), just run `make` in the `Verdict/Build` directory. If you used the Xcode generator on Mac OS X, simply open the Xcode project file in `Verdict/Build` click Xcode's build button. If you are on a Windows with MSVC, open the Visual Studio project file and click the build button.

2.5   Installing **Verdict**

If you used CMake, you should be able to build the install target. Otherwise, you will have to manually install Verdict– but this is a simple task since Verdict consists of a single header file named `verdict.h` and a single library. On

platforms with Makefiles, simply copy `verdict.h` to `/usr/local/include` or any other directory in your compiler's default search path. Then copy the file named `libverdict112.a`, `libverdict112.so`, or `libverdict112.dylib` (depending on your platform) to `/usr/local/lib` or some other directory in your link loader's default search path. On 64-bit Linux systems, you should use `/usr/local/lib64`.

Chapter 3

Application Programming Interface (API)

Verdict was designed with a C interface so that it can be used in a variety of applications.

Each metric has its own function. For example, the Hex Condition Number metric is:

```
double v_hex_condition(int num_nodes,
                       double node_coordiantes[][3])
```

It may be used as follows:

```
double coords[8][3];
...
double condition_value = v_hex_condition(8, coords);
```

The number of nodes is given for each element, and the implementation may be expanded to include higher order elements.

Each type of element has one function for getting multiple metrics at the same time. The following is a prototype to get multiple metrics for a hexahedron:

```
double v_hex_quality(int num_nodes,
                     double node_coordiantes[][3],
                     unsigned int request_flag,
                     struct HexMetricVals *metric_vals)
```

If one wants multiple metrics for an element, it is usually less computationally expensive to use this approach because some metrics share the same

computations. For example, computing the Jacobian and shape metrics of a hexahedron both use the Jacobian matrix. It may be used as follows:

```
double coords[8][3];
HexMetricVals vals;
double jacobian_value;
double shape_value;
int request = V_HEX_JACOBIAN | V_HEX_SHAPE;
...
v_hex_quality(8, coords, request, &vals);
double jacobian_value = vals.jacobian;
double shape_value = vals.shape;
```

Chapter 4

Triangle Quality Metrics

All the metrics in this section are defined on a triangular element as illustrated in Figure 4.1.
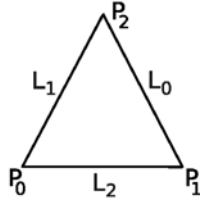
Figure 4.1: Numbering of vertices and edges on a triangular element.

Note that unlike all the other elements that follow, we name edge vectors of the triangle by the vertex opposite the edge so that

$$\begin{aligned}
\vec{L}_0 &= \vec{P}_2 - \vec{P}_1 \\
\vec{L}_1 &= \vec{P}_0 - \vec{P}_2 \\
\vec{L}_2 &= \vec{P}_1 - \vec{P}_0.
\end{aligned}$$

The triangle edge lengths are denoted as follows:

$$L_0 = \left\|\vec{L}_0\right\| \quad L_1 = \left\|\vec{L}_1\right\| \quad L_2 = \left\|\vec{L}_2\right\|$$

and the largest and smallest edge lenghts are, respectively,

$$L_{\min} = \min\left(L_0, L_1, L_2\right) \qquad L_{\max} = \max\left(L_0, L_1, L_2\right)$$

The area of a triangle is one half the magnitude of the cross product of any pair of adjacent edge vectors:

$$A = \frac{1}{2}\|\vec{L}_0 \times \vec{L}_1\| = \frac{1}{2}\|\vec{L}_1 \times \vec{L}_2\| = \frac{1}{2}\|\vec{L}_2 \times \vec{L}_0\|$$

In addition, we will let r be the inradius

$$r = \frac{2A}{\|\vec{L}_0\| + \|\vec{L}_1\| + \|\vec{L}_2\|}$$

and R the circumradius

$$R = \frac{\|\vec{L}_0\|\|\vec{L}_1\|\|\vec{L}_2\|}{2r\left(\|\vec{L}_0\| + \|\vec{L}_1\| + \|\vec{L}_2\|\right)}$$

of the triangle. These are respectively the radii of the inscribed and circumscribed circles of this triangle.

We will frequently use n to represent some arbitrary edge L_n or vertex P_n of the triangle. When referring to the next counterclockwise entry $n+1$ (or clockwise entry $n-1$), we take the result modulo 3 so that, for example, if $n = 1$, $n + 1 = 2$ and $n + 2 = 0$.

4.1 Area

This metric is simply the area as defined above

$$q = A.$$

Note that since A is non-negative, the current version of Verdict cannot detect inverted triangles. What are you doing with inverted triangles, anyway? There's only 3 vertices to keep track of!

	triangle area
Dimension:	L^2
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for equilateral unit triangle:	$\frac{\sqrt{3}}{4}$
Reference:	$-$
Verdict function:	`v_tri_area`

4.2   Aspect Ratio

The aspect ratio of a triangle is:

$$q = \frac{L_{\max}}{2\sqrt{3}r}.$$

Using (4), one can thus write it alternatively as

$$q = \frac{L_{\max}(L_0 + L_1 + L_2)}{4\sqrt{3}A}.$$

Note that in earlier versions of Verdict, triangle aspect ratio was used to call out what is now called the triangle aspect.

	triangle aspect ratio
Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[9]
Verdict function:	v_tri_aspect_ratio

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4.3   Aspect Frobenius

The aspect Frobenius is the sum of the edge lengths squared divided by the area and normalized so that a unit equilateral triangle has a value of 1.

$$q = \frac{\left\|\vec{L_0}\right\|^2 + \left\|\vec{L_1}\right\|^2 + \left\|\vec{L_2}\right\|^2}{4A\sqrt{3}}$$

Note that in earlier versions of Verdict, this metric was called the triangle aspect ratio.

triangle aspect Frobenius	
Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[9]
Verdict function:	v_tri_aspect_frobenius

4.4 Condition

The condition number of the weighted Jacobian matrix is

$$q = \frac{\left(\vec{L}_2 \cdot \vec{L}_2 + \vec{L}_1 \cdot \vec{L}_1 + \vec{L}_1 \cdot \vec{L}_2 \right)}{2A\sqrt{3}}.$$

Note that when $A = 0$, we set $q = DBL\_MAX$. In theory the condition number is invariant to which node it is computed at, but floating point truncation error can contribute to differences between values computed for each node. Verdict always uses the first vertex.

	triangle condition
Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[5, 6]
Verdict function:	v_tri_condition

4.5   Distortion

Let A be the area as defined in §4.1 and $A_m = \sqrt{3}$ be the area of a "master" triangle with vertices

$$
\begin{aligned}
\vec{P}_0 &= (\quad -1 \quad, \quad -\tfrac{\sqrt{3}}{3} \quad, \quad 0 \quad) \\
\vec{P}_1 &= (\quad 1 \quad, \quad -\tfrac{\sqrt{3}}{3} \quad, \quad 0 \quad) \\
\vec{P}_2 &= (\quad 0 \quad, \quad \tfrac{2\sqrt{3}}{3} \quad, \quad 0 \quad).
\end{aligned}
$$

Now define $|J|$ as the minimum value of the determinant of the Jacobian evaluated at all Gauss points of the element. The distortion is then

$$
q = \frac{|J|A_m}{A} = \frac{|J|\sqrt{3}}{A}.
$$

Distortion is a measure of how well-behaved the mapping from parameter space to world coordinates is.
Note that this metric is currently unsupported.

	triangle distortion
Dimension:	1
Acceptable Range:	$[0.5, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	Adapted from [11]
Verdict function:	v_tri_distortion

4.6 Edge Ratio

The edge ratio of a triangle is:

$$\frac{L_{\max}}{L_{\min}}.$$

triangle edge ratio	
Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[9]
Verdict function:	v_tri_edge_ratio

4.7 Maximum Angle

The maximum included angle of the triangle is

$$q = \max_{n \in \{0,1,2\}} \left\{ \arccos \left(\frac{\vec{L}_n \cdot \vec{L}_{n+1}}{\|\vec{L}_n\| \|\vec{L}_{n+1}\|} \right) \left(\frac{180°}{\pi} \right) \right\}$$

measured in degrees.

Note that if any edge vector has zero length, Verdict will return $q = 0°$.

<table>
<tr><td colspan="2">triangle maximum included angle</td></tr>
<tr><td>Dimension:</td><td>A^1</td></tr>
<tr><td>Acceptable Range:</td><td>$[60°, 90°]$</td></tr>
<tr><td>Normal Range:</td><td>$[60°, 180°]$</td></tr>
<tr><td>Full Range:</td><td>$[0°, 180°]$</td></tr>
<tr><td>q for equilateral unit triangle:</td><td>$60°$</td></tr>
<tr><td>Reference:</td><td>$-$</td></tr>
<tr><td>Verdict function:</td><td><code>v_tri_maximum_angle</code></td></tr>
</table>

4.8 Minimum Angle

The minimum included angle of the triangle is

$$q = \min_{n \in \{0,1,2\}} \left\{ \arccos \left(\frac{\vec{L}_n \cdot \vec{L}_{n+1}}{\|\vec{L}_n\| \|\vec{L}_{n+1}\|} \right) \left(\frac{180°}{\pi} \right) \right\}$$

measured in degrees.

