Beyond the Descartes Circle Theorem

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ABSTRACT

The Descartes circle theorem states that if four circles are mutually tangent in the plane, with disjoint interiors, then their curvatures (or "bends") $b_i = \frac{1}{r_i}$ satisfy the relation $(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2)$. We show that similar relations hold involving the centers of the four circles in such a configuration, coordinatized as complex numbers, yielding a complex Descartes Theorem. These relations have elegant matrix generalizations to the *n*-dimensional case, in each of Euclidean, spherical, and hyperbolic geometries. These include analogues of the Descartes circle theorem for spherical and hyperbolic space.

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1. Introduction

We call a configuration of four mutually tangent circles, in which no three circles have a common tangent, a "Descartes configuration". The possible arrangements ¹ of such configurations appear in Figure 1, where we allow certain "degenerate" arrangements where some of the circles are straight lines. Suppose the radii of the circles are r_1, r_2, r_3, r_4 . The reciprocals of these are the curvatures (or "bends") $b_j = 1/r_j$. A straight line is assigned infinite radius; then the "bend" is zero.

In 1643 Rene Descartes [12, pp. 45–50], in a letter to Princess Elizabeth of Bohemia, stated a relation connecting the four radii. This relation can be written as a quadratic equation connecting the four curvatures, namely:

Theorem 1.1 (Descartes Circle Theorem) For a Descartes configuration of four mutually tangent circles, their curvatures satisfy

$$\sum_{j=1}^{4} b_j^2 = \frac{1}{2} (\sum_{j=1}^{4} b_j)^2.$$
 (1.1)

¹An arrangement of concentric circles having a common tangent is not a Descartes configuration.

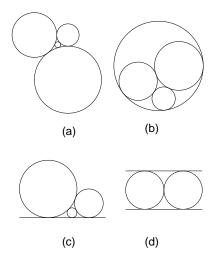


Figure 1: Descartes configurations

Descartes considered only the configuration (a) in Figure 1. He did not state the result in this form, but gave a more complicated relation algebraically equivalent to (1.1), and his sketched proof is incomplete. In 1826 Jakob Steiner [29, pp. 61–63] independently found the result and gave a complete proof. Another independent rediscovery with a complete proof was given in 1842 by H. Beecroft [4], and is described in Coxeter [8]. Many other proofs have been discovered (and rediscovered), some of which appear in Pedoe [23].

The Descartes circle theorem applies to all Descartes configurations of types (a)- (d), provided we define the curvatures to have appropriate signs, as follows. An oriented circle is a circle together with an assigned direction of unit normal vector, which can point inward or outward. If it has radius r then its oriented radius is r for an inward pointing normal and -r for an outward pointing normal. Its oriented curvature, (or "signed curvature") is $\frac{1}{r}$ for an inward pointing normal and $-\frac{1}{r}$ for an outward pointing normal. By convention, the interior of an oriented circle is its interior for an inward pointing normal and its exterior for an outward pointing normal. We define an oriented Descartes configuration to be a Descartes configuration with the circles having orientations which are compatible in the following sense: either (i) the interiors of all four oriented circles are disjoint, or (ii) the interiors are disjoint when all orientations are reversed. Each Descartes configuration has exactly two compatible orientations in this sense, one obtained from the other by reversing all orientations ². With these definitions, the Descartes Circle Theorem remains valid for all oriented Descartes configurations, using oriented curvatures.

In 1936 Sir Frederick Soddy (who earned a 1921 Nobel prize for discovering isotopes) published in Nature [27] a poem entitled "The Kiss Precise" in which he reported the result above and a generalization to three dimensions. The following year Thorold Gossett [13] contributed another stanza giving the general n-dimensional result. We extend the definition of a Descartes configuration to consist of n+2 mutually tangent (n-1)-spheres in \mathbb{R}^n in which

²The *inward pointing orientation* of a Descartes configuration is the one in which at least two oriented curvatures are strictly positive; the *outward pointing orientation* is one in which at least two curvatures are strictly negative.

all pairs of tangent (n-1)-spheres have distinct points of tangency, and orientation is done as above.

Theorem 1.2 (Soddy-Gossett Theorem) Given an oriented Descartes configuration in \mathbb{R}^n , if we let $b_i = 1/r_i$ be the oriented curvatures of the n + 2 mutually tangent spheres, then

$$\sum_{j=1}^{n+2} b_j^2 = \frac{1}{n} (\sum_{j=1}^{n+2} b_j)^2.$$
 (1.2)

The case n=3 of this result already appears in an 1886 paper of Lachlan [21, p. 498] and his proof is given in the 1916 book of Coolidge [7, p. 258]. Thus in calling this result the "Soddy-Gossett theorem" we are continuing the tradition that theorems are often not named for their first discoverers, cf. Stigler [31]. Proofs of the n-dimensional theorem appear in Pedoe [23] and Coxeter [9]. Pedoe observes that this result is actually a theorem ³ of real algebraic geometry, rather than of complex algebraic geometry, in dimensions 3 and above.

In this paper we present some very simple and elegant extensions of these results, which involve the centers of the circles. We show that there are relations, similar to (1.2), involving the centers, together with the curvatures, in the combination curvature×center. Furthermore, all these relations generalize to arrangements of n + 2 mutually tangent (n - 1)-spheres in n-dimensional Euclidean, spherical and hyperbolic spaces, and have a matrix formulation. In the process we recover spherical and hyperbolic analogues of the Soddy-Gossett Theorem; these were first obtained by Mauldon [22] in 1962. There is a vast literature on this subject, spanning two centuries, but (so far) we have not found our matrix formulations in the literature. In spirit the ideas trace back at least to Wilker [33, p. 390], see the remark at the end of §4.

2. The Complex Descartes Theorem

Given any three mutually tangent circles, with curvatures b_1, b_2, b_3 , there are exactly two other circles that are tangent to each of these; each gives a four-circle Descartes configuration. See Figure 2 for the possible arrangements of the resulting five circles; the three initial circles are given by dotted lines.

