

Exact Recovery of Community Structures Using DeepWalk and Node2vec

Yichi Zhang and Minh Tang

Abstract—Random-walk based network embedding algorithms like DeepWalk and node2vec are widely used to obtain Euclidean representation of the nodes in a network prior to performing downstream inference tasks. However, despite their impressive empirical performance, there is a lack of theoretical results explaining their large-sample behavior. In this paper, we study node2vec and DeepWalk through the perspective of matrix factorization. In particular we analyze these algorithms in the setting of community detection for stochastic blockmodel graphs (and their degree-corrected variants). By exploiting the row-wise uniform perturbation bound for leading singular vectors, we derive high-probability error bounds between the matrix factorization-based node2vec/DeepWalk embeddings and their true counterparts, uniformly over all node embeddings. Based on strong concentration results, we further show the *perfect* membership recovery by node2vec/DeepWalk, followed by K -means/medians algorithms. Specifically, as the network becomes sparser, our results guarantee that with large enough window size and vertices number, applying K -means/medians on the matrix factorization-based node2vec embeddings can, with high probability, correctly recover the memberships of all vertices in a network generated from the stochastic blockmodel (or its degree-corrected variants). The theoretical justifications are mirrored in the numerical experiments and real data applications, for both the original node2vec and its matrix factorization variant.

Index Terms—Stochastic blockmodel, network embedding, perfect community recovery, node2vec, DeepWalk, matrix factorization.

1 INTRODUCTION

Given a network \mathcal{G} , a popular approach for analyzing \mathcal{G} is to first map or embed its vertices into some low dimensional Euclidean space and then apply machine learning and statistical inference procedures in this space. Through this embedding process, multiple tasks could be conducted on the network such as community detection (e.g., [1], [2]), link prediction (e.g., [3]), node classification (e.g., [4], [5]) and network visualization (e.g., [6]). There has been a large and diverse collection of network embedding algorithms proposed in the literature, including those based on spectral embedding [7], [8], [9], multivariate statistical dimension reduction [10], [11], and neural network [12], [13], [14]. See [15], [16], [17] and [18] for recent surveys of network embedding and graph representation learning.

In recent years there has been significant interest in network embeddings based on random-walks. The most well-known examples include DeepWalk [4] and node2vec [19]. These algorithms are computationally efficient and furthermore yield impressive empirical performance in many different scientific applications including recommendation systems [20], biomedical natural language processing [21], human protein identification [22], traffic prediction [23] and city road layout modeling [24]. Nevertheless, despite their wide-spread use, there is still a lack of theoretical results on their large-sample properties. In particular it is unclear what the node embeddings represents as well as their behavior as the number of nodes increases.

Theoretical properties for DeepWalk, node2vec, and related algorithms had been studied previously in the computer science community. The focus here had been mostly

on the convergence of the *entries* of the co-occurrence matrix as the lengths and/or number of random walks go to infinity. For example, motivated by the analysis in [25] for word2vec, the authors of [26], [27] showed that DeepWalk and node2vec using the skip-gram model with negative sampling is equivalent to factorizing a matrix whose entries are obtained by taking the entry-wise logarithm of a co-occurrence matrix, provided that the embedding dimension d is sufficiently large (possibly exceeding the number of nodes n). These authors also derived the limiting form of the entries of this matrix as the length of the random walks goes to infinity. These results were further extended in [28] to yield finite-sample concentration bounds for the co-occurrence entries. Note, however, that the above cited works focused exclusively on the case of a fixed graph and thus do not provide results on the large sample behavior of these algorithms as n increases.

The statistical community, in contrast, had extensively studied the large-sample properties of graph embeddings based on matrix factorization. However the embedding algorithms considered are almost entirely based on singular value decomposition (SVD) of either the adjacency matrix or the Laplacian matrix and its normalized and/or regularized variants. For example, in the setting of the popular stochastic blockmodel random graphs, [7] and [8] derived consistency results for a truncated SVD of the normalized Laplacian matrix and the adjacency matrix. Subsequently [29], [30] strengthened these results by providing central limit theorems for the components of the eigenvectors of either the adjacency matrix or the normalized Laplacian matrix under the more general random dot product graphs model. As DeepWalk and node2vec are based on taking the entry-wise logarithm of a random-walk co-occurrence matrix, the techniques used in these cited results do not

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readily translate to this setting.

1.1 Contributions of the current paper

The current paper studies large-sample properties of random-walk based embedding algorithms. We first present convergence results for the embeddings of DeepWalk and node2vec in the case of stochastic blockmodel graphs and their degree-correctd variant. We then show that running K -means or K -medians on the resulting embeddings is sufficient for *exact* recovery of the latent community assignments. Our theoretical results thus provide a bridge between previous results in the computer science community and their statistics counterpart.

We emphasize that our focus on stochastic blockmodel graphs is done purely for ease of exposition. Indeed, most of our results continue to hold for the more general inhomogeneous Erdős-Rényi (IER) random graphs model [31], [32], provided that the edge probabilities are sufficiently homogeneous, i.e., the minimum and maximum values for the edge probabilities are of the same order (possibly converging to 0) as n increases; recall that IER is one of the most general model for edge independent random graphs. In particular we can show that the co-occurrence matrices constructed from the sampled networks is uniformly close (entrywise) to that for the true but unknown edge probabilities matrices. However, as IER random graphs need not possess low-dimensional structure (even when n increases), it is not clear what the embeddings obtained from these co-occurrence matrices represent. See Section 6 for further discussion.

We now outline our approach. The original node2vec and DeepWalk algorithms are based on optimizing a non-convex skip-gram model using stochastic gradient descent (SGD); this optimization problem has multiple local minima and the obtained embeddings can thus be numerically unstable (see e.g., [33]). We instead consider, for each embedding dimension d , the optimal low-rank approximation of an observed transformed co-occurrence matrix similar to that used in [25], [28], and recently [27], [34]. We first show that the entries of the co-occurrence matrix computed using the observed adjacency matrix is *uniformly* close to the entries of the co-occurrence matrix computed using the true but unknown edge probabilities matrix. This uniform bound implies that the entry-wise logarithm of the two co-occurrence matrices are also *uniformly* close and thus, with high probability, the co-occurrence matrix constructed using the observed graph is well-defined. In the case of stochastic blockmodel graphs the true edge probabilities matrix give rise to a (transformed) co-occurrence matrix with rank at most K where K is the number of blocks and thus for stochastic blockmodel graphs with $K \ll n$ blocks. By leveraging both classical (e.g., the celebrated Davis-Kahan theorem [35]) as well as recent results on matrix perturbations in the $2 \rightarrow \infty$ norm (e.g., [36], [37]), we show that the truncated low-rank representation of both matrices are *uniformly* close, i.e., the embeddings of the observed graph is, up to orthogonal transformation, approximately the same as that for the true edge probabilities matrix. Therefore, by running K -means or K -medians on the embeddings of the observed graph, we can with high probability recover the latent community structures for *every* vertices.

Our paper is organized as follows. In Section 2, we give a brief introduction of node2vec [19] and DeepWalk [4], and describe the matrix factorization perspective for these algorithms. In particular, DeepWalk can be treated as a special case of node2vec by setting the 2nd-order random-walk parameters (p, q) to be $(1, 1)$, which will be assumed in Section 3 for simplicity of theoretical analysis. In Section 3 we provide uniform entry-wise error bounds for the entries of the t -step random-walk transition matrix and their implications for community recovery. The theoretical results in Section 3 hold for both the dense and sparse regimes where the average degree grows linearly and sublinearly in the number of nodes, respectively. In Section 4 we present simulations to corroborate our theoretical results. In Section 5, we apply node2vec to three real-world network datasets and show its remarkable practical performances. We conclude the paper in Section 6 with a discussion of some open questions and potential improvements. All proofs of the stated results, associated technical lemmas, and additional numerical results are provided in the Supplementary File.

1.2 Notation

We first introduce some general notations that are used throughout this paper. For a given positive integer K , we denote by $[K]$ the set $\{1, 2, \dots, K\}$. We denote a graph on n vertices by $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ where $\mathbf{V} = \{v_i\}_{i=1}^n$ and $\mathbf{E} = \{e_{ii'}\}_{i,i'=1}^n$ are the vertices and edge sets, respectively. Unless specified otherwise, all graphs in this paper are assumed to be undirected, unweighted and loop-free. For each node v_i we denote by $\mathcal{N}(v_i)$ the set of nodes $v_{i'}$ adjacent to v_i . If \mathcal{G} is a graph on n vertices then its $n \times n$ adjacency matrix is denoted as $\mathbf{A} = [a_{ii'}]$. In the subsequent discussion we often assume that the upper triangular entries of \mathbf{A} are independent Bernoulli random variables with $\mathbb{E}[a_{ii'}] = p_{ii'}$ when $i' < i$. As \mathbf{A} is symmetric we also set $a_{ii'} = a_{i'i}$ for $i' > i$ and denote by $\mathbf{P} = [p_{ii'}]$ the corresponding $n \times n$ matrix of edge probabilities.

Given a graph \mathcal{G} with adjacency matrix \mathbf{A} , let $\mathbf{D}_\mathbf{A} = \text{diag}(d_1, \dots, d_n)$ be a diagonal matrix with $d_i = \sum_{i'=1}^n a_{ii'}$ as its i th diagonal element. Assuming \mathcal{G} is connected, we define a random walk on \mathcal{G} with a 1-step transition matrix $\tilde{\mathbf{W}} = \mathbf{AD}_\mathbf{A}^{-1}$. Correspondingly, when appropriate, we also define $\mathbf{W} = \mathbf{PD}_\mathbf{P}^{-1}$ where $\mathbf{D}_\mathbf{P} = \text{diag}(p_1, \dots, p_n)$ is the diagonal matrix with $p_i = \sum_{i'=1}^n p_{ii'}$.

We use $\|\cdot\|$, $\|\cdot\|_\mathbf{F}$, $\|\cdot\|_\infty$ and $\|\cdot\|_\text{max}$ to denote the spectral norm, Frobenius norm, maximum absolute row sum, and maximum entry-wise value of a matrix, respectively. We also use $\|\cdot\|_\text{max,off}$ and $\|\cdot\|_\text{max,diag}$ to denote the maximum value for the off-diagonal and diagonal entries of a matrix, i.e., for a square matrix $\mathbf{M} = [m_{ii'}]$,

$$\|\mathbf{M}\|_\text{max,off} = \max_{i \neq i'} |m_{ii'}|, \quad \|\mathbf{M}\|_\text{max,diag} = \max_i |m_{ii}|. \quad (1.1)$$

We use $|\cdot|$ to denote the absolute value of a real number as well as the cardinality of a finite set. The vectors $\mathbf{0}_d$ and $\mathbf{1}_d \in \mathbb{R}^d$ are d dimensional vectors with all elements equal to 0 and 1, respectively. The set of $d \times d'$ matrices with orthonormal columns is denoted as $\mathbb{O}_{d,d'}$ while the set of $d \times d$ orthogonal matrices is denoted as \mathbb{O}_d .

For two terms a and b , let $a \wedge b := \min\{a, b\}$. We write $a \lesssim b$ and $a \gtrsim b$ if there exists a constant c not depending

on a and b such that $a \leq cb$ and $a \geq cb$, respectively. If $a \lesssim b$ and $a \gtrsim b$ then $a \asymp b$. We say an event \mathcal{A} depending on n happens with high probability (whp) if $\mathbb{P}(\mathcal{A}) \geq 1 - \mathcal{O}(n^{-c})$ for some constant $c > 3$. Finally, for random sequences A_n, B_n , we write $A_n = O_{\mathbb{P}}(B_n)$ if A_n/B_n is bounded whp and $A_n = o_{\mathbb{P}}(B_n)$ if $A_n/B_n \rightarrow 0$ whp.

2 SUMMARY OF NODE2VEC AND SBM

In this section we first provide a brief overview of the node2vec algorithm. We then discuss the popular stochastic blockmodel (SBM) for random graphs. Finally we discuss a matrix factorization perspective to node2vec and show that, for a graph \mathcal{G} generated from a stochastic blockmodel, this matrix factorization approach leads to a low-rank approximation of an elementwise non-linear transformation of the random walk transition matrix for \mathcal{G} .

2.1 Node2vec with negative sampling

First introduced in [19], node2vec is a computationally efficient and widely-used algorithm for network embedding. Motivated by the ideas behind word2vec for text documents [38], node2vec generates sequences of nodes using random walks which are then feed into a skip-gram model [39] to yield the node embeddings. The original skip-gram model is quite computationally demanding for large networks and hence, in practice, usually replaced by a skip-gram with negative sampling (SGNS). The resulting algorithm is summarized below.

- 1) **(Sampling Random Paths):** First generates r independent 2nd order random walks on \mathcal{G} with each having a fixed length L . A 2nd order random walk of length L starting at v_i with parameters p and q is generated as follows. First let $v_1^{(i)} = v_i$. Next sample $v_2^{(i)}$ from $\mathcal{N}(v_1^{(i)})$ uniformly at random. Then for $3 \leq \ell \leq L$, sample $v_{\ell}^{(i)} \in \mathcal{N}(v_{\ell-1}^{(i)})$ with probability,

$$\mathbb{P}(v_{\ell}^{(i)} = v_0) = \begin{cases} \frac{1}{p} J(v_0) & \text{if } v_0 = v_{\ell-2}^{(i)}, \\ J(v_0) & \text{if } v_0 \in \mathcal{N}(v_{\ell-2}^{(i)}), \\ \frac{1}{q} J(v_0) & \text{if } v_0 \notin \mathcal{N}(v_{\ell-2}^{(i)}), \end{cases}$$

where $J(v_0)$ is given by

$$\frac{1}{J(v_0)} = p^{-1} + |\mathcal{N}(v_{\ell-2}^{(i)}) \cap \mathcal{N}(v_{\ell-1}^{(i)})| + q^{-1} |\mathcal{N}(v_{\ell-2}^{(i)})^c \cap \mathcal{N}(v_{\ell-1}^{(i)})| \quad (2.1)$$

The form of $J(v_0)$ allows for $v_{\ell}^{(i)}$ to have possibly unbalanced probabilities of reaching three different types of nodes in the neighborhood of $v_{\ell-1}^{(i)}$, namely (1) the previous node $v_{\ell-2}^{(i)}$; (2) nodes belonging to both the neighborhoods of $v_{\ell-2}^{(i)}$ and $v_{\ell-1}^{(i)}$; (3) nodes belonging only to the neighborhood of $v_{\ell-1}^{(i)}$ but not the neighborhood of $v_{\ell-2}^{(i)}$. The parameters $p > 0$ and $q > 0$ provide weights for these three different type of nodes and hence control the speed at which the random walk leaves the neighborhood of the original node v_i . In this paper we assume

that the starting vertex v_i of any random walk is sampled according to a stationary distribution $\mathbf{S} = (S_1, \dots, S_n)$ on \mathcal{G} with

$$\mathbb{P}(\text{Starting Vertex is } v_i) = S_i = \frac{d_i}{2|\mathbf{E}|} \quad (2.2)$$

for all $v_i \in \mathbf{V}$. For a given $i \in [n]$ we denote by r_i the number of random walks starting from v_i , $\ell_j^{(i)}$ as the j th random walk starting from v_i and $\mathcal{L}_i = \{\ell_j^{(i)}, j \in [r_i]\}$ as the set of all random walks starting from v_i .

Remark 1. We consider only the case of $p = q = 1$ for our theoretical analysis. The choice $p = q = 1$ is the default setting for node2vec as suggested in the original paper [19] and leads to a sampling scheme equivalent to that of DeepWalk [4]; the subsequent analysis thus also applies to DeepWalk.

- 2) **(Calculating C) :** Borrowing ideas from word2vec [38], node2vec creates a $n \times n$ node-context matrix $\mathbf{C} = [C_{ii'}]_{n \times n}$ whose ii' th entry records the number of times the pair $(v_i, v_{i'})$ appears among all random paths in $\bigcup_{i=1}^n \mathcal{L}_i$. More specifically, for a given window size (t_L, t_U) , $C_{ii'}$ is the number of times that $(v_i, v_{i'})$ appears within a sequence

$$\dots, v_i, \underbrace{\dots}_{t-1 \text{ vertices}}, v_{i'}, \dots \quad \text{or} \\ \dots, v_{i'}, \underbrace{\dots}_{t-1 \text{ vertices}}, v_i, \dots \quad (2.3)$$

among all random paths in $\bigcup_{i=1}^n \mathcal{L}_i$; here t is any integer satisfying $t_L \leq t \leq t_U \leq L - 1$

Remark 2. The original node2vec algorithm fixed $t_L = 1$ while in this paper we allow for varying t_L for a more flexible theoretical analysis. In Section 3 we show that different values for (t_L, t_U) could lead to different convergence rates for the embedding and furthermore appropriate values for (t_L, t_U) depend intrinsically on the sparsity of the network.

- 3) **(Skip-gram model with negative sampling) :** Given the $n \times n$ matrix \mathbf{C} and an embedding dimension d , node2vec uses the SGNS model to learn the node embedding matrix $\mathbf{F} \in \mathbb{R}^{n \times d}$ and the context embedding matrix $\mathbf{F}' \in \mathbb{R}^{n \times d}$. The i th row of \mathbf{F} is the d -dimensional embedding vector of node v_i . In slight contrasts to the original node2vec, in this paper we do not require the constraint $\mathbf{F} = \mathbf{F}'$. The objective function of SGNS model for a given \mathbf{C} is defined as

$$\begin{aligned} g(\mathbf{F}, \mathbf{F}') = & \sum_{ij} C_{ij} \left[\log \{\sigma(\mathbf{f}_i^\top \mathbf{f}_j')\} \right. \\ & \left. + \kappa \mathbb{E}_{\mathbf{f}_{\mathcal{N}} \sim \mathbf{P}_{\text{ns}}} [\log \{\sigma(-\mathbf{f}_i^\top \mathbf{f}'_{\mathcal{N}})\}] \right]. \end{aligned} \quad (2.4)$$

Here \mathbf{f}_i (resp. \mathbf{f}'_j) are the i (resp. j) row of \mathbf{F} (resp. \mathbf{F}'), κ is the ratio of negative to positive samples,

$$\mathbf{P}_{\text{ns}}(\mathbf{f}'_{\mathcal{N}}) = \frac{\sum_{i'=1}^n C_{\mathcal{N}i'}}{\sum_{i,i'} C_{ii'}}$$

is the empirical unigram distribution for the negative samples, and σ is the logistic function. The original node2vec algorithm solves for $(\hat{\mathbf{F}}, \hat{\mathbf{F}}')$ by minimizing Eq. (2.4) over $(\mathbf{F}, \mathbf{F}')$ using SGD. In this paper we use a matrix factorization approach, described in section 2.3, to find $(\hat{\mathbf{F}}, \hat{\mathbf{F}}')$.

2.2 Stochastic blockmodel

The stochastic blockmodel (SBM) of [40] is one of the most popular generative model for network data. It often serves as a benchmark for evaluating community detection algorithms [41]. Our theoretical analysis of node2vec/DeepWalk is situated in the context of this model. We parametrize a K -blocks SBM in terms of two parameters (\mathbf{B}, \mathbf{Z}) where $\mathbf{B} = [b_{uu'}]$ is a symmetric matrix of blocks connectivity and $\mathbf{Z} \in \{0, 1\}^{n \times K}$ is a matrix whose rows denote the block assignments for the nodes; we use $\tau(i) \in [K]$ to represent the community assignment for node i , i.e., the i th row of \mathbf{Z} contains a single 1 in the $\tau_k(i)$ th element and 0 everywhere else. Given \mathbf{B} and \mathbf{Z} , the edges $a_{ii'}$ of \mathcal{G} are *independent* Bernoulli random variables with $\mathbb{P}[a_{ii'} = 1] = B_{\tau(i), \tau(i')}$, i.e., the probability of connection between i and i' depends only on the communities assignment of i and i' . Denote by

$$\mathbf{P} = [p_{ii'}] = \mathbf{Z}\mathbf{B}\mathbf{Z}^\top \quad (2.5)$$

the matrix of edge probabilities. We denote a graph with adjacency matrix \mathbf{A} sampled from a stochastic blockmodel as $\mathbf{A} \sim \text{SBM}(\mathbf{B}, \mathbf{Z})$, and, for any stochastic blockmodel graph, we denote by n_k the number of vertices assigned to block k . We shall also assume, without loss of generality, that \mathbf{Z} is ordered by blocks:

$$\mathbf{Z} := \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{n_K} \end{pmatrix}. \quad (2.6)$$

In real-world applications the average degree of a networks usually grows at a slower rate than $\Theta(n)$. To model this phenomenon we introduce a sparse parameter ρ_n that can vanish as $n \rightarrow \infty$. For ease of exposition we use the following parametrization of \mathbf{B} that is commonly used in the literature (see e.g., [42]).

Assumption 1. There exists a fixed $K \times K$ matrix \mathbf{B}_0 such that $\mathbf{B} = \rho_n \mathbf{B}_0$ with $\rho_n \gtrsim n^{-\beta}$ for some $\beta \in [0, 1)$.

The parameter ρ_n scales the edge probabilities in \mathbf{B} . As $\rho_n \gtrsim n^{-\beta}$, the average degree of the nodes in \mathcal{G} grows at rate $n^{1-\beta}$ so that larger values of β lead to sparser network. It is well known that, for sufficiently large n , if \mathcal{G} satisfies Assumption 1 then \mathcal{G} is connected with high probability (see e.g. Section 7.1 of [31]). Then $\mathbf{P} = \mathbf{Z}\mathbf{B}\mathbf{Z}^\top$ has a $K \times K$ block structure and thus has rank at most K .

2.3 Node2vec and matrix factorization

In general, for a fixed given embedding dimension $d < n$, minimization of the objective function in Eq. (2.4) leads to a non-convex optimization problem and the potential convergence of SGD into local minima makes the asymptotic analysis of $\hat{\mathbf{F}}$ quite complicated. Indeed, almost all existing

results for non-convex optimization using gradient descent or SGD only guarantees convergence to a local minima provided that the initial estimate is sufficiently close to this local minima, see e.g., [43, Section 5] and [44]. We thus desire a different approach for finding $\hat{\mathbf{F}}$, namely one for which the form of $\hat{\mathbf{F}}$ is more readily apparent. One such approach is the use of matrix factorization. For example, in the context of word2vec embedding, [25] showed that minimization of Eq. (2.4) when \mathbf{C} is a word-context matrix is equivalent to a matrix factorization problem on some *elementwise* nonlinear transformation of \mathbf{C} and that this transformation can be related to the notion of pointwise mutual information between the words. Motivated by this line of inquiry, we consider a formulation of node2vec wherein $\hat{\mathbf{F}}\hat{\mathbf{F}}^\top$ is a low-rank approximation of some elementwise transformation $\tilde{\mathbf{M}}$ of $\hat{\mathbf{W}}$; recall that $\hat{\mathbf{W}}$ is the 1-step transition matrix for the canonical random walk on \mathcal{G} . We emphasize that this approach had been considered previously in [26] and recently by [27], [34]. The main contribution of our paper is in showing that this matrix factorization leads to consistent community recovery for stochastic blockmodel graphs.

We now describe the matrix $\tilde{\mathbf{M}}$. In the context of the word2vec algorithm, [25] showed that there exists some embedding dimension d such that the minimizer of Eq. (2.4) over $\mathbf{F} \in \mathbb{R}^{n \times d}$ and $\mathbf{F}' \in \mathbb{R}^{n \times d}$ satisfies

$$\hat{\mathbf{F}}\hat{\mathbf{F}}^\top = \tilde{\mathbf{M}}(\mathbf{C}, \kappa) := \left[\log \frac{C_{ij}(\sum_i C_{ij})}{\kappa(\sum_i C_{ij})(\sum_j C_{ij})} \right]_{n \times n} \quad (2.7)$$

Using the same idea for our analysis of node2vec, we first fix n and show that if the number of sampled random paths increases then $\tilde{\mathbf{M}}(\mathbf{C}, k)$ converges, elementwise, to a limiting matrix $\tilde{\mathbf{M}}_0$ defined below. Note that the entries of $\tilde{\mathbf{M}}_0$ can be interpreted as point-wise mutual information (PMI) between the nodes.

Theorem 1. Let n be fixed but arbitrary. Suppose \mathcal{G} is a connected graph on n vertices and t_U is large enough such that the entries of $\sum_{t=t_L}^{t_U} \hat{\mathbf{W}}^t$ are all positive. Applying the node2vec sampling strategy introduced in Section 2.1 on \mathcal{G} we have

$$\begin{aligned} \tilde{\mathbf{M}}(\mathbf{C}, \kappa) &\xrightarrow{a.s.} \tilde{\mathbf{M}}_0(\mathcal{G}, t_L, t_U, \kappa, L) \\ &:= \log \left\{ \frac{2|\mathbf{A}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t)\mathbf{D}_\mathbf{A}^{-1}\hat{\mathbf{W}}^t \right\} \end{aligned} \quad (2.8)$$

as the number of random paths $r = \sum_{i=1}^n r_i \rightarrow \infty$; recalling that $\hat{\mathbf{W}} = \mathbf{AD}_\mathbf{A}^{-1}$. The convergence of $\tilde{\mathbf{M}}(\mathbf{C}, \kappa)$ to $\tilde{\mathbf{M}}_0$ is elementwise and uniform over all entries of $\tilde{\mathbf{M}}(\mathbf{C}, \kappa)$. Here $|\mathbf{A}|$ denote the sum of the entries in \mathbf{A} and the constant γ is defined as

$$\gamma := \frac{1}{2}(L - t_L - t_U)(t_U - t_L + 1).$$

To reduce notation clutter, we will henceforth drop the dependency of $\tilde{\mathbf{M}}_0$ on the parameters $\mathcal{G}, t_L, t_U, \kappa, L$. As the value of r is chosen purely for computational expediency, i.e., smaller values of r require sampling fewer random walks, we will thus take the conceptual view that $r \rightarrow \infty$ so that $\tilde{\mathbf{M}}(\mathbf{C}, \kappa) \rightarrow \tilde{\mathbf{M}}_0$; note that $\tilde{\mathbf{M}}_0$ can be constructed explicitly from \mathbf{A} without needing to sample any random walk. Combining Eq. (2.7) and Theorem 1, we have that, for any fixed n , there exists an embedding dimension d such that for $r \rightarrow \infty$, the matrices $\hat{\mathbf{F}}$ and $\hat{\mathbf{F}}'$ are exact factors

for factorizing $\tilde{\mathbf{M}}_0$. Note that $\mathbf{D}_A^{-1}\hat{\mathbf{W}}^t$ is symmetric for any $t \geq 1$ and hence \mathbf{M}_0 is *symmetric*.

In practice one usually chooses $d \ll n$ to reduce the noise in the embeddings as well as combat the curse of dimensionality in downstream inference. Obviously if $d < n$ then exact factors $(\hat{\mathbf{F}}, \hat{\mathbf{F}}')$ for factorizing $\tilde{\mathbf{M}}_0$ might no longer exist (see e.g., [25]). The requirement that $\hat{\mathbf{F}}\hat{\mathbf{F}}'^\top = \tilde{\mathbf{M}}_0$ is, however, both misleading and unnecessary. Indeed, as the observed graph is but a single *noisy* sample generated from some true but unobserved edge probabilities matrix \mathbf{P} , what we really want to recover is the factorization induced by \mathbf{P} . More specifically, replacing $\hat{\mathbf{W}}^t$ and $|\mathbf{A}|$ with \mathbf{W}^t and $|\mathbf{P}|$ in $\tilde{\mathbf{M}}_0$, we define

$$\mathbf{M}_0 = \log \left\{ \frac{2|\mathbf{P}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t) \mathbf{D}_P^{-1} \mathbf{W}^t \right\} \quad (2.9)$$

as the underlying-truth counterpart of $\tilde{\mathbf{M}}_0$; note that, similar to $\tilde{\mathbf{M}}_0$, we had dropped the parameters associated with \mathbf{M}_0 for simplicity of notations. Under the SBM setting, the true signal matrices \mathbf{P} and \mathbf{M}_0 are both low-rank and hence an embedding dimension of $d = \text{rk}(\mathbf{M}_0) \ll n$ is sufficient to recover the factorization induced by \mathbf{M}_0 .

To be more precise, recall from Eq. (2.6) that for stochastic blockmodel graphs, the matrix \mathbf{P} has a $K \times K$ block structure. Thus both \mathbf{W}^t and $\mathbf{D}_P^{-1}\mathbf{W}^t$ also have $K \times K$ block structures. Eq. (2.9) then implies that \mathbf{M}_0 also has a $K \times K$ block structure and hence $\text{rank}(\mathbf{M}_0) \leq K$. Most importantly, the $K \times K$ block structure of \mathbf{M}_0 is also sufficient for recovering the community structure in \mathcal{G} . We will show in Section 3 that the relative error, in the *row-wise maximum* norm, between $\tilde{\mathbf{M}}_0$ and \mathbf{M}_0 converges to 0 as $n \rightarrow \infty$. This convergence, together with results for perturbation of eigenspaces, implies the existence of an embedding dimension $d \leq K$ for which the $n \times d$ matrices $\hat{\mathbf{F}}$ and $\hat{\mathbf{F}}'$ obtained by factorizing $\tilde{\mathbf{M}}_0$ lead to exact recovery of the community structure in \mathcal{G} .

Remark 3. If \mathbf{P} does not arise from a stochastic blockmodel graph then \mathbf{M}_0 need not have a low-rank structure. Nevertheless we can still consider a rank- d approximation to \mathbf{M}_0 for some $d < \text{rk}(\mathbf{M}_0)$. Furthermore, as we will clarify in Section 6, the bound for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max}$ in Section 3 also holds for general edge independent random graphs, provided that the entries of \mathbf{P} is reasonably homogeneous. Hence $\tilde{\mathbf{M}}_0$ has an approximate low-rank structure if and only if \mathbf{M}_0 also has an approximate low-rank structure.

In summary, motivated by the low-rank structure of \mathbf{M}_0 in the case of SBM graphs, we view the matrix factorization approach for node2vec as finding the best rank $d < n$ approximation $\hat{\mathcal{F}} \cdot \hat{\mathcal{F}}'^\top$ to $\tilde{\mathbf{M}}_0$ under Frobenius norm, i.e.,

$$(\hat{\mathcal{F}}, \hat{\mathcal{F}}') = \arg \min_{(\mathcal{F}, \mathcal{F}') \in \mathbb{R}^{n \times d} \cdot \mathbb{R}^{n \times d}} \|\tilde{\mathbf{M}}_0 - \mathcal{F} \cdot \mathcal{F}'^\top\|_{\text{F}}. \quad (2.10)$$

The minimizer of Eq. (2.10) is obtained by truncating the SVD of $\tilde{\mathbf{M}}_0$. More specifically, let

$$\tilde{\mathbf{M}}_0 = \hat{\mathbf{U}} \hat{\Sigma} \hat{\mathbf{V}}^\top \quad (2.11)$$

with a decreasing order of singular values in $\hat{\Sigma}$. Then for a given $d \leq \text{rk}(\mathbf{M}_0)$, let

$$\hat{\mathcal{F}} = \hat{\mathbf{U}}_d, \quad \hat{\mathcal{F}}' = \hat{\mathbf{V}}_d \hat{\Sigma}_d \quad (2.12)$$

where $\hat{\mathbf{U}}_d \in \mathbb{R}^{n \times d}$, $\hat{\mathbf{V}}_d \in \mathbb{R}^{n \times d}$ are the first d columns of $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$, respectively, and $\hat{\Sigma}_d \in \mathbb{R}^{d \times d}$ is the diagonal matrix containing the d largest singular values in $\hat{\Sigma}$.