Note that if any edge vector has zero length, Verdict will return $q = 360°$.

triangle minimum included angle

Dimension:	A^1
Acceptable Range:	$[30°, 60°]$
Normal Range:	$[0°, 60°]$
Full Range:	$[0°, 360°]$
q for equilateral unit triangle:	$60°$
Reference:	[9]
Verdict function:	v_tri_minimum_angle

4.9 Scaled Jacobian

First, let $L_{\max}$ be the product of the lengths of the 2 longest edges:

$$L_{\max} = \max\left\{\|\vec{L}_0\|\|\vec{L}_1\|, \|\vec{L}_0\|\|\vec{L}_2\|, \|\vec{L}_1\|\|\vec{L}_2\|\right\}$$

Let J' be the Jacobian of the triangle. If the triangle surface normal $\hat{n}$ is evaluated at the center of the triangle and $\hat{n} \cdot \left(\vec{L}_2 \times \vec{L}_1\right) < 0$, then take $J = -J'$. Otherwise take $J = J'$. The scaled Jacobian is then

$$q = \frac{2\sqrt{3}}{3} \frac{J}{L_{\max}}$$

which is normalized so that a unit equilateral triangle has value 1.
Note that if $L_{\max} \leq DBL\_MIN$, we set $q = 0$.

triangle scaled Jacobian	
Dimension:	1
Acceptable Range:	$[0.5, \frac{2\sqrt{3}}{3}]$
Normal Range:	$[-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[5]
Verdict function:	v_tri_scaled_jacobian

4.10 Radius Ratio

The radius ratio is:

$$\frac{R}{2r}.$$

triangle radius ratio	
Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[9]
Verdict function:	v_tri_radius_ratio

4.11 Relative Size Squared

Let R be ratio of the triangle area A to the average area $\overline{A}$ of an ensemble of triangles

$$R = \frac{A}{\overline{A}}$$

The relative size is the minimum of R and its inverse and the relative size squared is

$$q = \left(\min\left\{ R, \frac{1}{R} \right\} \right)^2 .$$

Note that if $R = 0$, we take $q = 0$.

triangle relative size squared	
Dimension:	1
Acceptable Range:	$[0.25, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for equilateral unit triangle:	Dependent on $\overline{A}$
Reference:	[6]
Verdict function:	v_tri_relative_size_squared

4.12 Shape

Let C be the condition number as defined in §4.4. Then the shape metric is simply

$$q = \frac{1}{C}$$

triangle relative size squared

Dimension:	1
Acceptable Range:	$[0.25, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for equilateral unit triangle:	1
Reference:	[6]
Verdict function:	v_tri_shape

4.13   Shape and Size

Let R be the relative size squared as defined in §4.11 and S be the shape as defined in §4.12. Then the "shape and size" metric is

$$q = RS$$

triangle shape and size	
Dimension:	1
Acceptable Range:	$[0.25, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for equilateral unit triangle:	Dependent on $\overline{A}$
Reference:	[6]
Verdict function:	v_tri_shape_and_size

Chapter 5

Quadrilateral Quality Metrics

All the metrics in this section are defined on a quadrilateral element with vertices shown in Figure 5.1. Furthermore, we define the following edge vectors for convenience. Note that each edge has two versions, one defined by its endpoints and another indexed by sequential integers:

$$
\begin{aligned}
\vec{L}_0 &= \vec{P}_1 - \vec{P}_0 & \vec{L}_{01} &= \vec{P}_1 - \vec{P}_0 \\
\vec{L}_1 &= \vec{P}_2 - \vec{P}_1 & \vec{L}_{12} &= \vec{P}_2 - \vec{P}_1 \\
\vec{L}_2 &= \vec{P}_3 - \vec{P}_2 & \vec{L}_{23} &= \vec{P}_3 - \vec{P}_2 \\
\vec{L}_3 &= \vec{P}_0 - \vec{P}_3 & \vec{L}_{30} &= \vec{P}_0 - \vec{P}_3.
\end{aligned}
$$

The quadrangle edge lengths are denoted as follows:

$$
L_0 = \left\|\vec{L}_0\right\| \quad L_1 = \left\|\vec{L}_1\right\| \quad L_2 = \left\|\vec{L}_2\right\| \quad L_3 = \left\|\vec{L}_3\right\|
$$

and the largest and smallest edge lenghts are, respectively,

$$
L_{\min} = \min\left(L_0, L_1, L_2, L_3\right) \qquad L_{\max} = \max\left(L_0, L_1, L_2, L_3\right)
$$

The diagonals of a quadrilateral are denoted

$$
\vec{D}_0 = \vec{P}_2 - \vec{P}_0 \qquad\qquad \vec{D}_1 = \vec{P}_3 - \vec{P}_1
$$

and the longest diagonal has length

$$
D_{\max} = \max\left\{\left\|\vec{D}_0\right\|, \left\|\vec{D}_1\right\|\right\}.
$$
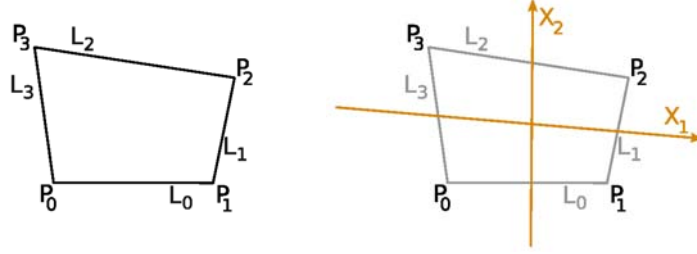
(a) Vertices of a quadrilateral.　　　(b) Principal axis vectors.

Figure 5.1: A quadrilateral showing notation used in metric definitions.

The principal axes are

$$
\begin{aligned}
\vec{X}_1 &= \left(\vec{P}_1 - \vec{P}_0\right) + \left(\vec{P}_2 - \vec{P}_3\right) \\
\vec{X}_2 &= \left(\vec{P}_2 - \vec{P}_1\right) + \left(\vec{P}_3 - \vec{P}_0\right)
\end{aligned}
$$

and the cross derivatives of the map from parametric to world space are oriented along

$$
\begin{aligned}
\vec{X}_{12} &= \left(\vec{P}_0 - \vec{P}_1\right) + \left(\vec{P}_2 - \vec{P}_3\right) = \\
\vec{X}_{21} &= \left(\vec{P}_0 - \vec{P}_3\right) + \left(\vec{P}_2 - \vec{P}_1\right).
\end{aligned}
$$

Each corner has a normal vector associated with it

$$
\begin{aligned}
\vec{N}_0 &= \vec{L}_3 \times \vec{L}_0 & \vec{N}_2 &= \vec{L}_1 \times \vec{L}_2 \\
\vec{N}_1 &= \vec{L}_0 \times \vec{L}_1 & \vec{N}_3 &= \vec{L}_2 \times \vec{L}_3
\end{aligned}
$$

and these vectors can be normalized to unit length:

$$
\begin{aligned}
\hat{n}_0 &= \frac{\vec{N}_0}{\left\|\vec{N}_0\right\|} & \hat{n}_2 &= \frac{\vec{N}_2}{\left\|\vec{N}_2\right\|} \\
\hat{n}_1 &= \frac{\vec{N}_1}{\left\|\vec{N}_1\right\|} & \hat{n}_3 &= \frac{\vec{N}_3}{\left\|\vec{N}_3\right\|}.
\end{aligned}
$$

In addition to corner normals, we can define a "center" normal

$$
\vec{N}_c = \vec{X}_1 \times \vec{X}_2
$$

and its unit-length companion

$$
\hat{n}_c = \frac{\vec{N}_c}{\left\|\vec{N}_c\right\|}
$$

In the event that the vertices of the quadrilateral are all contained in the same plane, all the unit normals will be equivalent (i.e., $\hat{n}_0 = \hat{n}_1 = \hat{n}_2 = \hat{n}_3 = \hat{n}_c$).
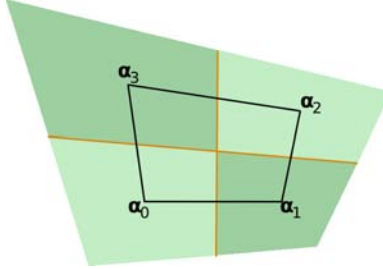
Figure 5.2: Areas associated with each quadrilateral vertex.

It is often useful to partition the quadrilateral into four areas, one associated with each vertex. These areas are denoted

$$\alpha_k = \hat{n}_c \cdot \vec{N}_k \qquad\qquad \forall k \in \{0, 1, 2, 3\}$$

and are shown in Figure 5.2. If $\vec{N}_c = \vec{0}$, then the signed corner areas are undefined, and all the metrics which depend on α_k are undefined. In this case, we set $\alpha_k = 0$ for $k = 0, 1, 2, 3$. When $\alpha_k \leq 0$ for any one or more k, the quadrilateral is degenerate. This occurs when an element is so small its edge length approach the machine epsilon or when its vertices are collinear or when its vertices define a concave quadrilateral.

5.1 Area

Signed area, defined as

$$q = \frac{1}{4} \sum_{i=0}^{3} \alpha_i$$

is useful for two purposes: first, the sign can indicate elements that have vertices ordered incorrectly or arranged in a concave pattern; and second, the magnitude can be used to identify elements that are too small for accurate analysis. Figure 5.3 shows how each vertex area contributes to the total area of a quadrilateral.
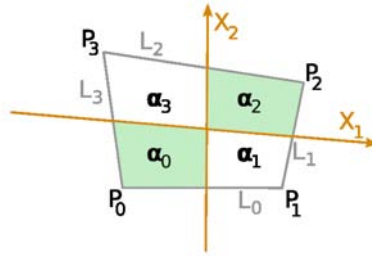
Figure 5.3: The areas associated with each vertex may be summed and weighted to get the area of the entire quadrilateral.

quadrilateral area	
Dimension:	L^2
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit square:	1
Reference:	–
Verdict function:	v_quad_area

5.2 Aspect Ratio

The aspect ratio of a quadrilateral is:

$$q = \frac{L_{\max}(L_0 + L_1 + L_2 + L_3)}{4A},$$

where A is the area of the quadrilateral.

