15-859 Assignment #3

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Task 1

Proof. Note that to prove claim for all $\mathbb{R}^d$ we just need to show for all $x \in \mathbb{R}^d$ with $\|x\|_1 = 1$ by a scaling argument. We first fix x with $\|x\|_1 = 1$ and try to show SAx good. We focus on just the ith row of SAx and observe that

$$\begin{aligned}
(SAx)_i &= \langle (Z_1/m, \cdots, Z_n/m)^T, Ax\rangle && Z_j \text{i.i.d. standard Cauchy} \\
&= \sum_{j=1}^{n} \frac{(Ax)_j}{m} Z_j \\
&= \left\| \frac{Ax}{m} \right\|_1 Z && \text{1 norm invariance} \\
\Rightarrow |(SAx)_i| &= \frac{1}{m}\|Ax\|_1 |Z|.
\end{aligned}$$

Using the cdf of Cauchy we have that

$$\begin{aligned}
\Pr[|Z| < 1-\epsilon] &= \frac{2}{\pi}\arctan(1-\epsilon) \\
&\le \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{\epsilon}{2}\right) \\
&\le \frac{1}{2} - \frac{\epsilon}{6}.
\end{aligned}$$

The inequality uses observation that arctan is concave on $x \geq 0$ and is thus upper bounded by the tangent line at $x = 1$. Further, for $\epsilon < \sqrt{3} - 1$, $\arctan(1+\epsilon)$ is lower bounded by line segment from $(1, \pi/4), (\sqrt{3}, \pi/3)$. Hence

$$\begin{aligned}
\Pr[|Z| < 1+\epsilon] &= \frac{2}{\pi}\arctan(1+\epsilon) \\
&\ge \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{\epsilon}{\sqrt{3}-1}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right) \\
&= \frac{1}{2} + \frac{2\epsilon}{\sqrt{3}-1}\frac{1}{12} \\
&\ge \frac{1}{2} + \frac{\epsilon}{6} \\
\Rightarrow \Pr[|Z| > 1+\epsilon] &\le \frac{1}{2} - \frac{\epsilon}{6}.
\end{aligned}$$

Then let X_1 denote the sum of m i.i.d. Bernoulli r.v. each of which is 1 iff $|Z| < 1 - \epsilon$. Let X_2

denote sum of m Bernoullis each of which is 1 iff $|Z| > 1+\epsilon$. Then

$$\begin{aligned}
\Pr[X_1 \geq m/2] &= \Pr[X_1 \geq (1+\delta)m(1/2-\epsilon/6)] && \delta = \frac{\epsilon/6}{1/2-\epsilon/6} \\
&\leq \Pr[X_1 \geq (1+\delta)\mathbb{E}X_1] && \text{by analysis above} \\
&\leq \exp\left(-\delta^2 \mathbb{E}X_1/(2+\delta)\right) && \text{Chernoff} \\
&\leq \exp\left(-\delta^2 m/4(2+\delta)\right) && \text{for small enough constant } \epsilon \\
&\leq \exp\left(-(\epsilon^2/9)m(1/12)\right) && \delta \in [\epsilon/3, 1] \text{ for small enough } \epsilon \\
&= \exp(-\epsilon^2 m/108).
\end{aligned}$$

Similarly we have $\Pr[X_2 \geq m/2] \leq \exp(-\epsilon^2 m/108)$. Then

$$\Pr[\|SAx\|_{\text{med}} < (1-\epsilon)\|Ax\|_1/m] \leq \Pr[X_1 \geq m/2] \leq \exp(-\epsilon^2 m/108).$$

Union bound gives

$$\Pr[\|SAx\|_{\text{med}} \notin (1\pm\epsilon)\|Ax\|_1/m] \leq 2\exp(-\epsilon^2 m/108).$$

Also, by hint we may assume there is a γ-net for $\{Ax : \|x\|_1 = 1\}$ of size $\gamma^{O(d)}$.

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Now, take an arbitrary vector x such that $\|x\|_1 = 1$. Take a vector y in the γ-net such that $\|Ax - y\|_1 \leq \gamma$. Then we have that

□

Task 2