Remark 4. The appropriate embedding dimension d for factorizing $\tilde{\mathbf{M}}_0$ depends on knowing $\text{rank}(\mathbf{M}_0)$, but the convergence of $\tilde{\mathbf{M}}_0$ to that of \mathbf{M}_0 does not require knowing $\text{rank}(\mathbf{M}_0)$. For ease of exposition we will assume that $\text{rank}(\mathbf{M}_0)$ is known; in practice it can be estimated consistently using an eigenvalue thresholding procedure provided that \mathbf{M}_0 has a low-rank structure. Finally, in the context of SBM graphs and their degree-corrected variant, community recovery using $\hat{\mathcal{F}}$ also depends on knowing K . For simplicity we also assume that K is known, noting that consistent estimates for K are provided in [45], [46].

3 THEORETICAL ANALYSIS

3.1 Entry-wise concentration of $\hat{\mathbf{W}}^t$ and $\tilde{\mathbf{M}}_0$

Recall that $\hat{\mathcal{F}}$ is obtained from the eigendecomposition of $\tilde{\mathbf{M}}_0$ while the true embedding is obtained from the eigendecomposition of \mathbf{M}_0 (see Eq. (2.9)). Therefore, before studying the community recovery using $\hat{\mathcal{F}}$, we first study the convergence of $\tilde{\mathbf{M}}_0$ to \mathbf{M}_0 . In particular we derive concentration bounds for $\tilde{\mathbf{M}}_0 - \mathbf{M}_0$ in both Frobenius and infinity norms. These bounds are facilitated by the following Theorem 2 which provides a precise uniform bound for the entry-wise difference between the t -step transition matrix $\hat{\mathbf{W}}^t$ and \mathbf{W}^t defined using the adjacency matrix \mathbf{A} and the edge probabilities matrix \mathbf{P} , respectively.

Theorem 2. Let $\mathcal{G} \sim \text{SBM}(\mathbf{B}, \mathbf{Z})$ where \mathbf{B} satisfies Assumption 1. We then have the following bounds.

- 1) (*Dense regime*) Suppose $\rho_n \asymp 1$. Then

$$\|\hat{\mathbf{W}} - \mathbf{W}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-1}), \quad (3.1)$$

$$\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{diag}} = \mathcal{O}_{\mathbb{P}}(n^{-1}), \quad (3.2)$$

$$\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{3/2}}\right), \quad (3.3)$$

Furthermore, for $t \geq 3$,

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{3/2}}\right), \quad (3.4)$$

- 2) (*Sparse regime*) Let $\rho_n \rightarrow 0$ with $\rho_n \gtrsim n^{-\beta}$ for some $\beta \in [0, 1]$. Then for $t \geq 4$ satisfying $\frac{t-3}{t-1} > \beta$ we have

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{3/2}\rho_n^{1/2}}\right). \quad (3.5)$$

In addition if $0 \leq \beta < 1/2$ then

$$\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{3/2}\rho_n}\right), \quad (3.6)$$

$$\|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{3/2}\rho_n}\right).$$

Remark 5. Throughout this paper we assume that $t_L \geq 2$ instead of $t_L \geq 1$ as used in the original node2vec formulation. The rationale for this assumption is as follows. Recall the definition of $\tilde{\mathbf{M}}_0$ in Eq. (2.8). If we allow t to starts from 1 in the sum $\sum_{t=t_L}^{t_U} (L-t) \cdot (\mathbf{D}_A^{-1}\hat{\mathbf{W}}^t)$ then the term $\hat{\mathbf{W}}$

might lead to a convergence rate of $\tilde{\mathbf{M}}_0$ to \mathbf{M}_0 that is slower than that given in Eq. (3.7). For example in the dense regime Eq. (3.1) and Eq. (3.3) show that the entries of $\hat{\mathbf{W}} - \mathbf{W}$ are of larger magnitude than the entries of $\hat{\mathbf{W}}^t - \mathbf{W}^t$ for $t \geq 2$.

Before discussing the convergence rate of $\tilde{\mathbf{M}}_0$ to \mathbf{M}_0 we first find a value of t_U such that, for large values of n , $\tilde{\mathbf{M}}_0$ is well defined with high probability. We note that the entries of $\{\mathbf{W}^t\}_{t \geq 1}$ are uniformly of order $\Theta(n^{-1})$. Then, under the dense regime, $t = 2$ is sufficient to guarantee that all the off-diagonal entries of $\hat{\mathbf{W}}^t$ are uniformly of order $\Omega(n^{-1} - n^{-3/2} \log^{1/2} n) = \Omega(n^{-1})$ with high probability (c.f. Eq. (3.2)) while $t = 3$ is sufficient to guarantee that all entries of $\hat{\mathbf{W}}^t$ are of order $\Omega(n^{-1})$ with high probability (c.f. Eq. (3.3)). If we are under the sparse regime with $\beta < 1/2$ then these same values of $t \geq 2$ are still sufficient to guarantee that the entries of $\hat{\mathbf{W}}^t$ are of order $\Omega(n^{-1})$ (c.f. Eq. (3.5) and Eq. (3.6)). Finally, if we are under the sparse regime with $\beta \geq 1/2$ then choosing $t \geq 4$ with $\frac{t-3}{t-1} > \beta$ is sufficient to guarantee that the entries $\hat{\mathbf{W}}^t$ are uniformly of order $\Omega(n^{-1} - n^{-3/2} \rho_n^{-1/2} \log^{1/2} n) = \Omega(n^{-1})$ with high probability. Now recall that the matrix $\tilde{\mathbf{M}}_0$ is of the form

$$\log \left\{ \frac{2|\mathbf{A}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t)\mathbf{D}_A^{-1}\hat{\mathbf{W}}^t \right\}$$

We therefore have, for $t_U \geq 3$ in the dense regime, $t_U \geq 2$ in the not too sparse regime of $\beta < 1/2$, or for $\frac{t_U-3}{t_U-1} > \beta$ in general, that the entries of the inner sum are bounded away from 0 with high probability. For the dense regime, the condition can be further relaxed to $t_U \geq 2$, as a dense graph has a diameter of 2 and thus all entries of $\hat{\mathbf{W}}^2$ are uniformly larger than 0 with high probability; see Theorem 10.10 in [31]. Therefore, with high probability, the elementwise logarithm is well-defined for all entries of $\tilde{\mathbf{M}}_0$. Given the existence of $\tilde{\mathbf{M}}_0$, the following result shows the convergence rate of $\tilde{\mathbf{M}}_0$ to \mathbf{M}_0 .

Theorem 3. Suppose $\mathcal{G} \sim \text{SBM}(\mathbf{B}, \Theta)$ satisfies Assumption 1, and $t_U \geq t_L \geq 2$ where t_L is chosen as described above. Then $\tilde{\mathbf{M}}_0$ is well-defined with high probability. Denote

$$\Delta = \max\{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F, \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_\infty\}.$$

We then have the following bounds.

- 1) **(Dense regime)** Let $\rho_n \asymp 1$. Then for $t_L \geq 2$ we have

$$\Delta = \mathcal{O}_{\mathbb{P}}(n^{1/2} \log^{1/2} n). \quad (3.7)$$

- 2) **(Sparse regime)** Let $\rho_n \rightarrow 0$ with $\rho_n \gtrsim n^{-\beta}$ for some $\beta \in [0, 1)$. Then for t_L satisfying $\frac{t_L-3}{t_L-1} > \beta$ we have

$$\Delta = \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1/2} \log^{1/2} n). \quad (3.8)$$

In addition if $0 \leq \beta < 1/2$ then for $t_L \geq 2$ we have

$$\Delta = \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1} \log^{1/2} n). \quad (3.9)$$

In both regimes we have $\|\mathbf{M}_0\|_F = \Theta(n)$ and $\|\mathbf{M}_0\|_\infty = \Theta(n)$.

Theorem 3 indicates that as β increases (equivalently, as ρ_n decreases) so that the graph \mathcal{G} becomes sparser, we could (1) still guarantee the existence of $\tilde{\mathbf{M}}_0$ when t_U is sufficiently large, and (2) control the convergence rate of $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$ relative to $\|\mathbf{M}_0\|_F$ by increasing t_L .

3.2 Subspace perturbations and exact recovery

Theorem 3 implies that $\tilde{\mathbf{M}}_0$ is close to \mathbf{M}_0 under both Frobenius and infinity norms, i.e., $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_\star/\|\mathbf{M}_0\|_\star = o_{\mathbb{P}}(1)$ for $\star \in \{F, \infty\}$ and sufficiently large n . Now, by Eq. (2.6), \mathbf{M}_0 has a $K \times K$ block structure and hence $\text{rk}(\mathbf{M}_0) \leq K$. Furthermore the eigenvectors of \mathbf{M}_0 associated with its non-zero eigenvalues is sufficient for recovering the community assignments induced by \mathbf{Z} . The following result, which follows from bounds for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_\infty$ given in Theorem 3 together with perturbations bounds for invariant subspaces using $2 \rightarrow \infty$ norm [36], shows that the embedding $\hat{\mathcal{F}}$ given by the leading eigenvectors of $\tilde{\mathbf{M}}_0$ is *uniformly* close to that of the leading eigenvectors of \mathbf{M}_0 . Therefore K -means or K -medians clustering on the rows of $\hat{\mathcal{F}}$ will recover the community membership for *every* nodes, i.e, attain strong or exact recovery of \mathbf{Z} .

Theorem 4. Under the condition of Theorem 3, let $\hat{\mathbf{U}}\hat{\Sigma}\hat{\mathbf{U}}^\top$ and $\mathbf{U}\Sigma\mathbf{U}^\top$ be the eigen-decomposition of $\tilde{\mathbf{M}}_0$ and \mathbf{M}_0 , respectively. Let $d = \text{rk}(\mathbf{M}_0)$ and note that \mathbf{U} is a $n \times d$ matrix. Let $\hat{\mathcal{F}} = \hat{\mathbf{U}}_d$ be the matrix formed by the columns of $\hat{\mathbf{U}}$ corresponding to the d largest-in-magnitude eigenvalues of \mathbf{M}_0 . For a $n \times d$ matrix \mathbf{Z} with rows Z_1, Z_2, \dots, Z_n let $\|\mathbf{Z}\|_{2 \rightarrow \infty}$ denote the maximum ℓ_2 norms of the $\{Z_i\}$, i.e.,

$$\|\mathbf{Z}\|_{2 \rightarrow \infty} = \max_i \|Z_i\|_2.$$

We then have the following results.

- (i) **(Dense regime)** Let $\rho_n \asymp 1$. Then for $t_L \geq 2$ we have

$$\begin{aligned} \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_F &= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{1/2}}\right) \\ \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} &= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n}\right). \end{aligned} \quad (3.10)$$

- (ii) **(Sparse regime)** Let $\rho_n \rightarrow 0$ with $\rho_n \gtrsim n^{-\beta}$ for some $\beta \in [0, 1/2)$. If $t_L \geq 2$, we have

$$\begin{aligned} \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_F &= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{1/2} \rho_n}\right) \\ \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} &= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n \rho_n}\right) \end{aligned} \quad (3.11)$$

- (iii) **(Sparse regime)** Let $\rho_n \rightarrow 0$ with $\rho_n \gtrsim n^{-\beta}$ for some $\beta \in [0, 1)$. If $\frac{t_L-3}{t_L-1} > \beta$, we have,

$$\begin{aligned} \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_F &= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{(n \rho_n)^{1/2}}\right) \\ \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} &= \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n \rho_n^{1/2}}\right). \end{aligned} \quad (3.12)$$

Given the above convergence rates, clustering the rows of $\hat{\mathcal{F}}$ using either K -means or K -medians will, with high probability, recover the memberships of every nodes in \mathcal{G} .

Remark 6. Settings (ii) and (iii) in Theorem 4 both consider the sparse regime but setting (ii) focuses on the case where $\rho_n = \omega(n^{-1/2})$ and exact recovery is achieved whenever $t_L \geq 2$ while setting (iii) considers the more general scenario of $\rho_n = \omega(n^{-\beta})$ for any fixed but arbitrary $\beta < 1$. We note

that for ease of exposition we had impose $\frac{t_L-3}{t_L-1} > \beta$ for setting (iii) but this condition can be relaxed to

$$\frac{t_L-2}{t_L} > \beta, \quad (3.13)$$

under which we still have $\tilde{\mathbf{M}}_0$ is well-defined with high probability, and have a more complicated bound of

$$\min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}}\mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \lesssim \mathcal{O}_{\mathbb{P}} \left\{ \frac{\log^{1/2} n}{n^{3/2} \rho_n^{1/2}} + (n \rho_n)^{-t_L/2} \right\}$$

(see (B.71)). The above bound is still sufficient to guarantee that running K -means or K -medians on the rows of $\hat{\mathcal{F}}$ will recover the memberships of every nodes in \mathcal{G} with high probability; see Section B.4 in the Supplementary File for a rigorous proof.

A recent preprint [34] which appeared on arXiv after the first version of our paper also studied community recovery using SVD-based DeepWalk/node2vec and they have a similar requirement for t_L as Eq. (3.13); see (3.1) in [34]. For comparison we note that [34] only derived the convergence rate of $\hat{\mathcal{F}}$ under Frobenius norm, and thereby prove a *weak* recovery result which allows at most $o(n^{1/2})$ nodes to be misclassified. In contrast the max-norm concentration of $\hat{\mathbf{W}}$ in Theorem 2 helps us derive a $2 \rightarrow \infty$ norm convergence for $\hat{\mathcal{F}}$, based on which we achieved the much stronger exact recovery (i.e., there are no mis-classified nodes). Finally we conjecture that Eq. (3.13) for t_L is sufficient but not necessary. Our simulation results in Section 4 agree with this conjecture and we leave its verification for future work.

Remark 7. The exact recovery results in Theorem 4 can also be extended to the case of degree-corrected SBM graphs [47], [48], [49]. Recall that the edge probabilities for a DCSBM is $\mathbf{P} = \Theta \mathbf{Z} \mathbf{B} \mathbf{Z}^T \Theta$ where $\Theta = \text{diag}(\theta_1, \dots, \theta_n)$ are the degree-correction parameters. DCSBM allows heterogeneous edge probabilities within each community and thus yields a more flexible model in comparison with SBM. Section A.4 and B.5 in the Supplementary File demonstrates how to extend the technical derivations for Theorem 4 to the DCSBM case provided that the $\{\theta_i\}$ are sufficiently homogeneous, i.e., that $\max_i \theta_i / \min_i \theta_i = \mathcal{O}(1)$.

4 SIMULATION

We now present simulations experiments for the matrix factorization perspective of node2vec/DeepWalk. These experiments complement our theoretical results in Section 3 and illustrate the interplay between the sparsity of the graphs, the choice of window sizes, and their combined effects on the nodes embedding.

4.1 Error bounds for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$

We first compare the large-sample empirical behavior of $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$ against the theoretical bounds given in Theorem 3. We shall simulate undirected graphs generated from a 2-blocks SBM with parameters

$$\mathbf{B}(\rho_n) := \begin{pmatrix} 0.8\rho_n & 0.3\rho_n \\ 0.3\rho_n & 0.8\rho_n \end{pmatrix}, \quad \boldsymbol{\pi} = (0.4, 0.6), \quad (4.1)$$

and sparsity $\rho_n \in \{1, 3n^{-1/3}, 3n^{-1/2}, 3n^{-2/3}\}$. While this two blocks setting is quite simple it nevertheless displays

the effect of the sparsity ρ_n and the window size (t_L, t_U) on the upper bound for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$.

For each value of n and sparsity ρ_n we run 100 independent replications where, in each replicate, we generate $\mathcal{G} \sim \text{SBM}(\mathbf{B}(\rho_n), \Theta_n)$ and calculate $\tilde{\mathbf{M}}_0$ for different choices of (t_L, t_U) . In particular we consider two types of window size, namely $t_U = t_L + 1$ and $t_U = t_L + 3$. While $t_U = t_L + 1$ is not commonly used in practice, for simulation purpose this choice clearly show the effects of the random walks' length t on the error $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$. In contrast the choice $t_U = t_L + 3$ is more realistic but also partially obfuscate the effect of t on $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$. Recall that, from the discussion prior to Theorem 3, sparser values of ρ_n requires larger values of t_U to guarantee that $\tilde{\mathbf{M}}_0$ is well-defined. The choices for $(\rho_n, n, (t_L, t_U))$ in the simulations are summarized below.

- If $\rho_n \geq 3n^{-1/2}$ then $n \in \{100, 200, 300, \dots, 1500\}$. We chose $2 \leq t_L \leq 7$ when $t_U = t_L + 1$ and chose $2 \leq t_L \leq 5$ when $t_U = t_L + 3$.
- If $\rho_n = 3n^{-2/3}$ then $n \in \{800, 900, \dots, 4000\}$. We chose $4 \leq t_L \leq 7$ when $t_U = t_L + 1$ and $3 \leq t_L \leq 5$ when $t_U = t_L + 3$.

We calculate two relative error criteria for $\tilde{\mathbf{M}}_0$, namely

$$\varepsilon_1(\tilde{\mathbf{M}}_0) = \frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F}{\|\mathbf{M}_0\|_F} \text{ and } \varepsilon_2(\tilde{\mathbf{M}}_0) = \frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F}{n^{1/2} \rho_n^{-1/2} \log^{1/2} n}.$$

We expect that, as n increases, the first criteria converges to 0 while the second criteria remains bounded.

Relative Error 1: We first confirm the convergence of $\varepsilon_1(\tilde{\mathbf{M}}_0)$ to 0. Figures 1 and D1 shows the means and 95% confidence intervals for $\varepsilon_1(\tilde{\mathbf{M}}_0)$ based on 100 Monte Carlo replicates for different values of $\rho_n, (t_L, t_U)$. These figures indicate the following general patterns as predicted by the theoretical results in Theorem 3.

- The error $\varepsilon_1(\tilde{\mathbf{M}}_0)$ is smallest in the dense case and deteriorates as the sparsity factor ρ_n decreases.
- The error also depends on (t_L, t_U) with larger values of $t_U - t_L$ leading to smaller $\varepsilon_1(\tilde{\mathbf{M}}_0)$.
- If the window size is too small, e.g., $(t_L, t_U) = (2, 3)$ or $(t_L, t_U) = (2, 5)$, then $\tilde{\mathbf{M}}_0$ is often times not well-defined.

Relative Error 2: Figures 1 and Figure D1 (see the Supplementary File) corroborate our theoretical results in Section 3. Nevertheless there are two additional questions we should consider. The first is whether or not the bound $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F = \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1/2} \log^{1/2} n)$ in Theorem 3 is tight and, if it is tight, the second is whether or not the condition $\frac{t_L-3}{t_L-1} > \beta$ is necessary to achieve this rate. Analogous to the previous two figures, Figures 2 and D2 show the means and 95% empirical confidence intervals for the relative error $\varepsilon_2(\tilde{\mathbf{M}}_0)$ over 100 Monte Carlo replicates for different values of ρ_n and (t_L, t_U) . From these simulations we can answers the above questions as follows.

- If $\rho_n \gtrsim n^{-\beta}$ is such that $\beta \leq \frac{t_L-3}{t_L-1}$ then $\varepsilon_2(\tilde{\mathbf{M}}_0)$ appears to converge to a constant as n increases. There is thus evidence that the rate $n^{1/2} \rho_n^{-1/2} \log^{1/2} n$ for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$ is optimal. Nevertheless if t_L is large relative to ρ_n , e.g., $\rho_n \in \{3n^{-1/3}, 3n^{-1/2}\}$ and

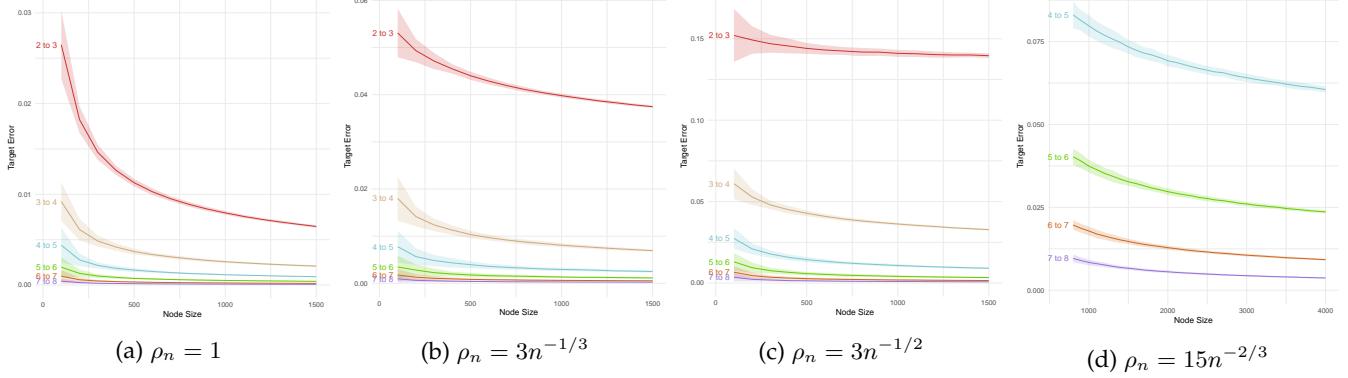


Fig. 1: Sample means and 95% empirical confidence intervals for $\varepsilon_1(\tilde{\mathbf{M}}_0)$ based on 100 Monte Carlo replicates for different values of n , ρ_n and (t_L, t_U) with $t_U - t_L = 1$.

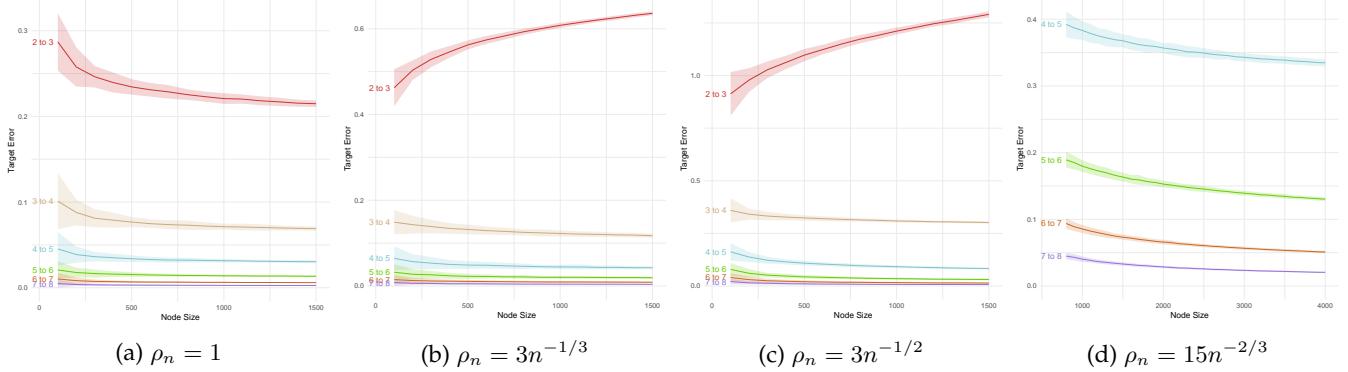


Fig. 2: Sample means and 95% empirical confidence intervals for $\varepsilon_2(\tilde{\mathbf{M}}_0)$ based on 100 Monte Carlo replicates for different values of n , ρ_n , and (t_L, t_U) with $t_U - t_L = 1$.

$t_L \geq 6$, then $\varepsilon_2(\tilde{\mathbf{M}}_0)$ appears to converge to 0 which suggests that for a fixed β the error rate for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$ can be smaller than $n^{1/2}\rho_n^{-1/2} \log^{1/2} n$; this might be due to the convergence of $\hat{\mathbf{W}}^t$ and \mathbf{W}^t towards the stationary distributions as t increases.

- For cases such as $(t_L, t_U) \in \{(3, 4), (3, 6)\}$ and $\rho_n = 3n^{-1/2}$ or $(t_L, t_U) \in \{(4, 5), (3, 6)\}$ and $\rho_n = 15n^{-2/3}$, the t_L 's do not satisfy $\frac{t_L-3}{t_L-1} > \beta$. Nevertheless $\varepsilon_2(\tilde{\mathbf{M}}_0)$ still appears to converge to a constant as n increases. This suggests that $\frac{t_L-3}{t_L-1} > \beta$ is sufficient but possibly not necessary for the bound in Eq. (3.8) to hold. On the other hand, for fixed n and ρ_n , the error $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$ generally decreases as $t_U - t_L$ increases.
- Finally if $(t_L, t_U) \in \{(2, 3), (2, 5)\}$ and $\rho_n \in \{3n^{-1/3}, 3n^{-1/2}\}$ then $\varepsilon_2(\tilde{\mathbf{M}}_0)$ increases with n . This supports the claim in Theorem 3 of a phase transition for the error rate of $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F$ as t_L increases.

In summary Figure 1 through Figure D2 supports the conclusion of Theorem 3. In particular the error rate in Theorem 3 is sharp and the condition $\frac{t_L-3}{t_L-1} > \beta$ is sufficient but perhaps not necessary.

4.2 Exact recovery of community structure

Theorem 4 together with Remark 6 showed that $\hat{\mathcal{F}}$ combined with K -means/medians can correctly recover the

memberships of all nodes in a SBM with high probability. We demonstrate this result for two-blocks SBMs with block probabilities being either $\mathbf{B}(\rho_n)$ as given in Eq. (4.1) or

$$\mathbf{B}^\natural(\rho_n) := \begin{pmatrix} 0.3\rho_n & 0.8\rho_n \\ 0.8\rho_n & 0.3\rho_n \end{pmatrix}.$$

Note that $\mathbf{B}(\rho_n)$ and $\mathbf{B}^\natural(\rho_n)$ corresponds to an assortative and a dis-assortative structure, respectively. Given specific setting of \mathbf{B} , n , ρ_n , we randomly sample 100 graphs where each vertex is randomly assigned to one of the two blocks with equal probability and evaluate the membership recovery performances of the original node2vec [19] (based on SGD) and node2vec using matrix factorization (as described [26], [27], [34] and this paper) followed by clustering using K -means. We set the window sizes to $t_U \in \{5, 8\}$ and choose $\kappa = 5$ and $L = 200$. For the original node2vec we also set $t_L = 1$ as the default and $r_1 = \dots = r_n = 200$, while for the SVD-based node2vec we set $t_L = t_U - 3$. We report in Tables 1 and 2 the proportions of times for the 100 simulated graphs that these two variants of the node2vec algorithm correctly recover the memberships of all nodes.

The numerical results show that as n increases, both the original and SVD-based node2vec are more likely to perfectly recover memberships of all nodes in the graph, under all different settings of ρ_n, \mathbf{B}, t_U . Furthermore the accuracy when $\rho_n = 3n^{-1/3}$ is considerably higher than that for $\rho_n = 3n^{-1/2}$. This is consistent with the results

n	SVD-based node2vec		Original node2vec	
	$t_U = 5$	$t_U = 8$	$t_U = 5$	$t_U = 8$
600	1.00	1.00	1.00	1.00
900	1.00	1.00	1.00	1.00
1500	1.00	1.00	1.00	1.00

TABLE 1: Proportions of times that SGD-based and SVD-based node2vec variants perfectly recover all nodes memberships. The graphs are generated from $\mathbf{B}(\rho_n)$ with sparsity $\rho_n = 3n^{-1/3}$ (left table) and $\rho_n = 3n^{-1/2}$ (right table).

n	SVD-based node2vec		original Node2vec	
	$t_U = 5$	$t_U = 8$	$t_U = 5$	$t_U = 8$
600	1.00	1.00	1.00	0.45
900	1.00	1.00	1.00	0.95
1500	1.00	1.00	1.00	1.00

n	SVD-based node2vec		original node2vec	
	$t_U = 5$	$t_U = 8$	$t_U = 5$	$t_U = 8$
600	0.30	0.32	0.01	0.05
900	0.57	0.55	0.07	0.11
1500	0.86	0.86	0.57	0.28

TABLE 2: Proportions of times that SGD-based and SVD-based node2vec variants perfectly recover all nodes memberships. The graphs are generated from $\mathbf{B}^\dagger(\rho_n)$ with sparsity $\rho_n = 3n^{-1/3}$ (left table) and $\rho_n = 3n^{-1/2}$ (right table).

in Theorem 4 as a smaller magnitude for ρ_n results in a slower convergence rate for $\hat{\mathcal{F}}$ under both the Frobenius and $2 \rightarrow \infty$ norms. In addition the exact recovery performance of SVD-based node2vec when $\rho_n \asymp n^{-1/2}$ and $(t_L, t_U) = (2, 5)$ suggests that the t_L threshold for Theorem 4 in Eq. (3.13) is possibly not sharp as $\frac{t_L-2}{t_L} = 0 < \beta = 1/2$. Finally we note that the SVD-based node2vec has better empirical performance than the original node2vec in these experiments as well as in the experiments for three-blocks SBMs and DCSBMs in Section 4.3. This is consistent with the discussion in Section 2. Indeed, the entries of $\tilde{\mathbf{M}}_0$ are the limit of those for the original node2vec when the number of sampled paths $r \rightarrow \infty$ and furthermore $\tilde{\mathbf{M}}_0$ has an approximately low-rank structure as n increases. In other words, at least for SBM and DCSBM graphs, we can view the original node2vec as a computationally efficient approach to approximate the embeddings based on SVD of $\tilde{\mathbf{M}}_0$.

4.3 Embedding performance

In this section we perform more numerical experiments to take a closer look at the finite-sample performance of community detection, using both the original and SVD-based node2vec embeddings. We consider both three-blocks SBM and three-blocks DCSBM. We will vary the sample size n , window sizes t_U , and sparsity ρ_n in these simulations and investigate the effect of these parameters on the community detection accuracy.

More specifically, for each simulation with a specified value of n and ρ_n , we run 100 Monte Carlo replications where, for each replicate, we apply both the original and SVD-based node2vec algorithms with different window sizes on the simulated random graph to obtain the embeddings followed by community detection using K -means on these embeddings. Let the true and estimated cluster labels be denoted by $\{\tau(i)\}_{i=1}^n$ and $\{\hat{\tau}(i)\}_{i=1}^n$. We calculate the accuracy of $\hat{\tau}$ as (here $\xi(\cdot)$ denotes an arbitrary permutation of $\{1, 2, \dots, K\}$),

$$\text{Accuracy} = \min_{\xi(\cdot)} \frac{\#\{i | \xi(\hat{\tau}(i)) \neq \tau(i)\}}{n}. \quad (4.2)$$

Before presenting the formal numerical results, we first fix $\rho_n = 3n^{-1/2}$, $n = 600$ and sample one random realization from both the SBM and the DCSBM to illustrate the node2vec embedding performances. These visualizations, which are depicted in Figure D6 and Figure D3 in the Supplementary File, provide us with some intuitions, namely that (i) the original and SVD-based node2vec variants yield similar embeddings (ii) for SVD-based node2vec, increasing the window size could help separate nodes from different communities and thereby improve the community detection accuracy; (iii) although the embeddings appear similar, K -means clustering yields more accurate membership recovery for the SVD-based node2vec compared to the original SVD-based node2vec embeddings. We now describe the settings of the network generation models used in these simulations.

Stochastic Blockmodel: We consider three-blocks SBMs with block probabilities being either

$$\mathbf{B}_1 = \begin{pmatrix} 0.8 & 0.4 & 0.3 \\ 0.4 & 0.7 & 0.5 \\ 0.3 & 0.5 & 0.9 \end{pmatrix} \text{ or } \mathbf{B}_2 = \begin{pmatrix} 0.8 & 0.5 & 0.5 \\ 0.5 & 0.8 & 0.5 \\ 0.5 & 0.5 & 0.8 \end{pmatrix}, \quad (4.3)$$

and block assignment probabilities $\boldsymbol{\pi} = (0.3, 0.3, 0.4)$.

