Fourier Series

Recall the Fourier series formula:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where the Fourier coefficients a_0, a_n and b_n are defined by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note, L is half the period of the function and n is a positive integer. Also note that a_0 is the average value of the $f(x)$.

Square Wave

The square wave function can be defined over the interval $[0, 2)$ as:

$$\text{Square Wave}(x) = \begin{cases} 1, & 0 \leq x < 1 \\ -1, & 1 \leq x < 2 \end{cases}$$

First we can determine a_0. By inspection we can expect it to be zero, but we shall still figure it out for practice. Note that $L = 1$ and we are considering the interval $[0, 2)$ instead of $[-1, 1)$ since the function is periodic:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2(1)}\int_{0}^{2} \text{Square Wave}(x)dx \\
&= \frac{1}{2}\left[\int_0^1 (1)dx + \int_1^2 (-1)dx\right] = \frac{1}{2}\left[x\Big|_0^1 - x\Big|_1^2\right] \\
&= \frac{1}{2}\Big[1 - 0 - (2-1)\Big] \\
&= 0
\end{aligned}$$

Next we shall look at a_n. Again by inspection we can argue that since the function is odd that it cosine components since those are even. Anyways... we shall still work it out for practice.

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{(1)}\int_0^2 \text{Square Wave}(x)\cos\left(\frac{n\pi x}{1}\right)dx \\
&= \int_0^1 (1)\cos(n\pi x)\,dx + \int_1^2 (-1)\cos(n\pi x)\,dx \\
&= \frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\sin(n\pi x)\Big|_1^2 \\
&= \frac{1}{n\pi}\Big(\sin(n\pi) - \sin(0)\Big) - \frac{1}{n\pi}\Big(\sin(2n\pi) - \sin(n\pi)\Big)
\end{aligned}$$

Since $\sin(0) = 0$, $\sin(n\pi) = 0$, and $\sin(2n\pi) = 0$ for all positive integer values of n, then

$$a_n = 0$$

Now working out b_n...

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{(1)}\int_{0}^{2} \text{Square Wave}(x)\sin\left(\frac{n\pi x}{1}\right)dx \\
&= \int_{0}^{1}(1)\sin(n\pi x) + \int_{1}^{2}(-1)\sin(n\pi x)\,dx \\
&= -\frac{1}{n\pi}\cos(n\pi x)\Big|_{0}^{1} + \frac{1}{n\pi}\cos(n\pi x)\Big|_{1}^{2} \\
&= -\frac{1}{n\pi}\Big(\cos(n\pi) - \cos(0)\Big) + \frac{1}{n\pi}\Big(\cos(2n\pi) - \cos(n\pi)\Big) \\
&= \frac{1}{n\pi}\Big(1 - \cos(n\pi) - \cos(n\pi) + 1\Big)
\end{aligned}$$

Note $\cos(n\pi) = -1, 1, -1, 1, \ldots$ which can be writen as $(-1)^n$, so then:

$$\begin{aligned}
b_n &= \frac{2}{n\pi}\Big(1 - (-1)^n\Big) \\
&= \frac{2}{n\pi}\Big\{2, 0, 2, 0, \ldots\Big\} \\
&= \frac{4}{n\pi}\Big\{1, 0, 1, 0, \ldots\Big\}
\end{aligned}$$

Therefore $b_n = 1$ for odd values of n and $b_n = 0$ for even values of n. To 'select' just the odd numbers, we can use $2n - 1$. So then we have:

$$b_n = \frac{4}{(2n-1)\pi}$$

And the final Fourier series is:

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\Big((2n-1)\pi x\Big)$$

Sawtooth Wave

The square wave function can be defined over the interval $[0, 2)$ as:

$$\text{Sawtooth Wave}(x) = \frac{1}{2}x$$

First we shall find a_0 or the average value:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2(1)}\int_{0}^{2}\frac{x}{2}dx \\
&= \frac{1}{4}\frac{x^2}{2}\Big|_{0}^{2} \\
&= \frac{1}{2}
\end{aligned}$$

Next we look at a_n...

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{(1)}\int_{0}^{2} \frac{x}{2}\cos\left(\frac{n\pi x}{(1)}\right)dx \\
&= \frac{1}{2}\int_{0}^{2} x\cos\left(n\pi x\right)dx \\
&= \frac{1}{2}\left[\frac{1}{n\pi}x\sin\left(n\pi x\right) - \frac{1}{n\pi}\int \sin\left(n\pi x\right)dx\right]_0^2 \\
&= \frac{1}{2}\left[\frac{1}{n\pi}x\sin\left(n\pi x\right) + \frac{1}{n^2\pi^2}\cos\left(n\pi x\right)\right]_0^2 \\
&= \frac{1}{2}\left[\frac{2n\pi\sin\left(2n\pi\right)+\cos\left(2n\pi\right)}{n^2\pi^2} - \frac{n\pi(0)\sin\left(0\right)+\cos\left(0\right)}{n^2\pi^2}\right] \\
&= \frac{1}{2}\left[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] \\
&= 0
\end{aligned}$$

And finally b_n...