The curvatures of these two new circles are the roots of the quadratic equation (1.1) (treating b_4 as the variable). Suppose these roots are b_4 and b'_4 . Both can be positive, as in Figure 2(a), or one may be negative as in Figure 2(b). From (1.1) we have

$$b_4 + b_4' = 2(b_1 + b_2 + b_3). (2.1)$$

Thus, starting from a Descartes configuration, we can select any one of the four circles and replace it by the other circle that is tangent to the remaining three; this gives a new Descartes configuration. The new curvature can be obtained from the original four by using (2.1). This construction can be repeated indefinitely. We arrive at a packing of circles that fills either (i) a single circle, as for example in Figure 3, or (ii) a strip between two parallel lines, or (iii) a

³The theorem depends on the fact that the number of real circles, simultaneously tangent to each of n+1 mutually tangent real circles with distinct tangents, is exactly two. The total number of complex circles with this tangency property is two in dimension n=2 but typically exceeds two in dimensions $n \geq 3$.

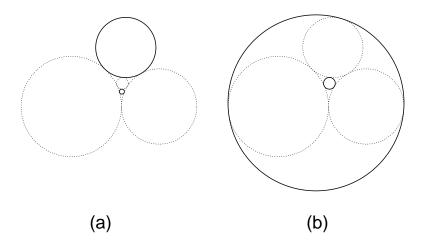


Figure 2: Circles Tangent to Three Tangent Circles

half-plane, or (iv) the whole plane. Such a figure is called an *Apollonian packing*, in honor of Apollonius of Perga, who considered (about 200 BC) the eight circles that are tangent to each of three given circles in general position, cf. Kasner and Supnick [20]. An Apollonian packing is completely specified by any three mutually tangent circles in it.

In constructing the Apollonian packing pictured in Figure 3, we started with four circles with vector of curvatures (-1, 2, 2, 3). Each circle has been labelled with its curvature; we notice that these are all integers. It is clear from (2.1) that once we have a Descartes configuration with all curvatures integral, then in this construction all the curvatures in the packing will be integers.

In 1998, one of us, while computing Figure 3, with the center of the outer circle located at the origin, noticed that the centers of all the circles are rational; in fact in this figure, if a circle has curvature b and center (x, y) then (it appeared) bx and by are always integers. Following this clue, we were led to the following generalization of (1.1), in which the centers are taken to be the complex numbers $z_j = x_j + iy_j$.

Theorem 2.1 (Complex Descartes Theorem) Any Descartes configuration of four mutually tangent circles, with curvatures b_i and centers $z_i = x_i + iy_i$ satisfies

$$\sum_{j=1}^{4} (b_j z_j)^2 = \frac{1}{2} (\sum_{j=1}^{4} b_j z_j)^2.$$
(2.2)

Notice that the relation (2.2) is of the same form as the original Descartes' relation (1.1). The complex Descartes theorem implies both the Descartes circle theorem (1.1) and a third relation

$$\sum_{j=1}^{4} b_j(b_j z_j) = \frac{1}{2} (\sum_{j=1}^{4} b_j) (\sum_{j=1}^{4} b_j z_j), \tag{2.3}$$

These results are obtained by replacing z_j by $z_j + w$ in (2.2), where w is an arbitrary complex number, and identifying coefficients of powers of w.

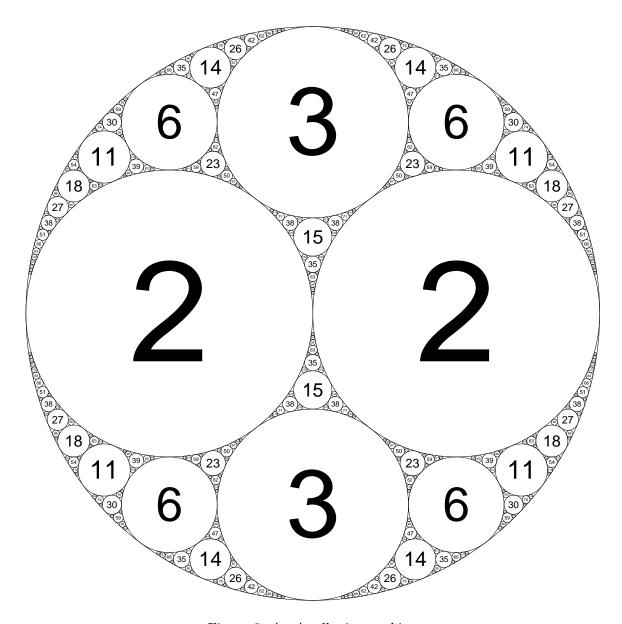


Figure 3: An Apollonian packing

The complex Descartes theorem also implies a relation similar to (2.1) connecting the centers of two circles, each of which is tangent to each of three given mutually tangent circles, namely:

$$b_4 z_4 + b_4' z_4' = 2(b_1 z_1 + b_2 z_2 + b_3 z_3). (2.4)$$

Thus in the iterative construction of an Apollonian packing that we described above, both the curvatures and the centers of the new circles can be obtained by simple linear operations (followed by divisions). This makes it very easy to draw figures such as Figure 3 using the computer.