Degree-Corrected Stochastic Blockmodel: DCSBMs are direct generalization of SBMs with the only difference being that each node i has a degree-correction parameter θ_i and that the probability of connection between nodes i and j is

$$p_{ij} = \theta_i \theta_j B_{\tau(i)\tau(j)}$$

instead of $p_{ij} = B_{\tau(i)\tau(j)}$ as in the case of SBMs. For more on DCSBMs and their inference, see [47], [48], [49]. We generate the degree correction parameters θ_i as

$$\theta_i = |Z_i| + 1 - (2\pi)^{-1/2}, \quad Z_1, \dots, Z_n \stackrel{\text{iid}}{\sim} \mathcal{N}(0, 0.25) \quad (4.4)$$

This procedure for generating θ_i is the same as that in [49].

For each simulated graph, we test both the original node2vec and the SVD-based node2vec with $t_U = 5, 6, 7$. Other settings of the node2vec algorithms are similar to Section 4.2. The simulation results for the SBMs and the DCSBMs are presented in Figure 4 and Figure D4 in the

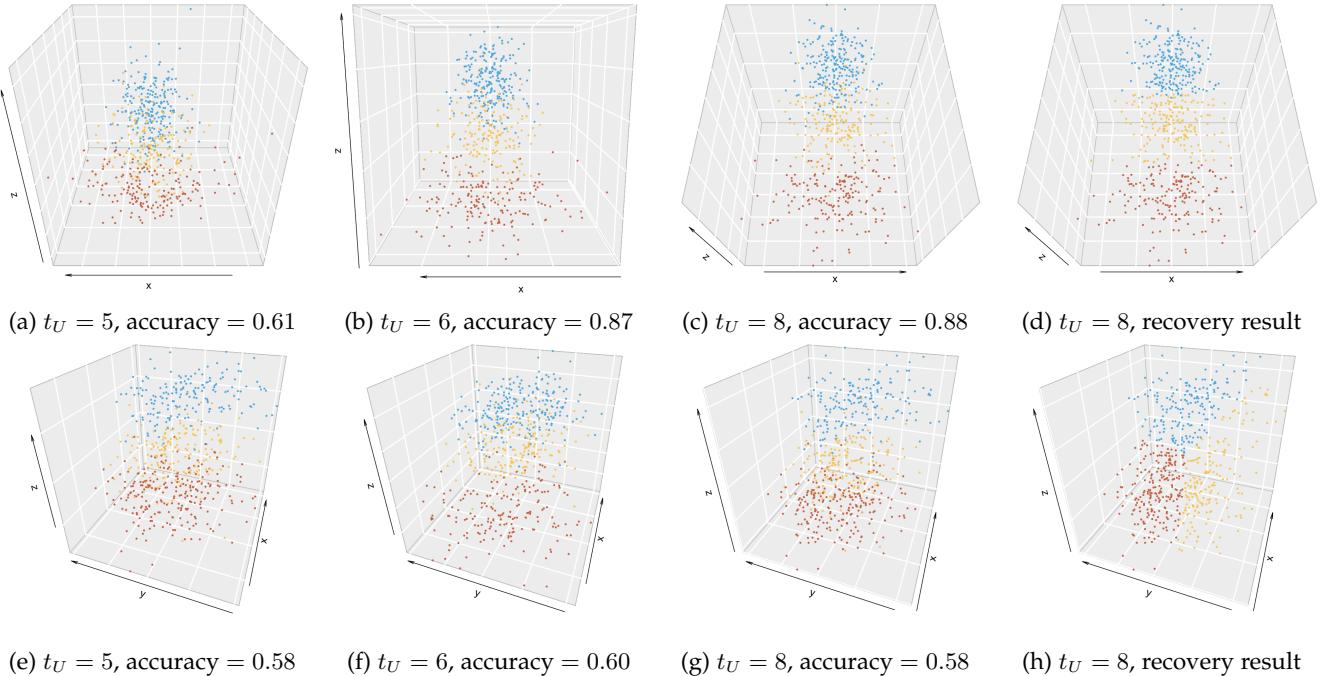


Fig. 3: Visualizations of the SVD-based node2vec embeddings (first row) and original node2vec embeddings (second row) for different choices of t_U . The plots are for a single realization of a SBM graph on $n = 600$ vertices with block probabilities matrix \mathbf{B}_1 (see eq. (4.3)), sparsity $\rho_n = 3n^{-1/2}$, and block assignment probabilities $\pi = (0.3, 0.3, 0.4)$. The embeddings in panels (a)–(c) and (e)–(g) are colored using the true membership assignments while the embeddings in panels (d) and (h) are colored using the K -means clustering. Accuracy of the recovered memberships (by K -means clustering) are also reported for panels (a)–(c) and (e)–(g).

Supplementary File. We now summarize the main trend in these figures.

- The box plots when $\rho_n = 1$ (dense regime) have large interquartile ranges because there are a few replicates where, due to sampling variability, the simulated graphs are quite noisy and the community detection algorithm has low accuracy while for most of the remaining graphs we achieved exact recovery using both the original and SVD-based node2vec algorithms. Furthermore if $\rho_n = 1$ then increasing the window size from $t_L = 2$ to $t_L > 2$ does not yield noticeable improvement in accuracy for the SVD-based node2vec. The condition $t_L \geq 2$ in Theorem 4 (i) is thus sufficient for exact recovery in the dense regime. On the other hand, when $\rho_n \neq 1$, we see that the accuracy increases with n as indicated by the large-sample results in Theorem 4.
- When $\rho_n \rightarrow 0$ faster (i.e., the network is more sparse), we need a larger n to achieve the same level of accuracy. This is consistent with Theorem 4 as the convergence rate for $\hat{\mathcal{F}}$ depends on $n\rho_n$.
- When $\mathbf{B} = \mathbf{B}_2$ the original node2vec and SVD-based node2vec have very similar accuracy and thus our theoretical analysis of SVD-based node2vec closely reflects the performance of the original node2vec.
- When $\mathbf{B} = \mathbf{B}_1$ the SVD-based node2vec has higher accuracy compared to the original node2vec. However the embeddings generated by these algorithms are still quite similar. A plausible reason for why the original node2vec has lower accuracy is because the

downstream K -means clustering is sub-optimal for these embeddings. For example, comparing panels (c) and (d) in Fig. D6 we can see that K -means clustering correctly recovers most of the membership assignments for embeddings from the SVD-based node2vec. In contrast, panels (g) and (h) in Fig. D6 show that K -means clustering is less accurate for embeddings from the original node2vec. Indeed, when replacing K -means with Gaussian mixtures model (GMM) [50], [51] in panels (g) and (h) of Fig. D6 we increase the clustering accuracy from 0.58 to 0.84 which is close to that of 0.88 for the SVD-based node2vec (see Fig. D5 of the Supplementary File).

5 APPLICATIONS TO REAL-WORLD NETWORKS

We test the membership recovery performance of node2vec on three real-world networks, namely, the Zachary's karate graph (henceforth, ZK) [52], political blogs graph (henceforth, PB) [53], and Wikipedia graph (henceforth, WIKI) [8]. In each of the three graphs, the memberships of all vertices have been assigned based on specific real-world meanings without missing. Both ZK and PB contain 2 communities, while WIKI contains 6 communities. ZK is connected with 34 vertices. By conventions [8], [47], we ignore the directions of edges and focus on the largest connected components of PB and WIKI, which contain 1222 and 1323 vertices, respectively. We refer interested readers to the references above for more detailed information about the three real-world network datasets.

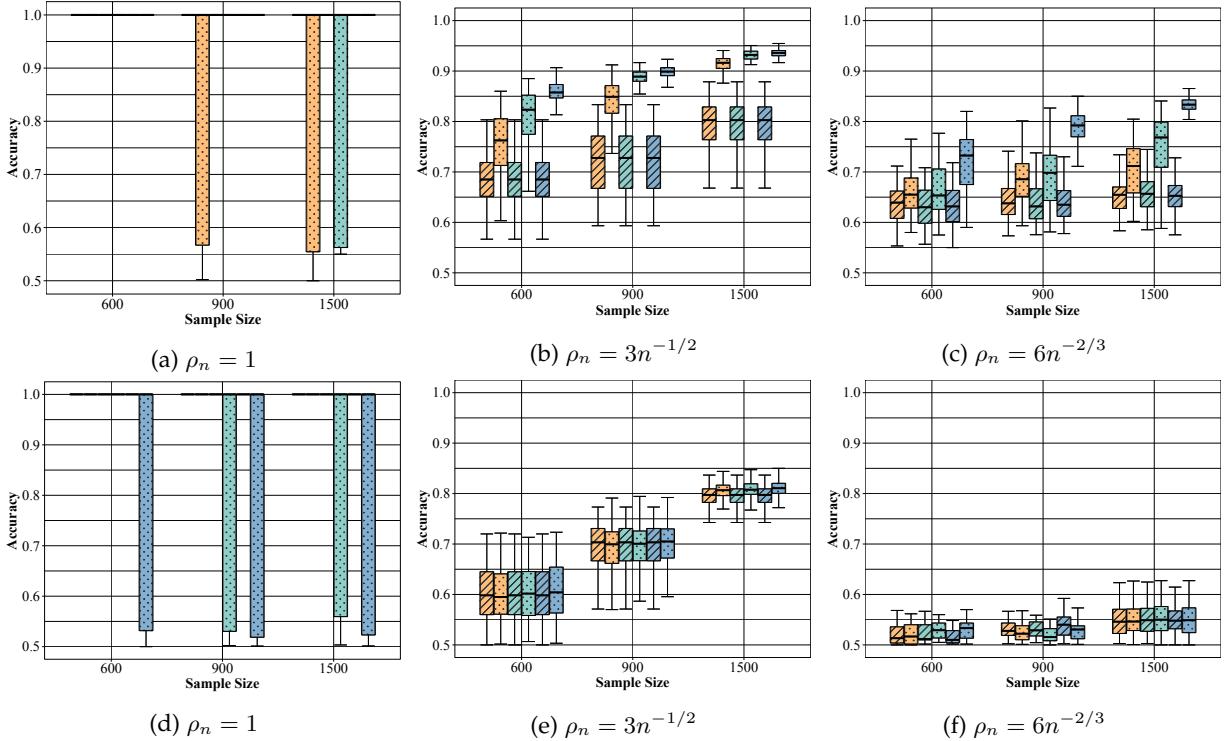


Fig. 4: Community detection accuracy of node2vec followed by K -means for SBM graphs. The boxplots of the accuracy for each value of n , ρ_n and t_U are based on 100 Monte Carlo replications. Boxplots with the slash pattern (resp. dot pattern) summarized the results for the original (resp. SVD-based) node2vec. Different colors (yellow, green, blue) represent the algorithms implemented for different choices of $t_U \in \{5, 6, 8\}$. The first and second row plot the results when the block probabilities for the SBM is B_1 and B_2 , respectively.

For each network dataset, we embed the vertices into the K -dimensional Euclidian space through both the SVD-based and original node2vec, and then cluster the embeddings by K -means to estimate the memberships of each vertex; K is chosen as the exact number of memberships in each graph. We test three window sizes $t_U \in \{10, 15, 20\}$. Similar to Section 4, we set $t_L = t_U - 5$ for the SVD-based node2vec and $t_L = 1$ for the original node2vec by default. To measure the membership recovery performances, we calculate the accuracies between the estimated memberships and the real memberships for ZK and PB; see the definition of accuracy in Eq. (4.2). For WIKI, because the criteria of accuracy becomes computationally inflexible, we alternatively use the adjusted rand index (ARI). Similar to the accuracy, $ARI = 1$ indicates the estimated memberships perfectly recover the real memberships, while $ARI = 0$ indicates the estimated memberships are assigned randomly. We also compare performances of node2vec algorithms with other popular spectral embedding algorithms, including the spectral clustering based on adjacency and normalized Laplacian [1], [7], [8], and the spectral clustering with projection onto the sphere [54]; for all methods we use K -means for the downstream clustering.

The recovery results are summarized in Table 3. The SVD-based and original node2vec algorithms have similar performances, which are generally better than or equivalently to other methods in all three datasets. In addition, we note the PB dataset is better modeled as a DCSBM [47]. Recall that, as shown in Remark 4, node2vec can

theoretically attain exact recovery for DCSBMs and hence the high-accuracy of node2vec on the PB dataset is expected. Similarly, [54] shows a valid theoretical guarantee of the spectral clustering with a spherical projection, when applying to the DCSBM graph. This can also be verified by the high accuracy of ASE+SP on PB as shown in Table 3.

6 DISCUSSION

In this paper we derive perturbation bounds and show exact recovery for the DeepWalk and node2vec (with $p = q = 1$) algorithms under the assumption that the observed graphs are instances of the stochastic blockmodel graphs. Our results are valid under both the dense and sparse regimes for sufficient large t_L and n . The simulation results corroborate our theoretical findings; in particular they show that increasing the sample size and window size can improve the community detection accuracy for both sparse SBM and DCSBM graphs.

We emphasize that our paper only include real data analysis on simple graphs with a small number of nodes just to illustrate the agreement between our theoretical results and the empirical performance of DeepWalk/node2vec. This is intentional as DeepWalk and node2vec are widely-used algorithms with numerous papers demonstrating their uses for analyzing real graphs in diverse applications. In contrast, our paper is one of a few that addresses the theory underpinning these algorithms and is, to the best of our knowledge, the first paper to establish consistency and exact

Network	SVD-based node2vec			Original node2vec			ASE	LSE	ASE+SP
	$t_U = 10$	$t_U = 15$	$t_U = 20$	$t_U = 10$	$t_U = 15$	$t_U = 20$			
ZK	0.97	0.97	0.97	0.97	0.97	0.97	1.00	0.97	0.97
PB	0.96	0.95	0.95	0.96	0.95	0.95	0.61	0.51	0.95

Network	SVD-based node2vec			Original node2vec			ASE	LSE	ASE+SP
	$t_U = 10$	$t_U = 15$	$t_U = 20$	$t_U = 10$	$t_U = 15$	$t_U = 20$			
WIKI	0.09	0.10	0.08	0.09	0.10	0.11	0.03	0.09	0.09

TABLE 3: The upper table reports the membership recovery accuracy of different embedding methods on the ZK and PB network datasets. The lower table reports the ARI of different embedding methods on the WIKI network dataset. ASE and LSE denote spectral clusterings using the truncated eigendecomposition of the adjacency and normalized Laplacian matrix [1], [7], [8], respectively. ASE+SP denote spectral clustering using the truncated eigendecomposition of the adjacency matrix together with a spherical projection step [54], [55].

recovery for SBMs and DCSBMs using these random-walk based embedding algorithms. Note that exact recovery for SBMs can also be achieved using other algorithms such as those based on semidefinite programming, variational Bayes, and spectral embedding; see [41], [56], [57] for a few examples.

There are several open questions for future research:

- 1) In this paper we only consider the case of $p = q = 1$ for node2vec embedding (recall that $p = q = 1$ is the default parameter values for node2vec). If $p \neq 1$ and/or $q \neq 1$ then the transformed co-occurrence matrix \mathbf{M}_0 can no longer be expressed in terms of the adjacency matrix \mathbf{A} or the transition matrix $\hat{\mathbf{W}}^t$; this renders the theoretical analysis for general values of p and q substantially more involved. One potential approach to this problem is to consider, similar to the notion of the *non-backtracking matrix* in community detection for sparse SBM [58], a transition matrix associated with the edges of \mathcal{G} as opposed to the transition matrix associated with the vertices in \mathcal{G} . Indeed, if $p \neq q$ then the transition probability from a vertex v to another vertex w depends also on the vertex, say u , preceding v in the random walk. i.e., the transition probability for (v, w) depends on the choice of (u, v) .
- 2) In this paper we focus on error bounds (in Frobenius and infinity norms) of node2vec/DeepWalk embedding for stochastic blockmodel graphs and their degree-corrected variant. An important question is whether or not stronger limit results are available for these algorithms. For example spectral embeddings of stochastic blockmodel graphs obtained via eigendecompositions of either the adjacency or the normalized Laplacian matrices are well-approximated by mixtures of multivariate Gaussians; see [29], [30] for more precise statements of these results and their implications for statistical inference in networks. It is thus natural to inquire if normal approximations also holds node2vec/Deepwalk. We ran several one-round simple simulations to visualize the embeddings of node2vec/DeepWalk when the graphs are sampled

from a SBM with

$$\mathbf{B} = \begin{pmatrix} 0.42 & 0.42 \\ 0.42 & 0.5 \end{pmatrix} \text{ and } \boldsymbol{\pi} = (0.4, 0.6). \quad (6.1)$$

The results are summarized in Fig. D6 in the Supplementary File. In particular when n is large these embeddings are also well-approximated by a mixture of multivariate Gaussians. We leave the theoretical justification of this phenomenon for future work.

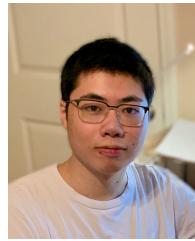
- 3) As we allude to in the introduction, for simplicity we only consider (degree-corrected) stochastic blockmodel graphs in this paper. For the more general inhomogeneous Erdős-Renyi random graphs model, we expect that Theorem 2 and Theorem 3 still hold, provided that the edge probabilities are sufficiently homogeneous, i.e., the minimum and maximum values for the edge probabilities values are of the same order as n increases. However, the error bounds in Theorem 4 might no longer apply since the entry-wise logarithmic transformation of the co-occurrence matrices can lead to the setting wherein \mathbf{M}_0 is no longer low-rank, e.g., the rank of \mathbf{M}_0 can be as large as n the number of vertices. Furthermore, even when \mathbf{M}_0 have an approximate low-rank structure, due to the logarithmic transformation there is still the question of how the embedding of \mathbf{M}_0 relates to the underlying latent structure in \mathbf{P} .
- 4) Finally, in this paper we mainly focus on the node2vec and DeepWalk embedding through matrix factorization (SVD-based node2vec), but also compare the SVD-based node2vec with the original node2vec in the numerical experiments. As we mentioned in the introduction the original node2vec algorithm uses (stochastic) gradient descent (GD/SGD) to optimize Eq. (2.4) and obtain the embeddings. As Eq. (2.4) is non-convex there can be a large number of local-minima, thereby making the theoretical analysis intractable unless we assume that the initial estimates for GD/SGD are sufficiently close to the global minima; see e.g., [43], [44] for some examples of results relating the closeness of the initial estimates and the convergence rate of GD/SGD. One popular initialization scheme for GD/SGD is via spectral methods and thus we

can consider using the SVD-based embedding $\hat{\mathcal{F}}$ as a “warm-start” for Eq. (2.4). We leave the precise convergence analysis of the resulting GD/SGD iterations to the interested reader. We note, however, that while this is certainly an interesting technical problem, the practical benefits might be limited. Indeed, the theoretical results in Section 3 guaranteed perfect recovery using $\hat{\mathcal{F}}$ while the empirical evaluations in Sections 4.2 and Section 4.3 suggest that clustering based on $\hat{\mathcal{F}}$ is comparable or even better than that of the original node2vec. In other words as the main objective is to recover the structure in M_0 induced by P , it is certainly possible that optimizing Eq. (2.4) does not lead to better inference performance due to the noise in using A as a replacement for P .

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APPENDIX A

PROOFS UNDER THE DENSE REGIME

In this section we provide proofs of our main theorems under the dense regime of $\rho_n = \Theta(1)$. This is done for ease of exposition as the proofs for the sparse regime follows the same conceptual ideas and proof structure but with substantially more involved and tedious technical derivations. We first list two basic lemmas that will be used repeatedly in the subsequent proofs. Proofs of these lemmas are deferred to Appendix C.

Lemma A1. Let $\mathbf{d} = (d_1, \dots, d_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$. Then for any integer $t > 0$ we have

$$\mathbf{1}_n^\top \hat{\mathbf{W}}^t = \mathbf{1}_n^\top \mathbf{W}^t = \mathbf{1}_n^\top, \quad \hat{\mathbf{W}}^t \mathbf{d} = \mathbf{d}, \quad \mathbf{W}^t \mathbf{p} = \mathbf{p}. \quad (\text{A.1})$$

Lemma A2. Under Assumption 1, we have

$$\begin{aligned} \|\mathbf{D}_\mathbf{A}\| &= \max_i d_i = \mathcal{O}_{\mathbb{P}}(n), \\ \|\mathbf{D}_\mathbf{A}^{-1}\| &= \max_i 1/d_i = \mathcal{O}_{\mathbb{P}}(1/n), \\ \|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{P}\| &= \max_i |d_i - p_i| = \mathcal{O}_{\mathbb{P}}(n^{1/2} \log^{1/2} n), \\ \|\mathbf{D}_\mathbf{A}^{-1} - \mathbf{D}_\mathbf{P}^{-1}\| &= \max_i |d_i^{-1} - p_i^{-1}| = \mathcal{O}_{\mathbb{P}}\left(\frac{\log^{1/2} n}{n^{3/2}}\right), \\ \|\hat{\mathbf{W}}^t\|_{\max} &= \mathcal{O}_{\mathbb{P}}(n^{-1}), \\ \max_{i,i'} w_{ii'}^{(t)} &\asymp 1/n, \quad \min_{i,i'} w_{ii'}^{(t)} \asymp 1/n, \end{aligned} \quad (\text{A.2})$$

for any fixed $t \geq 1$. Here $w_{ii'}^{(t)}$ is the ii' th entry of \mathbf{W}^t .

A.1 Proof of Theorem 2 (Dense regime)

We use the following three steps to bound Eq. (3.1)-Eq. (3.4) in turn.

Step 1 (Bounding $\|\hat{\mathbf{W}} - \mathbf{W}\|_{\max}$): We start with the decomposition

$$\hat{\mathbf{W}} - \mathbf{W} = \mathbf{A}\mathbf{D}_\mathbf{A}^{-1} - \mathbf{P}\mathbf{D}_\mathbf{P}^{-1} = \underbrace{\mathbf{A}\mathbf{D}_\mathbf{P}^{-1}\mathbf{D}_\mathbf{A}^{-1}(\mathbf{D}_\mathbf{P} - \mathbf{D}_\mathbf{A})}_{\Delta_1^{(1)}} + \underbrace{(\mathbf{A} - \mathbf{P})\mathbf{D}_\mathbf{P}^{-1}}_{\Delta_2^{(1)}}. \quad (\text{A.3})$$

For the first term, we have

$$\Delta_1^{(1)} = \mathbf{A}\mathbf{D}_\mathbf{P}^{-1}\mathbf{D}_\mathbf{A}^{-1}(\mathbf{D}_\mathbf{P} - \mathbf{D}_\mathbf{A}) = \left[\left| \frac{a_{ii'}}{d_{i'} \cdot p_{i'}} (d_{i'} - p_{i'}) \right| \right]_{n \times n}$$

and hence

$$\|\Delta_1^{(1)}\|_{\max} = \max_{i,i'} \left| \frac{a_{ii'}}{d_{i'} \cdot p_{i'}} (d_{i'} - p_{i'}) \right| \lesssim \max_{i'} \frac{1}{n} \left| \frac{d_{i'} - p_{i'}}{d_{i'}} \right|$$

by $|a_{ii'}| \leq 1$ and $c_0 < p_{ii'} < c_1$. Lemma A2 then implies

$$\|\Delta_1^{(1)}\|_{\max} \lesssim \frac{1}{n} \max_{i'} |d_{i'} - p_{i'}| \cdot \max_{i'} \frac{1}{d_{i'}} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n). \quad (\text{A.4})$$

For the second term we have, by Assumption 1, that

$$\|\Delta_2^{(1)}\|_{\max} = \max_{i,i'} \left| \frac{a_{ii'} - p_{ii'}}{p_{i'}} \right| \leq \max_{i'} \frac{1}{p_{i'}} = \mathcal{O}_{\mathbb{P}}(n^{-1}). \quad (\text{A.5})$$

Combining Eq. (A.4) and Eq. (A.5) yields

$$\|\hat{\mathbf{W}} - \mathbf{W}\|_{\max} \leq \|\Delta_1^{(1)}\|_{\max} + \|\Delta_2^{(1)}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-1}). \quad (\text{A.6})$$

Step 2 (Bounding $\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{diag}}, \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}}$): We first decompose $\hat{\mathbf{W}}^2 - \mathbf{W}^2$ as

$$\hat{\mathbf{W}}^2 - \mathbf{W}^2 = \underbrace{(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}}_{\Delta_1^{(2)}} + \underbrace{\hat{\mathbf{W}}(\hat{\mathbf{W}} - \mathbf{W})}_{\Delta_2^{(2)}}. \quad (\text{A.7})$$

As with Eq (A.3) we have,

$$\begin{aligned} \Delta_1^{(2)} &= (\mathbf{A}\mathbf{D}_\mathbf{A}^{-1} - \mathbf{P}\mathbf{D}_\mathbf{P}^{-1})\mathbf{W} \\ &= \{\mathbf{A}\mathbf{D}_\mathbf{P}^{-1}\mathbf{D}_\mathbf{A}^{-1}(\mathbf{D}_\mathbf{P} - \mathbf{D}_\mathbf{A}) + (\mathbf{A} - \mathbf{P})\mathbf{D}_\mathbf{P}^{-1}\}\mathbf{W} \\ &= \underbrace{\{\mathbf{A}\mathbf{D}_\mathbf{P}^{-1}\mathbf{D}_\mathbf{A}^{-1}(\mathbf{D}_\mathbf{P} - \mathbf{D}_\mathbf{A}) - \mathbf{A}\mathbf{D}_\mathbf{P}^{-2}(\mathbf{D}_\mathbf{P} - \mathbf{D}_\mathbf{A})\}\mathbf{W}}_{\Delta_1^{(2,1)}} + \underbrace{\mathbf{A}\mathbf{D}_\mathbf{P}^{-2}(\mathbf{D}_\mathbf{P} - \mathbf{D}_\mathbf{A})\mathbf{W}}_{\Delta_1^{(2,2)}} + \underbrace{(\mathbf{A} - \mathbf{P})\mathbf{D}_\mathbf{P}^{-1}\mathbf{W}}_{\Delta_1^{(2,3)}}. \end{aligned} \quad (\text{A.8})$$

The ii' 'th element of $\Delta_1^{(2,1)}$ is given by

$$\sum_{i^*=1}^n \frac{a_{ii^*}(p_{i^*} - d_{i^*})^2}{p_{i^*}^2 d_{i^*}} \cdot w_{i^* i'}. \quad (\text{A.9})$$

We therefore have, by Assumption 1 and Lemma A2,

$$\|\Delta_1^{(2,1)}\|_{\max} \leq \max_{i,i'} \sum_{i^*=1}^n \frac{a_{ii^*}(p_{i^*} - d_{i^*})^2}{p_{i^*}^2 d_{i^*}} w_{i^* i'} \leq n \max_i |p_i - d_i|^2 \left(\max_i \frac{1}{p_i^2 d_i} \right) (\max_{i,i'} w_{i^* i'}) = \mathcal{O}_{\mathbb{P}}(n^{-2} \log n). \quad (\text{A.10})$$

Similarly, for $\Delta_1^{(2,2)}$ we have

$$\|\Delta_1^{(2,2)}\|_{\max} = \max_{i,i'} \left| \sum_{i^*=1}^n \frac{a_{ii^*}}{p_{i^*}^2} (p_{i^*} - d_{i^*}) w_{i^* i'} \right| \leq n (\max_i |p_i - d_i|) \cdot \max_{i,i'} w_{i^* i'} \cdot \left(\max_i \frac{1}{p_i^2} \right) = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n). \quad (\text{A.11})$$

We now consider the term $\Delta_1^{(2,3)}$. We have

$$\|\Delta_1^{(2,3)}\|_{\max} = \max_{i,i'} \left| \sum_{i^*=1}^n \frac{(a_{ii^*} - p_{ii^*})}{p_{i^*}} w_{i^* i'} \right| \quad (\text{A.12})$$

Assumption 1 and Lemma A2 then imply

$$\max_{i',i^*} w_{i^* i'} / p_{i^*} \asymp n^{-2}, \text{ and } \min_{i',i^*} w_{i^* i'} / p_{i^*} \asymp n^{-2}.$$

and hence, by Bernstein inequality, we have

$$\max_{i,i'} \left| \sum_{i^*=1}^n \frac{(a_{ii^*} - p_{ii^*})}{p_{i^*}} w_{i^* i'} \right| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n). \quad (\text{A.13})$$

Combining the above bounds we obtain

$$\|\Delta_1^{(2)}\|_{\max} \leq \|\Delta_1^{(2,1)}\|_{\max} + \|\Delta_1^{(2,2)}\|_{\max} + \|\Delta_1^{(2,3)}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n). \quad (\text{A.14})$$

We now consider the term $\Delta_2^{(2)} = \hat{\mathbf{W}}(\hat{\mathbf{W}} - \mathbf{W})$. We have

$$\|\Delta_2^{(2)}\|_{\max} = \|(\mathbf{W} - \hat{\mathbf{W}})^2 - \mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} \leq \|(\mathbf{W} - \hat{\mathbf{W}})^2\|_{\max} + \|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max}. \quad (\text{A.15})$$

The same argument for bounding $\|\Delta_2^{(1)}\|_{\max}$ as given above also yields

$$\|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \sqrt{\log n}).$$

We then bound $\|(\mathbf{W} - \hat{\mathbf{W}})^2\|_{\max}$ through the following expansion

$$\begin{aligned} \|(\mathbf{W} - \hat{\mathbf{W}})^2\|_{\max} &= \max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{d_{i'}} \right) \right| \\ &= \max_{i,i'} \left| \sum_{i^*=1}^n \left\{ \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) + \left(\frac{a_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \right\} \left\{ \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) + \left(\frac{a_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{d_{i'}} \right) \right\} \right| \\ &\leq \underbrace{\max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{a_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right|}_{\delta_2^{(2,1)}} + \underbrace{\max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{a_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{d_{i'}} \right) \right|}_{\delta_2^{(2,2)}} \\ &\quad + \underbrace{\max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right|}_{\delta_2^{(2,3)}}. \end{aligned} \quad (\text{A.16})$$

Since $|a_{ii'}| \leq 1$ and $|a_{ii'} - p_{ii'}| \leq 1$, we have

$$\delta_2^{(2,1)} \leq \max_{i'} \left(\sum_{i^*=1}^n \left| \frac{1}{p_{i^*}} - \frac{1}{d_{i^*}} \right| \frac{1}{p_{i'}} \right) \lesssim n \cdot \frac{1}{n} \max_{i^*} \left(\frac{|p_{i^*} - d_{i^*}|}{p_{i^*} d_{i^*}} \right) = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n).$$

Similar reasoning also yields

$$\begin{aligned} \delta_2^{(2,2)} &\leq \max_{i,i'} \sum_{i^*=1}^n \left| \frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right| \cdot \left| \frac{1}{p_{i'}} - \frac{1}{d_{i'}} \right| \\ &\leq n \left(\max_{i^*} \frac{1}{p_{i^*}} + \max_{i^*} \frac{1}{d_{i^*}} \right) \cdot \max_{i'} \left| \frac{1}{p_{i'}} - \frac{1}{d_{i'}} \right| \lesssim n \cdot \{n^{-1} + \mathcal{O}_{\mathbb{P}}(1/n)\} \cdot \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n) = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n). \end{aligned}$$

We now bound $\delta_2^{(2,3)}$ by considering the diagonal and off-diagonal terms respectively. For the diagonal terms we have

$$\max_i \sum_{i^*=1}^n \frac{(a_{ii^*} - p_{ii^*})^2}{p_{i^*} p_i} \leq \max_i \sum_{i^*=1}^n \frac{1}{p_i p_{i^*}} = \mathcal{O}(n^{-1}) \quad (\text{A.17})$$

Eq. (A.14) together with the above bounds for $\delta_2^{(2,1)}$ through $\delta_2^{(2,3)}$ yield

$$\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{diag}} \leq \|\Delta_1^{(2)}\|_{\max} + \|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} + \delta_2^{(2,1)} + \delta_2^{(2,2)} + \max_i \sum_{i^*=1}^n \frac{(p_{ii^*} - a_{ii^*})^2}{p_{i^*} p_i} = \mathcal{O}_{\mathbb{P}}(n^{-1}). \quad (\text{A.18})$$