Note that, strictly speaking, the aspect ratio is usually defined for simplicial elements as the ratio of the maximum edge length to the inradius (*cf.* §4.2 and §6.6). However, a planar quadrilateral does not have, in general, an inscribed circle: such an incircle exists if and only if $L_0 + L_2 = L_1 + L_3$. Nonetheless, using the expression of the triangle aspect ratio as given in (4.2), that is, with no explicit reference to the inradius but only to the perimeter and the area, one can then directly extrapolate to obtain a meaningful definition of the quadrangle aspect ratio.

triangle aspect ratio	
Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for equilateral unit triangle:	1
Reference:	[8]
Verdict function:	v_quad_aspect_ratio

5.3 Condition

$$q = \frac{1}{2} \max \left\{ \frac{\left\|\vec{L}_0\right\|^2 + \left\|\vec{L}_3\right\|^2}{\alpha_0}, \frac{\left\|\vec{L}_1\right\|^2 + \left\|\vec{L}_0\right\|^2}{\alpha_1}, \frac{\left\|\vec{L}_2\right\|^2 + \left\|\vec{L}_1\right\|^2}{\alpha_2}, \frac{\left\|\vec{L}_3\right\|^2 + \left\|\vec{L}_2\right\|^2}{\alpha_3} \right\}$$

Note that if $\alpha_i < DBL\_MIN$, we set $q = DBL\_MAX$.

quadrilateral condition

Dimension:	1
Acceptable Range:	$[1, 4]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit square:	1
Reference:	[5]
Verdict function:	v_quad_condition

5.4 Distortion

Let A be the area as defined in §5.1 and $A_m = 4$ be the area of a "master" quadrilateral with vertices

$$
\begin{aligned}
\vec{P}_0 &= (\quad -1 \quad, \quad -1 \quad, \quad 0 \quad) \\
\vec{P}_1 &= (\quad 1 \quad, \quad -1 \quad, \quad 0 \quad) \\
\vec{P}_2 &= (\quad 1 \quad, \quad 1 \quad, \quad 0 \quad) \\
\vec{P}_3 &= (\quad -1 \quad, \quad 1 \quad, \quad 0 \quad).
\end{aligned}
$$

Now define $|J|$ as the minimum value of the determinant of the Jacobian evaluated at all Gauss points of the element. The distortion is then

$$
q = \frac{|J|A_m}{A} = \frac{4|J|}{A}.
$$

Distortion is a measure of how well-behaved the mapping from parameter space to world coordinates is.

quadrilateral distortion

Dimension:	1
Acceptable Range:	$[0.5, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit square:	1
Reference:	[11]
Verdict function:	v_quad_distortion

5.5 Edge Ratio

The edge ratio of a quadrilateral is the ratio of its longest and shortest edge lengths:

$$q = \frac{L_{\max}}{L_{\min}}.$$

<table>
<tr><td colspan="2" align="center">**quadrilateral edge ratio**</td></tr>
<tr><td>Dimension:</td><td>1</td></tr>
<tr><td>Acceptable Range:</td><td>$[1, 1.3]$</td></tr>
<tr><td>Normal Range:</td><td>$[1, DBL\_MAX]$</td></tr>
<tr><td>Full Range:</td><td>$[1, DBL\_MAX]$</td></tr>
<tr><td>q for unit square:</td><td>1</td></tr>
<tr><td>Reference:</td><td>[8]</td></tr>
<tr><td>Verdict function:</td><td>v_quad_edge_ratio</td></tr>
</table>

5.6 Jacobian

The minimum Jacobian computed at each vertex is used:

$$q = \min_{i \in \{0,1,2,3\}} \{\alpha_i\}$$

quadrilateral Jacobian

Dimension:	L^2
Acceptable Range:	$[0.DBL\_MAX]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit square:	1
Reference:	[5]
Verdict function:	`v_quad_jacobian`

5.7   Maximum Aspect Frobenius

For quadrilaterals, there is not a unique definition of the aspect Frobenius. Instead, we use the aspect Frobenius defined for triangles (see section §4.3). Consider the four triangles formed by pairs of neighboring quadrilateral edges. Given three counterclockwise, consecutively ordered quadrilateral vertices i, j, and k denote the triangular aspect frobenius F_{ijk}. To obtain a single value for the metric, we take the maximum of the four unique triangular aspects

$$q = \max\left(F_{301}, F_{012}, F_{123}, F_{230}\right).$$

quadrilateral maximum aspect frobenius

Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit square:	1
Reference:	[8]
Verdict function:	v_quad_max_aspect_frobenius

5.8   Maximum Angle

In order to properly compute the included angle, we'll need to correct for incorrectly oriented elements. Let

$$s_i = \begin{cases} 1 & \alpha_i < 0 \\ 0 & \alpha_i \geq 0 \end{cases}$$

The included angle between two neighboring edges is

$$\theta_i = (-1)^{s_i} \arccos\left(-\frac{\vec{L}_i \cdot \vec{L}_{i+1}}{\|\vec{L}_i\|\|\vec{L}_{i+1}\|}\right)\left(\frac{180}{\pi}\right) + 360°\, s_i$$

where $i \in \{0, 1, 2, 3\}$ and $\vec{L}_4 = \vec{L}_0$. We take the maximum of this quantity as the value of the metric:

$$q = \max_{i \in \{0,1,2,3\}} \{\theta_i\}$$

Note that if $\|\vec{L}_i\| \leq DBL\_MIN$ or $\|\vec{L}_{i+1}\| \leq DBL\_MIN$, Verdict returns $q = 0°$.

quadrilateral maximum included angle

Dimension:	A^1
Acceptable Range:	$[90°, 135°]$
Normal Range:	$[90°, 360°]$
Full Range:	$[0°, 360°]$
q for unit square:	$90°$
Reference:	$-$
Verdict function:	`v_quad_maximum_angle`

5.9 Maximum Edge Ratio

$$q = \max\left\{ \|\vec{X}_1\|/\|\vec{X}_2\|, \|\vec{X}_2\|/\|\vec{X}_1\| \right\}$$

Note that if $\|\vec{X}_1\|$ or $\|\vec{X}_2\| < DBL\_MIN$, we set $q = DBL\_MAX$.

quadrilateral maximum edge ratio

Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit square:	1
Reference:	[10]
Verdict function:	v_quad_max_edge_ratio

5.10 Mean Aspect Frobenius

For quadrilaterals, there is not a unique definition of the aspect Frobenius. Instead, we use the aspect Frobenius defined for triangles (see section §4.3). Consider the four triangles formed by pairs of neighboring quadrilateral edges. Given three counterclockwise, consecutively ordered quadrilateral vertices i, j, and k denote the triangular aspect frobenius F_{ijk}. To obtain a single value for the metric, we average the four unique triangular aspects

$$q = \frac{1}{4}\left(F_{301} + F_{012} + F_{123} + F_{230}\right).$$

quadrilateral mean aspect frobenius

Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit square:	1
Reference:	[8]
Verdict function:	v_quad_med_aspect_frobenius

5.11 Minimum Angle

In order to properly compute the included angle, we'll need to correct for incorrectly oriented elements. Let

$$s_i = \begin{cases} 1 & \alpha_i < 0 \\ 0 & \alpha_i \geq 0 \end{cases}$$

The included angle between two neighboring edges is

$$\theta_i = (-1)^{s_i} \arccos\left(-\frac{\vec{L}_i \cdot \vec{L}_{i+1}}{\|\vec{L}_i\|\|\vec{L}_{i+1}\|}\right)\left(\frac{180}{\pi}\right) + 360° s_i$$

where $i \in \{0, 1, 2, 3\}$ and $\vec{L}_4 = \vec{L}_0$. We take the minimum of this quantity as the value of the metric:

$$q = \min_{i \in \{0,1,2,3\}} \{\theta_i\}$$

Note that if $\|\vec{L}_i\| \leq DBL\_MIN$ or $\|\vec{L}_{i+1}\| \leq DBL\_MIN$, Verdict returns $q = 360°$.

quadrilateral minimum included angle

Dimension:	A^1
Acceptable Range:	$[45°, 90°]$
Normal Range:	$[0°, 90°]$
Full Range:	$[0°, 360°]$
q for unit square:	$90°$
Reference:	$-$
Verdict function:	v_quad_minimum_angle

5.12   Oddy

Let $\vec{L}_4 = \vec{L}_0$. The Oddy metric is then defined as

$$q = \max_{i \in \{0,1,2,3\}} \left\{ \frac{(\left\|\vec{L}_i\right\|^2 - \left\|\vec{L}_{i+1}\right\|^2)^2 + 4(\vec{L}_i \cdot \vec{L}_{i+1})^2}{2\left\|\vec{N}_{i+1}\right\|^2} \right\}.$$

This metric measures the maximum deviation of the metric tensor at the corners of the quadrilateral.

Note that if $\left\|\vec{N}_{i+1}\right\|^2 < DBL\_MIN$, we set $q = DBL\_MAX$.

quadrilateral Oddy

Dimension:	1
Acceptable Range:	$[0, 0.5]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit square:	0
Reference:	[7]
Verdict function:	v_quad_oddy

5.13 Radius Ratio

Let $h_{\max}$ be the maximum length of all edges and diagonals

$$h_{\max} = \max\left(L_{\max}, D_{\max}\right)$$

and $\mathcal{L}_2$ be the sum of the squares of all edge lengths

$$\mathcal{L}_2 = \sum_{i=0}^{3} \left\|\vec{L}_i\right\|^2$$

and $\mathcal{A}_i$ be the area of one of the 4 triangles formed by pairs of quadrilateral neighboring edges

$$\mathcal{A}_i = \left|\frac{\alpha_i}{2}\right|.$$

Then the radius ratio of a planar quadrilateral is

$$q = \frac{\mathcal{L}_2 h_{\max}}{\min_{i \in \{0,1,2,3\}} \mathcal{A}_i}.$$

quadrilateral radius ratio

Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit square:	1
Reference:	[8]
Verdict function:	v_quad_radius_ratio

5.14 Relative Size Squared

The relative size squared metric is defined as

$$q = \left(\min \left\{ \frac{A}{\overline{A}}, \frac{\overline{A}}{A} \right\} \right)^2$$

where A is the area of the element as defined in §5.1 and $\overline{A}$ is the average of A over all of the elements in the ensemble of elements being considered. It is the square of the minimum of the ratio of quad area to the average quad area and its inverse.