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{(1)}\int_{0}^{2} \frac{x}{2}\sin\left(\frac{n\pi x}{(1)}\right)dx \\
&= \frac{1}{2}\int_{0}^{2} x\sin\left(n\pi x\right)dx \\
&= \frac{1}{2}\left[-\frac{1}{n\pi}x\cos\left(n\pi x\right) + \frac{1}{n\pi}\int \cos\left(n\pi x\right)dx\right]_0^2 \\
&= \frac{1}{2}\left[-\frac{1}{n\pi}x\cos\left(n\pi x\right) + \frac{1}{n^2\pi^2}\sin\left(n\pi x\right)dx\right]_0^2 \\
&= \frac{1}{2}\left[\frac{\sin(2n\pi)-\sin(0)}{n^2\pi^2} - \frac{2\cos(2n\pi)-0}{n\pi}\right] \\
&= \frac{1}{2}\left[-\frac{2}{n\pi}\right] \\
&= -\frac{1}{n\pi}
\end{aligned}$$

Therefore the final solution is:

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{n\pi}\sin(n\pi x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\sin(n\pi x)$$

Parabolic Wave

The parabolic wave function can be defined over the interval $[-1, 1)$ as:

$$\text{Parabolic Wave}(x) = x^2$$

First we shall find a_0 since it is obvious that the average value is not zero:

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2(1)}\int_{-1}^{1} x^2 dx \\ &= \frac{1}{2}\left.\frac{x^3}{3}\right|_{-1}^{1} \\ &= \frac{1}{2}\left(\frac{1}{3}-\frac{-1}{3}\right) \\ &= \frac{1}{3} \end{aligned}$$

Next we look at a_n...

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{1}{(1)}\int_{-1}^{1} x^2\cos\left(\frac{n\pi x}{(1)}\right)dx \\ &= \int_{-1}^{1} x^2\cos(n\pi x)\,dx \\ &= \left[\frac{1}{n\pi}x^2\sin(n\pi x) - \int\frac{1}{n\pi}2x\sin(n\pi x)dx\right]_{-1}^{1} \\ &= \left[\frac{1}{n\pi}x^2\sin(n\pi x) - \frac{2}{n\pi}\left(\frac{-1}{n\pi}x\cos(n\pi x) - \int\frac{-1}{n\pi}\cos(n\pi x)dx\right)\right]_{-1}^{1} \\ &= \left[\frac{1}{n\pi}x^2\sin(n\pi x) + \frac{2}{n^2\pi^2}x\cos(n\pi x) - \frac{2}{n^3\pi^3}\sin(n\pi x)\right]_{-1}^{1} \\ &= \left(\frac{1}{n\pi}\sin(n\pi) + \frac{2}{n^2\pi^2}\cos(n\pi) - \frac{2}{n^3\pi^3}\sin(n\pi)\right) - \left(\frac{1}{n\pi}\sin(-n\pi) - \frac{2}{n^2\pi^2}\cos(-n\pi) - \frac{2}{n^3\pi^3}\sin(-n\pi)\right) \end{aligned}$$

Note that $\sin(n\pi)=0$ and $\sin(-n\pi)=0$; also since $\cos(x)=\cos(-x)$, then we have:

$$\begin{aligned} a_n &= \frac{4}{n^2\pi^2}\cos(n\pi) \\ &= \frac{4}{n^2\pi^2}\Big\{-1,1,-1,1,\dots\Big\} \\ &= \frac{4}{n^2\pi^2}(-1)^n \end{aligned}$$

And finally b_n...

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{1}{(1)}\int_{-1}^{1} x^2\sin\left(\frac{n\pi x}{(1)}\right)dx \\ &= \int_{-1}^{1} x^2\sin(n\pi x)dx \\ &= \left[\frac{-1}{n\pi}x^2\cos(n\pi x) - \int\frac{-1}{n\pi}2x\cos(n\pi x)dx\right]_{-1}^{1} \\ &= \left[\frac{-1}{n\pi}x^2\cos(n\pi x) + \frac{2}{n\pi}\left(\frac{1}{n\pi}x\sin(n\pi x) - \int\frac{1}{n\pi}\sin(n\pi x)dx\right)\right]_{-1}^{1} \\ &= \left[\frac{-1}{n\pi}x^2\cos(n\pi x) + \frac{2}{n^2\pi^2}x\sin(n\pi x) + \frac{1}{n^3\pi^3}\cos(n\pi x)\right]_{-1}^{1} \\ &= \left(\frac{-1}{n\pi}\cos(n\pi) + \frac{2}{n^2\pi^2}\sin(n\pi) + \frac{1}{n^3\pi^3}\cos(n\pi)\right) - \left(\frac{-1}{n\pi}\cos(-n\pi) - \frac{2}{n^2\pi^2}\sin(-n\pi) + \frac{1}{n^3\pi^3}\cos(-n\pi)\right) \end{aligned}$$

Again $\sin(n\pi)$ and $\sin(-n\pi)$ is 0 for all n; also $\cos(x) = \cos(-x)$, so then:

$$
\begin{aligned}
&= \frac{-1}{n\pi}\cos(n\pi) + 0 + \frac{1}{n^3\pi^3}\cos(n\pi) - \frac{-1}{n\pi}\cos(n\pi) - 0 - \frac{1}{n^3\pi^3}\cos(n\pi) \\
&= 0
\end{aligned}
$$

Therefore the final solution is:

$$
f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{-1^n}{n^2}\cos(n\pi x)
$$

Triangle Wave

The triangle wave function can be defined over the interval $[-1, 1)$ as:

$$
\text{Triangle Wave}(x) = \begin{cases} 2x+1 & -1 \le x < 0 \\ -2x+1 & 0 \le x < 1 \end{cases}
$$

First we shall find a_0, even though it is obvious it is zero...