We now consider the off-diagonal terms with $i \neq i'$ for $\delta_2^{(2,3)}$. First define

$$\zeta_{i^*}^{ii'} = \frac{1}{p_{i^*} p_{i'}} (p_{ii^*} - a_{ii^*})(p_{i^* i'} - a_{i^* i'}).$$

We now make an important observation that if $i \neq i'$ then the collection of random variables $\zeta_{i^*}^{ii'}$ for $i^* = 1, 2, \dots, n$ are independent mean 0 random variables. Indeed, when $i \neq i'$ then a_{ii^*} and $a_{i^* i'}$ are independent and hence

$$\mathbb{E} \zeta_{i^*}^{ii'} = \frac{1}{p_{i^*} p_{i'}} \mathbb{E}(p_{ii^*} - a_{ii^*}) \cdot \mathbb{E}(p_{i^* i'} - a_{i^* i'}) = 0.$$

We thus have

$$\max_{i \neq i'} \left| \sum_{i^*=1}^n \frac{(p_{ii^*} - a_{ii^*})(p_{i^* i'} - a_{i^* i'})}{p_{i^*} p_{i'}} \right| = \max_{i \neq i'} \left| \sum_{i^*=1}^n \zeta_{i^*}^{ii'} \right| = \max_{i \neq i'} \left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \zeta_{i^*}^{ii'} + \zeta_i^{ii'} + \zeta_{i'}^{ii'} \right|. \quad (\text{A.19})$$

Now fix a pair $\{i, i'\}$ with $i \neq i'$. Then by Bernstein inequality, we have, for any $\tilde{t} > 0$,

$$\mathbb{P}\left(\left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \zeta_{i^*}^{ii'} \right| > \tilde{t}\right) \leq 2 \exp\left(-\frac{\tilde{t}^2}{2\sigma_1^2 + \frac{M}{3}\tilde{t}}\right) \quad (\text{A.20})$$

where the variance proxy σ_1^2 is bounded as

$$\sigma_1^2 = \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \text{Var}(\zeta_{i^*}^{ii'}) = \frac{1}{p_{i^*}^2 p_{i'}^2} \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \{\text{Var}(p_{ii^*} - a_{ii^*}) \cdot \text{Var}(p_{i^* i'} - a_{i^* i'})\} \leq \frac{n}{16p_{i^*}^2 p_{i'}^2} \leq \frac{1}{16c_0^4 n^3},$$

and M is any constant bigger than $\max_{i^*} |\tilde{\zeta}_{i^*}^{ii'}|$. In particular

$$|\zeta_{i^*}^{ii'}| = \left| \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right| \leq \frac{1}{c_0^2 n^2}$$

and we can take $M = (c_0 n)^{-2}$. Let $\vartheta_n = n^{-3/2} \log^{1/2} n$. Plugging the above bounds for σ_1^2 and M into Eq. (A.20), we have, for any $C_1 > 0$,

$$\mathbb{P}\left(\left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \tilde{\zeta}_{i^*}^{ii'} \right| > C_1 \vartheta_n\right) \leq 2 \exp\left\{-\frac{(C_1 \vartheta_n)^2}{\frac{2}{16c_0^4 n^3} + \frac{1}{3c_0^2 n^2} (C_1 \vartheta_n)}\right\} = 2 \exp\left(-\frac{C_1 \log n}{\frac{1}{8c_0^4} + \frac{\log^{1/2} n}{n^{1/2} 3c_0^2 C_1}}\right) \lesssim n^{-8C_1 c_0^4}.$$

Now choose C_1 such that $8C_1 c_0^4 > 5$. We then have, by a union bound over all pairs $\{i, i'\}$ with $i \neq i'$, that

$$\mathbb{P}\left(\max_{i \neq i'} \left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \tilde{\zeta}_{i^*}^{ii'} \right| > C_1 n^{-3/2} \log^{1/2} n\right) \lesssim n^{2-8C_1 c_0^4} \lesssim n^{-3}.$$

We thus conclude

$$\max_{i \neq i'} \left| \sum_{i^*=1}^n \frac{(p_{ii^*} - a_{ii^*})(p_{i^* i'} - a_{i^* i'})}{p_{i^*} p_{i'}} \right| \leq \max_{i \neq i'} \left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \zeta_{i^*}^{ii'} \right| + 2M = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n).$$

A similar argument to Eq. (A.18) then yields

$$\begin{aligned} \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}} &\leq \|\Delta_1^{(2)}\|_{\max} + \|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} + \delta_2^{(2,1)} + \delta_2^{(2,2)} + \max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right| \\ &= \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n). \end{aligned} \quad (\text{A.21})$$

Step 3 (Bounding $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max}, t \geq 3$): We first consider $t = 3$. We have

$$\|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max} \leq \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}\|_{\max} + \|\mathbf{W}^2(\hat{\mathbf{W}} - \mathbf{W})\|_{\max} \quad (\text{A.22})$$

For the first term on the RHS of Eq. (A.22) we have, by Eq. (A.18), Eq. (A.21) and Lemma A2, that

$$\|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}\|_{\max} \leq n\|\hat{\mathbf{W}}\|_{\max}\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{off}} + \|\hat{\mathbf{W}}\|_{\max}\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{diag}} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n). \quad (\text{A.23})$$

For the second term in the RHS of Eq. (A.22) we use the same argument as that for bounding $\|(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}\|_{\max}$ in Step 2. In particular we have

$$\|\mathbf{W}^2(\hat{\mathbf{W}} - \mathbf{W})\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n). \quad (\text{A.24})$$

Combining Eq. (A.22) through Eq. (A.24) yields

$$\|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n). \quad (\text{A.25})$$

The case when $t = 4$ is analogous. More specifically,

$$\begin{aligned} \|\hat{\mathbf{W}}^4 - \mathbf{W}^4\|_{\max} &= \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^2\|_{\max} + \|\mathbf{W}^2(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\|_{\max} \\ &\leq n\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{off}}(\|\hat{\mathbf{W}}^2\|_{\max} + \|\mathbf{W}^2\|_{\max}) + \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{diag}}(\|\hat{\mathbf{W}}^2\|_{\max} + \|\mathbf{W}^2\|_{\max}) \\ &= \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n) \end{aligned} \quad (\text{A.26})$$

We now consider a general $t \geq 5$. We start with the decomposition

$$\hat{\mathbf{W}}^t - \mathbf{W}^t = (\hat{\mathbf{W}}^t - \mathbf{W}^2\hat{\mathbf{W}}^{t-2}) + (\mathbf{W}^2\hat{\mathbf{W}}^{t-2} - \mathbf{W}^t) = \underbrace{(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^{t-2}}_{\Delta_1^{(3)}} + \underbrace{\mathbf{W}^2(\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2})}_{\Delta_2^{(3)}},$$

We now have, by Lemma A2, Eq. (A.18), and Eq. (A.21), that

$$\|\Delta_1^{(3)}\|_{\max} \leq n\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{off}}\|\hat{\mathbf{W}}^{t-2}\|_{\max} + \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{diag}}\|\hat{\mathbf{W}}^{t-2}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n). \quad (\text{A.27})$$

Once again, by Lemma A2, we have

$$\|\Delta_2^{(3)}\|_{\max} \leq n\|\mathbf{W}^2\|_{\max}\|\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2}\|_{\max} \lesssim \|\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2}\|_{\max}. \quad (\text{A.28})$$

Combining Eq. (A.27) and Eq. (A.28), we have

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} \lesssim \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n) + \|\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2}\|_{\max} \quad (\text{A.29})$$

As t is finite, iterating the above argument yields

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} \lesssim \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n) + \|\hat{\mathbf{W}}^4 - \mathbf{W}^4\|_{\max} & (\text{when } t \text{ is even}) \\ \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n) + \|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max} & (\text{when } t \text{ is odd}) \end{cases}$$

Recalling Eq. (A.25) and Eq. (A.26), we conclude that

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\log^{1/2}n) \quad \text{for all } t \geq 3$$

as desired. \square

A.2 Proof of Theorem 3 (Dense regime)

We will only present the proof of bounding $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}}$ and $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty}$ here as the rates for $\|\mathbf{M}_0\|_{\text{F}}$ and $\|\mathbf{M}_0\|_{\infty}$ are derived in the proof of Theorem 4, Eq. (A.50). Under the dense regime of Assumption 1 and for a sufficiently large n , the diameter of \mathcal{G} is 2 with high probability, see e.g., Corollary 10.11 in [31]. Therefore, with high probability, all entries of $\hat{\mathbf{W}}^2$ are positive and hence

$$\tilde{\mathbf{M}}_0 = \log \left\{ \frac{2|\mathbf{A}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t) \cdot (\mathbf{D}_{\mathbf{A}}^{-1}\hat{\mathbf{W}}^t) \right\}$$

is well-defined with high probability. Next recall the definition of \mathbf{M}_0 given in Eq. (2.9). We have

$$\tilde{\mathbf{M}}_0 - \mathbf{M}_0 = \underbrace{\log \left\{ |\mathbf{A}| \sum_{t=t_L}^{t_U} (L-t) \cdot (\mathbf{D}_{\mathbf{A}}^{-1}\hat{\mathbf{W}}^t) \right\}}_{\mathbf{I}_{\mathbf{A}}} - \underbrace{\log \left\{ |\mathbf{P}| \sum_{t=t_L}^{t_U} (L-t) \cdot (\mathbf{D}_{\mathbf{P}}^{-1}\mathbf{W}^t) \right\}}_{\mathbf{I}_{\mathbf{P}}}. \quad (\text{A.30})$$

By the mean value theorem, the absolute value of ii' 'th entry in $\tilde{\mathbf{M}}_0 - \mathbf{M}_0$ is

$$\frac{1}{\alpha_{ii'}} \underbrace{\left| \sum_{t=t_L}^{t_U} (L-t) \cdot \left\{ |\mathbf{A}| \cdot \left(\frac{\hat{w}_{ii'}^{(t)}}{d_i} \right) - |\mathbf{P}| \cdot \left(\frac{w_{ii'}^{(t)}}{p_i} \right) \right\} \right|}_{I_{\mathbf{A}}^{ii'} - I_{\mathbf{P}}^{ii'}}$$

where $\alpha_{ii'} \in (I_{\mathbf{A}}^{ii'}, I_{\mathbf{P}}^{ii'})$ and $I_{\mathbf{A}}^{ii'}$ and $I_{\mathbf{P}}^{ii'}$ are the ii' th entry of $\mathbf{I}_{\mathbf{A}}$ and $\mathbf{I}_{\mathbf{P}}$, respectively. We therefore have

$$\begin{aligned}\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}} &\leq \max_{i \neq i'} \left(\frac{1}{\alpha_{ii'}} \right) \cdot \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}}, \\ \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}} &\leq \max_i \left(\frac{1}{\alpha_{ii'}} \right) \cdot \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}}.\end{aligned}$$

We now bound $\alpha_{ii'}^{-1}$, $\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}}$ and $\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}}$.

Step 1 (Bounding $\max_{ii'} \alpha_{ii'}^{-1}$): As $\alpha_{ii'} \in (I_{\mathbf{A}}^{ii'}, I_{\mathbf{P}}^{ii'})$ we have

$$\max_{i, i'} \frac{1}{\alpha_{ii'}} \leq \max_{i, i'} \left\{ \frac{1}{I_{\mathbf{A}}^{ii'}}, \frac{1}{I_{\mathbf{P}}^{ii'}} \right\}. \quad (\text{A.31})$$

We first bound $(I_{\mathbf{P}}^{ii'})^{-1}$. In particular

$$\frac{1}{I_{\mathbf{P}}^{ii'}} = \frac{1}{\sum_{t=t_L}^{t_U} (L-t)|\mathbf{P}| \cdot \left(\frac{w_{ii'}^{(t)}}{p_i} \right)} \leq \frac{p_i}{|\mathbf{P}| w_{ii'}^{(t_L)}}$$

as $L-t \geq 1$ for all $t_L \leq t \leq t_U \leq L-1$. Now there exists constants c_0 and c_1 such that $c_0 \leq p_{ii'} \leq c_1$ for all $(i, i') \in [n]^2$. Therefore, by Lemma A2, we have

$$\max_{i, i'} \frac{1}{I_{\mathbf{P}}^{ii'}} \lesssim \frac{n}{n^2 \cdot \frac{1}{n}} = 1. \quad (\text{A.32})$$

We then bound $(I_{\mathbf{A}}^{ii'})^{-1}$. We have

$$\frac{1}{I_{\mathbf{A}}^{ii'}} = \frac{1}{\sum_{t=t_L}^{t_U} (L-t)|\mathbf{A}| \cdot \left(\frac{\hat{w}_{ii'}^{(t)}}{d_i} \right)} \leq \frac{d_i}{|\mathbf{A}| \hat{w}_{ii'}^{(t_L)}}$$

First suppose $i \neq i'$. Then by Theorem 2, we have

$$\max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \sqrt{\log n})$$

which implies, whp, that

$$0 \leq w_{ii'}^{(t_L)} - \max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}| \quad \text{for all } i \neq i'. \quad (\text{A.33})$$

As $\min_{i, i'} w_{ii'}^{(t_L)} \asymp \max_{i, i'} w_{ii'}^{(t_L)} \asymp n^{-1}$ we also have $\hat{w}_{ii'}^{(t_L)} \asymp n^{-1}$ for all $i \neq i'$. Hence, by Lemma A2, we have

$$\max_{i \neq i'} \frac{1}{I_{\mathbf{A}}^{ii'}} \leq \max_{i \neq i'} \frac{d_i}{|\mathbf{A}| \{ \hat{w}_{ii'}^{(t_L)} \}} \lesssim \max_i d_i \cdot \max_i \frac{1}{nd_i} \cdot \max_{i \neq i'} \frac{1}{\hat{w}_{ii'}^{(t_L)}} = \mathcal{O}_{\mathbb{P}}(1). \quad (\text{A.34})$$

Now suppose that $i = i'$. Then for $t_L = 2$, we have

$$\frac{1}{\hat{w}_{ii}^{(2)}} = \frac{1}{\sum_{i'=1}^n \hat{w}_{ii'} \hat{w}_{i'i}} = \frac{1}{\sum_{i'=1}^n \frac{a_{ii'}}{d_{i'}} \frac{a_{i'i}}{d_i}} = \frac{d_i}{\sum_{i'=1}^n a_{ii'}/d_{i'}}.$$

Once again, by Lemma A2, we have

$$\max_i \frac{1}{\hat{w}_{ii}^{(2)}} \leq \frac{\max_i d_i}{\frac{1}{\max_i d_i} \sum_{i'=1}^n a_{ii'}} \leq (\max_i d_i)^2 \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(n),$$

and hence

$$\max_i \frac{1}{I_{\mathbf{A}}^{ii}} \lesssim \max_i d_i \cdot \max_i \frac{1}{nd_i} \cdot \max_i \frac{1}{\hat{w}_{ii}^{(2)}} = \mathcal{O}_{\mathbb{P}}(1).$$

If $t_L \geq 3$, then $\max_i |w_{ii}^{(t_L)} - \hat{w}_{ii}^{(t_L)}| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \log^{1/2} n)$ (see Theorem 2) and hence, using the same argument as that for deriving Eq. (A.34), we also have $\max_i (I_{\mathbf{A}}^{ii})^{-1} = \mathcal{O}_{\mathbb{P}}(1)$. In summary, we have

$$\max_{i, i'} \frac{1}{\alpha_{ii'}} = \mathcal{O}_{\mathbb{P}}(1). \quad (\text{A.35})$$

Step 2 (Bounding $\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}}$): We start with the inequality

$$\max_{i \neq i'} |I_{\mathbf{A}}^{ii'} - I_{\mathbf{P}}^{ii'}| = \max_{i \neq i'} \left| \sum_{t=t_L}^{t_U} (L-t) \left\{ |\mathbf{A}| \left(\frac{\hat{w}_{ii'}^{(t)}}{d_i} \right) - |\mathbf{P}| \left(\frac{w_{ii'}^{(t)}}{p_i} \right) \right\} \right| \leq \sum_{t=t_L}^{t_U} (L-t) \max_{i \neq i'} \left| |\mathbf{A}| \left(\frac{\hat{w}_{ii'}^{(t)}}{d_i} \right) - |\mathbf{P}| \left(\frac{w_{ii'}^{(t)}}{p_i} \right) \right|. \quad (\text{A.36})$$

We now bound each of the summand in the RHS of the above display. Consider a fixed value of $t \geq 2$. We have

$$\max_{i \neq i'} \left| |\mathbf{A}| \frac{\hat{w}_{ii'}^{(t)}}{d_i} - |\mathbf{P}| \frac{w_{ii'}^{(t)}}{p_i} \right| \leq (|\mathbf{A}| - |\mathbf{P}|) \cdot \max_{i \neq i'} \left| \frac{w_{ii'}^{(t)}}{d_i} \right| + |\mathbf{P}| \max_{i \neq i'} \left| \frac{(p_i - d_i) w_{ii'}^{(t)}}{p_i d_i} \right| + |\mathbf{A}| \max_{i \neq i'} \left| \frac{\hat{w}_{ii'}^{(t)} - w_{ii'}^{(2)}}{d_i} \right|. \quad (\text{A.37})$$

By Lemma A2, the first term in RHS of Eq. (A.37) is bounded as

$$(|\mathbf{A}| - |\mathbf{P}|) \max_{i \neq i'} \left| \frac{w_{ii'}^{(t)}}{d_i} \right| \leq n \max_i |d_i - p_i| \cdot \max_{i, i'} w_{ii'}^{(t)} \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n). \quad (\text{A.38})$$

The second term in the RHS of Eq. (A.37) is also bounded by Lemma A2 as

$$|\mathbf{P}| \max_{i \neq i'} \left| \frac{(p_i - d_i) w_{ii'}^{(t)}}{p_i d_i} \right| \leq |\mathbf{P}| \cdot \max_i \left| \frac{p_i - d_i}{p_i d_i} \right| \cdot \max_{i, i'} w_{ii'}^{(t)} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n) \quad (\text{A.39})$$

The third term is bounded by Theorem 2 and Lemma A2 as

$$|\mathbf{A}| \max_{i \neq i'} \left| \frac{\hat{w}_{ii'}^{(t)} - w_{ii'}^{(t)}}{d_i} \right| \leq |\mathbf{A}| \max_{i \neq i'} |\hat{w}_{ii'}^{(t)} - w_{ii'}^{(t)}| \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n).$$

Collecting the above terms and summing over the finite values of $t_L \leq t \leq t_U$, we obtain

$$\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n). \quad (\text{A.40})$$

Step 3 (Bounding $\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}}$): With a similar argument as Step 2, we consider

$$\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}} \leq \sum_{t=2}^{t_U} (L-t) \max_i \left| |\mathbf{A}| \left(\frac{\hat{w}_{ii}^{(t)}}{d_i} \right) - |\mathbf{P}| \left(\frac{w_{ii}^{(t)}}{p_i} \right) \right|. \quad (\text{A.41})$$

We once again bound each summand in the above display. Similar to Eq. (A.37), we have

$$\max_i \left| |\mathbf{A}| \frac{\hat{w}_{ii'}^{(t)}}{d_i} - |\mathbf{P}| \frac{w_{ii'}^{(t)}}{p_i} \right| \leq (|\mathbf{A}| - |\mathbf{P}|) \max_i \left| \frac{w_{ii}^{(t)}}{d_i} \right| + |\mathbf{P}| \max_i \left| \frac{(p_i - d_i) w_{ii}^{(t)}}{p_i d_i} \right| + |\mathbf{A}| \max_i \left| \frac{\hat{w}_{ii}^{(t)} - w_{ii}^{(t)}}{d_i} \right|. \quad (\text{A.42})$$

The first two terms in the RHS of Eq. (A.42) is bounded via $\mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n)$; see the arguments for Eq. (A.38) and Eq. (A.39). For the third term, we consider the cases $t = 2$ and $t > 2$ separately. For $t = 2$, we have

$$|\mathbf{A}| \max_i \left| \frac{\hat{w}_{ii}^{(2)} - w_{ii}^{(2)}}{d_i} \right| \leq |\mathbf{A}| \max_i |\hat{w}_{ii}^{(2)} - w_{ii}^{(2)}| \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(1).$$

In contrast, for $t > 2$, we have

$$|\mathbf{A}| \max_i \left| \frac{\hat{w}_{ii}^{(t)} - w_{ii}^{(t)}}{d_i} \right| = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n)$$

Combining the above terms, we have

$$\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}} = \begin{cases} \mathcal{O}_{\mathbb{P}}(1) & \text{if } t_L = 2, \\ \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n) & \text{if } t_L \geq 3. \end{cases} \quad (\text{A.43})$$

Step 4 (Bounding $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}}$): In summary, Eq. (A.35), Eq. (A.40) and Eq. (A.43) imply

$$\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}} \leq \max_{i \neq i'} \left(\frac{1}{\alpha_{ii'}} \right) \cdot \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n), \quad (\text{A.44})$$

$$\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}} \leq \max_i \left(\frac{1}{\alpha_{ii}} \right) \cdot \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}} = \mathcal{O}_{\mathbb{P}}(1). \quad (\text{A.45})$$

We thus conclude that

$$\begin{aligned} \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}} &\leq \left(n^2 \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}}^2 + n \cdot \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}}^2 \right)^{1/2} = \mathcal{O}_{\mathbb{P}}(n^{1/2} \log^{1/2} n), \\ \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty} &\leq n \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}} + \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}} = \mathcal{O}_{\mathbb{P}}(n^{1/2} \log^{1/2} n). \end{aligned} \quad (\text{A.46})$$

as desired. \square .

A.3 Proof of Theorem 4 (Dense Regime)

Recall that n_k denote the number of vertices assigned to block k . For ease of exposition we shall assume that

$$n_k = n\pi_k, \quad \text{for all } k \in [K]. \quad (\text{A.47})$$

The case where the n_k are random with $\mathbb{E}[n_k] = n\pi_k$ is, except for a few slight modifications, identical. Note that $\tilde{\mathbf{M}}_0$ and \mathbf{M}_0 are symmetric matrices. Indeed

$$(\mathbf{D}_A^{-1}\hat{\mathbf{W}}^t)^T = (\mathbf{D}_A^{-1}\mathbf{A}\mathbf{D}_A^{-1} \cdots \mathbf{D}_A^{-1}\mathbf{A}\mathbf{D}_A^{-1})^T = \mathbf{D}_A^{-1}\hat{\mathbf{W}}^t$$

and similarly for $(\mathbf{D}_P^{-1}\mathbf{W}^t)^T = (\mathbf{D}_P^{-1}\mathbf{W}^t)$. Recall from Eq. (2.6) that $\mathbf{P} = \mathbf{Z}\mathbf{B}\mathbf{Z}^\top$ has a $K \times K$ blocks structure. Now let $\mathbf{B} = \mathbf{V}\mathbf{\Upsilon}\mathbf{V}^\top$ be the eigendecomposition of \mathbf{B} where \mathbf{V} is a $K \times \text{rank}(\mathbf{B})$ matrix with orthonormal columns, i.e., $\mathbf{V}^\top\mathbf{V} = \mathbf{I}$ and $\mathbf{\Upsilon}$ is a diagonal matrix containing the non-zero eigenvalues of \mathbf{B} . We then have, for any $t \geq 1$,

$$\mathbf{D}_P^{-1}\mathbf{W}^t = \mathbf{D}_P^{-1}(\mathbf{Z}\mathbf{B}\mathbf{Z}^\top\mathbf{D}_P^{-1})^t = \mathbf{D}_P^{-1}(\mathbf{Z}\mathbf{V}\mathbf{\Upsilon}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1})^t = \mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V}(\mathbf{\Upsilon}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V})^{t-1}\mathbf{\Upsilon}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1} \quad (\text{A.48})$$

Now $\mathbf{Z}\mathbf{D}_P^{-1}\mathbf{Z}$ is a $K \times K$ diagonal matrix and is positive definite. Therefore $\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V}$ is also positive definite. Let $\mathcal{D} := \mathbf{\Upsilon}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V}$. Note that \mathcal{D} is invertible and all of its eigenvalues are real-valued as it is similar to the matrix $(\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V})^{1/2}\mathbf{\Upsilon}(\mathbf{V}^\top\mathbf{Z}\mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V})^{1/2}$. We then have

$$\mathbf{M}'_0 := \frac{2|\mathbf{P}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t)\mathbf{D}_P^{-1}\mathbf{W}^t = \frac{2|\mathbf{P}|}{\kappa\gamma} \mathbf{D}_P^{-1}\mathbf{Z}\mathbf{V} \underbrace{\left(\sum_{t=t_L}^{t_U} (L-t)\mathcal{D}^{t-1} \right)}_{f(\mathcal{D})} \mathbf{\Upsilon}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{D}_P^{-1}.$$

Recall that $t_L \leq t_U \leq L$. Then $f(\mathcal{D})$ is singular if and only if there is an eigenvalue λ of \mathcal{D} such that $\sum_{t=t_L}^{t_U} (L-t)\lambda^{t-1} = 0$. As the set of roots for any fixed polynomial equation has measure 0, we conclude that $f(\mathcal{D})$ is invertible for almost every \mathcal{D} , or equivalently that the set of matrix \mathbf{B} whose induced $f(\mathcal{D})$ is singular has Lebesgue measure 0 in the space of $K \times K$ symmetric matrices. Note that $f(\mathcal{D})$ is guaranteed to be invertible whenever \mathbf{B} is positive semidefinite.

We thus assume, without loss of generality, that $f(\mathcal{D})$ is invertible. Then $f(\mathcal{D})\mathbf{\Upsilon}$ is also invertible and hence $\mathbf{V}f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top$ is a symmetric $K \times K$ matrix with distinct rows. Indeed, suppose that there exists two rows k and k' of $\mathbf{V}f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top$ that are the same. Let V_k and $V_{k'}$ be the k th and k' th row vectors for V . Then the k th and k' th rows of $\mathbf{V}f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top$ are $V_k f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top$ and $V_{k'} f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top$. We then have

$$\begin{aligned} V_k f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top &= V_{k'} f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top \implies (V_k - V_{k'}) f(\mathcal{D})\mathbf{\Upsilon}\mathbf{V}^\top \mathbf{V} \mathbf{\Upsilon} f(\mathcal{D})^\top (V_k - V_{k'})^\top = 0 \\ &\implies (V_k - V_{k'}) f(\mathcal{D})\mathbf{\Upsilon}^2 f(\mathcal{D})^\top (V_k - V_{k'})^\top = 0 \end{aligned}$$

which is only possible if $V_k - V_{k'} = 0$ as $f(\mathcal{D})\mathbf{\Upsilon}^2 f(\mathcal{D})^\top$ is positive definite. As \mathbf{B} has distinct rows, the rows of V are also distinct and hence $V_k - V_{k'} = 0$ if and only if $k = k'$. The matrix \mathbf{M}'_0 thus have the same $K \times K$ block structure as \mathbf{P} . Recall that the entries of \mathbf{M}_0 are the logarithm of the entries of \mathbf{M}'_0 and hence \mathbf{M}_0 is of the form

$$\mathbf{M}_0 = \begin{pmatrix} \xi_{11} \mathbf{1}_{n\pi_1} \mathbf{1}_{n\pi_1}^\top & \dots & \xi_{1K} \mathbf{1}_{n\pi_1} \mathbf{1}_{n\pi_K}^\top \\ \vdots & \vdots & \vdots \\ \xi_{K1} \mathbf{1}_{n\pi_K} \mathbf{1}_{n\pi_1}^\top & \dots & \xi_{KK} \mathbf{1}_{n\pi_K} \mathbf{1}_{n\pi_K}^\top \end{pmatrix} = \mathbf{Z}\mathbf{\Xi}\mathbf{Z}^\top,$$

where $\mathbf{\Xi} = (\xi_{ii'})_{K \times K}$ is a symmetric matrix with distinct rows. The non-zero eigenvalues of \mathbf{M}_0 coincides with that of

$$\mathbf{Z}^\top \mathbf{Z} \mathbf{\Xi} = n \text{diag}(\pi_1, \pi_2, \dots, \pi_K) \mathbf{\Xi}. \quad (\text{A.49})$$

As $\text{diag}(\pi_1, \dots, \pi_K)$ is fixed, it is left to study $\mathbf{\Xi}$. Since the entries in \mathbf{P} are $\Theta(1)$, we have $|\mathbf{P}| = \Theta(n^2)$, and entries in \mathbf{D}_P^{-1} and \mathbf{W}^t are of order $\Theta(n^{-1})$ (see Lemma A2). Therefore each entry of \mathbf{M}'_0 is of order $\Theta(1)$. Indeed, with some more careful book-keeping, one can show that under Eq. (A.47) each entry of \mathbf{M}'_0 can take on one of K^2 possible values (with these values not depending on n). Thus $\mathbf{\Xi}$ is a fixed matrix not depending on n and hence, by Eq. (A.49), all non-zero singular values of \mathbf{M}_0 grows at order n since $\text{diag}(\pi_1, \pi_2, \dots, \pi_K) \cdot \mathbf{\Xi}$ is fixed. In summary we have

$$\sigma_d(\mathbf{M}_0) = \Theta(n).$$

In addition as \mathbf{M}_0 has rank at most K , we have

$$\|\mathbf{M}_0\|_{\text{F}} \asymp n, \quad \|\mathbf{M}_0\|_\infty \asymp n \quad (\text{A.50})$$

Therefore, by the Davis-Kahan Theorem [35], [59] and Theorem 3 (dense regime), we have

$$\min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{\text{F}} \leq \frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}}}{\sigma_d(\mathbf{M}_0)} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n). \quad (\text{A.51})$$

To show the $2 \rightarrow \infty$ norm concentration of $\hat{\mathcal{F}}$, we need to further bound $\|\mathbf{U}\|_{2 \rightarrow \infty}$. Since \mathbf{M}_0 shares the exact same $K \times K$ block structure as \mathbf{B} (with \mathbf{B} assumed to have distinct rows), we can follow the same argument as that in Section 5.1 in [37]

and show that $[\mathbf{U}]_i = [\mathbf{U}]_{i'}$ if and only if $\mathbf{k}(i) = \mathbf{k}(i')$ which also implies $\|\mathbf{U}\|_{2 \rightarrow \infty} \lesssim n^{-1/2}$. Summarizing above results, by Theorem 3 (dense regime) and Theorem 4.2 in [36], we have

$$\min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \leq 14 \left(\frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_\infty}{\sigma_d(\mathbf{M}_0)} \right) \|\mathbf{U}\|_{2 \rightarrow \infty} = \mathcal{O}_{\mathbb{P}}(n^{-1} \log^{1/2} n), \quad (\text{A.52})$$

with high probability. We thus have

$$n^{1/2} \cdot \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n).$$

Hence, by combining Lemma 5.1 and the arguments in Section 5.1 of [37], we can show that running K -medians or K -means on the rows of $\hat{\mathbf{U}}$ will, with high probability, correctly recover the memberships of every nodes in \mathcal{G} . \square

A.4 Strong recovery for DCSBM

We now extend the proof of Theorem 4 to the setting of DCSBM. Recall that the edge probabilities matrix of a DCSBM is of the form $\mathbf{P} = \Theta \mathbf{Z} \mathbf{B} \mathbf{Z}^\top \Theta$ where $\Theta = \text{diag}(\theta_1, \dots, \theta_n)$ contains the degree correction factors. Substituting the above \mathbf{P} into Eq. (A.48) we obtain

$$\mathbf{D}_P^{-1} \mathbf{W}^\top = \mathbf{D}_P^{-1} (\Theta \mathbf{Z} \mathbf{B} \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1})^t = \mathbf{D}_P^{-1} \Theta \mathbf{Z} \mathbf{V} (\mathbf{Y} \mathbf{V}^\top \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1} \Theta \mathbf{Z} \mathbf{V})^{t-1} \mathbf{Y} \mathbf{V}^\top \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1} \quad (\text{A.53})$$