Note that if $\overline{A} < DBL\_MIN$ or $A < DBL\_MIN$, we take $q = 0$.

quadrilateral relative size squared

Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit square:	Dependent on $\overline{A}$
Reference:	[6]
Verdict function:	v_quad_relative_size_squared

5.15 Scaled Jacobian

The scaled Jacobian is the minimum of the Jacobian at each corner divided by the lengths of the 2 edge vectors (which is the minimum sine of the included angles):

$$q = \min \left\{ \frac{\alpha_0}{\left\|\vec{L}_0\right\|\left\|\vec{L}_3\right\|}, \frac{\alpha_1}{\left\|\vec{L}_1\right\|\left\|\vec{L}_0\right\|}, \frac{\alpha_2}{\left\|\vec{L}_2\right\|\left\|\vec{L}_1\right\|}, \frac{\alpha_3}{\left\|\vec{L}_3\right\|\left\|\vec{L}_2\right\|} \right\}.$$

Note that if any edge has $L < DBL\_MIN$, we take $q = 0$.

quadrilateral scaled Jacobian	
Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[-1, 1]$
Full Range:	$[-1, 1]$
q for unit square:	1
Reference:	[5]
Verdict function:	v_quad_scaled_jacobian

5.16 Shape

The shape metric is 2 divided by the condition number of the Jacobian matrix:

$$q = 2\min\left\{\frac{\alpha_0}{\left\|\vec{L}_0\right\|^2 + \left\|\vec{L}_3\right\|^2}, \frac{\alpha_1}{\left\|\vec{L}_1\right\|^2 + \left\|\vec{L}_0\right\|^2}, \frac{\alpha_2}{\left\|\vec{L}_2\right\|^2 + \left\|\vec{L}_1\right\|^2}, \frac{\alpha_3}{\left\|\vec{L}_3\right\|^2 + \left\|\vec{L}_2\right\|^2}\right\}.$$

Note that if $\alpha_i < DBL\_MIN$ or any edge has length $L < DBL\_MIN$, we set $q = 0$.

quadrilateral shape	
Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit square:	1
Reference:	[6]
Verdict function:	v_quad_shape

5.17 Shape and Size

Let R be the relative size squared as defined in §5.14 and S be the shape as defined in §5.16. The shape and size metric is the product of these two numbers:

$$q = RS.$$

quadrilateral shape and size

Dimension:	1
Acceptable Range:	$[0.2, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit square:	Dependent on $\overline{A}$
Reference:	[6]
Verdict function:	v_quad_shape_and_size

5.18   Shear

The shear metric

$$q = \min \left\{ \frac{\alpha_0}{\left\|\vec{L}_0\right\|\left\|\vec{L}_3\right\|}, \frac{\alpha_1}{\left\|\vec{L}_1\right\|\left\|\vec{L}_0\right\|}, \frac{\alpha_2}{\left\|\vec{L}_2\right\|\left\|\vec{L}_1\right\|}, \frac{\alpha_3}{\left\|\vec{L}_3\right\|\left\|\vec{L}_2\right\|} \right\}$$

is the same as the scaled Jacobian, except that it has a truncated range. Note that if $\alpha_i < DBL\_MIN$ or any edge has length $L < DBL\_MIN$, we set $q = 0$.

quadrilateral shear	
Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit square:	1
Reference:	[6]
Verdict function:	v_quad_shear

5.19 Shear and Size

Let R be the relative size squared as defined in §5.14 and H be the shear as defined in §5.18. The shear and size metric is the product of these two numbers:

$$q = RH$$

quadrilateral shear and size

Dimension:	1
Acceptable Range:	$[0.2, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit square:	Dependent on $\overline{A}$
Reference:	[6]
Verdict function:	v_quad_shear_and_size

5.20 Skew

First define normalized principal axes

$$
\begin{aligned}
\hat{X}_1 &= \frac{\vec{X}_1}{\left\|\vec{X}_1\right\|} \\
\hat{X}_2 &= \frac{\vec{X}_2}{\left\|\vec{X}_2\right\|}.
\end{aligned}
$$

The skew is then

$$
q = |\hat{X}_1 \cdot \hat{X}_2|.
$$

A geometric intepretation of the skew is that it measures the angle between the principal axes. In fact, it is the absolute value of the cosine of the angle between the principal axes.

Note that if $\left\|\vec{X}_1\right\|$ or $\left\|\vec{X}_2\right\| < DBL\_MIN$, we set $q = 0$.

<table>
<tr><td colspan="2" align="center">quadrilateral skew</td></tr>
<tr><td>Dimension:</td><td>1</td></tr>
<tr><td>Acceptable Range:</td><td>$[0.5, 1]$</td></tr>
<tr><td>Normal Range:</td><td>$[0, 1]$</td></tr>
<tr><td>Full Range:</td><td>$[0, 1]$</td></tr>
<tr><td>q for unit square:</td><td>1</td></tr>
<tr><td>Reference:</td><td>Adapted from [10]</td></tr>
<tr><td>Verdict function:</td><td><code>v_quad_skew</code></td></tr>
</table>

5.21 Stretch

The stretch is

$$q = \frac{\sqrt{2} \min_{i \in \{0,1,2,3\}} \{L_i\}}{D_{\max}}$$

Note that if $D_{\max} < DBL\_MIN$, we take $q = DBL\_MAX$.

quadrilateral stretch

Dimension:	1
Acceptable Range:	$[0.25, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, DBL\_MAX]$
q for unit square:	1
Reference:	[3]
Verdict function:	`v_quad_stretch`

5.22 Taper

Taper is the maximum ratio of cross derivative magnitude to principal axis magnitude:

$$q = \frac{\left\|\vec{X}_{12}\right\|}{\min\left\{\left\|\vec{X}_1\right\|, \left\|\vec{X}_2\right\|\right\}}$$

Note that if $\left\|\vec{X}_1\right\|$ or $\left\|\vec{X}_2\right\| < DBL\_MIN$, we set $q = DBL\_MAX$.

quadrilateral taper	
Dimension:	1
Acceptable Range:	$[0, 0.7]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit square:	0
Reference:	Adapted from [10]
Verdict function:	v_quad_taper

5.23 Warpage

Warpage is defined as

$$q = 1 - \min\left\{ (\hat{n}_0 \cdot \hat{n}_2)^3, (\hat{n}_1 \cdot \hat{n}_3)^3 \right\}$$

which is the cosine of the minimum dihedral angle formed by planes intersecting in diagonals (to the fourth power).
Note that if $\left\| \vec{N}_k \right\| < DBL\_MIN$ for any k, we set $q = DBL\_MAX$.

quadrilateral warpage	
Dimension:	1
Acceptable Range:	$[0, 0.7]$
Normal Range:	$[0, 2]$
Full Range:	$[0, DBL\_MAX]$
q for unit square:	0
Reference:	−
Verdict function:	v_quad_warpage

Chapter 6

Tetrahedral Quality Metrics

All the metrics in this section are defined on a tetrahedral element with vertices shown in Figure 6.1. Furthermore, we define the following edge vectors for convenience

$$
\begin{array}{llll}
\vec{L}_0 & = & \vec{P}_1 - \vec{P}_0 & \qquad \vec{L}_3 & = & \vec{P}_3 - \vec{P}_0 \\
\vec{L}_1 & = & \vec{P}_2 - \vec{P}_1 & \qquad \vec{L}_4 & = & \vec{P}_3 - \vec{P}_1 \\
\vec{L}_2 & = & \vec{P}_0 - \vec{P}_2 & \qquad \vec{L}_5 & = & \vec{P}_3 - \vec{P}_2
\end{array}
$$
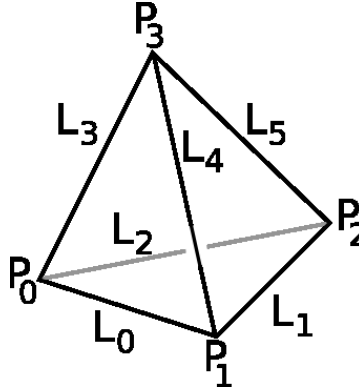
Figure 6.1: Vertices of a tetrahedron.

The tetrahedron edge lengths are denoted as follows:

$$L_0 = \left\|\vec{L}_0\right\| \quad L_1 = \left\|\vec{L}_1\right\| \quad L_2 = \left\|\vec{L}_2\right\| \quad L_3 = \left\|\vec{L}_3\right\| \quad L_4 = \left\|\vec{L}_4\right\| \quad L_5 = \left\|\vec{L}_5\right\|$$

and the largest and smallest edge lenghts are, respectively,

$$L_{\min} = \min\left(L_0, L_1, L_2, L_3, L_4, L_5\right) \qquad L_{\max} = \max\left(L_0, L_1, L_2, L_3, L_4, L_5\right)$$

The volume can then be defined in terms of the edge vectors as

$$V = \frac{\left(\vec{L}_2 \times \vec{L}_0\right) \cdot \vec{L}_3}{6}.$$

In addition, we will respectively denote R and r the circumradius and the inradius of the tetrahedron, *i.e.*, respectively, the radii of the circumscribed and inscribed spheres of this tetrahedron. Note that the inradius is

$$r = \frac{3V}{A}$$

where A is the surface area of the tetrahedron:

$$A = \frac{1}{2}\left(\left\|\vec{L}_2 \times \vec{L}_0\right\| + \left\|\vec{L}_3 \times \vec{L}_0\right\| + \left\|\vec{L}_4 \times \vec{L}_1\right\| + \left\|\vec{L}_3 \times \vec{L}_2\right\|\right),$$

and that the the circumradius is

$$R = \frac{\left\|\left\|\vec{L}_3\right\|^2 \left(\vec{L}_2 \times \vec{L}_0\right) + \left\|\vec{L}_2\right\|^2 \left(\vec{L}_3 \times \vec{L}_0\right) + \left\|\vec{L}_0\right\|^2 \left(\vec{L}_3 \times \vec{L}_2\right)\right\|}{12V}.$$

Sometimes, we will to refer to the edge vectors indexed by their endpoints:

$$
\begin{aligned}
\vec{L}_{01} &= \vec{L}_0 & \qquad \vec{L}_{03} &= \vec{L}_3 \\
\vec{L}_{12} &= \vec{L}_1 & \qquad \vec{L}_{13} &= \vec{L}_4 \\
\vec{L}_{20} &= \vec{L}_2 & \qquad \vec{L}_{23} &= \vec{L}_5
\end{aligned}
$$

6.1 Edge Ratio

The edge ratio of a tetrahedron is:

$$\frac{L_{\max}}{L_{\min}}.$$

tetrahedral edge ratio

Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	−
Verdict function:	`v_tet_edge_ratio`

6.2 Aspect Beta

This metric measures the radius ratio (*cf.* §6.12) of a positively-oriented tetrahedron. Note that it is equal to the tetrahedral radius ratio.

