STA 630 - Homework 2

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1 Problem 1 - Hoff 3.10

Change of variables problem. Let $\phi = g(\theta)$ be a monotone function with an inverse h, such that $\theta = h(\psi)$.

If $p_\theta(\theta)$ is the density of θ, then the density of ψ is given by the following.

$$p_\psi(\psi) = p_\theta(h(\psi)) \left| \frac{d}{d\psi} h(\psi) \right| \tag{1}$$

1.1 Part A

Let $\theta \sim beta(a, b)$, and let $\psi = log[\theta/(1-\theta)]$. Find the density of ψ and plot it for the case where $a = b = 1$.

We will start by finding the density and finding the form of $h(\psi)$ in this case.

$$\psi = g(\theta) = log \left[\frac{\theta}{1-\theta} \right] \tag{2}$$

$$exp(log(\theta/(1-\theta))) = \theta/(1-\theta) = exp(\psi) \tag{3}$$

$$\theta = \frac{exp(\psi)}{1 + exp(\psi)} \tag{4}$$

Now that we have the inverse, we can find its derivative and use this as the Jacobian in the change of variables formula.

$$h(\psi) = \frac{exp(\psi)}{1 + exp(\psi)} \tag{5}$$

$$\frac{d}{d\psi}h(\psi) = \frac{exp(\psi)}{(1+exp(\psi))^2} \tag{6}$$

This comes from a straightforward application of the quotient rule.

Finally, we can plug this into the change of variables formula to get the density of ψ, considering that θ is beta-distributed with parameters a and b.

We'll first delineate the density of θ with the accompanying parameters before applying the change of variables formula.

$$p_\theta(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \tag{7}$$

Where the denominator is, as typical, the beta function.

We can now write our density function with the change of variables formula, with respect to ψ.

$$p_\theta(h(\psi)) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left(\frac{exp(\psi)}{1+exp(\psi)}\right)^{a-1}\left(1-\frac{exp(\psi)}{1+exp(\psi)}\right)^{b-1}\frac{exp(\psi)}{(1+exp(\psi))^2} \tag{8}$$

Simplifying the density function allows us to put it in a more interpretable form. After cancellations, we get the following.

$$p_\psi(\psi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{exp(\psi)}{1+exp(\psi)}^{a+b} \tag{9}$$

This is the density function of ψ given that θ is beta-distributed with parameters a and b.

We can plot this density function using the following code, recognizing this as a beta density function.

Put another way, we recognize this as having a beta distribution with parameters a + 1 and b + 1.

Code and plot will go here...

1.2 Part B

Let $\theta \sim gamma(a,b)$. Let $\psi = log(\theta)$. Find the density of ψ.

We will do this problem in a very similar way, and start by finding the inverse function.

$$\psi = g(\theta) = log(\theta) \tag{10}$$

$$\theta = exp(\psi) \tag{11}$$

Here, $\theta = h(\psi)$, and we can find the derivative of this function.

$$\frac{d}{d\psi} h(\psi) = exp(\psi) \tag{12}$$

We can now plug this into the change of variables formula to get the density of ψ, using the gamma distribution as our model.

$$p_\theta(h(\psi)) = \frac{b^a}{\Gamma(a)} exp(-b exp(\psi)) exp(\psi) \tag{13}$$