The relations in the complex Descartes theorem can be expressed in a more elegant form using the matrix

in which $\mathbf{1}_n$ denotes a column of n 1's , and \mathbf{Q}_2 is the coefficient matrix of the Descartes quadratic form

$$Q_2(x_1, x_2, x_3, x_4) := \mathbf{x}^T \mathbf{Q}_2 \mathbf{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - \frac{1}{2} (x_1 + x_2 + x_3 + x_4)^2.$$

The subscript 2 in \mathbb{Q}_2 refers to the dimensionality of the space we are considering,

If $\mathbf{b} = (b_1, b_2, b_3, b_4)^T$ denotes the column vector of curvatures, and $\mathbf{c} = (b_1 z_1, b_2 z_2, b_3 z_3, b_4 z_4)^T$, then the Descartes theorem asserts that

$$\mathbf{b}^T \mathbf{Q}_2 \mathbf{b} = 0, \tag{2.6}$$

and the complex Descartes theorem asserts that

$$\mathbf{c}^T \mathbf{Q}_2 \mathbf{c} = 0. \tag{2.7}$$

The complex Descartes Theorem does not completely characterize Descartes configurations in the Euclidean plane. There is a slightly stronger result which does, namely:

Theorem 2.2 (Extended Descartes Theorem) Given a configuration of four oriented circles with non-zero curvatures (b_1, b_2, b_3, b_4) and centers $\{(x_i, y_i) : 1 \le i \le 4\}$, let M be the 4×3 matrix

$$\mathbf{M} := \begin{bmatrix} b_1 & b_1 x_1 & b_1 y_1 \\ b_2 & b_2 x_2 & b_2 y_2 \\ b_3 & b_3 x_3 & b_3 y_3 \\ b_4 & b_4 x_4 & b_4 y_4 \end{bmatrix}. \tag{2.8}$$

Then this configuration is an oriented Descartes configuration if and only if

$$\mathbf{M}^T \mathbf{Q}_2 \mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2.9}$$

If one or two curvatures b_i are zero, and the corresponding centers are infinite, then \mathbf{M} can be defined in such a way that this matrix identity remains true.

The complex Descartes theorem follows from this result by applying it to the vector $\mathbf{c} = \mathbf{x} + i\mathbf{y}$, where \mathbf{x} and \mathbf{y} are the second and third columns of \mathbf{M} . The extended Descartes Theorem gracefully generalizes to n-dimensions, which we turn to next.

3. Descartes Configurations in n- Dimensional Euclidean Space

An n-dimensional oriented Descartes configuration consists of n+2 mutually tangent oriented (n-1)- spheres S_i in n-dimensional space \mathbb{R}^n , having distinct tangencies, with the orientations compatible in the sense that all interiors are disjoint, either with the given orientation or with the reversal of all orientation vectors. Here we suppose that $n \geq 2$; the one-dimensional case is treated in the concluding section. We often regard a hyperplane as a limiting case of a sphere, having zero curvature, with orientation given by a unit normal vector. In what follows an "oriented sphere" includes the hyperplane case unless otherwise stated.

The Soddy-Gossett theorem (1.2) relates the curvatures of such a configuration of mutually tangent n-spheres, and can be written

$$Q_n(\mathbf{b}) := \mathbf{b}^T \mathbf{Q}_n \mathbf{b} = 0,$$

where $\mathbf{b} = (b_1, \dots, b_{n+2})^T$ and $Q_n(\mathbf{x}) = \mathbf{x}^T \mathbf{Q}_n \mathbf{x}$ is the *n*-dimensional Descartes quadratic form whose associated symmetric $(n+2) \times (n+2)$ matrix \mathbf{Q}_n is

$$\mathbf{Q}_n := I_{n+2} - \frac{1}{n} \mathbf{1}_{n+2} \mathbf{1}_{n+2}^T. \tag{3.1}$$

The Soddy-Gossett theorem has a converse.

Theorem 3.1 (Converse to Soddy-Gosset Theorem) If $\mathbf{b} = (b_1, ..., b_{n+2})^T$ is a nonzero real column vector that satisfies

$$\mathbf{b}^T \mathbf{Q}_n \mathbf{b} = 0, \tag{3.2}$$

then there exists an oriented Descartes configuration whose oriented curvature vector is **b**. Furthermore any two oriented Descartes configurations having the same oriented curvature vector are congruent; that is, there is a Euclidean motion taking one to the other.

A Euclidean motion is one that preserves angles and distances; it includes reflections, which reverse orientations. We do not know an easy proof of this result; a proof appears in [16].

The geometry of Descartes configurations is encoded in the curvature vector **b**. If all b_i are non-zero and $\sum_{j=1}^{n+2} b_j > 0$, then one of the following holds: (i) all of b_1 , b_2 ,..., b_{n+2} are positive; (ii) n+1 are positive and one is negative; (iii) n+1 are positive and one is zero; or (iv) n are positive and equal and the other two are zero. These four cases correspond respectively to the following configurations of mutually tangent spheres: (i) n+1 spheres, with another in the curvilinear simplex that they enclose; (ii) n+1 spheres inscribed inside another larger sphere; (iii) n+1 spheres with one hyperplane (the (n+2)-nd "sphere"), tangent to each of them; (iv) n equal spheres with two common parallel tangent planes.

Definition 3.1. Given an oriented sphere S in \mathbb{R}^n , its *curvature-center coordinates* consist of the (n+1)-vector $\mathbf{m}(S)$ given by

$$\mathbf{m}(S) = (b, bx_1, ..., bx_n) \tag{3.3}$$

in which b is the signed curvature of S (assumed nonzero) and $\mathbf{x}(S) = \mathbf{x} = (x_1, x_2, ..., x_n)$ is its center. For the "degenerate case" of an oriented hyperplane H its curvature-center coordinates $\mathbf{m}(H)$ are defined to be

$$\mathbf{m}(S) = (0, \mathbf{h}) \tag{3.4}$$

where $\mathbf{h} := (h_1, h_2, \dots, h_n)$ is the unit normal vector giving the orientation of the hyperplane.

Curvature-center coordinates are not quite a global coordinate system, because they do not always uniquely specify an oriented sphere. Given $\mathbf{m} \in \mathbb{R}^{n+1}$, if its first coordinate $a \neq 0$ then there exists a unique sphere having $\mathbf{m} = \mathbf{m}(S)$. But if a = 0, the hyperplane case, there is a hyperplane if and only if $\sum h_i^2 = 1$, and in that case there is a pencil of hyperplanes having the given value \mathbf{m} , which differ from each other by a translation.