For a positively-oriented tetrahedron, the aspect β is the quotient of these two radii normalized by $\frac{1}{3}$ so that an equilateral tetrahedron has quality of 1:

$$
\begin{aligned}
q &= \frac{R}{3r} \\
&= \frac{\left| \|\vec{L}_3\|^2 \left(\vec{L}_2 \times \vec{L}_0 \right) + \|\vec{L}_2\|^2 \left(\vec{L}_3 \times \vec{L}_0 \right) + \|\vec{L}_0\|^2 \left(\vec{L}_3 \times \vec{L}_2 \right) \right| A}{108 V^2}.
\end{aligned}
$$

Note that if the tetrahedron has negative orientation, we set $q = DBL\_MAX$.

tetrahedral aspect β	
Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[2]
Verdict function:	v_tet_aspect_beta

6.3 Aspect Delta

Aspect δ is a dimensionless number defined as the smallest ratio of the height of a vertex above its opposing triangle (see Figure 6.2) to the square root of the area the triangle across all vertices of the tetrahedron. In general, take
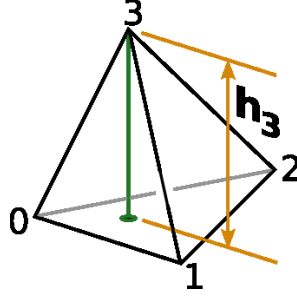
Figure 6.2: An illustration of the height h_3 of vertex 3.

(i, j, k, ℓ) to be a permutation of $\{0, 1, 2, 3\}$ (i.e., $(i, j, k, \ell) \in \mathfrak{S}_4$) and $\left\| \vec{L}_{ab} \right\|$ to be the length of the edge connecting vertices a and b. Then aspect ratio δ may be written

$$q = \min_i \left\{ C \frac{h_i}{\sqrt{A_{jk\ell}}} \right\}$$

where $A_{jk\ell}$ is the area of the triangle opposite vertex i and $C = \frac{\sqrt[4]{108}}{4} \approx 0.805927$ chosen so that an equilateral tetrahedron has $q = 1$. PATRAN [1] also speaks of a "normalized" aspect ratio defined as

$$q_{alt} = 1 - q = 1 - \min_i \left\{ C \frac{h_i}{\sqrt{A_{jk\ell}}} \right\}$$

which is 0 for an equilateral tetrahedron.

tetrahedral aspect δ	
Dimension:	1
Acceptable Range:	$[0.1, DBL\_MAX]$
Normal Range:	$(0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[1]
Verdict function:	Not supported

6.4   Aspect Frobenius

The edge matrix of the tetrahedral element is defined as follows:

$$T_0 = (\vec{L_0}\ \vec{L_1}\ \vec{L_2})$$

and let W be the edge matrix of the reference regular tetrahedron. Consider the matrix that maps W into T_0:

$$A_0 = T_0 W^{-1}.$$

The Frobenius norm of A_0 is

$$|A_0|_F = \sqrt{\mathrm{tr}(A_0^T A_0)},$$

and the Frobenius condition number is the condition number associated with this norm.

The aspect Frobenius of the element is defined as the normalized (equal to 1 when the element is regular) Frobenius condition number of A_0.

tetrahedral aspect Frobenius

Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[5]
Verdict function:	v_tet_aspect_frobenius

6.5   Aspect Gamma

This metric compares root-mean-square edge length to volume. The root-mean-square edge length is

$$R = \sqrt{\frac{\sum_{i=0}^{5} \left\| \vec{L}_i \right\|^2}{6}}$$

and so, normalizing the metric to a value of 1 for equilateral tetrahedra, we have

$$q = \frac{R^3 \sqrt{2}}{12|V|}.$$

Note that if $|V| < DBL\_MIN$, we set $q = DBL\_MAX$.

tetrahedral aspect γ	
Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[2]
Verdict function:	v_tet_aspect_gamma

6.6 Aspect Ratio

The aspect ratio of a tetrahedron K is:

$$\frac{L_{\max}}{2\sqrt{6}r}.$$

tetrahedral aspect ratio

Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[4]
Verdict function:	v_tet_aspect_ratio

6.7 Collapse Ratio

The collapse ratio is a dimensionless number defined as the smallest ratio of the height of a vertex above its opposing triangle to the longest edge of that opposing triangle across all vertices of the tetrahedron. Figure 6.2 shows how the ratio is computed for a single vertex (vertex 3). Assuming that edge $0 - 2$ is the longest edge of triangle $0 - 1 - 2$, the collapse ratio for vertex 3 becomes:

$$q_{ex} = \frac{h_3}{\left\| \vec{L}_{02} \right\|}.$$

In general, take (i, j, k, ℓ) to be a permutation of $\{0, 1, 2, 3\}$ (i.e., $(i, j, k, \ell) \in \mathfrak{S}_4$) and $\left\| \vec{L}_{ab} \right\|$ to be the length of the edge connecting vertices a and b. Then the collapse ratio may be written

$$q = \min_{i} \left\{ \frac{h_i}{\max \left\{ \left\| \vec{L}_{jk} \right\|, \left\| \vec{L}_{k\ell} \right\|, \left\| \vec{L}_{\ell j} \right\| \right\}} \right\}.$$

The collapse ratio is intended to identify tetrahedra whose vertices are nearly planar (slivers). Note that q approaches zero as vertex 3 in Figure 6.2 approaches the plane defined by $0 - 1 - 2$. However, this metric can be misleading when the vertex with the smallest projected height (say 3 without loss of generality) is not projected interior to triangle $0 - 1 - 2$. In this case, it is possible for $0 - 1 - 2$ to have a small area (which increases q) but for edges $0 - 3$, $1 - 3$, and $2 - 3$ to be very long compared to those of triangle $0 - 1 - 2$. Thus slivers can have arbitrarily high collapse ratios.

tetrahedral collapse ratio	
Dimension:	1
Acceptable Range:	$[0.1, DBL\_MAX]$
Normal Range:	$(0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit equilateral tetrahedron:	$\frac{\sqrt{6}}{3}$
Reference:	[1]
Verdict function:	v_tet_collapse_ratio

6.8 Condition

First, define

$$\begin{aligned}
\vec{C}_1 &= \vec{L}_0 \\
\vec{C}_2 &= \left(-2\vec{L}_2 - \vec{L}_0\right)/\sqrt{3} \\
\vec{C}_3 &= \left(3\vec{L}_3 + \vec{L}_2 - \vec{L}_0\right)/\sqrt{6}
\end{aligned}$$

$$C_{det} = \vec{C}_1 \cdot (\vec{C}_2 \times \vec{C}_3),$$

and

$$\begin{aligned}
T_1 &= \vec{C}_1 \cdot \vec{C}_1 + \vec{C}_2 \cdot \vec{C}_2 + \vec{C}_3 \cdot \vec{C}_3 \\
T_2 &= (\vec{C}_1 \times \vec{C}_2) \cdot (\vec{C}_1 \times \vec{C}_2) + (\vec{C}_2 \times \vec{C}_3) \cdot (\vec{C}_2 \times \vec{C}_3) + (\vec{C}_1 \times \vec{C}_3) \cdot (\vec{C}_1 \times \vec{C}_3)
\end{aligned}$$

The condition metric is then defined as follows:

$$q = \frac{\sqrt{T_1 T_2}}{3 C_{det}}.$$

Note that if If $C_{det} \leq DBL\_MIN$, we set $q = DBL\_MAX$.

tetrahedral condition

Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[5]
Verdict function:	v_tet_condition

6.9 Distortion

The distortion is a measure of how well-behaved the mapping from parameter space to world coordinates is. The parameter space is defined using a "master" tetrahedron with vertices

$$
\begin{array}{rlcccc}
\vec{P}_0 & = (& -1 & , & -\frac{\sqrt{3}}{3} & , & -\frac{2\sqrt{6}}{9} &) \\
\vec{P}_1 & = (& 1 & , & -\frac{\sqrt{3}}{3} & , & -\frac{2\sqrt{6}}{9} &) \\
\vec{P}_2 & = (& 0 & , & \frac{2\sqrt{3}}{3} & , & -\frac{2\sqrt{6}}{9} &) \\
\vec{P}_3 & = (& 0 & , & 0 & , & \frac{4\sqrt{6}}{9} &)
\end{array}
$$

and volume V_m. The behavior of the map is measured by sampling the determinant of the Jacobian at Gauss points $G = \{g_k\}$. The minimum of these is then used to scale the ratio of the "master" tetrahedron to the tetrahedron of interest:

$$
q = \frac{\min_k\{\det(J_{g_k})\}V_m}{V}
$$

Note that if $V < DBL\_MIN$, we set $q = DBL\_MAX$. This metric is currently unsupported.

	tetrahedral distortion
Dimension:	1
Acceptable Range:	$[0.5, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit equilateral tetrahedron:	0
Reference:	Adapted from [11]
Verdict function:	v_tet_distortion

6.10 Jacobian

This metric is defined as follows:

$$q = \left(\vec{L}_2 \times \vec{L}_0 \right) \cdot \vec{L}_3$$

	tetrahedral Jacobian
Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit equilateral tetrahedron:	$\frac{\sqrt{2}}{2}$
Reference:	[6]
Verdict function:	`v_tet_jacobian`

6.11 Minimum Angle

The (nonoriented) dihedral angle of two faces of the tetrahedron that are adjacent along edge i ($0 \leq i \leq 5$), is, measured in degrees,

$$\alpha_i = \frac{180°}{\pi} \arccos\left(\vec{n_{i1}} \cdot \vec{n_{i2}}\right),$$

where $\vec{n_{i1}}$ and $\vec{n_{i2}}$ are unit vectors normal to the two tetrahedron faces that are adjacent to edge i. Subsequently, the minimum (nonoriented) dihedral angle of the tetrahedron, measured in degrees, is

$$q = \min_{i \in \{0,1,2,3,4,5\}} \alpha_i.$$

tetrahedral minimum dihedral angle

Dimension:	A^1
Acceptable Range:	$[40°, \frac{180°}{\pi} \arccos \frac{1}{3}]$
Normal Range:	$[0°, \frac{180°}{\pi} \arccos \frac{1}{3}]$
Full Range:	$[0°, 360°]$
q for unit equilateral tetrahedron:	$\frac{180°}{\pi} \arccos \frac{1}{3} \approx 70.528779°$
Reference:	$-$
Verdict function:	v_tet_minimum_angle

6.12   Radius Ratio

This metric is commonly known as the radius ratio since it is the normalized ratio of the radius of the inscribed sphere to the radius of the circumsphere. Note that it is equal to the tetrahedral aspect β for positively-oriented tetrahedra.