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2(1)}\left(\int_{-1}^{0}(2x+1)dx + \int_{0}^{1}(-2x+1)dx\right) \\
&= \frac{1}{2}\left(\left[x^2+x\right]_{-1}^{0} + \left[-x^2+x\right]_{0}^{1}\right) \\
&= \frac{1}{2}\left((0+0) - (1-1) + (-1+1) - (0+0)\right) \\
&= 0
\end{aligned}
$$

Next we look at a_n...

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{(1)}\int_{-1}^{1} \text{Triangle Wave}(x)\cos\left(\frac{n\pi x}{(1)}\right)dx \\
&= \int_{-1}^{0}(2x+1)\cos(n\pi x)dx + \int_{0}^{1}(-2x+1)\cos(n\pi x)dx \\
&= \left[2\int_{-1}^{0} x\cos(n\pi x)dx + \int_{-1}^{0}\cos(n\pi x)dx\right] + \left[-2\int_{0}^{1} x\cos(n\pi x)dx + \int_{0}^{1}\cos(n\pi x)dx\right] \\
&= \left.\frac{2(x\pi n\sin(n\pi x) + \cos(n\pi x))}{n^2\pi^2}\right|_{-1}^{0} + \left.\frac{1}{n\pi}\sin(n\pi x)\right|_{-1}^{0} - \left.\frac{2(x\pi n\sin(n\pi x) + \cos(n\pi x))}{n^2\pi^2}\right|_{0}^{1} + \left.\frac{1}{n\pi}\sin(n\pi x)\right|_{0}^{1}
\end{aligned}
$$

Recall that $\sin(n\pi) = 0$, so then

$$
\begin{aligned}
&= \frac{2(0+\cos(0)) - 2(0+\cos(-n\pi))}{n^2\pi^2} + 0 - \frac{2(0+\cos(n\pi)) - 2(0+\cos(0))}{n^2\pi^2} + 0 \\
&= \frac{2-2\cos(n\pi)}{n^2\pi^2} - \frac{2\cos(n\pi)-2}{n^2\pi^2} \\
&= \frac{4}{n^2\pi^2}\Big(1-\cos(n\pi)\Big) \\
&= \frac{4}{n^2\pi^2}\Big\{2, 0, 2, 0, \ldots\Big\} = \frac{8}{n^2\pi^2} \qquad \text{(When } n \text{ is odd)}
\end{aligned}
$$

And finally b_n...

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{(1)}\int_{-1}^{1} \textsf{Triangle Wave}(x)\sin\left(\frac{n\pi x}{(1)}\right)dx \\
&= \int_{-1}^{0}(2x+1)\sin(n\pi x)dx + \int_{0}^{1}(-2x+1)\sin(n\pi x)dx \\
&= \left[2\int_{-1}^{0} x\sin(n\pi x)dx + \int_{-1}^{0}\sin(n\pi x)dx\right] + \left[-2\int_{0}^{1} x\sin(n\pi x)dx + \int_{0}^{1}\sin(n\pi x)dx\right] \\
&= \left.\frac{2(\sin(n\pi x) - xn\pi\cos(n\pi x))}{n^2\pi^2}\right|_{-1}^{0} - \left.\frac{1}{n\pi}\sin(n\pi x)\right|_{-1}^{0} - \left.\frac{2(\sin(n\pi x) - xn\pi\cos(n\pi x))}{n^2\pi^2}\right|_{0}^{1} - \left.\frac{1}{n\pi}\sin(n\pi x)\right|_{0}^{1} \\
&= \frac{2(0-0) - 2(0+n\pi\cos(n\pi))}{n^2\pi^2} - 0 - \frac{2(0-n\pi\cos(n\pi)) - 2(0-0)}{n^2\pi^2} - 0 \\
&= \frac{-2(n\pi\cos(n\pi))}{n^2\pi^2} + \frac{2(n\pi\cos(n\pi))}{n^2\pi^2} \\
&= 0
\end{aligned}$$

Therefore the final solution is:

$$\begin{aligned}
f(x) &= \sum_{n\in\text{Odds}}^{\infty} \frac{8}{n^2\pi^2}\cos(n\pi x) \\
&= \frac{8}{\pi^2}\sum_{n\in\text{Odds}}^{\infty} \frac{\cos(n\pi x)}{n^2} \\
&= \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x)}{(2n-1)^2}
\end{aligned}$$