Theorem 3.2 (Euclidean Generalized Descartes Theorem) Given a configuration of n+2 oriented spheres $S_1, S_2, \ldots S_{n+2}$ in \mathbb{R}^n (allowing hyperplanes), let \mathbf{M} be the $(n+2) \times (n+1)$ matrix whose j-th row are the curvature-center coordinates $\mathbf{m}(S_j)$, of the j-th sphere. If this configuration is an oriented Descartes configuration then

$$\mathbf{M}^T \mathbf{Q}_n \mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 2I_n \end{bmatrix} = diag(0, 2, 2, ..., 2).$$
(3.5)

Conversely, any real solution M to this equation is the matrix of a unique oriented Descartes configuration.

The curvature-center coordinate matrix **M** of an oriented Descartes configuration determines it uniquely even if it contains hyperplanes, because the other spheres in the configuration give enough information to fix the locations of the hyperplanes. This result contains the Soddy-Gossett theorem as its (1,1)- coordinate. We derive the "if" part of this theorem from the next result, proved in §5. However the converse part of this theorem seems more difficult, and we do not prove it here. A proof appears in [16].

We proceed to a further generalization, which extends the $(n+2) \times (n+1)$ matrix \mathbf{M} to an $(n+2) \times (n+2)$ matrix \mathbf{W} obtained by adding an additional column. This augmented matrix incorporates information about two oriented Descartes configurations, the original one and one obtained from it by inversion in the unit sphere, as we now explain. The definition of \mathbf{W} may seem pulled out of thin air, but in the next two sections we will show that it naturally arises from an analogous result in spherical geometry, which is how we discovered it.

In *n*-dimensional Euclidean space, the operation of inversion in the unit sphere replaces the point \mathbf{x} by $\mathbf{x}/|\mathbf{x}|^2$, where $|\mathbf{x}|^2 = \sum_{j=1}^n x_j^2$. Consider a general oriented sphere S with center \mathbf{x} and oriented radius r. Then inversion in the unit sphere takes S to the sphere \bar{S} with center $\bar{\mathbf{x}} = \mathbf{x}/(|\mathbf{x}|^2 - r^2)$ and oriented radius $\bar{r} = r/(|\mathbf{x}|^2 - r^2)$. Note that if $|\mathbf{x}|^2 > r^2$, \bar{S} has the same orientation as S. In all cases,

$$\frac{\mathbf{x}}{r} = \frac{\bar{\mathbf{x}}}{\bar{r}} \tag{3.6}$$

and

$$\bar{b} = \frac{|\mathbf{x}|^2}{r} - r. \tag{3.7}$$

Definition 3.2. Given an oriented sphere S in \mathbb{R}^n , its augmented curvature-center coordinates are the (n+2)-vector

$$\mathbf{w}(S) := (\bar{b}, b, bx_1, ..., bx_n) = (\bar{b}, \mathbf{m}), \tag{3.8}$$

in which $\bar{b} = b(\bar{S})$, is the curvature of the sphere or hyperplane \bar{S} obtained by inversion of S in the unit sphere, and \mathbf{m} are its curvature-center coordinates. For hyperplanes we define

$$\mathbf{w}(H) := (\bar{b}, 0, h_1, ..., h_n) = (\bar{b}, \mathbf{m}), \tag{3.9}$$

where \bar{b} is the oriented curvature of the sphere or hyperplane \bar{H} obtained by inversion of H in the unit sphere.

Augmented curvature-center coordinates provide a global coordinate system: no two distinct oriented spheres have the same coordinates. The only case to resolve is when S is a hyperplane, i.e. b=0. The relation (3.6) shows that $(\bar{b},bx_1,...,bx_n)$ are the curvature-center coordinates of \bar{S} , and if $\bar{b} \neq 0$, this uniquely determines \bar{S} , and then, by inversion in the unit circle, S. In the remaining case $b=\bar{b}=0$ then $S=\bar{S}$ is the unique hyperplane passing through the origin whose unit normal is given by the remaining coordinates.

Given a collection $(S_1, S_2, ..., S_{n+2})$ of n+2 oriented spheres (possibly hyperplanes) in \mathbb{R}^n , the augmented matrix **W** associated with it is the $(n+2) \times (n+2)$ matrix whose j-th row is given by the augmented curvature-center coordinates $\mathbf{w}(S_j)$ of the j-th sphere.

The action of inversion in the unit sphere has a particularly simple interpretation in augmented matrix coordinates. If \mathbf{W} is the augmented matrix associated to a Descartes configuration, and if \mathbf{W}' is the augmented matrix associated to its inversion in the unit sphere, then

$$\mathbf{W} = \mathbf{W}' \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix}, \tag{3.10}$$

a result which follows from (3.6).

Theorem 3.3 (Augmented Euclidean Descartes Theorem) An oriented Descartes configuration of n+2 spheres $\{S_i: 1 \leq i \leq n+2\}$ in \mathbb{R}^n has an augmented matrix \mathbf{W} which satisfies

$$\mathbf{W}^{T}\mathbf{Q}_{n}\mathbf{W} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_{n} \end{bmatrix}.$$
 (3.11)

Conversely, any real solution W to this matrix equation is the augmented matrix of a unique oriented Descartes configuration.

The augmented Euclidean Descartes theorem includes as special cases the "if" direction of each of the theorems stated so far, and represents our final stage of generalization of the Descartes circle theorem in Euclidean space. In particular, the "if" part of the Euclidean generalized Descartes theorem is just (3.11) with the first row and column deleted. In the converse direction ⁴ this theorem gives a parametrization of all oriented Descartes configurations, and it is a "moduli space" for such configurations given as an affine real-algebraic variety.

We discovered the augmented Euclidean Descartes theorem in studying analogues of the Descartes theorem in non-Euclidean geometries. In the next section we formulate and prove such an analogue in spherical geometry; then in §5 we deduce the augmented Euclidean Descartes theorem from it.