The radius ratio is the quotient of these two radii normalized by $\frac{1}{3}$ so that an equilateral tetrahedron has quality of 1:

$$
\begin{aligned}
q &= \frac{R}{3r} \\
&= \frac{\left| \|\vec{L}_3\|^2 \left(\vec{L}_2 \times \vec{L}_0 \right) + \|\vec{L}_2\|^2 \left(\vec{L}_3 \times \vec{L}_0 \right) + \|\vec{L}_0\|^2 \left(\vec{L}_3 \times \vec{L}_2 \right) \right| A}{108 V^2}.
\end{aligned}
$$

Note that if $|V| < DBL\_MIN$, we set $q = DBL\_MAX$.

tetrahedral radius ratio

Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[2]
Verdict function:	v_tet_radius_ratio

6.13 Relative Size Squared

This metric measures the size of a tetrahedron relative to an ensemble containing it using volume. Take $\overline{V}$ to be the average volume of the tetrahedra in the ensemble being analyzed and define

$$R = \frac{V}{\overline{V}}$$

Then the quality is defined as

$$q = \left[\min\left(R, \frac{1}{R} \right) \right]^2.$$

Note that if $\overline{V} < DBL\_MIN$ or if $R \leq DBL\_MIN$, we set $q = 0$.

tetrahedral relative size squared	
Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit equilateral tetrahedron:	N/A
Reference:	[6]
Verdict function:	`v_tet_relative_size_squared`

6.14 Scaled Jacobian

Let J be the Jacobian as defined in §6.10

$$\lambda_1 = \left\|\vec{L}_0\right\|\left\|\vec{L}_2\right\|\left\|\vec{L}_3\right\|$$

$$\lambda_2 = \left\|\vec{L}_0\right\|\left\|\vec{L}_1\right\|\left\|\vec{L}_4\right\|$$

$$\lambda_3 = \left\|\vec{L}_1\right\|\left\|\vec{L}_2\right\|\left\|\vec{L}_5\right\|$$

$$\lambda_4 = \left\|\vec{L}_3\right\|\left\|\vec{L}_4\right\|\left\|\vec{L}_5\right\|$$

$$\lambda_{\max} = \max\left\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, J\right\}$$

$$q = \frac{J\sqrt{2}}{\lambda_{\max}}$$

Note that if $\lambda_{\max} < DBL\_MIN$, we set $q = DBL\_MAX$.

tetrahedral scaled Jacobian

Dimension:	1
Acceptable Range:	$[\frac{1}{2}, \frac{\sqrt{2}}{2}]$
Normal Range:	$[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit equilateral tetrahedron:	1
Reference:	[5]
Verdict function:	`v_tet_scaled_jacobian`

6.15 Shape

Let J be the Jacobian as defined in §6.10. We define the shape quality metric as

$$q = \frac{3\left(J\sqrt{2}\right)^{2/3}}{\frac{3}{2}\left(\vec{L}_0 \cdot \vec{L}_0 + \vec{L}_2 \cdot \vec{L}_2 + \vec{L}_3 \cdot \vec{L}_3\right) - \left(\vec{L}_0 \cdot -\vec{L}_2 + \vec{L}_0 \cdot \vec{L}_3 + -\vec{L}_2 \cdot \vec{L}_3\right)}$$

Note that if $J < DBL\_MIN$, $q = 0$. If $\frac{3}{2}\left(\vec{L}_0 \cdot \vec{L}_0 + \vec{L}_2 \cdot \vec{L}_2 + \vec{L}_3 \cdot \vec{L}_3\right) - \left(\vec{L}_0 \cdot -\vec{L}_2 + \vec{L}_0 \cdot \vec{L}_3 + -\vec{L}_2 \cdot \vec{L}_3\right) < DBL\_MIN$, we set $q = 0$.

tetrahedral shape	
Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit equilateral tetrahedron:	1
Reference:	[6]
Verdict function:	v_tet_shape

6.16   Shape and Size

Let S be the shape as defined in §6.15 and R be the relative size squared as defined in §6.13. Then the shape and size metric is

$$q = SR$$

tetrahedral shape and size	
Dimension:	1
Acceptable Range:	$[0.2, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit equilateral tetrahedron:	Dependent on $\overline{V}$
Reference:	[6]
Verdict function:	v_tet_shape_and_size

6.17   Volume

The tetrahedron volume metric is simply

$$q = V.$$

tetrahedral volume

Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[-DBL\_MAX, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit equilateral tetrahedron:	$\frac{\sqrt{2}}{12}$
Reference:	[2]
Verdict function:	`v_tet_volume`

Chapter 7

Hexahedral Quality Metrics

All the metrics in this section are defined on a hexahedral element as shown in Figure 7.1. Unless noted otherwise, hexahedra are assumed to have planar faces.
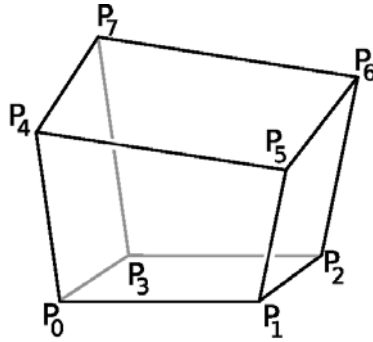
Figure 7.1: A prototypical hexahedral finite element.

We index the edges as follows. Note that order of the edges does **not** match VTK.

$$
\begin{array}{rclcrclcrcl}
\vec{L}_0 & = & \vec{P}_1 - \vec{P}_0 & \quad & \vec{L}_4 & = & \vec{P}_4 - \vec{P}_0 & \quad & \vec{L}_8 & = & \vec{P}_5 - \vec{P}_4 \\
\vec{L}_1 & = & \vec{P}_2 - \vec{P}_1 & & \vec{L}_5 & = & \vec{P}_5 - \vec{P}_1 & & \vec{L}_9 & = & \vec{P}_6 - \vec{P}_5 \\
\vec{L}_2 & = & \vec{P}_3 - \vec{P}_2 & & \vec{L}_6 & = & \vec{P}_6 - \vec{P}_2 & & \vec{L}_{10} & = & \vec{P}_7 - \vec{P}_6 \\
\vec{L}_3 & = & \vec{P}_3 - \vec{P}_0 & & \vec{L}_7 & = & \vec{P}_7 - \vec{P}_3 & & \vec{L}_{11} & = & \vec{P}_7 - \vec{P}_4
\end{array}
$$

The tetrahedron edge lengths are denoted as follows:

$$
L_0 = \left\| \vec{L}_0 \right\| \quad \ldots \quad L_{11} = \left\| \vec{L}_{11} \right\|
$$

and the largest and smallest edge lenghts are, respectively,

$$
\begin{aligned}
L_{\min} &= \min\{L_0, \dots, L_{11}\} \\
L_{\max} &= \max\{L_0, \dots, L_{11}\}.
\end{aligned}
$$

Hexahedra have four diagonal vectors:

$$
\begin{aligned}
\vec{D}_0 &= \vec{P}_6 - \vec{P}_0 & \vec{D}_2 &= \vec{P}_4 - \vec{P}_2 \\
\vec{D}_1 &= \vec{P}_7 - \vec{P}_1 & \vec{D}_3 &= \vec{P}_5 - \vec{P}_3
\end{aligned}
$$

$$
\begin{aligned}
D_{\min} &= \min\left\{\|\vec{D}_0\|, \|\vec{D}_1\|, \|\vec{D}_2\|, \|\vec{D}_3\|\right\} \\
D_{\max} &= \max\left\{\|\vec{D}_0\|, \|\vec{D}_1\|, \|\vec{D}_2\|, \|\vec{D}_3\|\right\}
\end{aligned}
$$

The principal axes are

$$
\begin{aligned}
\vec{X}_1 &= \left(\vec{P}_1 - \vec{P}_0\right) + \left(\vec{P}_2 - \vec{P}_3\right) + \left(\vec{P}_5 - \vec{P}_4\right) + \left(\vec{P}_6 - \vec{P}_7\right) \\
\vec{X}_2 &= \left(\vec{P}_3 - \vec{P}_0\right) + \left(\vec{P}_2 - \vec{P}_1\right) + \left(\vec{P}_7 - \vec{P}_4\right) + \left(\vec{P}_6 - \vec{P}_5\right) \\
\vec{X}_3 &= \left(\vec{P}_4 - \vec{P}_0\right) + \left(\vec{P}_5 - \vec{P}_1\right) + \left(\vec{P}_6 - \vec{P}_2\right) + \left(\vec{P}_7 - \vec{P}_3\right)
\end{aligned}
$$

The cross derivatives are then

$$
\begin{aligned}
\vec{X}_{12} = \vec{X}_{21} &= \left(\vec{P}_2 - \vec{P}_3\right) - \left(\vec{P}_1 - \vec{P}_0\right) + \left(\vec{P}_6 - \vec{P}_7\right) - \left(\vec{P}_5 - \vec{P}_4\right) \\
\vec{X}_{13} = \vec{X}_{31} &= \left(\vec{P}_5 - \vec{P}_1\right) - \left(\vec{P}_4 - \vec{P}_0\right) + \left(\vec{P}_6 - \vec{P}_2\right) - \left(\vec{P}_7 - \vec{P}_3\right) \\
\vec{X}_{23} = \vec{X}_{32} &= \left(\vec{P}_7 - \vec{P}_4\right) - \left(\vec{P}_3 - \vec{P}_0\right) + \left(\vec{P}_6 - \vec{P}_5\right) - \left(\vec{P}_2 - \vec{P}_1\right)
\end{aligned}
$$