The matrix $\mathbf{Z}^\top \Theta \mathbf{D}_P^{-1} \Theta \mathbf{Z}$ is now a $K \times K$ positive definite diagonal matrix and hence $\mathbf{V}^\top \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1} \Theta \mathbf{Z} \mathbf{V}$ is also positive definite. Let $\mathcal{D} = \mathbf{Y} \mathbf{V}^\top \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1} \Theta \mathbf{Z} \mathbf{V}$. We then have

$$\mathbf{M}'_0 := \frac{2|\mathbf{P}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t) \mathbf{D}_P^{-1} \mathbf{W}^t = \frac{2|\mathbf{P}|}{\kappa\gamma} \mathbf{D}_P^{-1} \Theta \mathbf{Z} \mathbf{V} f(\mathcal{D}) \mathbf{Y} \mathbf{V}^\top \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1}$$

Once again the set of matrices \mathbf{B} for which $f(\mathcal{D})$ is singular has Lebesgue measure 0 in the space of $K \times K$ matrices and hence we can assume, without loss of generality, that $f(\mathcal{D})$ is invertible. Using the same reasoning as that in the proof of Theorem 4 we conclude that \mathbf{M}'_0 is of the form

$$\mathbf{M}'_0 = \mathbf{D}_P^{-1} \Theta \mathbf{Z} \Xi' \mathbf{Z}^\top \Theta \mathbf{D}_P^{-1}$$

where Ξ' is a $K \times K$ matrix with distinct rows. Now for any vertices i we have

$$p_i = \sum_j p_{ij} = \sum_j \theta_i \theta_j b_{\tau(i), \tau(j)} = \theta_i \sum_{k=1}^K \sum_{\tau(j)=k} \theta_j b_{\tau(i), k}$$

and hence $\theta_i/p_i = \theta_{i'}/p_{i'}$ whenever i and i' belong to the same community. The i th entry of $\mathbf{D}_P^{-1} \Theta$ is θ_i/p_i and thus

$$\mathbf{D}_P^{-1} \Theta \mathbf{Z} = \mathbf{Z} \tilde{\Theta}$$

for some $K \times K$ positive definite diagonal matrix $\tilde{\Theta}$. In summary we have

$$\mathbf{M}'_0 = \mathbf{D}_P^{-1} \Theta \mathbf{Z} \Xi' \mathbf{Z}^\top \mathbf{D}_P^{-1} = \mathbf{Z} \tilde{\Theta} \Xi' \tilde{\Theta} \mathbf{Z}^\top.$$

where $\tilde{\Theta} \Xi' \tilde{\Theta}$ is a $K \times K$ matrix with distinct rows. Again recall that the entries of \mathbf{M}_0 are the logarithm of the entries of \mathbf{M}'_0 and hence \mathbf{M}_0 is of the form

$$\mathbf{M}_0 = \mathbf{Z} \Xi \mathbf{Z}^\top,$$

for some $K \times K$ matrix Ξ . Let $\hat{\mathcal{F}}$ be the truncated SVD of $\tilde{\mathbf{M}}_0$ and \mathbf{U} be the leading singular vectors of \mathbf{M}_0 . We can then follow the remaining steps in the proof of Theorem 4 to show that

$$\begin{aligned} \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_F &\leq \frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_F}{\sigma_d(\mathbf{M}_0)} = \mathcal{O}_{\mathbb{P}}(n^{-1/2} \log^{1/2} n), \\ \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} &\leq 14 \left(\frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_\infty}{\sigma_d(\mathbf{M}_0)} \right) \|\mathbf{U}\|_{2 \rightarrow \infty} = \mathcal{O}_{\mathbb{P}}(n^{-1} \log^{1/2} n), \end{aligned}$$

noting that the involved high-probability bounds still hold for DCSBM as long as $\max_i \theta_i / \min_i \theta_i = \mathcal{O}(1)$ and $\theta_i \in (0, 1)$, because we are using entry-wise arguments in the main proofs. Thus clustering the rows of $\hat{\mathcal{F}}$ will, with high probability, exactly recover the memberships of every nodes in \mathcal{G} . \square

APPENDIX B

PROOFS UNDER THE SPARSE REGIME

The approach used in the proofs of our main theorems under the sparse regime is similar to that in the dense regime. However, if we simply use the same technique as in Section A then we only obtain a convergence rate of $\mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-3/2} \log^{1/2} n\}$ for $\|\hat{\mathbf{W}}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}}$ and $\|\hat{\mathbf{W}}^t - \hat{\mathbf{W}}^t\|_{\max}$ (with $t \geq 3$), which is too loose. More specifically the bounds for $\|\tilde{\mathbf{M}}_0 - \mathbf{M}\|_{\text{F}}$ and $\|\tilde{\mathbf{M}}_0 - \mathbf{M}\|_{\infty}$ Eq. (A.33) is currently valid only when $\rho_n = \omega(n^{-1/3} \log^{1/3} n)$. Before getting into the formal proofs, we first state Lemma B1 and Lemma B2 as the main technical results for bounding $\|\hat{\mathbf{W}}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}}$ and $\|\hat{\mathbf{W}}^t - \hat{\mathbf{W}}^t\|_{\max}$ for $t \geq 3$ under the sparse regime. We summarize the motivation behind these lemmas below.

- Lemma B1 is an analogue of Lemma A2 and is used repeatedly for bounding several important terms that frequently appear in our proofs.
- In Section A we show that the bound for $\|\hat{\mathbf{W}}^t - \hat{\mathbf{W}}^t\|_{\max}$ when $t \geq 3$ is of the same order as that for $\|\hat{\mathbf{W}}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}}$. For the sparse regime these bounds are generally of different magnitude as $n\rho_n$ increases. This difference is the main distinguish feature between the two regimes. Therefore, we derive a more accurate bound for $\|\hat{\mathbf{W}}^t - \hat{\mathbf{W}}^t\|_{\max}$ when $t \geq 3$ in Step 4 and Step 5 of the proof of Theorem 3 (sparse regime) presented below. The main challenge is in controlling the term $\|\mathbf{A}^t\|_{\max}$ as given in Lemma B2. Thus Lemma B2 is the main technical contribution of this section and might be of independent interest.

For ease of exposition we will only present the proof of Lemma B2 in this section; the proofs of the other lemmas are deferred to Section C.

Lemma B1. *Under Assumption 1 we have, for any $t \geq 1$ and sufficiently large n ,*

$$\|\mathbf{D}_A\| = \max_{1 \leq i \leq n} d_i = \mathcal{O}_{\mathbb{P}}(n\rho_n), \quad (\text{B.1})$$

$$\|\mathbf{D}_A^{-1}\| = \max_{1 \leq i \leq n} 1/d_i = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\}, \quad (\text{B.2})$$

$$\|\mathbf{D}_A - \mathbf{D}_P\| = \max_i |d_i - p_i| = \mathcal{O}_{\mathbb{P}}(\sqrt{n\rho_n \log n}), \quad (\text{B.3})$$

$$\|\mathbf{D}_A^{-1} - \mathbf{D}_P^{-1}\| = \max_i |d_i^{-1} - p_i^{-1}| = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-3/2} \sqrt{\log n}\}, \quad (\text{B.4})$$

$$\|\hat{\mathbf{W}}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\}, \quad (\text{B.5})$$

$$\max_{i,i'} w_{ii'}^{(\eta)}, \min_{i,i'} w_{ii'}^{(\eta)} \asymp 1/n, \quad (\text{B.6})$$

Remark B1. The bounds for $\|\hat{\mathbf{W}}^t\|_{\max}$ given in Eq. (B.5) is generally sub-optimal for $t \geq 2$. Nevertheless we use this bound purely as a stepping stone in proving Theorem 3. Once Theorem 3 is established we can improve the bound for $\|\hat{\mathbf{W}}^t\|_{\max}$ by applying triangle inequality incorporated with the tight bounds of $\|\mathbf{W}^t\|_{\max}$ and $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max}$.

Lemma B2. *Under Assumption 1, suppose ρ_n satisfies $\rho_n \rightarrow 0$ and,*

$$n^{-1/2} \log^{\beta_2} n \lesssim \rho_n \quad (\text{B.7})$$

for some $\beta_2 > 1/2$. Then we have

$$\begin{aligned} \|\mathbf{A}^2\|_{\max, \text{off}} &= \mathcal{O}_{\mathbb{P}}(n\rho_n^2), \\ \|\mathbf{A}^t\|_{\max} &= \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{t/2} + n^{t-1} \rho_n^t\}, \end{aligned} \quad (\text{B.8})$$

for any fixed $t \geq 3$. Furthermore, if $\rho_n \rightarrow 0$ and $n^{\frac{2-t}{t}} \lesssim \rho_n$ for some $t \geq 3$, then above bounds can be sharpened to

$$\|\mathbf{A}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t).$$

Proof. Suppose ρ_n satisfies Eq. (B.7) for some $\beta_2 > 1/2$. Define

$$\zeta_{i^*}^{ii'} := a_{ii^*} a_{i'i^*}, \quad \zeta_{ii'} := \max_{i,i'} \sum_{i^* \neq i, i'} a_{ii^*} a_{i'i^*} = \max_{i,i'} \sum_{i^* \neq i, i'} \zeta_{i^*}^{ii'}. \quad (\text{B.9})$$

Given $i \neq i'$, the $\{\zeta_{i^*}^{ii'}\}_{i^* \in [n], i^* \neq i, i'}$ are a set of independent Bernoulli variables with $c_2^2 \rho_n^2 \leq \mathbb{P}(\zeta_{i^*}^{ii'} = 1) \leq c_3^2 \rho_n^2$. A similar argument to that for deriving Eq. (C.4) yields (recall that $n^{-1/2} \log^{\beta_2} n \lesssim \rho_n$)

$$\log \left\{ \mathbb{P} \left(\frac{\zeta_{ii'}}{n-1} \leq \frac{c_2^2}{2} \rho_n^2 \right) \right\} \leq \log \left\{ \mathbb{P} \left(\zeta_{ii'} \leq \frac{1}{2} \mathbb{E} \zeta_{ii'} \right) \right\} \lesssim -C_1 n \rho_n^2 \lesssim -C_1 \log^{2\beta_2} n \quad (\text{B.10})$$

for all $i \neq i'$; here $C_1 \geq 0$ is a constant not depending on n or ρ_n . Eq. (B.10) together with a union bound then implies

$$\mathbb{P} \left\{ \max_{i \neq i'} \zeta_{ii'} \leq \frac{c_2^2}{2} n \rho_n^2 \right\} \leq n^2 \max_{i \neq i'} \left\{ \mathbb{P} \left(\zeta_{ii'} \leq \frac{c_2^2}{2} n \rho_n^2 \right) \right\} \lesssim \exp \left(2 \log n - C_1 \log^{2\beta_2} n \right) \rightarrow 0$$

as $n \rightarrow \infty$. We thus have $\max_{i \neq i'} \sum_{i^* \neq i, i'} a_{ii^*} a_{i'i^*} = \mathcal{O}_{\mathbb{P}}(n\rho_n^2)$ and hence

$$\|\mathbf{A}^2\|_{\max, \text{off}} = \max_{i \neq i'} \sum_{i^*=1}^n a_{ii^*} a_{i'i^*} \leq \max_{i \neq i'} \sum_{i^* \neq i, i'} a_{ii^*} a_{i'i^*} + 2 = \mathcal{O}_{\mathbb{P}}(n\rho_n^2).$$

We next consider the case when $t \geq 3$ and ρ_n satisfies $n^{\frac{2-t}{t}} \lesssim \rho_n$. We have

$$\|\mathbf{A}^t\|_{\max} \leq \|\mathbf{P}^t\|_{\max} + \|\mathbf{A}^t - \mathbf{P}^t\|_{\max}.$$

Under Assumption 1 we have

$$\|\mathbf{P}^t\|_{\max} \leq n \|\mathbf{P}^{t-1}\|_{\max} \|\mathbf{P}\|_{\max} \leq n^2 \|\mathbf{P}^{t-2}\|_{\max} \|\mathbf{P}\|_{\max}^2 \leq \dots \leq n^{t-1} \|\mathbf{P}\|_{\max}^t = \mathcal{O}(n^{t-1} \rho_n^t).$$

We now focus on bounding $\|\mathbf{A}^t - \mathbf{P}^t\|_{\max}$. Consider the following expansion for $\mathbf{A}^t - \mathbf{P}^t$

$$\mathbf{A}^t - \mathbf{P}^t = (\mathbf{A}^{t-1} - \mathbf{P}^{t-1})(\mathbf{A} - \mathbf{P}) + \mathbf{P}^{t-1}(\mathbf{A} - \mathbf{P}) + \sum_{b_0=1}^{t-1} \mathbf{A}^{t-1-b_0}(\mathbf{A} - \mathbf{P})\mathbf{P}^{b_0}. \quad (\text{B.11})$$

Let $\mathbf{E} = \mathbf{A} - \mathbf{P}$. Applying the same expansion to $\mathbf{A}^{t-1} - \mathbf{P}^{t-1}, \dots, \mathbf{A}^2 - \mathbf{P}^2$, we obtain

$$\begin{aligned} \mathbf{A}^t - \mathbf{P}^t &= (\mathbf{A}^{t-1} - \mathbf{P}^{t-1})\mathbf{E} + \mathbf{P}^{t-1}\mathbf{E} + \sum_{b_0=1}^{t-1} \mathbf{A}^{t-1-b_0}(\mathbf{A} - \mathbf{P})\mathbf{P}^{b_0} \\ &= \left\{ \sum_{b_1=0}^{t-2} \mathbf{A}^{t-2-b_1} \mathbf{E} \mathbf{P}^{b_1} \right\} \mathbf{E} + \mathbf{P}^{t-1}\mathbf{E} + \sum_{b_0=1}^{t-1} \mathbf{A}^{t-1-b_0} \mathbf{E} \mathbf{P}^{b_0} \\ &= (\mathbf{A}^{t-2} - \mathbf{P}^{t-2})\mathbf{E}^2 + \sum_{b_1=1}^{t-2} \mathbf{A}^{t-2-b_1} \mathbf{E} \mathbf{P}^{b_1} \mathbf{E} + \sum_{b_0=1}^{t-1} \mathbf{A}^{t-1-b_0} \mathbf{E} \mathbf{P}^{b_0} + \sum_{b'=1}^2 \mathbf{P}^{t-b'} \mathbf{E}^{b'} \\ &= \dots = \mathbf{E}^t + \underbrace{\sum_{c=0}^{t-1} \sum_{b=1}^{t-c-1} \mathbf{A}^{t-c-b-1} \mathbf{E} \mathbf{P}^b \mathbf{E}^c}_{\mathbf{L}_1} + \underbrace{\sum_{b'=1}^{t-1} \mathbf{P}^{t-b'} \mathbf{E}^{b'}}_{\mathbf{L}_2}. \end{aligned} \quad (\text{B.12})$$

Now for any summand appearing in \mathbf{L}_1 , if both $c \neq 0$ and $t - c - b - 1 \neq 0$ then

$$\begin{aligned} \|\mathbf{A}^{t-c-b-1} \mathbf{E} \mathbf{P}^b \mathbf{E}^c\|_{\max} &\leq \|\mathbf{A}\|_1 \cdot \|\mathbf{A}^{t-c-b-2} \mathbf{E} \mathbf{P}^b \mathbf{E}^c\|_{\max} \\ &\leq \dots \leq \|\mathbf{A}\|_1^{t-c-b-1} \|\mathbf{E} \mathbf{P}^b \mathbf{E}^c\|_{\max} \\ &\leq \|\mathbf{A}\|_1^{t-c-b-1} \|\mathbf{E} \mathbf{P}^b \mathbf{E}^{c-1}\|_{\max} (\|\mathbf{A}\|_1 + \|\mathbf{P}\|_1) \leq \|\mathbf{A}\|_1^{t-c-b-1} \|\mathbf{E} \mathbf{P}^b\|_{\max} (\|\mathbf{A}\|_1 + \|\mathbf{P}\|_1)^c. \end{aligned} \quad (\text{B.13})$$

Here $\|\mathbf{M}\|_1$ denote the maximum of the absolute column sum of a matrix \mathbf{M} . The bound in Eq. (B.13) also holds when $c = 0$ or $t - c - b - 1 = 0$. A similar argument yields

$$\|\mathbf{E} \mathbf{P}^b\|_{\max} \leq \|\mathbf{P}\|_{\max} (\|\mathbf{A}\|_1 + \|\mathbf{P}\|_1) \|\mathbf{P}\|_1^{b-1}. \quad (\text{B.14})$$

Observe that $\|\mathbf{A}\|_1 = \max_i d_i$ and $\|\mathbf{P}\|_1 = \max_i \sum_j p_{ij}$. Combining Eq. (B.13), Eq. (B.14) and Lemma B1, we have

$$\|\mathbf{L}_1\|_{\max} \leq \sum_{a=0}^{t-1} \sum_{b=1}^{t-c-1} \|\mathbf{A}^{t-c-b-1} \mathbf{E} \mathbf{P}^b \mathbf{E}^c\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t).$$

Similarly, we also have $\|\mathbf{L}_2\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t)$. We therefore have

$$\|\mathbf{A}^t\|_{\max} \leq \|\mathbf{P}^t\|_{\max} + \|\mathbf{E}^t\|_{\max} + \|\mathbf{L}_1\|_{\max} + \|\mathbf{L}_2\|_{\max} = \|\mathbf{E}^t\|_{\max} + \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t). \quad (\text{B.15})$$

Finally we bound $\|\mathbf{E}^t\|_{\max}$. Note that $\|\mathbf{M}\|_{\max} \leq \|\mathbf{M}\|_2$ for any matrix \mathbf{M} . Now, under Assumption 1, the maximal expected degree of \mathcal{G} is of order $n\rho_n \gtrsim n^{1-\beta} \gg \log^4 n$. We can thus apply the spectral norm concentration in [60] to obtain

$$\|\mathbf{E}^t\|_{\max} \leq \|\mathbf{A} - \mathbf{P}\|_2^t = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{t/2}). \quad (\text{B.16})$$

We thus have, after a bit of algebra, that $\|\mathbf{E}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t)$ whenever $n^{\frac{2-t}{t}} \lesssim \rho_n$. Combining Eq. (B.15), we have $\|\mathbf{A}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t)$ whenever $n^{\frac{2-t}{t}} \lesssim \rho_n \prec 1$ holds. \square

B.1 Proof of Theorem 3 (Sparse Regime)

The proof is organized as follows. In Step 1 through Step 4, we bound $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max}$ under the general sparse condition as specified in Assumption 1. These arguments are generalizations of the corresponding arguments in the proof of Theorem 2 in Section A.1. In Step 5, we provide an improved bound for $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max}$ when $t \geq 3$ and $\frac{t-3}{t-1} > \beta$. For ease of exposition we omitted some of the more mundane technical details from the current proof and refer the interested reader to Section C.4.

Step 1 (Bounding $\|\hat{\mathbf{W}} - \mathbf{W}\|_{\max}$): Similar to Step 1 in the proof of Theorem 2, we have

$$\hat{\mathbf{W}} - \mathbf{W} = \mathbf{A}\mathbf{D}_A^{-1} - \mathbf{P}\mathbf{D}_P^{-1} = \underbrace{\mathbf{A}\mathbf{D}_P^{-1}\mathbf{D}_A^{-1}(\mathbf{D}_P - \mathbf{D}_A)}_{\Delta_1^{(1)}} + \underbrace{(\mathbf{A} - \mathbf{P})\mathbf{D}_P^{-1}}_{\Delta_2^{(1)}} \quad (\text{B.17})$$

and hence

$$\|\Delta_1^{(1)}\|_{\max} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-3/2} \log^{1/2} n\}, \quad \|\Delta_2^{(1)}\|_{\max} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\}.$$

We therefore have

$$\|\hat{\mathbf{W}} - \mathbf{W}\|_{\max} \leq \|\Delta_1^{(1)}\|_{\max} + \|\Delta_2^{(1)}\|_{\max} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-3/2} \log^{1/2} n\} + \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\}. \quad (\text{B.18})$$

See Section C.4.1 for additional details in deriving the above inequalities.

Step 2 (Bounding $\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{diag}}$): Similar to Step 2 in the proof of Theorem 2,

$$\begin{aligned} \hat{\mathbf{W}}^2 - \mathbf{W}^2 &= \underbrace{(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}}_{\Delta_1^{(2)}} + \underbrace{\hat{\mathbf{W}}(\hat{\mathbf{W}} - \mathbf{W})}_{\Delta_2^{(2)}}, \\ \Delta_1^{(2)} &= \underbrace{\{\mathbf{A}\mathbf{D}_P^{-1}\mathbf{D}_A^{-1}(\mathbf{D}_P - \mathbf{D}_A) - \mathbf{A}\mathbf{D}_P^{-2}(\mathbf{D}_P - \mathbf{D}_A)\}\mathbf{W}}_{\Delta_1^{(2,1)}} + \underbrace{\mathbf{A}\mathbf{D}_P^{-2}(\mathbf{D}_P - \mathbf{D}_A)\mathbf{W}}_{\Delta_1^{(2,2)}} + \underbrace{(\mathbf{A} - \mathbf{P})\mathbf{D}_P^{-1}\mathbf{W}}_{\Delta_1^{(2,3)}}. \end{aligned} \quad (\text{B.19})$$

The bounds in Section C.4.2 imply

$$\|\Delta_1^{(2)}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n), \quad \|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n).$$

Furthermore, we also have

$$\|\Delta_2^{(2)}\|_{\max} = \|(\mathbf{W} - \hat{\mathbf{W}})(\mathbf{W} - \hat{\mathbf{W}}) - \mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} \leq \|(\mathbf{W} - \hat{\mathbf{W}})^2\|_{\max} + \|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max}. \quad (\text{B.20})$$

Replacing $\|\cdot\|_{\max}$ with $\|\cdot\|_{\max,\text{diag}}$ in Eq. (A.16), and following the same argument as that for Eq. (A.16) and Eq. (A.17), we have, by Lemma B1,

$$\|(\mathbf{W} - \hat{\mathbf{W}})^2\|_{\max,\text{diag}} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\}, \quad (\text{B.21})$$

and thus $\|\mathbf{W}^2 - \hat{\mathbf{W}}^2\|_{\max,\text{diag}} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1}\}$.

Step 3 (Bounding $\|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max,\text{off}}$): Similar to Eq. (A.16),

$$\begin{aligned} \|(\mathbf{W} - \hat{\mathbf{W}})^2\|_{\max,\text{off}} &= \max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{p_{i^*i'}}{p_{i'}} - \frac{a_{i^*i'}}{d_{i'}} \right) \right| \\ &\leq \underbrace{\max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{a_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{p_{i^*i'}}{p_{i'}} - \frac{a_{i^*i'}}{p_{i'}} \right) \right|}_{\delta_{2,\text{off}}^{(2,1)}} + \underbrace{\max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{a_{i^*i'}}{p_{i'}} - \frac{a_{i^*i'}}{d_{i'}} \right) \right|}_{\delta_{2,\text{off}}^{(2,2)}} \\ &\quad + \underbrace{\max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^*i'}}{p_{i'}} - \frac{a_{i^*i'}}{p_{i'}} \right) \right|}_{\delta_{2,\text{off}}^{(2,3)}}. \end{aligned} \quad (\text{B.22})$$

We then have the bounds

$$\delta_{2,\text{off}}^{(2,1)} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log n\}, \quad \delta_{2,\text{off}}^{(2,2)} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log n\}, \quad \delta_{2,\text{off}}^{(2,3)} = \mathcal{O}_{\mathbb{P}}\{n^{-3/2} \rho_n^{-1} \log^{1/2} n\}. \quad (\text{B.23})$$

For succinctness we only derive the bound for $\delta_{2,\text{off}}^{(2,1)}$ here. The bounds for $\delta_{2,\text{off}}^{(2,2)}$ and $\delta_{2,\text{off}}^{(2,3)}$ are derived similarly; see Section C.4.3 for more details.

Claim 1. Under the setting of Theorem 3 we have $\delta_{2,\text{off}}^{(2,1)} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log n\}$.

Proof. We start by writing

$$\sum_{i^*=1}^n \left(\frac{a_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{p_{i^*i'}}{p_{i'}} - \frac{a_{i^*i'}}{p_{i'}} \right) = \frac{1}{p_{i'}} \sum_{i^*=1}^n a_{ii^*} \left\{ \frac{(d_{i^*} - p_{i^*})^2}{p_{i^*}^2 d_{i^*}} + \frac{p_{i^*} - d_{i^*}}{p_{i^*}^2} \right\} (a_{i^*i'} - p_{i^*i'}), \quad (\text{B.24})$$

and hence

$$\delta_{2,\text{off}}^{(2,1)} \leq \underbrace{\max_{i \neq i'} \left| \frac{1}{p_{i'}} \sum_{i^*=1}^n a_{ii^*} \frac{(d_{i^*} - p_{i^*})^2}{p_{i^*}^2 d_{i^*}} (p_{i^* i'} - a_{i^* i'}) \right|}_{\xi_{ii'}} + \underbrace{\max_{i \neq i'} \left| \frac{1}{p_{i'}} \sum_{i^*=1}^n a_{ii^*} \left(\frac{d_{i^*} - p_{i^*}}{p_{i^*}^2} \right) (p_{i^* i'} - a_{i^* i'}) \right|}_{\zeta_{ii'}}. \quad (\text{B.25})$$

For the term $\xi_{ii'}$, by Lemma B1, we have

$$\begin{aligned} \max |\xi_{ii'}| &\leq \left(\max_i \frac{1}{p_i} \right) \left\{ \max_{i^*} \left| \frac{(d_{i^*} - p_{i^*})^2}{p_{i^*}^2 d_{i^*}} \right| \right\} \left(2 + \max_{i \neq i'} \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n |a_{ii^*}| |p_{i^* i'} - a_{i^* i'}| \right) \\ &\leq \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-3} \log n\} \left[2 + \max_{i \neq i'} \left\{ i^* : a_{ii^*} = a_{i' i^*} = 1 \right\} + \|\mathbf{P}\|_{\max} \cdot \max_{i \neq i'} \left\{ i^* : a_{ii^*} = 1, a_{i' i^*} = 0 \right\} \right] \\ &\leq \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-3} \log n\} \left(2 + \max_i d_i + \|\mathbf{P}\|_{\max} \max_i d_i \right) = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log n\}. \end{aligned}$$

For the term $\zeta_{ii'}$, first let $e_{ii'} = a_{ii'} - p_{ii'}$. Then expanding $d_{i^*} - p_{i^*}$ as $\sum_{i^{**}} e_{i^* i^{**}}$, we have

$$\zeta_{ii'} = \frac{1}{p_{i'}} \sum_{i^*=1}^n \frac{a_{ii^*}}{p_{i^*}^2} \left(\sum_{i^{**} \notin \{i', i\}} -e_{i^* i^{**}} e_{i^* i'} - \sum_{i^{**} \in \{i', i\}} e_{i^* i^{**}} e_{i^* i'} \right). \quad (\text{B.26})$$

Now, for fixed i, i' , since $a_{ii} = p_{ii} = 0$ for all i , we have

$$\begin{aligned} \frac{1}{p_{i'}} \sum_{i^*=1}^n \frac{a_{ii^*}}{p_{i^*}^2} \sum_{i^{**} \notin \{i', i\}} e_{i^* i^{**}} e_{i^* i'} &= \frac{1}{p_{i'}} \sum_{i^* \notin \{i, i'\}} \sum_{i^{**} \notin \{i', i, i^*\}} \frac{1}{p_{i^*}^2} a_{ii^*} e_{i^* i^{**}} e_{i^* i'} \\ &= \frac{1}{p_{i'}} \sum_{(i^*, i^{**}) \in \mathcal{T}(i, i')} \left(\frac{a_{ii^*} e_{i^* i'}}{p_{i^*}^2} + \frac{a_{ii^{**}} e_{i^{**} i'}}{p_{i^{**}}^2} \right) e_{i^* i^{**}} := \mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'}). \end{aligned} \quad (\text{B.27})$$

Here $\mathcal{T}(i, i') = \{(i^*, i^{**}) | i^* < i^{**}, i^* \notin \{i, i'\}, i^{**} \notin \{i, i'\}\}$ and $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$. Let us now, in addition to conditioning on \mathbf{P} , also condition on both \mathbf{a}_i and $\mathbf{a}_{i'}$. Then the sum for $\mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'})$ in Eq. (B.27) is a sum of *independent*, mean zero random variables, i.e., once we conditioned on \mathbf{P}, \mathbf{a}_i , and $\mathbf{a}_{i'}$, the $e_{i^* i^{**}}$ for $(i^*, i^{**}) \in \mathcal{T}(i, i')$ are independent. We therefore have

$$\begin{aligned} \text{Var} \left(\frac{1}{p_{i'}} \sum_{(i^*, i^{**}) \in \mathcal{T}(i, i')} \left(\frac{a_{ii^*} e_{i^* i'}}{p_{i^*}^2} + \frac{a_{ii^{**}} e_{i^{**} i'}}{p_{i^{**}}^2} \right) e_{i^* i^{**}} \middle| \mathbf{a}, \mathbf{a}' \right) &= \frac{1}{p_{i'}^2} \sum_{(i^*, i^{**}) \in \mathcal{T}(i, i')} \left(\frac{a_{ii^*} e_{i^* i'}}{p_{i^*}^2} + \frac{a_{ii^{**}} e_{i^{**} i'}}{p_{i^{**}}^2} \right)^2 \text{Var}[e_{i^*, i^{**}}] \\ &\lesssim (n\rho_n)^{-6} \cdot (n\rho_n) \cdot (d_i + d_{i'}) \end{aligned} \quad (\text{B.28})$$

Furthermore, we also have

$$\left| \frac{1}{p_{i'}} \left(\frac{a_{ii^*} e_{i^* i'}}{p_{i^*}^2} + \frac{a_{ii^{**}} e_{i^{**} i'}}{p_{i^{**}}^2} \right) e_{i^* i^{**}} \right| \lesssim (n\rho_n)^{-3}.$$

Therefore, by Bernstein inequality, for any $c' > 0$ there exists a constant $C' > 0$ such that, for all sufficiently large n ,

$$\mathbb{P} \left\{ \left| \mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'}) \right| \geq C'(n\rho_n)^{-5/2} (d_i + d_{i'})^{1/2} \log^{1/2} n \mid \mathbf{a}_i, \mathbf{a}_{i'} \right\} \leq n^{-c'} \quad (\text{B.29})$$

We can now uncondition with respect to \mathbf{a}_i and $\mathbf{a}_{i'}$. More specifically, for any t , we have

$$\mathbb{P} \left\{ \left| \mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'}) \right| \geq \tilde{t} \right\} \leq \mathbb{P} \left\{ \left| \mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'}) \right| \geq \tilde{t} \mid \max\{d_i, d_{i'}\} < Cn\rho_n \right\} \mathbb{P} \left(\max\{d_i, d_{i'}\} < Cn\rho_n \right) + \mathbb{P} \left(\max\{d_i, d_{i'}\} \geq Cn\rho_n \right),$$

where C is any arbitrary positive constant. Now let c be arbitrary. Then by Lemma B1 and Eq. (B.29), together with taking $\tilde{t} = C'(n\rho_n)^{-2} \log^{1/2} n$ for some constant C' , we have

$$\mathbb{P} \left\{ \left| \mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'}) \right| \geq C'(n\rho_n)^{-2} \log^{1/2} n \right\} \leq 2n^{-c}.$$

A union bound over the $\binom{n}{2}$ possible choices of \mathbf{a}_i and $\mathbf{a}_{i'}$ yields

$$\max_{i \neq i'} \left| \mathcal{S}(\mathbf{a}_i, \mathbf{a}_{i'}) \right| = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log^{1/2} n\}. \quad (\text{B.30})$$