4. Spherical Geometry

The standard model for spherical geometry \mathbb{S}^n is the unit *n*-sphere S^n embedded in \mathbb{R}^{n+1} as the surface

$$S^{n} := \{ y : y_0^2 + y_1^2 + \dots + y_n^2 = 1 \}$$

$$(4.1)$$

with the Riemannian metric induced from the Euclidean metric in \mathbb{R}^{n+1} by restriction. In this model, the distance between two points of \mathbb{S}^n is simply the angle α between the radii that join the origin of \mathbb{R}^{n+1} to the representatives of these points on S^n . This distance α always satisfies $0 \le \alpha \le \pi$.

A sphere C in this geometry is the locus of points equidistant (at distance α say) from a point in \mathbb{S}^n called its center. The quantity $\alpha = \alpha(C)$ is the spherical radius or angular radius of C; it is the angle at the origin $\mathbf{0}$ of \mathbb{R}^{n+1} between a ray from $\mathbf{0}$ to the center of C and a ray from $\mathbf{0}$ to any point of C. Note that there are two choices for the center (and the angular radius) of a given sphere; these two choices form a pair of antipodal points of S^n . The choice of a center amounts to orienting the sphere. In this model the interior of a sphere is a spherical cap, cut off by the intersection of the sphere S^n with a hyperplane in \mathbb{R}^{n+1} , so (by abuse of language) we will also call an oriented sphere a spherical cap.

The two spherical caps determined by a given sphere are called *complementary* and the sum of their angular radii is π . We define the *interior* of an oriented sphere to contain all points of S^n on the same side of the hyperplane as the center of the sphere. If we describe a hyperplane by a linear form

$$\mathbf{F}(y) = \sum_{i=0}^{n} f_i y_i - f,\tag{4.2}$$

normalized by the requirement

$$\sum_{i=0}^{n} f_i^2 = 1,$$

⁴In the converse direction the Augmented Euclidean Descartes theorem is not as strong as the converse in the Euclidean generalized Descartes theorem, nor does it imply the converse to the Soddy-Gossett theorem; these results require separate proofs.

this provides an orientation by defining a positive half-space F(y) > 0. The sphere has center $\mathbf{f} := (f_0, f_1, \dots, f_n)$ and has positive radius if and only if |f| < 1. The radius α satisfies $\cos \alpha = f$, and the interior of the spherical cap it determines is the region where the linear form is positive. A spherical cap can be specified either by a pair (\mathbf{f}, α) or by the pair $(-\mathbf{f}, \alpha - \pi)$, while $(-\mathbf{f}, \pi - \alpha)$ determines the complementary spherical cap.

A spherical Descartes configuration consists of n+2 mutually tangent spherical caps on the surface of the unit n-sphere, such that either (i) the interiors of all spherical caps are mutually disjoint, or (ii) the interiors of all complementary spherical caps are mutually disjoint.

Theorem 4.1 (Spherical Soddy-Gossett Theorem) Given a spherical Descartes configuration of n+2 mutually tangent spherical caps C_i on the n-dimensional unit sphere S^n embedded in \mathbb{R}^{n+1} , with spherical radius α_j subtended by the j-th cap, then these spherical radii satisfy the relation

$$\sum_{i=1}^{n+2} (\cot \alpha_i)^2 = \frac{1}{n} (\sum_{i=1}^{n+2} \cot \alpha_i)^2 - 2.$$
 (4.3)

This theorem was found by Mauldon [22, Theorem 4] In 1962, as part of a more general result allowing non-tangent spheres. He also established a converse: to each real solution of (4.3) there corresponds some spherical Descartes configuration, and two spherical Descartes configuration with the same data in (4.3) are congruent configurations in spherical geometry.

The spherical Soddy-Gossett theorem is intrinsic, i.e. it depends only on the Riemannian metric for spherical geometry, and not on the coordinate system used to describe the manifold. However we shall establish it as a special case of a result that does depend on a particular choice of coordinate system. If C is a spherical cap with center $\mathbf{y} = (y_0, y_1, y_2, ... y_{n+1})$, and angular radius α , we define its *spherical curvature-center coordinates* $\mathbf{w}_+(C)$ to be the row vector

$$\mathbf{w}_{+}(C) := (\cot \alpha, \frac{y_0}{\sin \alpha}, \frac{y_1}{\sin \alpha}, \dots, \frac{y_n}{\sin \alpha}). \tag{4.4}$$

No two spherical caps have the same coordinates \mathbf{w}_+ , since α is uniquely determined by the first coordinate, and then the y_j are uniquely determined using the other coordinates.

To any configuration of n+2 caps C_1, \ldots, C_{n+2} we associate the $(n+2) \times (n+2)$ spherical curvature-center coordinate matrix \mathbf{W}_+ whose jth row is $\mathbf{w}_+(C_i)$.

Theorem 4.2 (Spherical Generalized Descartes Theorem) Given a configuration of n+2 oriented spherical caps C_j that is a spherical Descartes configuration, then the $(n+2) \times (n+2)$ matrix \mathbf{W}_+ whose j-th row is the spherical curvature-center coordinates of C_j satisfies

$$\mathbf{W}_{+}^{T}\mathbf{Q}_{n}\mathbf{W}_{+} = \begin{bmatrix} -2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2I_{n} \end{bmatrix} = diag(-2, 2, 2, ..., 2).$$

$$(4.5)$$

Conversely, any real matrix \mathbf{W}_+ that satisfies this equation is the spherical curvature-center coordinate matrix of some spherical Descartes configuration.