We can define a series of 3×3 Jacobian matrices on a given hexahedron using the edge vectors $\vec{L}_i$ to form columns of each matrix:

$$
\begin{aligned}
A_0 &= (\quad \vec{L}_0 \quad , \quad \vec{L}_3 \quad , \quad \vec{L}_4 \quad) \\
A_1 &= (\quad \vec{L}_1 \quad , \quad -\vec{L}_0 \quad , \quad \vec{L}_5 \quad) \\
A_2 &= (\quad \vec{L}_2 \quad , \quad -\vec{L}_1 \quad , \quad \vec{L}_6 \quad) \\
A_3 &= (\quad -\vec{L}_3 \quad , \quad -\vec{L}_2 \quad , \quad \vec{L}_7 \quad) \\
A_4 &= (\quad \vec{L}_{11} \quad , \quad \vec{L}_8 \quad , \quad -\vec{L}_4 \quad) \\
A_5 &= (\quad -\vec{L}_8 \quad , \quad \vec{L}_9 \quad , \quad -\vec{L}_5 \quad) \\
A_6 &= (\quad -\vec{L}_9 \quad , \quad \vec{L}_{10} \quad , \quad -\vec{L}_6 \quad) \\
A_7 &= (\quad -\vec{L}_{10} \quad , \quad -\vec{L}_{11} \quad , \quad -\vec{L}_7 \quad) \\
A_8 &= (\quad \vec{X}_1 \quad , \quad \vec{X}_2 \quad , \quad \vec{X}_3 \quad)
\end{aligned}
$$

These matrices will be useful in calculating the volume and condition number of a hexahedron. Some operations we will need to perform on these matrices

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can be reduced to simple vector operations. Since this is how we define the matrices, these forms of the operations serve as efficient implementations. Let A be one of these 3×3 matrix defined by the column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, i.e., $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$. Then,

$$|A|^2 = \left\|\vec{v}_1\right\|^2 + \left\|\vec{v}_2\right\|^2 + \left\|\vec{v}_3\right\|^2$$

and

$$|\mathrm{adj}(A)|^2 = \left\|\vec{v}_1 \times \vec{v}_2\right\|^2 + \left\|\vec{v}_2 \times \vec{v}_3\right\|^2 + \left\|\vec{v}_3 \times \vec{v}_1\right\|^2.$$

We then define α to be the determinant of A

$$\alpha = \det(A) = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$$

and we denote the determinant of a specific one of the A_i as

$$\alpha_i = \det(A_i)$$

where $i \in \{0, 1, \ldots, 8\}$. Finally, we define normalized versions of the Jacobian matrices

$$\hat{A} = \left(\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|} , \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|} , \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|} \right)$$

and their normalized determinants

$$\hat{\alpha}_u = \det\left(\hat{A}_i\right).$$

7.1 Diagonal

This metric is the ratio of the minimum diagonal length to the maximum diagonal length:

$$q = \frac{D_{\min}}{D_{\max}}.$$

Note that if $D_{\max} < DBL\_MIN$, we set $q = DBL\_MAX$.

hexahedral diagonal

Dimension:	1
Acceptable Range:	$[0.65, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[1, DBL\_MAX]$
q for unit cube:	1
Reference:	–
Verdict function:	`v_hex_diagonal`

7.2 Dimension

This metric was specifically designed in the context of Sandia's Pronto code, for stable time step calculation. It is defined as follows:

$$q = \frac{V}{2\nabla V}$$

hexahedral dimension

Dimension:	L^1
Acceptable Range:	application-dependent
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit cube:	1
Reference:	Adapted from [12]
Verdict function:	v_hex_dimension

7.3 Distortion

Given a set of Gauss points $G = \{g_k\}$ for a hexahedron, let

$$|J| = \min_{g_k} \{\det(J_{g_k})\}$$

be the minimum determinant of the Jacobian when evaluated at each Gauss point g_k. Then the distortion is

$$q = \frac{|J| V_m}{V}$$

where $V_m = 8$ is the volume of a "master" hexahedron defined by the vertices

$$
\begin{aligned}
\vec{P_0} &= (\ -1\ ,\ -1\ ,\ -1\ ) & \vec{P_4} &= (\ -1\ ,\ -1\ ,\ 1\ ) \\
\vec{P_1} &= (\ \ 1\ ,\ -1\ ,\ -1\ ) & \vec{P_5} &= (\ \ 1\ ,\ -1\ ,\ 1\ ) \\
\vec{P_2} &= (\ \ 1\ ,\ \ 1\ ,\ -1\ ) & \vec{P_6} &= (\ \ 1\ ,\ \ 1\ ,\ 1\ ) \\
\vec{P_3} &= (\ -1\ ,\ \ 1\ ,\ -1\ ) & \vec{P_7} &= (\ -1\ ,\ \ 1\ ,\ 1\ )
\end{aligned}
$$

and V is the volume of the hexahedron being evaluated. See §7.19 for details on computing the hex volume V.

	hexahedral distortion
Dimension:	L^3
Acceptable Range:	$[0.5, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit cube:	1
Reference:	Adapted from [11]
Verdict function:	v_hex_distortion

7.4 Edge Ratio

The edge ratio quality metric is the ratio of the longest to shortest edge of a hexahedron:

$$q = \frac{L_{\max}}{L_{\min}}.$$

hexahedral edge ratio

Dimension:	1
Acceptable Range:	–
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit cube:	1
Reference:	–
Verdict function:	`v_hex_edge_ratio`

7.5   Jacobian

This is the minimum determinant of the Jacobian matrix evaluated at each corner and the center of the element:

$$q = \min\left\{ \{\alpha_i\}_{i=0}^{7}, \frac{\alpha_8}{64} \right\}.$$

This can also be interpreted as the minimum pointwise volume of local map at the 8 corners and the center of the hexahedron.

hexahedral Jacobian

Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit cube:	1
Reference:	[5]
Verdict function:	v_hex_jacobian

7.6 Maximum Edge Ratio

Given principal axes with lengths L_f and L_g, the aspect ratio is defined as the largest ratio of those lengths

$$A_{fg} = \max \left\{ \frac{L_f}{L_g}, \frac{L_g}{L_f} \right\}.$$

Since a hexahedron has 3 principal axes, we take the largest of all pairwise combinations of axes.

$$q = \max \left\{ A_{\left\| \vec{X}_1 \right\| \left\| \vec{X}_2 \right\|}, A_{\left\| \vec{X}_1 \right\| \left\| \vec{X}_3 \right\|}, A_{\left\| \vec{X}_2 \right\| \left\| \vec{X}_3 \right\|} \right\}$$

Note that if $\left\| \vec{X}_1 \right\|$ or $\left\| \vec{X}_2 \right\|$ or $\left\| \vec{X}_3 \right\| < DBL\_MIN$, we set $q = DBL\_MAX$.

hexahedral maximum edge ratio

Dimension:	1
Acceptable Range:	$[1, 1.3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit cube:	1
Reference:	Adapted from [12]
Verdict function:	`v_hex_max_edge_ratio`

7.7 Maximum Aspect Frobenius

For hexahedra, there is not a unique definition of the aspect Frobenius. Instead, we use the aspect Frobenius defined for tetrahedra (see section §6.4), but choose the reference W element to be right isosceles at the hexahedral corner. Consider the eight tetrahedra formed by edges incident to the corner of a hexahedron. Given a corner vertex i and its three adjacent vertices j, k, and ℓ ordered in a clockwise manner (so that $ijk\ell$ is a positively oriented tetrahedron), denote the tetrahedral aspect frobenius of that corner as $F_{ijk\ell}$. To obtain a single value for the metric, we take the maximum value of the eight unique tetrahedral aspects

$$q = \max \left(F_{0134}, F_{1205}, F_{2316}, F_{3027}, F_{4750}, F_{5461}, F_{6572}, F_{7643} \right).$$

In the past, this metric was called the condition number and computed in terms of the Jacobian matrices A_i and their determinants α_i as in §7. We provide that method of computation below for reference purposes. First, define

$$\kappa(A_i) = |A_i| \left| A_i^{-1} \right| = \frac{|A_i| \left| \mathrm{adj}(A_i) \right|}{\alpha_i}.$$

There are 9 of these matrices and we evaluate the condition number at each and take a third of the maximum:

$$q = \frac{1}{3} \max \left\{ \kappa(A_0), \kappa(A_1), \ldots, \kappa(A_8) \right\}$$

The first 8 matrices represent the condition at the corners and the last represents the condition number at the element's center. Note that if $\alpha_i \leq DBL\_MIN$, for any i, then $q = DBL\_MAX$.

hexahedral maximum aspect frobenius	
Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit cube:	1
Reference:	[5]
Verdict function:	v_hex_max_aspect_frobenius or v_hex_condition*

* indicates a function that is deprecated and may be removed in future versions of Verdict.