Finally we also have

$$\max_{i, i'} \left| \frac{1}{p_{i'}} \sum_{i^*=1}^n \frac{a_{ii^*}}{p_{i^*}^2} \sum_{i^{**} \in \{i, i'\}} e_{i^* i^{**}} e_{i^* i'} \right| \lesssim \max_i d_i \cdot \left(\max_i 1/p_i \right)^3 = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2}\}. \quad (\text{B.31})$$

In summary, we have $\max_{i \neq i'} \zeta_{ii'} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log^{1/2} n\}$ and hence

$$\delta_{2,\text{off}}^{(2,1)} \leq \max_{ii'} \xi_{ii'} + \max_{ii'} \zeta_{ii'} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-2} \log n\}. \quad (\text{B.32})$$

□

Given the previous claim together with an argument similar to that for Eq. (A.18), we obtain

$$\begin{aligned} \|\mathbf{W}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}} &\leq \|\Delta_1^{(2)}\|_{\max} + \|\mathbf{W}(\mathbf{W} - \hat{\mathbf{W}})\|_{\max} + \delta_{2, \text{off}}^{(2,1)} + \delta_{2, \text{off}}^{(2,2)} + \delta_{2, \text{off}}^{(2,3)} \\ &= \mathcal{O}_{\mathbb{P}}\left[\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\sqrt{\log n}\right\}\right] \\ &= \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{-3/2}\rho_n^{-1}\log^{1/2} n) & \text{if } 0 \leq \beta < 1/2 \\ \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-2}\log n) & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{B.33})$$

Step 4 (Bounding $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max}$ for $t \geq 3$): The following argument is a generalization of the argument in **Step 3** of Section A.1. We first consider $t = 3$. We have

$$\|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max} \leq \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}\|_{\max} + \|\mathbf{W}^2(\hat{\mathbf{W}} - \mathbf{W})\|_{\max} \quad (\text{B.34})$$

For the first term in the RHS, by Eq. (B.21) and Lemma B1, we have

$$\begin{aligned} \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}\|_{\max} &\leq \max_i d_i \cdot \|\hat{\mathbf{W}}\|_{\max} \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}} \\ &= \mathcal{O}_{\mathbb{P}}(1) \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}\left(\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\log^{1/2} n\right\}\right). \end{aligned} \quad (\text{B.35})$$

For the second term in the RHS of Eq. (B.34), we apply the similar technique for $\Delta_1^{(2,1)}$ in Step 2 and deduce

$$\|\mathbf{W}^2(\hat{\mathbf{W}} - \mathbf{W})\|_{\max} = \mathcal{O}_{\mathbb{P}}(1) \|\mathbf{W}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}}. \quad (\text{B.36})$$

Combining Eq. (B.34), Eq. (B.35) and Eq. (B.36), we have

$$\|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\log^{1/2} n\right\}\right). \quad (\text{B.37})$$

Now for $t = 4$, we have

$$\hat{\mathbf{W}}^4 - \mathbf{W}^4 = (\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^2 + \mathbf{W}^2(\hat{\mathbf{W}}^2 - \mathbf{W}^2) = (\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^2 - (\mathbf{W}^2 - \hat{\mathbf{W}}^2)^2 + \hat{\mathbf{W}}^2(\hat{\mathbf{W}}^2 - \mathbf{W}^2). \quad (\text{B.38})$$

For $(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^2$, by Eq. (B.35) and Lemma B1

$$\begin{aligned} \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^2\|_{\max} &\leq \max_i d_i \cdot \|\hat{\mathbf{W}}\|_{\max} \cdot \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}\|_{\max} \\ &= \mathcal{O}_{\mathbb{P}}(n\rho_n \cdot (n\rho_n)^{-1}) \cdot \mathcal{O}_{\mathbb{P}}(1) \|\mathbf{W}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}(1) \|\mathbf{W}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}}. \end{aligned} \quad (\text{B.39})$$

Similarly, we have $\|\hat{\mathbf{W}}^2(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\|_{\max} = \mathcal{O}_{\mathbb{P}}(1) \|\mathbf{W}^2 - \hat{\mathbf{W}}^2\|_{\max, \text{off}}$. Furthermore,

$$\|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)^2\|_{\max} \leq n \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{off}}^2 + \|\hat{\mathbf{W}}^2 - \mathbf{W}^2\|_{\max, \text{diag}}^2 = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-2}\log n). \quad (\text{B.40})$$

Combining Eq. (B.38), Eq. (B.39) and Eq. (B.40), we obtain

$$\begin{aligned} \|\hat{\mathbf{W}}^4 - \mathbf{W}^4\|_{\max} &\leq \mathcal{O}_{\mathbb{P}}(1) \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^2\|_{\max} + \|(\mathbf{W}^2 - \hat{\mathbf{W}}^2)^2\|_{\max} \\ &= \mathcal{O}_{\mathbb{P}}\left(\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\log^{1/2} n\right\}\right). \end{aligned} \quad (\text{B.41})$$

For any $t \geq 5$, we can write $\hat{\mathbf{W}}^t - \mathbf{W}^t$ as

$$\hat{\mathbf{W}}^t - \mathbf{W}^t = \underbrace{(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^{t-2}}_{\Delta_1^{(3)}} + \underbrace{\mathbf{W}^2(\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2})}_{\Delta_2^{(3)}}.$$

We first consider $\Delta_1^{(3)}$. By Lemma B1 and Eq. (B.35)

$$\begin{aligned} \|\Delta_1^{(3)}\|_{\max} &\leq \max_i d_i \cdot \|\hat{\mathbf{W}}\|_{\max} \cdot \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}^{t-3}\|_{\max} \\ &\leq \dots \leq \left(\max_i d_i \|\hat{\mathbf{W}}\|_{\max}\right)^{t-3} \|(\hat{\mathbf{W}}^2 - \mathbf{W}^2)\hat{\mathbf{W}}\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\log^{1/2} n\right\}\right). \end{aligned} \quad (\text{B.42})$$

For $\Delta_2^{(3)}$, by Lemma B1

$$\|\Delta_2^{(3)}\|_{\max} \leq n \cdot \|\mathbf{W}^2\|_{\max} \|\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2}\|_{\max} = \mathcal{O}(1) \|\hat{\mathbf{W}}^{t-2} - \mathbf{W}^{t-2}\|_{\max}. \quad (\text{B.43})$$

Similar to Eq. (A.29), we obtain

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\log^{1/2} n\right\}\right) + \begin{cases} \mathcal{O}(1) \|\hat{\mathbf{W}}^4 - \mathbf{W}^4\|_{\max} & \text{if } t \text{ is even} \\ \mathcal{O}(1) \|\hat{\mathbf{W}}^3 - \mathbf{W}^3\|_{\max}, & \text{if } t \text{ is odd} \end{cases} \quad (\text{B.44})$$

Eq. (B.37) and Eq. (B.41) then implies

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(\max\left\{(n\rho_n)^{-2}\log n, n^{-3/2}\rho_n^{-1}\log^{1/2} n\right\}\right) = \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{-3/2}\rho_n^{-1}\log^{1/2} n) & \text{if } 0 \leq \beta < 1/2 \\ \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-2}\log n) & \text{otherwise.} \end{cases}$$

Step 5 (Bounding $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max}$ for $t \geq 4$ and $\frac{t-3}{t-1} \geq \beta$): Similar to Eq. (B.11) and Eq. (B.12) in the proof of Lemma B2, we write

$$\hat{\mathbf{W}}^t - \mathbf{W}^t = (\hat{\mathbf{W}} - \mathbf{W})^t + \sum_{r=0}^{t-1} \sum_{s=1}^{t-r-1} \hat{\mathbf{W}}^{t-r-s-1} (\hat{\mathbf{W}} - \mathbf{W}) \mathbf{W}^s (\hat{\mathbf{W}} - \mathbf{W})^r + \sum_{r'=1}^{t-1} \mathbf{W}^{t-r'} (\hat{\mathbf{W}} - \mathbf{W})^{r'}. \quad (\text{B.45})$$

In Step 2, we shown $\|(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\rho_n^{-1/2}\sqrt{\log n})$. Let $d_{\max} = \max_i d_i$. A similar argument to that for L_1 in the proof of Lemma B2 yields, for $0 \leq r \leq t-1$ and $1 \leq s \leq t-r-1$, that

$$\begin{aligned} \|\hat{\mathbf{W}}^{t-r-s-1} (\hat{\mathbf{W}} - \mathbf{W}) \mathbf{W}^s (\hat{\mathbf{W}} - \mathbf{W})^r\| &\leq (d_{\max} \|\hat{\mathbf{W}}\|_{\max})^{t-r-s-1} \|(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}^s\|_{\max} (n\|\mathbf{W}\|_{\max} + d_{\max} \|\hat{\mathbf{W}} - \mathbf{W}\|_{\max})^r \\ &= \mathcal{O}_{\mathbb{P}}\left(\|(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}^s\|_{\max}\right) \\ &= \mathcal{O}_{\mathbb{P}}\left(n^{s-1} \|(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}\|_{\max} \|\mathbf{W}\|_{\max}^{s-1}\right) \\ &= \mathcal{O}_{\mathbb{P}}\left(\|(\hat{\mathbf{W}} - \mathbf{W})\mathbf{W}\|_{\max}\right) = \mathcal{O}_{\mathbb{P}}\left(n^{-3/2}\rho_n^{-1/2}\log^{1/2} n\right), \end{aligned}$$

where the second inequality follows from Lemma B1 and Eq. (B.18) and the last inequality follows from Lemma B1. As t is finite, we have

$$\left\| \sum_{r=0}^{t-1} \sum_{s=1}^{t-r-1} \hat{\mathbf{W}}^{t-r-s-1} (\hat{\mathbf{W}} - \mathbf{W}) \mathbf{W}^s (\hat{\mathbf{W}} - \mathbf{W})^r \right\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(n^{-3/2}\rho_n^{-1/2}\log^{1/2} n\right).$$

Similarly,

$$\left\| \sum_{r'=1}^{t-1} \mathbf{W}^{t-r'} (\hat{\mathbf{W}} - \mathbf{W})^{r'} \right\|_{\max} = \mathcal{O}_{\mathbb{P}}\left(n^{-3/2}\rho_n^{-1/2}\log^{1/2} n\right).$$

Recalling Eq. (B.45), we obtain

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} \leq \|(\hat{\mathbf{W}} - \mathbf{W})^t\|_{\max} + \mathcal{O}_{\mathbb{P}}\left(n^{-3/2}\rho_n^{-1/2}\log^{1/2} n\right). \quad (\text{B.46})$$

Now we focus on $(\hat{\mathbf{W}} - \mathbf{W})^t$. We start with the polynomial expansion

$$(\hat{\mathbf{W}} - \mathbf{W})^t = \left\{ \mathbf{A}(\mathbf{D}_A^{-1} - \mathbf{D}_P^{-1}) + (\mathbf{A} - \mathbf{P})\mathbf{D}_P^{-1} \right\}^t = \left\{ (\mathbf{A} - \mathbf{P})\mathbf{D}_P^{-1} \right\}^t + \sum_{\substack{\mathbf{c} \in \{1,2\}^t \\ \mathbf{c} \neq (1,\dots,1)}} \prod_{r=1}^t \Xi_{c_r}.$$

Here c_r represents the r th coordinate of $\mathbf{c} = (c_1, \dots, c_t) \in \{1,2\}^t$ and

$$\Xi_{c_r} = \begin{cases} (\mathbf{A} - \mathbf{P})\mathbf{D}_P^{-1}, & \text{if } c_r = 1 \\ \mathbf{A}(\mathbf{D}_A^{-1} - \mathbf{D}_P^{-1}), & \text{if } c_r = 2. \end{cases}$$

We note that there are $2^t - 1$ distinct $\mathbf{c} \neq (1, 1, \dots, 1)$. Now for any given \mathbf{c} , let $r^* = r^*(\mathbf{c})$ be the smallest value of r such that $c_r = 2$. We emphasize that r^* depends on \mathbf{c} ; however, for simplicity of notation we make this dependency implicit. We further denote

$$(\Xi_{c_r}^1, \Xi_{c_r}^2) := \begin{cases} (\mathbf{A}\mathbf{D}_A^{-1}, -\mathbf{A}\mathbf{D}_P^{-1}), & \text{if } c_r = 1, \\ (\mathbf{A}\mathbf{D}_P^{-1}, -\mathbf{P}\mathbf{D}_P^{-1}), & \text{if } c_r = 2, \end{cases}$$

so that $\Xi_{c_r} = \Xi_{c_r}^1 + \Xi_{c_r}^2$. Given \mathbf{c} , we could write

$$\prod_{r=1}^t \Xi_{c_r} = \prod_{j < r^*} (\Xi_{c_j}^1 + \Xi_{c_j}^2) \cdot \Xi_{c_{r^*}} \cdot \prod_{k > r^*} (\Xi_{c_k}^1 + \Xi_{c_k}^2) = \sum_{\mathbf{m} \in \{1,2\}^t} \underbrace{\prod_{j < r^*} \Xi_{c_j}^{m_j} \cdot \Xi_{c_{r^*}} \cdot \prod_{k > r^*} \Xi_{c_k}^{m_k}}_{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}} \quad (\text{B.47})$$

Now for any $c_r \in \{1,2\}$ and $m \in \{1,2\}$, the ii' th entry of $\Xi_{c_r}^m$ is $\xi_{ii'}^{c_r, m} \cdot \vartheta_{ii'}^{c_r, m}$ where

$$(\xi_{ii'}^{c_r, m}, \vartheta_{ii'}^{c_r, m}) = \begin{cases} (a_{ii'}, 1/d_{i'}) & \text{if } c_r = 1 \text{ and } m = 1, \\ (a_{ii'}, -1/p_{i'}) & \text{if } c_r = 1 \text{ and } m = 2, \\ (a_{ii'}, 1/p_{i'}) & \text{if } c_r = 2 \text{ and } m = 1, \\ (p_{ii'}, -1/p_{i'}) & \text{if } c_r = 2 \text{ and } m = 2. \end{cases} \quad (\text{B.48})$$

Using the above notations, we can now write the ii' th entry of $\mathfrak{S}_{\mathbf{m}, \mathbf{c}}$ as

$$\mathfrak{S}_{ii'}^{\mathbf{m}, \mathbf{c}} = \sum_{i \in \{1, \dots, n\}^{t-1}} \left(\prod_{j < r^*} \xi_{i_{j-1} i_j}^{c_j, m_j} \vartheta_{i_{j-1} i_j}^{c_j, m_j} \right) \cdot a_{i_{r^*} - 1 i_{r^*}} \left(\frac{1}{d_{i_{r^*}}} - \frac{1}{p_{i_{r^*}}} \right) \left(\prod_{k > r^*} \xi_{i_{k-1} i_k}^{c_k, m_k} \cdot \vartheta_{i_{k-1} i_k}^{c_k, m_k} \right),$$

where, with a slight abuse of notation, we denote $\mathbf{i} = (i_1, \dots, i_{t-1}) \in \{1, \dots, n\}^{t-1}$, $i_0 = i$ and $i_t = i'$. We thus have

$$\begin{aligned} |\mathfrak{S}_{ii'}^{\mathbf{m}, \mathbf{c}}| &\leq \sum_i \left| a_{i_{r^*} - 1 i_{r^*}} \prod_{j \neq r^*} \xi_{i_{j-1} i_j}^{c_j, m_j} \right| \cdot \left| \left(\frac{1}{d_{i_{r^*}}} - \frac{1}{p_{i_{r^*}}} \right) \prod_{j \neq r^*} \vartheta_{i_{j-1} i_j}^{c_j, m_j} \right| \\ &\leq \|\mathbf{D}_{\mathbf{A}}^{-1}\|^{s(\mathbf{c}, \mathbf{m})} \|\mathbf{D}_{\mathbf{P}}^{-1}\|^{t-1-s(\mathbf{c}, \mathbf{m})} \cdot \|\mathbf{D}_{\mathbf{A}}^{-1} - \mathbf{D}_{\mathbf{P}}^{-1}\| \left(\sum_i a_{i_{r^*} - 1 i_{r^*}} \prod_{j \neq r^*} \xi_{i_{j-1} i_j}^{c_j, m_j} \right). \end{aligned}$$

Here $s(\mathbf{c}, \mathbf{m})$ is the number of indices r with $r \neq r^*$ and $(c_r, m_r) = (1, 1)$. Now, by Lemma B1 and Assumption 1, we have

$$\|\mathfrak{S}_{\mathbf{m}, \mathbf{c}}\|_{\max} = \mathcal{O}_{\mathbb{P}} \left\{ (n\rho_n)^{-1/2-t} (\log n)^{1/2} \cdot \max_{ii'} \left(\sum_i a_{i_{r^*} - 1 i_{r^*}} \prod_{j \neq r^*} \xi_{i_{j-1} i_j}^{c_j, m_j} \right) \right\}. \quad (\text{B.49})$$

with the convention $i_0 = i$ and $i_t = i'$. Now define a matrix-valued function $\xi_{\mathbf{A}, \mathbf{P}}(\cdot)$ by

$$\xi_{\mathbf{A}, \mathbf{P}}(c_r, m_r) = \begin{cases} \mathbf{A} & \text{if } (c_r, m_r) \in \{(1, 1), (1, 2), (2, 1)\}, \\ \mathbf{P} & \text{if } (c_r, m_r) = (2, 2), \end{cases}$$

Also define

$$\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}} = \left\{ \prod_{j < r^*} \xi_{\mathbf{A}, \mathbf{P}}(c_j, m_j) \right\} \cdot \mathbf{A} \cdot \left\{ \prod_{k > r^*} \xi_{\mathbf{A}, \mathbf{P}}(c_k, m_k) \right\}.$$

Then by the definition of the $\xi_{ii'}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}$ in Eq. (B.48), we have

$$\|\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}\|_{\max} = \max_{ii'} \left(\sum_i a_{i_{r^*} - 1 i_{r^*}} \prod_{j \neq r^*} \xi_{i_{j-1} i_j}^{c_j, m_j} \right). \quad (\text{B.50})$$

We now consider two cases to bound $\|\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}\|_{\max}$.

Case 1: Suppose that for the given \mathbf{m}, \mathbf{c} , there exists at least one index $r \neq r^*$ with $(c_r, m_r) = 2$, i.e., the matrix \mathbf{P} appears at least once among the collection of $\xi_{\mathbf{A}, \mathbf{P}}(c_r, m_r)$ for $r \neq r^*$. Then $\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}$ must have the form

$$\underbrace{\dots}_{g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}}, \quad \text{or} \quad \underbrace{\dots}_{g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \mathbf{A} \mathbf{P} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}}. \quad (\text{B.51})$$

Note that it is possible that $g(\mathbf{c}, \mathbf{m}) = 0$ or $g(\mathbf{c}, \mathbf{m}) = t-2$ as \mathbf{A} could be either the first or last matrix in the product $\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}$. Consider the first form in Eq. (B.51). We have

$$\begin{aligned} \|\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}\|_{\max} &= \left\| \underbrace{\dots}_{g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \right\|_{\max} \\ &\stackrel{(1)}{\leq} \begin{cases} n \max_{i, i'} p_{ii'} \cdot \left\| \underbrace{\dots}_{g(\mathbf{c}, \mathbf{m})-1 \text{ matrices}} \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \right\| \\ (\text{if first matrix is } \mathbf{P}) \\ \max_i d_i \cdot \left\| \underbrace{\dots}_{g(\mathbf{c}, \mathbf{m})-1 \text{ matrices}} \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \right\| \\ (\text{if first matrix is } \mathbf{A}) \end{cases} \\ &\stackrel{(1)}{=} \mathcal{O}_{\mathbb{P}} \left(n\rho_n \left\| \underbrace{\dots}_{g(\mathbf{c}, \mathbf{m}-1) \text{ matrices}} \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \right\| \right) \\ &\stackrel{(1)}{\leq} \dots \stackrel{(1)}{\leq} \mathcal{O}_{\mathbb{P}} \left((n\rho_n)^{g(\mathbf{c}, \mathbf{m})} \left\| \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-2-g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \right\| \right) \\ &\stackrel{(2)}{\leq} \mathcal{O}_{\mathbb{P}} \left((n\rho_n)^{g(\mathbf{c}, \mathbf{m})+1} \left\| \mathbf{P} \mathbf{A} \underbrace{\dots}_{t-3-g(\mathbf{c}, \mathbf{m}) \text{ matrices}} \right\| \right) \\ &\stackrel{(2)}{\leq} \dots \stackrel{(2)}{\leq} \mathcal{O}_{\mathbb{P}} \left((n\rho_n)^{t-2} \|\mathbf{P} \mathbf{A}\|_{\max} \right) \leq \mathcal{O}_{\mathbb{P}} \left((n\rho_n)^{t-2} \max_{i, i'} p_{ii'} \max_i d_i \right) = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t). \end{aligned} \quad (\text{B.52})$$

In the above inequality, all relationships with $\stackrel{(1)}{\leq}$ and $\stackrel{(2)}{\leq}$ are when we removed either \mathbf{P} or \mathbf{A} from before and after the term $\mathbf{P} \mathbf{A}$, respectively. An identical argument also yields

$$\|\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{\mathbf{m}, \mathbf{c}}}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t) \quad (\text{B.53})$$

for the second form in Eq. (B.51).

Case 2: Suppose now that, for all $r \neq r^*$, we have $(c_r, m_r) \neq (2, 2)$, i.e., $\xi_{\mathbf{A}, \mathbf{P}}(c_r, m_r) = \mathbf{A}$ for all $r \neq r^*$. Then $\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{m,c}} = \mathbf{A}^t$ and, by the fact that $\frac{t-3}{t-1} > \beta$ already yields $n^{\frac{2-t}{t}} \lesssim \rho_n$ hold, we have by Lemma B2,

$$\|\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{m,c}}\|_{\max} = \|\mathbf{A}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{t-1} \rho_n^t). \quad (\text{B.54})$$

Combining Eq. (B.47), Eq. (B.49), Eq. (B.53) and Eq. (B.54), we have

$$\begin{aligned} \left\| \prod_{r=1}^t \Xi_{c_r} \right\|_{\max} &\leq \sum_{\mathbf{m} \in \{1, 2\}^t} \|\mathfrak{S}_{\mathbf{m}, c}\|_{\max} \\ &= \mathcal{O}_{\mathbb{P}}\left\{2^t \|\xi_{\mathbf{A}, \mathbf{P}}^{\mathfrak{S}_{m,c}}\|_{\max} (n \rho_n)^{-1/2-t} \log^{1/2} n\right\} \\ &= \mathcal{O}_{\mathbb{P}}\left((n \rho_n)^{-1/2-t} \log^{1/2} n\right) \cdot \mathcal{O}_{\mathbb{P}}\left(n^{t-1} \rho_n^t\right) = \mathcal{O}_{\mathbb{P}}\left\{n^{-3/2} \rho_n^{-1/2} \log^{1/2} n\right\}. \end{aligned} \quad (\text{B.55})$$

We then have, by Eq. (B.46),

$$\begin{aligned} \|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} &\leq \|(\hat{\mathbf{W}} - \mathbf{W})^t\|_{\max} + \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n) \\ &\leq \|\{(\mathbf{A} - \mathbf{P})\mathbf{D}_{\mathbf{P}}^{-1}\}^t\|_{\max} + \sum_{\substack{\mathbf{c} \in \{1, 2\}^t \\ \mathbf{c} \neq (1, \dots, 1)}} \left\| \prod_{r=1}^t \Xi_{c_r} \right\|_{\max} + \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n) \\ &\leq \|\{(\mathbf{A} - \mathbf{P})\mathbf{D}_{\mathbf{P}}^{-1}\}^t\|_{\max} + \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n). \end{aligned} \quad (\text{B.56})$$

The last inequality in the above display follows from the fact that the number of distinct \mathbf{c} is $2^t - 1$ which is finite and does not depend on n . Finally we focus on $\|\{(\mathbf{A} - \mathbf{P})\mathbf{D}_{\mathbf{P}}^{-1}\}^t\|_{\max}$. An argument similar to that for $\|(\mathbf{A} - \mathbf{P})^t\|_{\max}$ in the proof of Lemma B2 yields

$$\begin{aligned} \|\{(\mathbf{A} - \mathbf{P})\mathbf{D}_{\mathbf{P}}^{-1}\}^t\|_{\max} &\leq \|\mathbf{A} - \mathbf{P}\|_2^t \cdot \|\mathbf{D}_{\mathbf{P}}^{-1}\|_2^t \\ &\lesssim (n \rho_n)^{-t/2} \|\mathbf{A} - \mathbf{P}\|_2^t \quad (\text{By Lemma B1}) \\ &= \mathcal{O}_{\mathbb{P}}\{(n \rho_n)^{-t/2}\} \quad (\text{By Eq. (B.16)}). \end{aligned} \quad (\text{B.57})$$

It could be directly checked that the rate $(n \rho_n)^{-t/2} \lesssim n^{-3/2} \rho_n^{-1/2} \log^{1/2} n$ when $\frac{t-3}{t-1} > \beta$. Thus, together with Eq. (B.55), we conclude

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n) \quad (\text{B.58})$$

for $t \geq 4$ and $\frac{t-3}{t-1} > \beta$, as desired. \square

B.2 Proof of Theorem 3 (Sparse Regime)

We will only present the proof of bounding $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}}$ and $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty}$ here as the rates for $\|\mathbf{M}_0\|_{\text{F}}$ and $\|\mathbf{M}_0\|_{\infty}$ are derived in the proof of Theorem 4 (c.f. Eq. (B.65)). For sufficient large $t_U > 0$ such that, w.h.p. $\tilde{\mathbf{M}}_0$ is well defined, we use the same notations of $\mathbf{I}_{\mathbf{A}}, \mathbf{I}_{\mathbf{P}}$ as defined in Eq. (A.30). Then we also have

$$\begin{aligned} \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}} &\leq \max_{i, i'} \left(\frac{1}{\alpha_{ii'}} \right) \cdot \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}}, \\ \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}} &\leq \max_{i, i'} \left(\frac{1}{\alpha_{ii'}} \right) \cdot \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}}, \end{aligned} \quad (\text{B.59})$$

where $\alpha_{ii'} \in (\min\{I_{\mathbf{A}}^{ii'}, I_{\mathbf{P}}^{ii'}\}, \max\{I_{\mathbf{A}}^{ii'}, I_{\mathbf{P}}^{ii'}\})$, $I_{\mathbf{A}}^{ii'}, I_{\mathbf{P}}^{ii'}$ are ii' th entries of $\mathbf{I}_{\mathbf{A}}, \mathbf{I}_{\mathbf{P}}$. We now extend the argument in the dense regime to the sparse regime.

Step 1 (Bounding $\max_{i, i'} \alpha_{ii'}^{-1}$): We also have $\alpha_{ii'}$ is between $I_{\mathbf{A}}^{ii'}$ and $I_{\mathbf{P}}^{ii'}$ and $\max_{i, i'} \frac{1}{\alpha_{ii'}} \leq \max_{i, i'} \left\{ \frac{1}{I_{\mathbf{A}}^{ii'}}, \frac{1}{I_{\mathbf{P}}^{ii'}} \right\}$. By Lemma B1 and Assumption 1, we have

$$\max_{i, i'} \frac{1}{I_{\mathbf{P}}^{ii'}} = \max_{i, i'} \frac{1}{\sum_{t=t_L}^{t_U} (L-t) \left(\sum_{ii'} p_{ii'} \right) \cdot \left(\frac{w_{ii'}^{(t)}}{p_i} \right)} \leq \max_{i, i'} \frac{p_i}{(L-t_L)(\sum_{ii'} p_{ii'}) (w_{ii'}^{(t_L)})} \lesssim \frac{n \rho_n}{n^2 \rho_n \cdot \frac{1}{n}} = 1.$$

We also have $\frac{1}{I_{\mathbf{A}}^{ii'}} \leq \frac{d_i}{(L-t_L)(\sum_{ii'} a_{ii'}) (w_{ii'}^{(t_L)})}$ and we consider off-diagonal and diagonal cases separately.