The (1,1)-entry of the matrix relation (4.5) is the spherical Soddy-Gossett theorem above. This theorem has a remarkably simple proof, which is based on two preliminary lemmas. Let \mathbf{J}_n be the $(n+2)\times(n+2)$ matrix

$$\mathbf{J}_n = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} = \operatorname{diag}(-1, 1, ..., 1). \tag{4.6}$$

Lemma 4.3. (i) For any (n+2)-vector \mathbf{w}_+ , there is a spherical cap C with $\mathbf{w}_+(C) = \mathbf{w}_+$ if and only if

$$\mathbf{w}_{+}\mathbf{J}_{n}\mathbf{w}_{+}^{T} = 1. \tag{4.7}$$

(ii) The spherical caps C and C' are externally tangent if and only if

$$\mathbf{w}_{+}(C)\mathbf{J}_{n}\mathbf{w}_{+}(C')^{T} = -1. \tag{4.8}$$

Proof. (i). If \mathbf{w}_+ comes from a spherical cap with center \mathbf{y} and angular radius α , then

$$\mathbf{w}_{+}\mathbf{J}_{n}\mathbf{w}_{+}^{T} = \frac{-(\cos\alpha)^{2} + \sum_{j=0}^{n} y_{j}^{2}}{(\sin\alpha)^{2}} = \frac{1 - (\cos\alpha)^{2}}{(\sin\alpha)^{2}} = 1$$

so (4.7) holds.

Conversely, if (4.7) holds, then one recovers a unique α with $0 < \alpha < \pi$ by setting $\cot \alpha := (\mathbf{w}_+)_1$, and one then defines a vector $\mathbf{y} = (y_0, ..., y_{n+1})$ via $y_j := \frac{(\mathbf{w}_+)_j}{\sin \alpha}$, noting that $\sin \alpha \neq 0$. The equation (4.7) now implies that $|\mathbf{y}|^2 = 1$, so \mathbf{y} lies on the unit sphere, and we have determined a spherical cap giving the vector \mathbf{w}_+ .

(ii). Two spherical caps with centers \mathbf{y}, \mathbf{y}' with angular radii α, α' are externally tangent if and only if the angle between their centers, viewed from the origin in \mathbb{R}^{n+1} is $\alpha + \alpha'$. Since \mathbf{y} and \mathbf{y}' are unit vectors, this holds if and only if

$$\mathbf{y}(\mathbf{y}')^T = \cos(\alpha + \alpha'),$$

Now

$$\mathbf{w}_{+}(C)\mathbf{J}_{n}\mathbf{w}_{+}(C')^{T} = \frac{1}{\sin\alpha\sin\alpha'}(-\cos\alpha\cos\alpha' + \mathbf{y}(\mathbf{y}')^{T})$$

and this gives (4.8), using $\cos(\alpha + \alpha') = \cos \alpha \cos \alpha' - \sin \alpha \sin \alpha'$. \Box

Lemma 4.4. If **A**, **B** are symmetric non-singular $n \times n$ matrices and $\mathbf{W}\mathbf{A}\mathbf{W}^T = \mathbf{B}$, then $\mathbf{W}^T\mathbf{B}^{-1}\mathbf{W} = \mathbf{A}^{-1}$.

Proof. The matrix **W** is non-singular since **B** is non-singular. Now invert both sides, then multiply on the left by \mathbf{W}^T and on the right by \mathbf{W} . \square

Proof of the Spherical Generalized Descartes Theorem. If the caps C_j touch externally, we have from Lemma 4.3 that

$$\mathbf{W}_{+}\mathbf{J}_{n}\mathbf{W}_{+}^{T} = 2\mathbf{I}_{n+2} - \mathbf{1}_{n+2}\mathbf{1}_{n+2}^{T} = 2\mathbf{Q}_{n}^{-1}.$$
(4.9)

Then applying Lemma 4.4 (with $\mathbf{A} = \mathbf{J}_n$ and $\mathbf{W} = \mathbf{W}_+$) we obtain

$$\mathbf{W}_{+}^{T}\mathbf{Q}_{n}\mathbf{W}_{+} = 2\mathbf{J}_{n}^{-1} = 2\mathbf{J}_{n}.$$
(4.10)

Conversely, (4.10) implies (4.9), by Lemma 4.4. Looking at the diagonal elements of $\mathbf{W}_{+}\mathbf{J}_{n}\mathbf{W}_{+}^{T}$, which are all 1's, Lemma 4.3(i) guarantees that the *j*-th row of \mathbf{W}_{+} is a vector $\mathbf{w}_{+}(C_{j})$ for some (uniquely determined) spherical cap C_{j} , and the off-diagonal elements all being -1 shows by Lemma 4.3(ii) that the caps touch externally pairwise, so form a spherical Descartes configuration. \square

Remark. Wilker [33, pp. 388-390] came tantalizingly close to obtaining the spherical generalized Descartes theorem. He termed a spherical Descartes configuration a "cluster", and introduced spherical curvature-center coordinates. In a remark he noted our Lemma 4.3 and stated equation (4.9). However he did not invert his formula, via Lemma 4.4 and so failed to formulate a result in terms of the Descartes quadratic form.

5. Stereographic Projection and the Augmented Euclidean Descartes Theorem

We derive the augmented Euclidean Descartes theorem from the spherical Generalized Descartes theorem, using stereographic projection. The resulting derivation is reversible, so the spherical Generalized Descartes theorem and the augmented Euclidean Descartes theorem may be viewed as equivalent results.

Consider the unit sphere in \mathbb{R}^{n+1} , given by $\sum_{i=0}^{n} y_i^2 = 1$. Points on this sphere can be mapped into the plane $y_0 = 0$ by stereographic projection from the "south pole" (-1, 0, ..., 0), see Figure 4. (The hyperboloid in the figure will be used later.)