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7.8   Mean Aspect Frobenius

For hexahedra, there is not a unique definition of the aspect Frobenius. Instead, we use the aspect Frobenius defined for tetrahedra (see section §6.4), but choose the reference W element to be right isosceles at the hexahedral corner. Consider the eight tetrahedra formed by edges incident to the corner of a hexahedron. Given a corner vertex i and its three adjacent vertices j, k, and ℓ ordered in a clockwise manner (so that $ijk\ell$ is a positively oriented tetrahedron), denote the tetrahedral aspect frobenius of that corner as $F_{ijk\ell}$. To obtain a single value for the metric, we average the eight unique tetrahedral aspects

$$q = \frac{1}{8} \left(F_{0134} + F_{1205} + F_{2316} + F_{3027} + F_{4750} + F_{5461} + F_{6572} + F_{7643} \right).$$

hexahedral mean aspect frobenius

Dimension:	1
Acceptable Range:	$[1, 3]$
Normal Range:	$[1, DBL\_MAX]$
Full Range:	$[1, DBL\_MAX]$
q for unit cube:	1
Reference:	$-$
Verdict function:	v_hex_med_aspect_frobenius

7.9 Oddy

First we define the Oddy O in terms of the Jacobian matrices A_i from §7:

$$O(A_i) = \frac{\left|A_i^t A_i\right|^2 - \frac{1}{3}\left|A_i\right|^4}{\alpha_i^{\frac{4}{3}}}.$$

The metric value is then the maximum Oddy over all the corners and the element center

$$q = \max_{i \in \{0,1,\ldots,8\}} \{O(A_i)\}.$$

This can be interpreted as the maximum deviation of the metric tensor $(A_i^t A_i)$ from the identity matrix, evaluated at the corners and element center. Note that if $\alpha_i \leq DBL\_MIN$ for any i, we set $q = DBL\_MAX$.

hexahedral Oddy

Dimension:	1
Acceptable Range:	$[0, 0.5]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit cube:	0
Reference:	Adapted from [7]
Verdict function:	v_hex_oddy

7.10 Relative Size Squared

Consider the ratio D of the hex volume to the average volume of an ensemble of hexahedra:

$$D = \frac{\sum_{i=0}^{7} \alpha_i}{8\overline{V}} = \frac{\alpha_8}{64\overline{V}}.$$

The relative size the minimum of D and its inverse; and the relative size squared is

$$q = \left(\min\left\{ D, \frac{1}{D} \right\} \right)^2.$$

Note that if $\overline{V} < DBL\_MIN$ or $D \leq DBL\_MIN$, we set $q = 0$.

hexahedral relative size squared

Dimension:	1
Acceptable Range:	$[0.5, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit cube:	Dependent on $\overline{V}$
Reference:	[6]
Verdict function:	v_hex_relative_size_squared

7.11 Scaled Jacobian

This metric is the minimum determinant of the Jacobian matrix evaluated at each corner and the center of the element, divided by the corresponding edge lengths.

$$q = \min_{i \in \{0,1,\ldots,8\}} \{\hat{\alpha}_i\}.$$

Note that if $L_{\min}^2 \leq DBL\_MIN$, we set $q = DBL\_MAX$.

hexahedral scaled Jacobian

Dimension:	1
Acceptable Range:	$[0.5, 1]$
Normal Range:	$[-1, 1]$
Full Range:	$[-1, DBL\_MAX]$
q for unit cube:	1
Reference:	[5]
Verdict function:	v_hex_scaled_jacobian

7.12 Shape

The shape metric is 3 divided by the minimum mean ratio of the Jacobian matrix evaluated at the element corners:

$$q = 3 \min_{i \in \{0,1,\dots,8\}} \left\{ \frac{\alpha_i^{\frac{2}{3}}}{|A_i|^2}, \right\}.$$

Note that if $\alpha_i \leq DBL\_MIN$ or $|A_i|^2 \leq DBL\_MIN$ for any i, we set $q = 0$.

hexahedral shape

Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit cube:	1
Reference:	[6]
Verdict function:	v_hex_shape

7.13   Shape and Size

Let R be the relative size squared as defined in §7.10 and S be the shape as defined in §7.12. The "shape and size" metric is the the product of these two numbers:

$$q = RS$$

hexahedral shape and size

Dimension:	1
Acceptable Range:	$[0.2, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit cube:	Dependent on $\overline{V}$
Reference:	[6]
Verdict function:	v_hex_shape_and_size

7.14 Shear

The shear metric is the minimum of the Jacobian matrix evaluated at the element corners divided by the product of the length of the 3 edge vectors meeting at that corner:

$$q = \min_{i \in \{0,1,\ldots,8\}} \{\hat{\alpha}_i\}.$$

Note that if $\hat{\alpha}_i \leq DBL\_MIN$ for any i or if ${L_{\min}}^2 \leq DBL\_MIN$, we set $q = 0$.

hexahedral shear

Dimension:	1
Acceptable Range:	$[0.3, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit cube:	1
Reference:	[6]
Verdict function:	v_hex_shear

7.15   Shear and Size

Let R be the relative size squared as defined in §7.10 and H be the shear as defined in §7.14. The "shear and size" metric is the the product of these two numbers:

$$q = RH$$

hexahedral shear and size

Dimension:	1
Acceptable Range:	$[0.2, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, 1]$
q for unit cube:	Dependent on $\overline{V}$
Reference:	[6]
Verdict function:	`v_hex_shear_and_size`

7.16   Skew

To compute the skew, we'll need to compute normalized versions of the principal axes:

$$
\begin{aligned}
\hat{X}_1 &= \frac{\vec{X}_1}{\left\|\vec{X}_1\right\|} \\
\hat{X}_2 &= \frac{\vec{X}_2}{\left\|\vec{X}_2\right\|} \\
\hat{X}_3 &= \frac{\vec{X}_3}{\left\|\vec{X}_3\right\|}
\end{aligned}
$$

Skew measures the degree to which a pair of vectors are parallel using the dot product. This means we have three skews to consider for a hexahedron, each of which is the absolute value of the cosine of the angle between two principal axes:

$$
\begin{aligned}
skew_{12} &= \left|\hat{X}_1 \cdot \hat{X}_2\right| \\
skew_{13} &= \left|\hat{X}_1 \cdot \hat{X}_3\right| \\
skew_{23} &= \left|\hat{X}_2 \cdot \hat{X}_3\right|.
\end{aligned}
$$

The metric is then the maximum of these skews

$$
q = \max\left\{skew_{12}, skew_{13}, skew_{23}\right\}
$$

Note that if $\left\|\vec{X}_1\right\|$ or $\left\|\vec{X}_2\right\|$ or $\left\|\vec{X}_3\right\| \leq DBL\_MIN$, we set $q = DBL\_MAX$.

hexahedral skew

Dimension:	1
Acceptable Range:	$[0, 0.5]$
Normal Range:	$[0, 1]$
Full Range:	$[0, DBL\_MAX]$
q for unit cube:	0
Reference:	Adapted from [12]
Verdict function:	v_hex_skew

7.17 Stretch

The stretch is the ratio of the minimum edge length to the maximum diagonal, normalized so that a unit cube has a value of 1:

$$q = \sqrt{3}\frac{L_{\min}}{D_{\max}}.$$

Note that if $D_{\max} < DBL\_MIN$, we set $q = DBL\_MAX$.

hexahedral stretch

Dimension:	1
Acceptable Range:	$[0.25, 1]$
Normal Range:	$[0, 1]$
Full Range:	$[0, DBL\_MAX]$
q for unit cube:	1
Reference:	Adapted from [3]
Verdict function:	v_hex_stretch

7.18 Taper

Taper measures the maximum ratio of a cross-derivative to its shortest associated principal axis. Given a pair of principal axes f and g, the taper is

$$T_{fg} = \frac{\left\|\vec{X}_{fg}\right\|}{\min\left\{\left\|\vec{X}_f\right\|, \left\|\vec{X}_g\right\|\right\}}$$

The metric is then the maximum taper of any cross-derivative

$$q = \max\left\{T_{12}, T_{13}, T_{23}\right\}$$

Note that if $\left\|\vec{X}_1\right\|$ or $\left\|\vec{X}_2\right\|$ or $\left\|\vec{X}_3\right\| < DBL\_MIN$, we set $q = DBL\_MAX$.

hexahedral taper

Dimension:	1
Acceptable Range:	$[0, 0.5]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[0, DBL\_MAX]$
q for unit cube:	0
Reference:	Adapted from [12]
Verdict function:	v_hex_taper

7.19   Volume

The volume of a hexahedron is simply

$$q = \frac{\alpha_8}{64}.$$

Physically, this is the product of the magnitudes of the 3 principal axes.

	hexahedral volume
Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[0, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit cube:	2
Reference:	–
Verdict function:	`v_hex_volume`

Chapter 8

Other Element Quality Metrics

In addition to triangular, quadrilateral, tetrahedral, and hexahedral elements, Verdict also provides volume computation for other element types, namely: pyramids (with quadrilateral base), wedges, and knives, respectively illustrated in Figures 8.1, 8.2, and 8.3. Note that vtkMeshQuality does not support these element types.
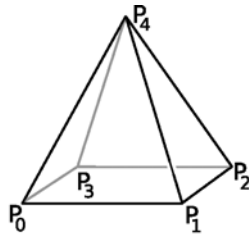

Figure 8.1: Numbering of vertices and edges on a pyramidal element.

The volume V of any of these elements is calculated by decomposing them in a tetrahedral partition, as follows:

- pyramids are divided into 2 tetrahedra,

- wedges are divided into 3 tetrahedra, and

- knives are divided into 4 tetrahedra.

Further, we define a *unit pyramid* as a pyramid whose triangular faces are equilateral triangles with unit edge length. Note that this entails that the
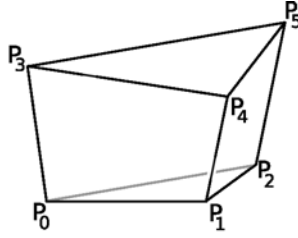
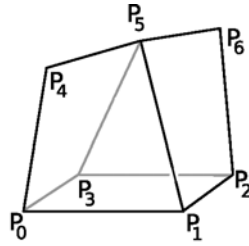Figure 8.2: Numbering of vertices and edges on a wedge element.

Figure 8.3: Numbering of vertices and edges on a knife element.

quadrilateral face is a unit square, thus making the unit pyramid a special case of regular pyramid.

Finally, we define a *unit wedge* as a wedge whose quadrilateral faces are unit squares. Note that this entails that the 2 triangular faces are equilateral triangles with unit edge length, thus making the unit wedge a special case of right triangular prism.

8.1 Pyramid Volume

The pyramid volume metric is simply

$$q = V.$$

volume

Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[-DBL\_MAX, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit element:	$\frac{1}{3\sqrt{2}}$
Reference:	$-$
Verdict function:	`v_pyramid_volume`

8.2 Wedge Volume

The wedge volume metric is simply

$$q = V.$$

volume

Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[-DBL\_MAX, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit element:	$\frac{\sqrt{3}}{4}$
Reference:	$-$
Verdict function:	`v_wedge_volume`

8.3 Knife Volume

The knife volume metric is simply

$$q = V.$$

volume

Dimension:	L^3
Acceptable Range:	$[0, DBL\_MAX]$
Normal Range:	$[-DBL\_MAX, DBL\_MAX]$
Full Range:	$[-DBL\_MAX, DBL\_MAX]$
q for unit element:	N/A
Reference:	–
Verdict function:	`v_knife_volume`

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