(1) When $i \neq i'$: By Theorem 3 and Lemma B1, we have $\min_{i, i'} w_{ii'}^{(t_L)}, \max_{i, i'} w_{ii'}^{(t_L)} \asymp n^{-1}$ for fixed $2 \leq t_L$ and

$$\max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}| / \min_{i, i'} w_{ii'}^{(t_L)} = \begin{cases} \mathcal{O}_{\mathbb{P}}\left(n^{-3/2} \rho_n^{-1} \log^{1/2} n / (1/n)\right) = o_{\mathbb{P}}(1) & (\text{when } t_L \geq 2 \text{ and } \beta < \frac{1}{2}) \\ \mathcal{O}_{\mathbb{P}}\left(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n / (1/n)\right) = o_{\mathbb{P}}(1) & (\text{when } t_L \geq 4 \text{ and } \beta < \frac{t_L-3}{t_L-1}) \end{cases} \quad (\text{B.60})$$

which implies

$$0 < \min_{i,i'} w_{ii'}^{(t_L)} - \max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}| \asymp \min_{i,i'} w_{ii'}^{(t_L)} = \mathcal{O}_{\mathbb{P}}(1/n).$$

Furthermore we have

$$\begin{aligned} \max_{i \neq i'} \frac{1}{w_{ii'}^{(t_L)} - \max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}|} &= \frac{1}{\min_{i \neq i'} (w_{ii'}^{(t_L)}) - \max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}|} = \mathcal{O}_{\mathbb{P}}(n), \\ \max_{i \neq i'} \frac{1}{I_{\mathbf{A}}^{ii'}} &\leq \max_{i \neq i'} \frac{d_i}{(L-t_L)(\sum_{ii'} a_{ii'})(\hat{w}_{ii'}^{(t_L)})} \leq \max_i d_i \cdot \max_i \frac{1}{nd_i} \cdot \max_{i \neq i'} \frac{1}{w_{ii'}^{(t_L)} - \max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}|} = \mathcal{O}_{\mathbb{P}}(1). \end{aligned}$$

(2) When $i = i'$: Similarly, when $t_L = 2$, we have $\max_i \frac{1}{\hat{w}_{ii}^{(2)}} = \max_i \frac{d_i}{\sum_{i'=1}^n a_{ii'}/d_i} \leq (\max_i d_i)^2 \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(n\rho_n)$. Thus

$$\max_i \frac{1}{I_{\mathbf{A}}^{ii}} \lesssim \max_i d_i \cdot \max_i \frac{1}{nd_i} \cdot \max_i \frac{1}{\hat{w}_{ii}^{(2)}} = \mathcal{O}_{\mathbb{P}}(\rho_n).$$

When $t_L \geq 3$, since $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max, \text{diag}} = \mathcal{O}_{\mathbb{P}}(n^{-3/2}\rho_n^{-1/2}\sqrt{\log n})$, a similar argument as the $i \neq i'$ case shows

$$\max_i \frac{1}{I_{\mathbf{A}}^{ii}} \leq \max_i \frac{d_i}{(L-t_L)(\sum_{ii'} a_{ii})[(\hat{w}_{ii}^{(t_L)} - w_{ii}^{(t_L)}) + w_{ii}^{(t_L)}]} = \mathcal{O}_{\mathbb{P}}(1).$$

In summary we have both $\max_{i,i'} \frac{1}{I_{\mathbf{A}}^{ii'}} = \mathcal{O}_{\mathbb{P}}(1)$ and $\max_{i,i'} \frac{1}{I_{\mathbf{P}}^{ii'}} = \mathcal{O}_{\mathbb{P}}(1)$ under the conditions of Theorem 3 (sparse regime). We thus conclude

$$\max_{i,i'} \frac{1}{\alpha_{ii'}} = \mathcal{O}_{\mathbb{P}}(1). \quad (\text{B.61})$$

Step 2 (Bound of $\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}}$ and $\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}}$): We consider two cases for (t_L, β)

Case 1 ($t_L \geq 2$ and $\beta < 0.5$): For $t \geq 2$ we have

$$\begin{aligned} \mathbf{(a).} \max_{i,i'} \left| \left(\sum_{ii'} a_{ii'} - \sum_{ii'} p_{ii'} \right) \frac{w_{ii'}^{(t)}}{d_i} \right| &\leq n \max_{i \neq i'} |d_i - p_i| \cdot \max_{i,i'} w_{ii'}^{(t)} \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1/2}\sqrt{\log n}), \\ \mathbf{(b).} \max_{i,i'} \left| \sum_{ii'} p_{ii'} \left(\frac{1}{d_i} - \frac{1}{p_i} \right) w_{ii'}^{(t)} \right| &= \left| \sum_{ii'} p_{ii'} \right| \cdot \max_{i \neq i'} \left| \frac{p_i - d_i}{p_i d_i} \right| \cdot \max_{i,i'} w_{ii'}^{(t)} = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1/2}\sqrt{\log n}), \\ \mathbf{(c).} \max_{i \neq i'} \left| \left(\sum_{ii'} a_{ii'} \right) \frac{\hat{w}_{ii'}^{(t)} - w_{ii'}^{(t)}}{d_i} \right| &\leq n \max_i d_i \cdot \max_{i \neq i'} |\hat{w}_{ii'}^{(t)} - w_{ii'}^{(t)}| \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}\sqrt{\log n}), \\ \mathbf{(d).} \max_i \left| \left(\sum_{ii'} a_{ii'} \right) \frac{\hat{w}_{ii}^{(t)} - w_{ii}^{(t)}}{d_i} \right| &\leq n \max_i d_i \cdot \max_i |\hat{w}_{ii}^{(t)} - w_{ii}^{(t)}| \cdot \max_i \frac{1}{d_i} = \mathcal{O}_{\mathbb{P}}(\rho_n^{-1}), \end{aligned} \quad (\text{B.62})$$

by Lemma B1, Theorem 3 and Eq. (3.6). Thus a similar argument as the proof under dense regime gives

$$\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}} = \mathcal{O}_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}\sqrt{\log n}), \quad \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}} = \mathcal{O}_{\mathbb{P}}(\rho_n^{-1}). \quad (\text{B.63})$$

Case 2 ($t_L \geq 4$ and $\beta < \frac{t_L-3}{t_L-1}$ holds): We note that **(a)** and **(b)** in Eq. (B.62) do not change for $t \geq t_L \geq 3$. By Eq. (B.8) we further have

$$\max_{i,i'} \left| \left(\sum_{ii'} a_{ii'} \right) \frac{\hat{w}_{ii'}^{(t)} - w_{ii'}^{(t)}}{d_i} \right| = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2}\sqrt{\log n}\} \quad (\text{B.64})$$

for all $t \geq t_L \geq 3$, which implies, in this case,

$$\|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{off}}, \|\mathbf{I}_{\mathbf{A}} - \mathbf{I}_{\mathbf{P}}\|_{\max, \text{diag}} = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2}\sqrt{\log n}\}.$$

Step 3 (Bounding $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}}$ and $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty}$): From Eq. (B.59), Eq. (B.61), Eq. (B.63) and Eq. (B.64), we obtain

$$\begin{aligned} \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}} &= \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}\sqrt{\log n}) & \text{when } t_L \geq 2 \text{ and } \beta < \frac{1}{2}, \\ \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2}\sqrt{\log n}\} & \text{when } t_L \geq 4 \text{ and } \beta < \frac{t_L-3}{t_L-1}, \end{cases} \\ \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}} &= \begin{cases} \mathcal{O}_{\mathbb{P}}(\rho_n^{-1}) & \text{when } t_L \geq 2 \text{ and } \beta < \frac{1}{2}, \\ \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2}\sqrt{\log n}\} & \text{when } t_L \geq 4 \text{ and } \beta < \frac{t_L-3}{t_L-1}. \end{cases} \end{aligned}$$

and hence

$$\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}} \leq \left(n^2 \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}}^2 + n \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}}^2 \right)^{1/2} = \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1} \log^{1/2} n) & \text{when } t_L \geq 2 \text{ and } \beta < \frac{1}{2}, \\ \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1/2} \log^{1/2} n) & \text{when } t_L \geq 4 \text{ and } \beta < \frac{t_L-3}{t_L-1}, \end{cases}$$

$$\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty} \leq n \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{off}} + \|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\max, \text{diag}} = \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1} \log^{1/2} n) & \text{when } t_L \geq 2 \text{ and } \beta < \frac{1}{2}, \\ \mathcal{O}_{\mathbb{P}}(n^{1/2} \rho_n^{-1/2} \log^{1/2} n) & \text{when } t_L \geq 4 \text{ and } \beta < \frac{t_L-3}{t_L-1}. \end{cases}$$

as desired \square .

B.3 Proof of Theorem 4 (Sparse Regime)

Same as the dense regime, we shall assume $n_k = n\pi_k$, for all $k \in [K]$. From Eq. (2.9), one key observation is that when we write (1) $\mathbf{B} = \rho_n \mathbf{B}_0$ and \mathbf{B}_0 is a constant matrix; (2) $n_k = n\pi_k$, for all $k \in [K]$; the \mathbf{M}_0 built on \mathbf{B} or \mathbf{B}_0 are identical and do not depend on n . We therefore have

$$\mathbf{M}_0 = \begin{pmatrix} \xi_{11} \mathbf{1}_{n\pi_1} \mathbf{1}_{n\pi_1}^T & \dots & \xi_{1K} \mathbf{1}_{n\pi_1} \mathbf{1}_{n\pi_K}^T \\ \vdots & \ddots & \vdots \\ \xi_{K1} \mathbf{1}_{n\pi_K} \mathbf{1}_{n\pi_1}^T & \dots & \xi_{KK} \mathbf{1}_{n\pi_K} \mathbf{1}_{n\pi_K}^T \end{pmatrix} = \boldsymbol{\Theta} \boldsymbol{\Xi} \boldsymbol{\Theta}^T$$

where $\boldsymbol{\Xi} := (\xi_{ii'})_{K \times K}$ is a fixed matrix. The remaining steps in the proof of Theorem 4 (dense regime) still hold, e.g.,

$$\|\mathbf{M}_0\|_{\text{F}} = \Theta(n), \quad \|\mathbf{M}_0\|_{\infty} = \Theta(n), \quad \sigma_d(\mathbf{M}_0) = \Theta(n), \quad \|\mathbf{U}\|_{2 \rightarrow \infty} \lesssim n^{-1/2}. \quad (\text{B.65})$$

Applying Theorem 3 (sparse regime) and similar to Eq. (A.51) and Eq. (A.52), one has

$$\min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{\text{F}} \leq \frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\text{F}}}{\sigma_d(\mathbf{M}_0)} = \begin{cases} \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2} \rho_n^{-1/2} \log^{1/2} n\} & \text{if } t_L \geq 2 \text{ \& } \beta < \frac{1}{2}, \\ \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2} \log^{1/2} n\} & \text{if } t_L \geq 4 \text{ \& } \beta < \frac{t_L-3}{t_L-1}, \end{cases} \quad (\text{B.66})$$

$$\min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \leq 14 \left(\frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty}}{\sigma_d(\mathbf{M}_0)} \right) \|\mathbf{U}\|_{2 \rightarrow \infty} = \begin{cases} \mathcal{O}_{\mathbb{P}}(n^{-1} \rho_n^{-1} \log^{1/2} n) & \text{if } t_L \geq 2 \text{ \& } \beta < \frac{1}{2}, \\ \mathcal{O}_{\mathbb{P}}(n^{-1} \rho_n^{-1/2} \log^{1/2} n) & \text{if } t_L \geq 4 \text{ \& } \beta < \frac{t_L-3}{t_L-1}. \end{cases} \quad (\text{B.67})$$

Finally it is easy to check that if $t_L \geq 2$ and $\beta < 1/2$ then

$$n^{1/2} \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \lesssim (n\rho_n)^{-1/2} \rho_n^{-1/2} \log^{1/2} n \lesssim n^{\beta-1/2} \log^{1/2} n \rightarrow 0$$

with high probability. When $t_L \geq 4$ and $\beta < \frac{t_L-3}{t_L-1}$ then

$$n^{1/2} \cdot \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \lesssim (n\rho_n)^{-1/2} \log^{1/2} n \rightarrow 0 \quad (\text{B.68})$$

with high probability. Similar to the dense regime, the above $2 \rightarrow \infty$ results together with the technical arguments in [37] show that applying the K -medians algorithm to cluster the rows of $\hat{\mathcal{F}}$ will recover the memberships for all nodes with high probability. \square

B.4 Proof of Remark 6

We now show that the condition $\beta < \frac{t_L-2}{t_L}$, which is weaker than that assumed in Theorem 4, is actually sufficient to achieve exact recovery under the sparse regime of $\rho_n \gtrsim n^{-\beta}$ for $\beta \in (0, 1]$. We will follow the same proof strategy as that presented earlier with the main changes being that we perform more careful book-keeping when bounding the important terms in Section B.1 and Section B.2.

More specifically all the arguments in Section B.1 are still valid up to (and including) Eq. (B.57). The rate shown in Eq. (B.57) is, however, only negligible compared to the term $\mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n)$ in Eq. (B.56) for $\frac{t_L-3}{t_L-1} > \beta$, i.e., if we only assume $\frac{t_L-2}{t_L} > \beta$ then we have to include the error term in Eq. (B.57) by replacing Eq. (B.58) the following bound

$$\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = \mathcal{O}_{\mathbb{P}} \left\{ \frac{\log^{1/2} n}{n^{3/2} \rho_n^{1/2}} + (n\rho_n)^{-t/2} \right\}. \quad (\text{B.69})$$

Noting above equation implies that $\|\hat{\mathbf{W}}^t - \mathbf{W}^t\|_{\max} = o_{\mathbb{P}}(1)$ as implied by the following Eq. (B.70), then similar to the discussion in Section 3.1, \mathbf{M}_0 is thus well-defined with high probability. We can then follow the same arguments as that presented in Section B.2, taking care to replace the bound in Eq. (B.58) with that in Eq. (B.69). Eq. (B.69) to the Proof of Theorem 3 in Section B.2. In particular Eq. (B.60) now becomes

$$\max_{i \neq i'} |w_{ii'}^{(t_L)} - \hat{w}_{ii'}^{(t_L)}| / \min_{i, i'} w_{ii'}^{(t_L)} = \mathcal{O}_{\mathbb{P}} \left\{ \frac{\log^{1/2} n}{n^{1/2} \rho_n^{1/2}} + n \cdot (n\rho_n)^{-t_L/2} \right\} = o_{\mathbb{P}}(1) \quad (\text{B.70})$$

as $(\log^{1/2} n)/(n^{1/2}\rho_n^{1/2}) \lesssim \log^{1/2} n^{(\beta-1)/2} \rightarrow 0$ and $n \cdot (n\rho_n)^{-t_L/2} \lesssim n^{\{2-(1-\beta)t_L\}/2} \rightarrow 0$ for $\beta < \frac{t_L-2}{t_L}$. Meanwhile Eq. (B.64) becomes

$$\max_{i,i'} \left| \left(\sum_{ii'} a_{ii'} \right) \frac{\hat{w}_{ii'}^{(t)} - w_{ii'}^{(t)}}{d_i} \right| = \mathcal{O}_{\mathbb{P}} \left\{ \frac{\log^{1/2} n}{n^{1/2}\rho_n^{1/2}} + n \cdot (n\rho_n)^{-t_L/2} \right\}$$

and in turn Eq. (B.65) becomes

$$\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty} = \mathcal{O}_{\mathbb{P}} \left\{ \frac{n^{1/2} \log^{1/2} n}{\rho_n^{1/2}} + n^2 \cdot (n\rho_n)^{-t_L/2} \right\}.$$

Finally, using the above bound $\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty}$, we can proceed with the same proof of Theorem 4 in Section B.3 and obtain, in place of Eq. (B.67), the bound

$$\min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \leq 14 \left(\frac{\|\tilde{\mathbf{M}}_0 - \mathbf{M}_0\|_{\infty}}{\sigma_d(\mathbf{M}_0)} \right) \|\mathbf{U}\|_{2 \rightarrow \infty} = \mathcal{O}_{\mathbb{P}} \left\{ \frac{\log^{1/2} n}{n^{3/2}\rho_n^{1/2}} + (n\rho_n)^{-t_L/2} \right\}. \quad (\text{B.71})$$

Eq. (B.68) then becomes

$$n^{1/2} \cdot \min_{\mathbf{T} \in \mathbb{O}_d} \|\hat{\mathcal{F}} \cdot \mathbf{T} - \mathbf{U}\|_{2 \rightarrow \infty} \lesssim n^{-1/2} \rho_n^{-1/2} \log^{1/2} n + n^{-t_L/2+1} \rho_n^{-t_L/2} \lesssim o(1) + n^{\beta t_L/2-t_L/2+1} = o(1), \quad (\text{B.72})$$

The technical arguments in [37] once again show that clustering the rows of $\hat{\mathcal{F}}$ using K -medians will, with high probability recovers the memberships of all nodes. \square

B.5 Strong recovery for DCSBM (Sparse Regime)

The exact same arguments as that presented in Section A.4 also apply to the sparse regime where $\rho_n \gtrsim n^{-\beta}$ for some $\beta \in [0, 1]$. Indeed, the matrix \mathbf{M}'_0 and \mathbf{M}_0 in Section A.4 does not depend on the sparsity ρ_n . Rather ρ_n only affects the convergence rate of $\tilde{\mathbf{M}}_0$ to \mathbf{M}_0 which in turns lead to a slower convergence rate for $\hat{\mathcal{F}}$ to \mathbf{U} . This convergence rate is, however, still the same as that in Theorem 4 provided that the θ_i are homogeneous (so that $\max_i \theta_i / \min_i \theta_i \lesssim 1$). Clustering the rows of $\hat{\mathcal{F}}$ will thus, with high probability, recover the membership of all nodes.

APPENDIX C ADDITIONAL PROOFS AND DERIVATIONS

C.1 Proof of Lemma A1

(1): The sum of i th row of \mathbf{AD}_A^{-1} is

$$\sum_{i=1}^n \frac{a_{ii'}}{d_{i'}} = \frac{\sum_{i=1}^n a_{ii'}}{\sum_{i=1}^n a_{ii'}} = 1.$$

So $\mathbf{1}_n^\top \cdot \hat{\mathbf{W}} = \mathbf{1}_n^\top \cdot \mathbf{AD}_A^{-1} = \mathbf{1}_n^\top$ and

$$\begin{aligned} \mathbf{1}_n^\top \cdot \hat{\mathbf{W}}^t &= \mathbf{1}_n^\top \underbrace{\mathbf{AD}_A^{-1} \dots \mathbf{AD}_A^{-1}}_{t \text{ items}} \\ &= (\mathbf{1}_n^\top \cdot \mathbf{AD}_A^{-1}) \underbrace{\mathbf{AD}_A^{-1} \dots \mathbf{AD}_A^{-1}}_{t-1 \text{ items}} = \mathbf{1}_n^\top \underbrace{\mathbf{AD}_A^{-1} \dots \mathbf{AD}_A^{-1}}_{t-1 \text{ items}} = \dots = \mathbf{1}_n^\top. \end{aligned}$$

With similar argument we have $\mathbf{1}_n^\top \mathbf{W}^t = \mathbf{1}_n^\top$.

(2): The i th element of $\hat{\mathbf{W}}\mathbf{d}$ is

$$\sum_{j=1}^n \frac{a_{ji'}}{d_{i'}} d_{i'} = d_i$$

and hence $\hat{\mathbf{W}}\mathbf{d} = \mathbf{d}$. This implies $\hat{\mathbf{W}}^t\mathbf{d} = \mathbf{d}$. The same argument yields $\mathbf{W}^t\mathbf{p} = \mathbf{p}$. \square

C.2 Proof of Lemma A2

(1) $\|\mathbf{D}_A\|, \|\mathbf{D}_A^{-1}\|$: With Chernoff bound, since $c_0 \leq p_{ii'} \leq c_1$ for any fixed i we can get

$$\begin{aligned}\mathbb{P}\left(\frac{d_i}{n} > \frac{3c_1}{2}\right) &\leq \mathbb{P}\left(d_i > \frac{3}{2}p_i\right) \lesssim \exp(-C_0 \cdot n) \\ \mathbb{P}\left(\frac{1}{d_i} / \frac{1}{n} > \frac{2}{c_0}\right) &\leq \mathbb{P}\left(d_i \leq \frac{p_i}{2}\right) \lesssim \exp(-C_1 \cdot n)\end{aligned}$$

for some $C_0, C_1 > 0$ by taking appropriate constant in general Chernoff bound. So we have

$$\begin{aligned}\mathbb{P}\left(\frac{\max_{1 \leq i \leq n} d_i}{n} \leq \frac{3c_1}{2}\right) &\lesssim n \exp(-C_0 \cdot n) \rightarrow 0 \\ \mathbb{P}\left\{\left(\max_{1 \leq i \leq n} \frac{1}{d_i}\right) / \frac{1}{n} \leq \frac{2}{c_0}\right\} &\lesssim n \exp(-C_1 \cdot n) \rightarrow 0\end{aligned}$$

as $n \rightarrow \infty$. We therefore have $\|\mathbf{D}_A\| = \max_{1 \leq i \leq n} d_i = \mathcal{O}_{\mathbb{P}}(n)$, $\|\mathbf{D}_A^{-1}\| = \max_{1 \leq i \leq n} 1/d_i = \mathcal{O}_{\mathbb{P}}(1/n)$.

(2) $\|\mathbf{D}_A - \mathbf{D}_P\|$: For a given i the a_{i1}, \dots, a_{in} are independent and $\mathbb{E}a_{ii'} = p_{ii'}$ for any $1 \leq i, i' \leq n$. Also $|a_{ii'} - p_{ii'}| \leq 1$. By Bernstein inequality, we have

$$\mathbb{P}\left(\left|\sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'}\right| > \tilde{t}\right) \leq 2 \exp\left(-\frac{\tilde{t}^2}{2\sigma_0^2 n + \frac{2}{3}\tilde{t}}\right),$$

where $\sigma_0^2 = \frac{1}{n} \sum_{j=1}^n \text{Var}(a_{ii'} - p_{ii'}) = \frac{1}{n} \sum_{j=1}^n p_{ii'}(1 - p_{ii'})$. So we have $\sigma_0^2 \leq \frac{1}{n} \sum_{j=1}^n (p_{ii'} + 1 - p_{ii'})^2/4 = 1/4$ and

$$\mathbb{P}\left(\left|\sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'}\right| > \tilde{t}\right) \leq 2 \exp\left(-\frac{\tilde{t}^2}{\frac{1}{2}n + \frac{2}{3}\tilde{t}}\right). \quad (\text{C.1})$$

Take $\tilde{t} = c\sqrt{n \log n}$ in Eq. (C.1), we have

$$\mathbb{P}\left(\left|\sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'}\right| > c\sqrt{n \log n}\right) \leq 2 \exp\left(-\frac{c^2 n \log n}{\frac{1}{2}n + \frac{2}{3}c\sqrt{n \log n}}\right) \lesssim \exp(-2c^2 \log n) = \left(\frac{1}{n}\right)^{2c^2}.$$

Combining all the events among $i \in \{1, \dots, n\}$, we get

$$\begin{aligned}\mathbb{P}\left(\|\mathbf{D}_A - \mathbf{D}_P\| > c\sqrt{n \log n}\right) &= \mathbb{P}\left(\max_{1 \leq i \leq n} \left|\sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'}\right| > c\sqrt{n \log n}\right) \\ &= \mathbb{P}\left(\bigcup_{i=1}^n \left\{\left|\sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'}\right| > c\sqrt{n \log n}\right\}\right) \lesssim n \cdot \left(\frac{1}{n}\right)^{2c^2} = \left(\frac{1}{n}\right)^{2c^2-1},\end{aligned}$$

which implies $\|\mathbf{D}_A - \mathbf{D}_P\| = \mathcal{O}_{\mathbb{P}}(\sqrt{n \log(n)})$.

(3) $\|\mathbf{D}_A^{-1} - \mathbf{D}_P^{-1}\|$: With the results in (1) and (2), we have

$$\|\mathbf{D}_A^{-1} - \mathbf{D}_P^{-1}\| = \max_i \left| \frac{1}{d_i} - \frac{1}{p_i} \right| = \max_i \left| \frac{p_i - d_i}{d_i p_i} \right| \leq \|\mathbf{D}_A - \mathbf{D}_P\| \cdot \|\mathbf{D}_A^{-1}\| \cdot \|\mathbf{D}_P^{-1}\| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \sqrt{\log n}).$$

(4) $\|\hat{\mathbf{W}}^t\|_{\max}, \max_{i,i'} w_{ii'}^{(t)} \asymp 1/n$ and $\min_{i,i'} w_{ii'}^{(t)} \asymp 1/n$: By result in (1) and \mathbf{A}, \mathbf{P} are bounded by 1 elementwisely, it is easy to see that

$$\|\hat{\mathbf{W}}\|_{\max} = \|\mathbf{A}\mathbf{D}_A^{-1}\|_{\max} \leq \|\mathbf{A}\|_{\max} \cdot \|\mathbf{D}_A^{-1}\| \leq \max_{1 \leq i \leq n} 1/d_i = \mathcal{O}_{\mathbb{P}}(n^{-1}). \quad (\text{C.2})$$

For any $t \geq 1$,

$$\|\hat{\mathbf{W}}^t\|_{\max} \leq n \|\hat{\mathbf{W}}^{t-1}\|_{\max} \|\hat{\mathbf{W}}\|_{\max} \leq n^2 \|\hat{\mathbf{W}}^{t-2}\|_{\max} \|\hat{\mathbf{W}}\|_{\max}^2 \leq \dots \leq n^{t-1} \|\hat{\mathbf{W}}\|_{\max}^t = \mathcal{O}_{\mathbb{P}}(n^{-1}).$$

Since $c_0 \leq p_{ii'} \leq c_1$, we could also see

$$\frac{c_0^t}{c_1^t n} \leq w_{ii'}^{(t)} \leq \frac{c_1^t}{c_0^t n} \quad (\text{C.3})$$

for any given $t \geq 1$ and $1 \leq i, i' \leq n$, which implies $\min_{i,i'} w_{ii'}^{(t)}, \max_{i,i'} w_{ii'}^{(t)} \asymp n^{-1}$. \square

C.3 Proof of Lemma B1

(1) Bounding $\|\mathbf{D}_\mathbf{A}\|, \|\mathbf{D}_\mathbf{A}^{-1}\|$: As $c_2\rho_n \leq p_{ii'} \leq c_3\rho_n$ and $\mathbb{E}d_i = p_i$, we have by Chernoff bounds that

$$\begin{aligned} \log \left\{ \mathbb{P} \left(\frac{d_i}{n} > \frac{3c_3}{2}\rho_n \right) \right\} &\leq \log \left\{ \mathbb{P} \left(d_i > \frac{3}{2}p_i \right) \right\} \leq -C_0 n \rho_n, \\ \log \left\{ \mathbb{P} \left(\frac{n}{d_i} > \frac{1}{2c_2\rho_n} \right) \right\} &\leq \log \left(\mathbb{P} \left(d_i \leq \frac{p_i}{2} \right) \right) \leq -C_0 n \rho_n, \end{aligned} \quad (\text{C.4})$$

for any $i \in \{1, 2, \dots, n\}$ where $C_0 > 0$ is a constant not depending on i or n . Eq. (C.4) together with a union bound imply

$$\begin{aligned} \mathbb{P} \left(\frac{\max_i d_i}{n} > \frac{3c_3\rho_n}{2} \right) &\leq n \max_i \left\{ \mathbb{P} \left(d_i > \frac{3p_i}{2} \right) \right\} \leq \exp(\log n - C_0 n \rho_n) \rightarrow 0, \\ \mathbb{P} \left(\max_i \frac{n}{d_i} > \frac{1}{2c_2\rho_n} \right) &\leq n \max_i \left\{ \mathbb{P} \left(d_i \leq \frac{p_i}{2} \right) \right\} \leq \exp(\log n - C_0 n \rho_n) \rightarrow 0, \end{aligned} \quad (\text{C.5})$$

as $n \rightarrow \infty$. So $\|\mathbf{D}_\mathbf{A}\| = \max_i d_i = \mathcal{O}_{\mathbb{P}}(n\rho_n)$, $\|\mathbf{D}_\mathbf{A}^{-1}\| = \max_i 1/d_i = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1})$.

(2) Bounding $\|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{P}\|, \|\mathbf{D}_\mathbf{A}^{-1} - \mathbf{D}_\mathbf{P}^{-1}\|$: For a given vertex i the $\{a_{i1}, \dots, a_{in}\}$ are independent Bernoulli random variables with $\mathbb{E}a_{ii'} = p_{ii'}$. Therefore, by Bernstein inequality, we have

$$\mathbb{P} \left(\left| \sum_{i'=1}^n a_{ii'} - \sum_{i'=1}^n p_{ii'} \right| > \tilde{t} \right) \leq 2 \exp \left(- \frac{\tilde{t}^2}{2\sigma_0^2 n + \frac{2}{3}\tilde{t}} \right), \quad (\text{C.6})$$

where $\sigma_0^2 = n^{-1} \sum_{i'} \text{Var}(a_{ii'} - p_{ii'}) = n^{-1} \sum_{i'} p_{ii'}(1 - p_{ii'})$. As $\min_{i,i'} p_{ii'} \asymp \rho_n$ and $\max_{i,i'} p_{ii'} \asymp \rho_n$ where $\rho_n \rightarrow 0$, we have $\sigma_0^2 \asymp \rho_n$. Letting $\tilde{t} = C(n\rho_n \log n)^{1/2}$ in Eq. (C.6), we have

$$\frac{\sigma_0^2 n}{\tilde{t}} \asymp \left(\frac{\rho_n}{n^{-1} \log n} \right)^{1/2} \rightarrow \infty$$

by Assumption 1. We therefore have

$$\frac{\tilde{t}^2}{2\sigma_0^2 n + \frac{2}{3}\tilde{t}} \asymp \frac{\tilde{t}^2}{2\sigma_0^2 n} \asymp C^2 \log n.$$

There thus exists some constant $C > 0$ such that

$$\mathbb{P} \left(\left| \sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'} \right| > C(n\rho_n \log n)^{1/2} \right) \leq 2 \exp(-2 \log n)$$

and hence

$$\mathbb{P} \left(\max_{1 \leq i \leq n} \left\{ \left| \sum_{j=1}^n a_{ii'} - \sum_{j=1}^n p_{ii'} \right| \right\} > C(n\rho_n \log n)^{1/2} \right) \leq 2n \exp(-2 \log n) \rightarrow 0 \quad (\text{C.7})$$

as $n \rightarrow \infty$, which implies $\|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{P}\| = \mathcal{O}_{\mathbb{P}}((n\rho_n \log n)^{1/2})$. Combining the above results we obtain

$$\|\mathbf{D}_\mathbf{A}^{-1} - \mathbf{D}_\mathbf{P}^{-1}\| = \max_i \left| \frac{1}{d_i} - \frac{1}{p_i} \right| = \max_i \left| \frac{p_i - d_i}{d_i p_i} \right| \leq \|\mathbf{D}_\mathbf{A} - \mathbf{D}_\mathbf{P}\| \cdot \|\mathbf{D}_\mathbf{A}^{-1}\| \cdot \|\mathbf{D}_\mathbf{P}^{-1}\| = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-3/2} \log^{1/2} n).$$

(3) Bounding $\|\hat{\mathbf{W}}^t\|_{\max}, \max_{i,i'} w_{ii'}^{(t)}$ and $\min_{i,i'} w_{ii'}^{(t)}$: As the elements of \mathbf{A} and \mathbf{P} are non-negative and bounded above by 1, we have

$$\|\hat{\mathbf{W}}\|_{\max} = \|\mathbf{A}\mathbf{D}_\mathbf{A}^{-1}\|_{\max} \leq \|\mathbf{A}\|_{\max} \|\mathbf{D}_\mathbf{A}^{-1}\| \leq \max_i d_i^{-1} = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1}). \quad (\text{C.8})$$

We thus have, for any $t \geq 1$, that

$$\|\hat{\mathbf{W}}^t\|_{\max} \leq (\max_i d_i) \|\hat{\mathbf{W}}^{t-1}\|_{\max} \|\hat{\mathbf{W}}\|_{\max} \leq (\max_i d_i)^2 \|\hat{\mathbf{W}}^{t-2}\|_{\max} \|\hat{\mathbf{W}}\|_{\max}^2 \leq \dots \leq (\max_i d_i)^{t-1} \|\hat{\mathbf{W}}\|_{\max}^t = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1}).$$

Finally, as $c_2\rho_n \leq p_{ii'} \leq c_3\rho_n$ for all $1 \leq i, i' \leq n$, we have

$$\frac{c_2^t}{c_3^t n} \leq w_{ii'}^{(t)} \leq \frac{c_3^t}{c_2^t n}$$

for all $t \geq 1$. As t is finite, this implies $\min_{i,i'} w_{ii'}^{(t)} \asymp n^{-1}$ and $\max_{i,i'} w_{ii'}^{(t)} \asymp n^{-1}$. \square

C.4 Technical details in section B.1

C.4.1 Bounding $\|\Delta_1^{(1)}\|_{\max}$ and $\|\Delta_2^{(1)}\|_{\max}$

We first write

$$\|\Delta_1^{(1)}\|_{\max} = \max_{i,i'} \left| \frac{a_{ii'}}{d_{i'} \cdot p_{i'}} (d_{i'} - p_{i'}) \right| \lesssim \max_{i'} \frac{1}{n\rho_n} \left| \frac{d_{i'} - p_{i'}}{d_{i'}} \right|$$

by $|a_{ii'}| \leq 1$ and Assumption 1. Applying Lemma B1, we obtain

$$\|\Delta_1^{(1)}\|_{\max} \lesssim \frac{1}{n\rho_n} \max_{i'} |d_{i'} - p_{i'}| \cdot \max_{i'} \frac{1}{d_{i'}} = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-3/2} \log^{1/2} n) \quad (\text{C.9})$$

$$\|\Delta_2^{(1)}\|_{\max} = \max_{i,i'} \left| \frac{a_{ii'} - p_{ii'}}{p_{i'}} \right| \leq \max_{i'} \frac{1}{p_{i'}} = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1}). \quad (\text{C.10})$$

Since $\rho_n \gtrsim \frac{\log n}{n}$, we have $(n\rho_n)^{-3/2} \log^{1/2} n \lesssim (n\rho_n)^{-1}$.

