15-859 Assignment #1

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Task 1

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1. *Proof.* Let $z_i = x_i y_i$ then observe that

$$\begin{aligned}\langle x^{\otimes p}, y^{\otimes p}\rangle &= \sum_{i_1, i_2, \cdots i_p} \prod_{l\in[p]} x_{i_l} y_{i_l} \\ &= \sum_{i_1, i_2, \cdots, i_p} \prod_{l\in[p]} z_{i_l} \\ &= (z_1 + \cdots + z_d)^p \\ &= \langle x, y\rangle^p. \end{aligned}$$

□

2. *Proof.* Let A_i denote ith row of A. Then for each $i \in [n]$, we have $M_i = A_i^{\otimes p/2}$. Then indeed M is a $n \times d^{p/2}$ matrix. It is also clear that

$$\begin{aligned}||Mx^{\otimes p/2}||_2^2 &= \sum_{i\in[n]} \langle M_i, x^{\otimes p/2}\rangle^2 \\ &= \sum_{i\in[n]} \langle A_i, x\rangle^{p/2\times 2} \\ &= ||Ax||_p^p. \end{aligned}$$

To construct M, each entry is a $p/2$ product, so it takes $O(nd^{p/2}(p/2)) = n \cdot \text{poly}(d)$ time. Also, by discussion in class / problem statement, we may let S be a random $s \times n$ Gaussian matrix, where $s = O(d^{p/2}/\epsilon^2)$, such that with probability at least 9/10,

$$\begin{aligned}&\forall x \in \mathbb{R}^{d^(p/2)}, ||SMx||_2^2 = (1 \pm \epsilon)||Mx||_2^2 \\ \Rightarrow &\forall x \in \mathbb{R}^d, ||SMx^{\otimes p/2}||_2^2 = (1 \pm \epsilon)||Mx^{\otimes p/2}||_2^2 \\ &\qquad\qquad\qquad\qquad\quad = (1 \pm \epsilon)||Ax||_p^p. \end{aligned}$$

Finally, to compute SM we need only to perform $O(d^{p/2}/\epsilon^2)d^{p/2}$ dot products of vectors of length n, which takes in total $n \cdot \text{poly}(d)$ time. □

3. *Proof.* In the special case when A is Vandermonde, we no longer need all $d^{p/2}$ columns of M because of repetitive entries. Let S_k denote indices in $A_i^{\otimes p}$ for which the entries equal y_i^k. Then let M' have rows $(1, y_i, \cdots, y_i^{p(d-1)})$. Let y be a $p(d-1)+1$ length vector such that $y_k = \sum_{l\in S_k} x_l^{\otimes p/2}$. Then using same theorem in class / what the problem statement mentions, we can have matrix S appropriatedly scaled Gaussian with $O(p(d-1)/\epsilon^2) = O(dp/\epsilon^2)$ rows such that with at least 9/10 probability,

$$\begin{aligned}||SMx^{\otimes p/2}||_2^2 = ||SM'y||_2^2 &= (1 \pm \epsilon)||M'y||_2^2 \\ &= (1 \pm \epsilon)||Mx^{\otimes p/2}||_2^2 \\ &= (1 \pm \epsilon)||Ax||_p^p. \end{aligned}$$

□

Task 2