This mapping $(y_0,...,y_n) \rightarrow (x_1,...,x_n)$ is given by

$$x_j = \frac{y_j}{1 + y_0}, \quad 1 \le j \le n.$$

The spherical cap C with center $(p_0,...,p_n)$ and angular radius α is the intersection of the unit sphere with the plane

$$\sum_{j=1}^{n} p_j y_j = \cos \alpha.$$

The sterographic projection of this cap in the hyperplane $y_0 = 0$ is the (Euclidean) sphere S with center $(x_1, ..., x_n)$ and radius r, where

$$x_j = \frac{p_j}{p_0 + \cos \alpha}, \quad 1 \le j \le n,$$

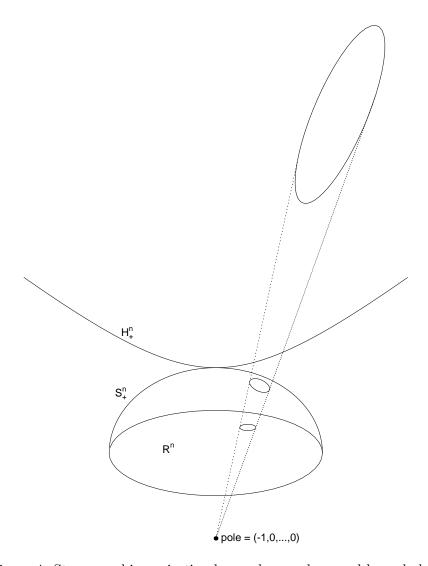


Figure 4: Stereographic projection-hyperplane, sphere and hyperboloid

and

$$r = \frac{\sin \alpha}{p_0 + \cos \alpha}.$$

If the boundary of the cap C contains the south pole, the corresponding sphere S has infinite radius, i.e. it is a hyperplane.

Proof of the Augmented Euclidean Descartes Theorem. The spherical coordinates of the spherical cap C are given by the row-vector

$$\mathbf{w}_{+}(C) = (\cot \alpha, \frac{p_0}{\sin \alpha}, \frac{p_1}{\sin \alpha}, \dots, \frac{p_n}{\sin \alpha}),$$

We relate this to the augmented Euclidean coordinate vector $\mathbf{w}(S)$ associated with the corresponding projected sphere S in the plane $y_0 = 0$, given by (3.8). We have $x_j/r = p_j/\sin \alpha$, $b = 1/r = \cot \alpha + \frac{p_0}{\sin \alpha}$, and we find

$$\bar{b} = \cot \alpha - \frac{p_0}{\sin \alpha}.$$

Thus

$$\mathbf{w}(S) = (\cot \alpha - \frac{p_0}{\sin \alpha}, \cot \alpha + \frac{p_0}{\sin \alpha}, \frac{p_1}{\sin \alpha}, ..., \frac{p_n}{\sin \alpha}) = \mathbf{w}_+(C)\mathbf{G},$$

where

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{5.1}$$

Suppose we have a configuration of n+2 spherical caps $C_1, ..., C_{n+2}$ on the unit sphere. These stereographically project into a configuration of Euclidean spheres $S_1, ..., S_{n+2}$ in the equatorial plane $y_0 = 0$, and conversely every configuration of Euclidean spheres lifts to a configuration of spherical caps. The map sends spherical Descartes configurations to Euclidean Descartes configurations. We assemble the corresponding rows $\mathbf{w}_+(C_j)$, $\mathbf{w}(S_j)$ into matrices \mathbf{W}_+ and \mathbf{W}_+ respectively. Then

$$\mathbf{W} = \mathbf{W}_{+}\mathbf{G},\tag{5.2}$$

and, using the Spherical Generalized Descartes Theorem 4.2, we have

$$\mathbf{W}^T \mathbf{Q}_n \mathbf{W} = \mathbf{G}^T \mathbf{W}_+^T \mathbf{Q}_n \mathbf{W}_+ \mathbf{G} = \mathbf{G}^T \operatorname{diag}(-2, 2, ..., 2) \mathbf{G} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_n \end{bmatrix},$$

which proves the augmented Euclidean Descartes Theorem 3.3. \Box

6. Hyperbolic Geometry

There are many models of hyperbolic space \mathbb{H}^n , of which the three most common are the (Poincaré) unit ball model, the half space model, and the hyperboloid model. (In two dimensions we say "unit disk" and "half-plane" for the first two models.) The unit ball and half-space models are described in many places, e.g. Beardon [3] and Berger [5, Chapter 19]. The hyperboloid model, which is less well known, but which is in some ways simpler than the others, is described in Beardon [3, Section 3.7], Reynolds [24] and Ryan [25]. The unit ball and half-space models are embedded in \mathbb{R}^n , though with different metrics, while the hyperboloid model is embedded in \mathbb{R}^{n+1} , endowed with a Minkowski metric. Here we need only the unit ball and hyperboloid models. A sphere in hyperbolic n-space \mathbb{H}^n is defined as the locus of points equidistant (in hyperbolic metric) from some fixed point in \mathbb{H}^n , the center.

The unit ball model consists of the points (y_1, \ldots, y_n) in \mathbb{R}^n with $\sum_{j=1}^n y_i^2 < 1$, with the ideal boundary being $\sum_{j=1}^n y_j^2 = 1$. In this model, the hyperbolic metric is

$$ds^{2} = (dy_{1}^{2} + \dots + dy_{n}^{2})/(1 - \sum_{j=1}^{n} y_{j}^{2})^{2}.$$

and the hyperbolic distance between two points y, y' satisfies

$$\cosh(d(\mathbf{y}, \mathbf{y}')) = \left((1 + \sum_{j=1}^{n} y_j^2)(1 + \sum_{j=1}^{n} y_j'^2) - 4\sum_{j=1}^{n} y_j y_j' \right) / \left((1 - \sum_{j=1}^{n} y_j^2)(1 - \sum_{j=1}^{n} y_j'^2) \right). (6.1)$$

In this model a hyperbolic sphere (of finite radius) is a Euclidean sphere contained strictly inside the unit ball; however its hyperbolic center and hyperbolic radius usually differ from the Euclidean ones.