C.4.2 Bounding $\|\Delta_1^{(2)}\|_{\max}$

The ii' th element of $\Delta_1^{(2,1)}$ is given by

$$\sum_{i^*=1}^n \left(\frac{a_{ii^*}}{d_{i^*} p_{i^*}} (p_{i^*} - d_{i^*}) - \frac{a_{ii^*}}{p_{i^*}^2} (p_{i^*} - d_{i^*}) \right) \cdot w_{i^* i'} = \sum_{i^*=1}^n \frac{a_{ii^*} (p_{i^*} - d_{i^*})^2 w_{i^* i'}}{p_{i^*}^2 d_{i^*}}.$$

We therefore have, by Lemma B1, that

$$\begin{aligned} \|\Delta_1^{(2,1)}\|_{\max} &\leq \max_{i,i'} \sum_{i^*=1}^n \frac{a_{ii^*} (p_{i^*} - d_{i^*})^2 w_{i^* i'}}{p_{i^*}^2 d_{i^*}} \\ &\leq \max_i d_i \left(\max_i |p_i - d_i| \right)^2 \cdot \left(\max_i \frac{1}{p_i} \right)^2 \left(\max_i \frac{1}{d_i} \right) (\max_{i,i'} w_{i^* i'}) = \mathcal{O}_{\mathbb{P}}(n^{-2} \rho_n^{-1} \log n) \end{aligned}$$

Similar to Eq. (A.11) and Eq. (A.13), we also have

$$\begin{aligned} \|\Delta_1^{(2,2)}\|_{\max} &= \max_{i,i'} \left(\left| \sum_{i^*=1}^n \frac{a_{ii^*}}{p_{i^*}^2} (p_{i^*} - d_{i^*}) w_{i^* i'} \right| \right) \leq \max_i d_i \left(\max_i |p_i - d_i| \right) \cdot \max_{i,i'} w_{i^* i'} \left(\max_i \frac{1}{p_i} \right)^2 = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n) \\ \|\Delta_1^{(2,3)}\|_{\max} &= \max_{i,i'} \left| \sum_{i^*=1}^n \frac{a_{ii^*} - p_{ii^*}}{p_{i^*}} w_{i^* i'} \right| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n). \end{aligned}$$

By Assumption 1, we have $n^{-3/2} \rho_n^{-1/2} \log^{1/2} n \gtrsim n^{-2} \rho_n^{-1} \log n$. Combining the above results we obtain

$$\|\Delta_1^{(2)}\|_{\max} \leq \|\Delta_1^{(2,1)}\|_{\max} + \|\Delta_1^{(2,2)}\|_{\max} + \|\Delta_1^{(2,3)}\|_{\max} = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1/2} \log^{1/2} n). \quad (\text{C.11})$$

C.4.3 Bounding $\delta_{2,\text{off}}^{(2,2)}$ and $\delta_{2,\text{off}}^{(2,3)}$

Step 1 (Bounding $\delta_{2,\text{off}}^{(2,2)}$): By Lemma B1, we have

$$\begin{aligned} \delta_{2,\text{off}}^{(2,2)} &= \max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) \left(\frac{a_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{d_{i'}} \right) \right| \\ &\leq \max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) a_{i^* i'} \right| \cdot \max_{i'} \left| \frac{1}{p_{i'}} - \frac{1}{d_{i'}} \right| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-3/2} \log^{1/2} n) \cdot \max_{i,i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) a_{i^* i'} \right|. \end{aligned}$$

Now we focus on $\sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) a_{i^* i'}$. We have

$$\begin{aligned} \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{d_{i^*}} \right) a_{i^* i'} &= \sum_{i^*=1}^n \left(\frac{d_{i^*}}{p_{i^*}} + 1 - \frac{d_{i^*}}{p_{i^*}} \right) \left(\frac{p_{ii^*} d_{i^*} - a_{ii^*} p_{i^*}}{p_{i^*} d_{i^*}} \right) a_{i^* i'} \\ &= \sum_{i^*=1}^n \left(\frac{p_{ii^*} d_{i^*} - a_{ii^*} p_{i^*}}{p_{i^*}^2} \right) a_{i^* i'} + \sum_{i^*=1}^n \left(1 - \frac{d_{i^*}}{p_{i^*}} \right) \cdot \left(\frac{p_{ii^*} d_{i^*} - a_{ii^*} p_{i^*}}{p_{i^*} d_{i^*}} \right) a_{i^* i'} \\ &= \sum_{i^*=1}^n \sum_{i^{**}=1}^n \left(\frac{a_{i^* i^{**}} p_{ii^*} - p_{i^* i^{**}} a_{ii^*}}{p_{i^*}^2} \right) a_{i^* i'} + \sum_{i^*=1}^n \frac{(p_{ii^*} d_{i^*} - a_{ii^*} p_{i^*})(p_{i^*} - d_{i^*}) a_{i^* i'}}{p_{i^*}^2 d_{i^*}}. \end{aligned} \quad (\text{C.12})$$

For the first term on the RHS of Eq. (C.12), we have

$$\begin{aligned}
\sum_{i^*=1}^n \sum_{i^{**}=1}^n \left(\frac{a_{i^* i^{**}} p_{ii^*} - p_{i^* i^{**}} a_{ii^*}}{p_{i^*}^2} \right) a_{i^* i'} &= \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \sum_{\substack{i^{**}=1 \\ i^{**} \neq i, i'}}^n \left(\frac{a_{i^* i^{**}} p_{ii^*} - p_{i^* i^{**}} a_{ii^*}}{p_{i^*}^2} \right) a_{i^* i'} \\
&= \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \frac{a_{i^* i'}}{p_{i^*}^2} \sum_{\substack{i^{**}=1 \\ i^{**} \neq i, i'}}^n a_{i^* i^{**}} p_{ii^*} - p_{i^* i^{**}} a_{ii^*} \\
&= \sum_{(i^*, i^{**}) \in \mathcal{T}(i, i')} \left(\frac{a_{i^* i'} p_{ii^*}}{p_{i^*}^2} + \frac{a_{i^{**} i'} p_{ii^{**}}}{p_{i^{**}}^2} \right) a_{i^* i^{**}} - \left(\frac{a_{i^* i'} a_{ii^*}}{p_{i^*}^2} + \frac{a_{i^{**} i'} a_{ii^{**}}}{p_{i^{**}}^2} \right) p_{i^* i^{**}} \\
&=: \mathcal{S}_2(\mathbf{a}_i, \mathbf{a}_{i'}).
\end{aligned}$$

Here $\mathcal{T}(i, i') = \{(i^*, i^{**}) | i^* < i^{**}, i^* \notin \{i, i'\}, i^{**} \notin \{i, i'\}\}$ and $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$.

Conditioning on the event that $\|\mathbf{D}_A\| = \max_i d_i \leq cn\rho_n$ with some $c > 0$ and $\mathbf{a}_i, \mathbf{a}_{i'}$, it is easy to check that the summands in $\mathcal{S}_2(\mathbf{a}_i, \mathbf{a}_{i'})$ are independent mean 0 random variables with $\text{Var}[\mathcal{S}_2(\mathbf{a}_i, \mathbf{a}_{i'})] \asymp n^{-2}$. Similar arguments to Eq. (B.28)-Eq. (B.30) yield

$$\max_{i, i'} \left| \sum_{i^*, i^{**}=1}^n \left(\frac{a_{i^* i^{**}} p_{ii^*} - p_{i^* i^{**}} a_{ii^*}}{p_{i^*}^2} \right) a_{i^* i'} \right| = \mathcal{O}_{\mathbb{P}}(n^{-1}).$$

In addition, by Lemma B1, we also have

$$\max_{i, i'} \left| \sum_{i^*=1}^n \frac{(p_{ii^*} d_{i^*} - a_{ii^*} p_{i^*})(p_{i^*} - d_{i^*})}{p_{i^*}^2 d_{i^*}} a_{i^* i'} \right| = \mathcal{O}_{\mathbb{P}}\{(n\rho_n)^{-1/2} \log^{1/2} n\},$$

as $\max_{i, i'} |p_{ii'} d_i - a_{ii'} p_i| \leq \max_{i, i'} p_{ii'} d_i + p_i = \mathcal{O}_{\mathbb{P}}(n\rho_n)$. Combining the above results we obtain

$$\delta_{2, \text{off}}^{(2,2)} = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-2} \log n).$$

Step 2 (Bounding $\delta_{2, \text{off}}^{(2,3)}$): We first note that

$$\delta_{2, \text{off}}^{(2,3)} = \max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right| = \max_{i \neq i'} \left| \sum_{i^*=1}^n \frac{1}{p_{i^*} p_{i'}} (p_{ii^*} - a_{ii^*})(p_{i^* i'} - a_{i^* i'}) \right|. \quad (\text{C.13})$$

Denote $\tilde{\zeta}_{i^*}^{ii'} := \frac{1}{p_{i^*} p_{i'}} (p_{ii^*} - a_{ii^*})(p_{i^* i'} - a_{i^* i'})$. We then have

$$\sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) = \left(\sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \tilde{\zeta}_{i^*}^{ii'} \right) + \tilde{\zeta}_i^{ii'} + \tilde{\zeta}_{i'}^{ii'}. \quad (\text{C.14})$$

Next observe that the $\{\tilde{\zeta}_{i^*}^{ii'}\}$ for $i^* \notin \{i, i'\}$ are independent mean 0 random variables. Hence

$$\begin{aligned}
\frac{1}{n-2} \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \text{Var}(\tilde{\zeta}_{i^*}^{ii'}) &= \frac{1}{n-2} \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \text{Var} \left\{ \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right\} \\
&= \frac{1}{(n-2)} \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \frac{1}{p_{i^*}^2 p_{i'}^2} \text{Var}(p_{ii^*} - a_{ii^*}) \cdot \text{Var}(p_{i^* i'} - a_{i^* i'}) \asymp \frac{1}{n^4 \rho_n^2}.
\end{aligned} \quad (\text{C.15})$$

We also have

$$\max_{i \neq i'} |\tilde{\zeta}_{i^*}^{ii'}| = \left| \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^* i'}}{p_{i'}} - \frac{a_{i^* i'}}{p_{i'}} \right) \right| \leq \left(\frac{1}{n\rho_n c_2} \right)^2 \asymp (n\rho_n)^{-2}. \quad (\text{C.16})$$

Therefore, by Bernstein inequality, for any given $c' > 0$ there exists a constant $C' > 0$ not depending on n such that

$$\mathbb{P} \left(\max_{i \neq i'} \left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \tilde{\zeta}_{i^*}^{ii'} \right| > C' n^{-3/2} \rho_n^{-1} \log^{1/2} n \right) \leq n^{-c'}$$

In summary we have

$$\max_{i \neq i'} \left| \sum_{\substack{i^*=1 \\ i^* \neq i, i'}}^n \tilde{\zeta}_{i^*}^{ii'} \right| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1} \log^{1/2} n).$$

By Eq. (C.16), we also have $\max_{i,i'} |\tilde{\zeta}_{i^*}^{ii'}| = \mathcal{O}((n\rho_n)^{-2})$ and thus

$$\begin{aligned} \delta_{2,\text{off}}^{(2,3)} &= \max_{i \neq i'} \left| \sum_{i^*=1}^n \left(\frac{p_{ii^*}}{p_{i^*}} - \frac{a_{ii^*}}{p_{i^*}} \right) \left(\frac{p_{i^*i'}}{p_{i'}} - \frac{a_{i^*i'}}{p_{i'}} \right) \right| \\ &\leq \max_{i \neq i'} \left| \sum_{\substack{i^*=1 \\ i^* \neq i,i'}}^n \tilde{\zeta}_{i^*}^{ii'} \right| + \max_{i,i'} \left| \tilde{\zeta}_i^{ii'} \right| + \max_{i,i'} \left| \tilde{\zeta}_{i'}^{ii'} \right| = \mathcal{O}_{\mathbb{P}}(n^{-3/2} \rho_n^{-1} \log^{1/2} n). \end{aligned} \quad (\text{C.17})$$

C.5 Proof of Theorem 1

To give a detailed analysis for all components in $C_{ii'}$, we firstly denote $C_{ii'}^{(t)}$ as the times that the structure,

$$\dots, v_i, \underbrace{\dots}_{t-1 \text{ terms}}, v_{i'}, \dots \quad (\text{C.18})$$

with a fixed t appears among all random paths in $\bigcup_{i=1}^n \mathcal{L}_i$. As $C_{ii'}$ counts all structures defined in Eq. (2.3), we have

$$C_{ii'} = \sum_{t=t_L}^{t_U} C_{ii'}^{(t)} + \sum_{t=t_L}^{t_U} C_{i'i}^{(t)}.$$

Let $C_{k,i,i'}^{(t)}$ denote the number of times the following structure appears among all random paths in $\bigcup_{i=1}^n \mathcal{L}_i$,

$$\underbrace{\dots}_{k \text{ nodes}}, v_i, \underbrace{\dots}_{t-1 \text{ nodes}}, v_{i'}, \dots \quad (\text{C.19})$$

We then have

$$C_{ii'}^{(t)} = \sum_{i^*=0}^{L-t-1} C_{i^*,i,i'}^{(t)}. \quad (\text{C.20})$$

Let $\{R_i\}_{i \geq 1}$ represents a stationary simple random walk on \mathcal{G} . Since all random paths are stationary and independent simple random walks over \mathcal{G} , the strong law of large numbers for Markov chain implies

$$\lim_{r \rightarrow \infty} \frac{1}{r} C_{i^*,i,i'}^{(t)} = \mathbb{P}(R_{i^*+1} = v_i) \cdot \mathbb{P}(R_{i^*+t+1} = v_{i'} | R_{i^*+1} = v_i) = S_i \cdot \mathbb{P}(R_{t+1} = v_{i'} | R_1 = v_i) = \frac{d_i}{2|\mathbf{A}|} \cdot \hat{w}_{i'i}^{(t)} \quad (\text{C.21})$$

almost surely. Furthermore we also have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{r} C_{ii'}^{(t)} &= \lim_{r \rightarrow \infty} \frac{1}{r} \sum_{i^*=0}^{L-t-1} C_{i^*,i,i'}^{(t)} = \frac{(L-t)d_i}{2|\mathbf{A}|} \hat{w}_{i'i}^{(t)}, \\ \lim_{r \rightarrow \infty} \frac{1}{r} C_{ii'} &= \lim_{r \rightarrow \infty} \frac{1}{r} \left(\sum_{t=t_L}^{t_U} C_{ii'}^{(t)} + \sum_{t=t_L}^{t_U} C_{i'i}^{(t)} \right) = \sum_{t=t_L}^{t_U} (L-t) \left(\frac{d_i}{2|\mathbf{A}|} \hat{w}_{i'i}^{(t)} + \frac{d_{i'}}{2|\mathbf{A}|} \hat{w}_{ii'}^{(t)} \right). \end{aligned}$$

almost surely. Combining the above two convergences, we have

$$\lim_{r \rightarrow \infty} \frac{1}{r} \sum_{i=1}^n C_{ii'} = \sum_{t=t_L}^{t_U} (L-t) \left(\frac{1}{2|\mathbf{A}|} \sum_{i=1}^n d_i \hat{w}_{i'i}^{(t)} + \frac{d_{i'}}{2|\mathbf{A}|} \sum_{i=1}^n \hat{w}_{ii'}^{(t)} \right) = \sum_{t=t_L}^{t_U} (L-t) \left(\frac{d_{i'}}{2|\mathbf{A}|} + \frac{d_{i'}}{2|\mathbf{A}|} \right) = \frac{\gamma d_{i'}}{|\mathbf{A}|} \quad (\text{C.22})$$

almost surely, where we denote $\gamma := \frac{(2L-t_L-t_U)(t_U-t_L+1)}{2}$. Note that the second equality in Eq. (C.22) is due to Lemma A1. Similar reasoning yields

$$\lim_{r \rightarrow \infty} \frac{1}{r} \sum_{i'=1}^n C_{i'i} = \frac{\gamma d_i}{|\mathbf{A}|}, \quad \lim_{r \rightarrow \infty} \frac{1}{r} \sum_{i=1}^n \sum_{i'=1}^n C_{ii'} = 2\gamma \quad (\text{C.23})$$

almost surely. Now for (t_L, t_U) satisfying that all entries in $\sum_{t=t_L}^{t_U} \hat{\mathbf{W}}^t$ are positive, the $i'i$ 'th entry in $\tilde{\mathbf{M}}(\mathbf{C}, \kappa)$ satisfies

$$\begin{aligned} \log \left(\frac{C_{ii'} \cdot (\sum_{i,i'} C_{ii'})}{\kappa \sum_i C_{ii'} \cdot \sum_{i'} C_{ii'}} \right) &= \log \left(\frac{(C_{ii'}/r) \cdot (\kappa \sum_{i,i'} C_{ii'}/r)}{\sum_i (C_{ii'}/r) \cdot \sum_{i'} (C_{ii'}/r)} \right) \\ &\xrightarrow{\text{a.s.}} \log \left[\frac{|\mathbf{A}|}{\kappa \gamma} \sum_{t=t_L}^{t_U} (L-t) \left(\frac{\hat{w}_{i'i}^{(t)}}{d_{i'}} + \frac{\hat{w}_{ii'}^{(t)}}{d_i} \right) \right] \\ &= \log \left[\frac{2|\mathbf{A}|}{\kappa \gamma} \sum_{t=t_L}^{t_U} (L-t) \left(\frac{\hat{w}_{ii'}^{(t)}}{d_i} \right) \right], \end{aligned} \quad (\text{C.24})$$

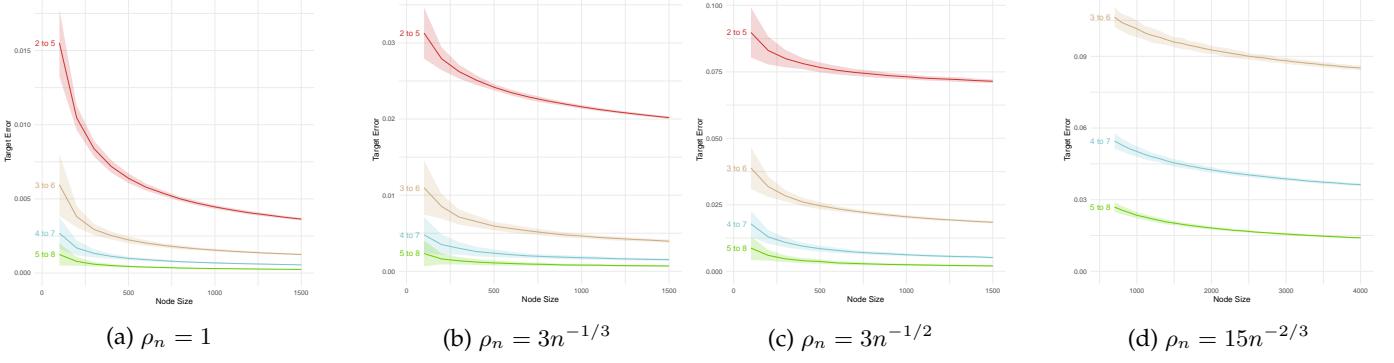


Fig. D1: Sample means and 95% empirical confidence intervals for $\varepsilon_1(\tilde{\mathbf{M}}_0)$ based on 100 Monte Carlo replicates for different settings of $n, \rho_n, (t_L, t_U)$. Here we set $t_U - t_L = 3$.

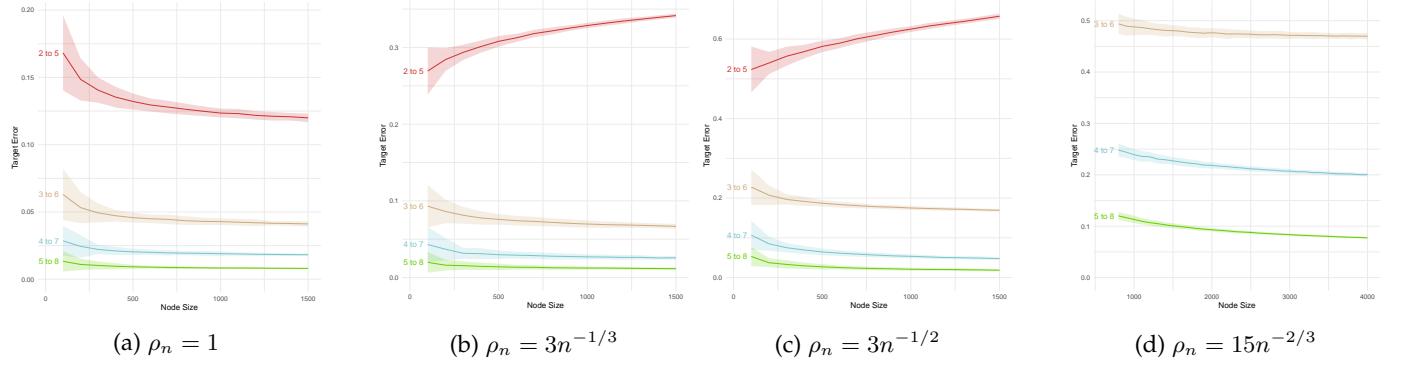


Fig. D2: Sample means and 95% empirical confidence intervals for $\varepsilon_2(\tilde{\mathbf{M}}_0)$ based on 100 Monte Carlo replicates for different values of n, ρ_n , and (t_L, t_U) with $t_L - t_U = 3$.

where the last equality is because $\hat{\mathbf{W}}^t$ is a transition matrix that satisfies the detailed balance condition. Writing Eq. (C.24) in matrix form, we obtain

$$\tilde{\mathbf{M}}(\mathbf{C}, k) \xrightarrow{a.s.} \log \left[\frac{2|\mathbf{A}|}{\kappa\gamma} \sum_{t=t_L}^{t_U} (L-t) \mathbf{D}_{\mathbf{A}}^{-1} \hat{\mathbf{W}}^t \right]$$

as desired. □

APPENDIX D ADDITIONAL FIGURES

This section contains the figures of all additional simulation results for Section 4 and Section 6 in the main paper.

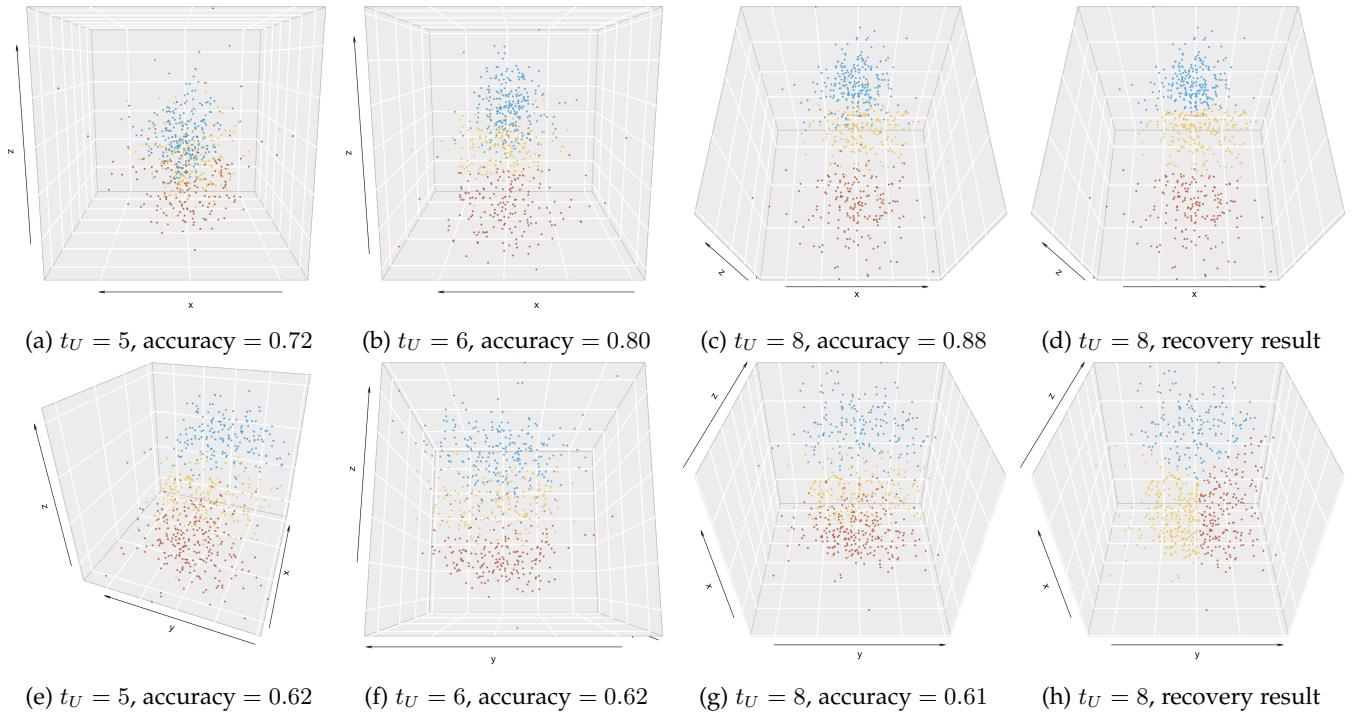


Fig. D3: Visualizations of the SVD-based node2vec embeddings (first row) and original node2vec embeddings (second row) for different choices of t_U . The plots are for a single realization of a DCSBM graph on $n = 600$ vertices with block probabilities matrix \mathbf{B}_1 (see eq. (4.3)), sparsity $\rho_n = 3n^{-1/2}$, and block assignment probabilities $\pi = (0.3, 0.3, 0.4)$. The embeddings in panels (a)–(c) and (e)–(g) are colored using the true membership assignments while the embeddings in panels (d) and (h) are colored using the K -means clustering. Accuracy of the recovered memberships (by K -means clustering) are also reported for panels (a)–(c) and (e)–(g).

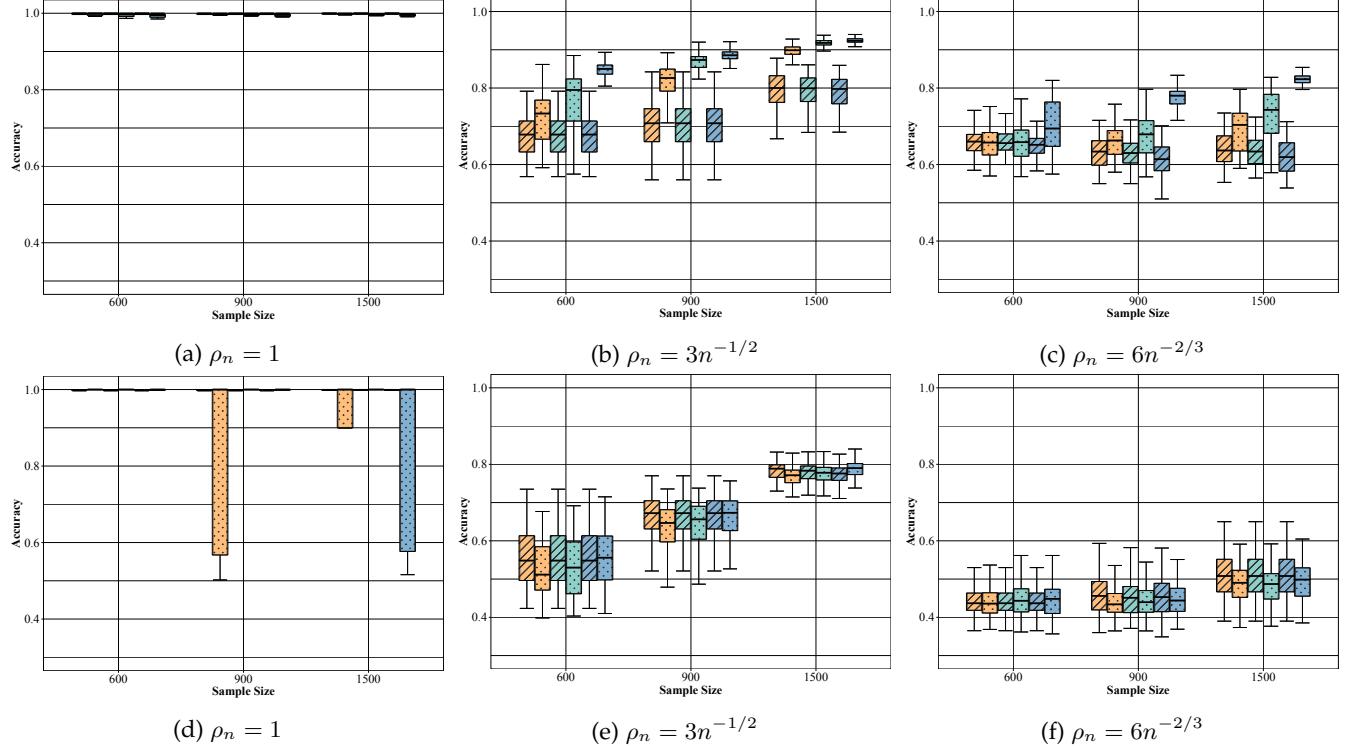


Fig. D4: Community detection accuracy of node2vec followed by K -means for DCSBM graphs. The boxplots of the accuracy for each value of n , ρ_n and t_U are based on 100 Monte Carlo replications. Boxplots with the slash pattern (resp. dot pattern) summarized the results for the original (resp. SVD-based) node2vec. Different colors (yellow, green, blue) represent the algorithms implemented for different choices of $t_U \in \{5, 6, 8\}$. The first and second row plot the results when the block probabilities for the DCSBM is \mathbf{B}_1 and \mathbf{B}_2 (see Eq. (4.3)) respectively.

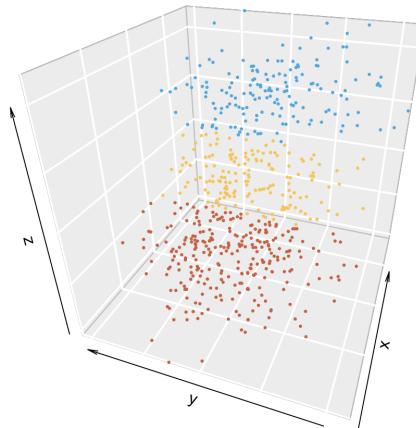


Fig. D5: The colors of embeddings represent the recovered memberships of corresponding vertices, by applying the Gaussian mixture model-based (GMM) clustering on the embeddings shown in Fig. D6 (g). Comparing Fig. D5 and Fig. D6 (g), one can see that GMM clustering correctly recovers most of vertices' memberships (accuracy of 0.84).

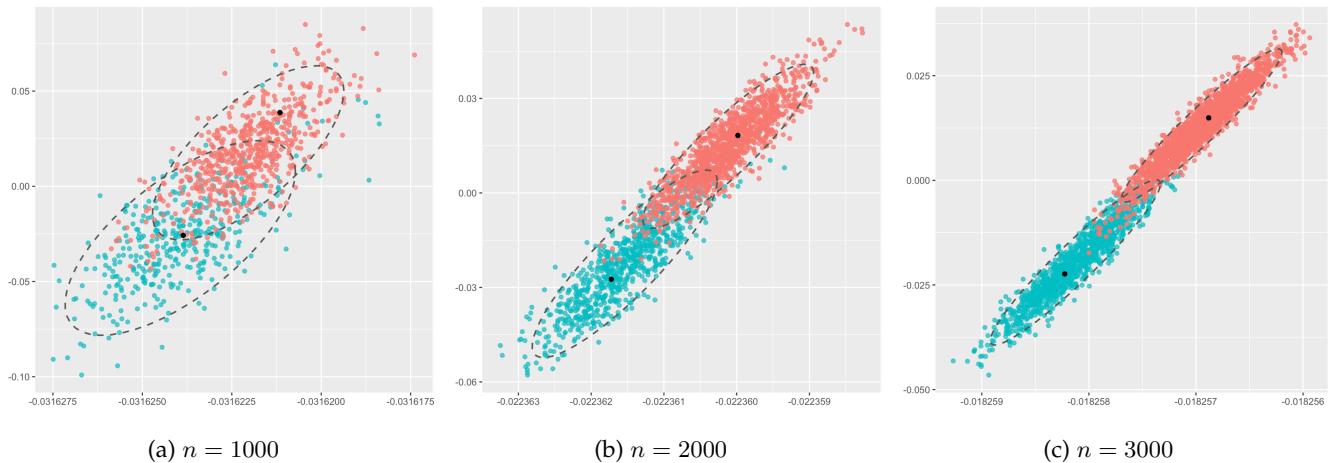


Fig. D6: Scatter plots of node2vec/DeepWalk embeddings for a two-blocks SBM with \mathbf{B} and π as defined in Eq. (6.1) as n varies. The points are colored according to their community membership. The dashed ellipses are the 95% level curves for the *block-conditional* empirical distributions. The two black points are the two distinct embedding vectors obtained by factorizing \mathbf{M}_0 ; note that these points had been transformed by the appropriate orthogonal matrices so as to align them with the node2vec/DeepWalk embedding obtained from the observed graphs.