Points in the hyperboloid model are represented in \mathbb{R}^{n+1} as points on the upper sheet H^n_+ given by $u_0 > 0$ of the two sheeted-hyperboloid H^n_+ cut out by the equation

$$u_0^2 = 1 + u_1^2 + \dots + u_n^2,$$

where $H^n_{\pm} = H^n_{+} \cup H^n_{-}$ with $H^n_{-} = -H^n_{+}$. However the Riemannian metric is not that induced from the Euclidean metric on \mathbb{R}^{n+1} , but rather is induced from the Minkowski metric

$$ds^2 = -du_0^2 + du_1^2 + \dots + du_n^2$$

on this space, cf. Beardon [3, p. 49]. The formula for the hyperbolic distance $d(\mathbf{u}, \mathbf{u}')$ in this metric is given by

$$\cosh(d(\mathbf{u}, \mathbf{u}')) = u_0 u_0' - u_1 u_1' - \dots - u_n u_n', \tag{6.2}$$

see Reynolds [24, formula (6.10)]. One can go between the hyperboloid model and the ball model by the change of variables

$$y_j = \frac{u_j}{1 + u_0}, \quad \text{for } 1 \le j \le n,$$

and in the opposite direction by

$$u_0 = \frac{2}{\Lambda} - 1$$
 and $u_j = \frac{2y_j}{\Lambda}$, $1 \le j \le n$,

where

$$\Delta = 1 - \sum_{j=1}^{n} u_j^2.$$

From (6.2) we see that in this model a hyperbolic sphere is represented by the intersection of H_+^n with a hyperplane $\mathbf{G}(u) = 0$, where

$$\mathbf{G}(u) = g_0 u_0 - \sum_{i=1}^{n} g_i u_i - g,$$
(6.3)

where g > 1 and we require **G** to be normalized by the requirement that

$$g_0^2 = 1 + \sum_{i=1}^n g_i^2, \tag{6.4}$$

i.e. the point $\mathbf{g} := (g_0, g_1, \dots, g_n)$ lies on H^n_+ . This intersection is typically a (Euclidean) ellipsoid. The center of the sphere is \mathbf{g} , and its radius d satisfies $\cosh d = g$. As in the spherical case, we define the interior of the hyperbolic sphere to be the region on the same side of the plane $\mathbf{G}(u) = 0$ as the center.

If we consider a general hyperplane $\mathbf{G}(u) = 0$ which has the normalized form (6.4), intersecting the complete hyperboloid $H_+^n \cup H_-^n$, the intersection may be empty, a single point, an ellipsoid (or sphere) on either sheet, a paraboloid on either sheet, or a two-sheeted hyperboloid. For later use we term these possibilities virtual hyperbolic spheres, except for the empty set or a point. We also assign them an orientation given by the sign of the constant term g in the associated linear form $\mathbf{G}(u)$. (Replacing $\mathbf{G}(u)$ by $-\mathbf{G}(u)$ reverses the orientation.) Only the points in H_+^n correspond to real points in \mathbb{H}^n . In the two-dimensional case parabolas on the upper sheet correspond to horocycles; in the unit disc model these are (Euclidean) circles that are tangent to the bounding circle. They have infinite radius. In the disc model their centers are on the bounding circle, and in the hyperboloid model their centers are at infinity. The boundary of the disc model corresponds to a circle at infinity in the hyperboloid model.

An oriented hyperbolic Descartes configuration is any set of n+2 mutually tangent oriented hyperbolic (n-1)-spheres in \mathbb{H}^n , having the property that either (i) all interiors of the spheres are disjoint, or (ii) the interiors of each pair of spheres intersect in a nonempty open set. In the following result we also allow (oriented) Descartes configurations which include those virtual hyperbolic spheres of infinite radius which in the ball model correspond to Euclidean spheres lying entirely inside the closed ball and tangent to its boundary. The ideal boundary itself forms a single limiting (n-1)-sphere in this sense.

Theorem 6.1 (Hyperbolic Soddy-Gossett Theorem) The oriented hyperbolic radii $\{s_j : 1 \leq j \leq n+2\}$ of an oriented Descartes configuration of n+2 spheres in hyperbolic space \mathbb{H}^n satisfy the relation

$$\sum_{j=1}^{n+2} (\coth s_j)^2 = \frac{1}{n} (\sum_{j=1}^{n+2} \coth s_j)^2 + 2.$$
 (6.5)

This result was found by Mauldon [22]. The hyperbolic Soddy-Gossett theorem is intrinsic, depending only on the hyperbolic metric. We derive it as a special case of a result which does depend on a specific coordinate system, namely that given above for the the hyperboloid model.

If S is a hyperbolic sphere in H^n_+ with center $\mathbf{u} = (u_0, u_1, u_2, ..., u_{n+1})$, and hyperbolic radius s_j , we define its hyperbolic curvature-center coordinates $\mathbf{w}_-(S)$ to be the row vector

$$\mathbf{w}_{-}(S) := (\coth s, \frac{u_0}{\sinh s}, \frac{u_1}{\sinh s}, ..., \frac{u_n}{\sinh s}). \tag{6.6}$$

To a configuration of n+2 hyperbolic spheres S_1, \ldots, S_{n+2} we associate the $(n+2) \times (n+2)$ matrix \mathbf{W}_- whose jth row is $\mathbf{w}_-(S_j)$.

Theorem 6.2 (Hyperbolic Generalized Descartes Theorem) Given a configuration of (n+2) oriented hyperbolic spheres which is a hyperbolic Descartes configuration, then the associated matrix \mathbf{W}_{-} whose rows are the hyperbolic curvature-center coordinates of the spheres satisfies

$$\mathbf{W}_{-}^{T}\mathbf{Q}_{n}\mathbf{W}_{-} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2I_{n} \end{bmatrix} = diag(2, -2, 2, ..., 2).$$
(6.7)

The converse of Theorem 6.2 does not hold, because some matrices \mathbf{W}_{-} satisfying (6.7) do not correspond to hyperbolic Descartes configurations. We can obtain a converse by allowing "virtual Descartes configurations" that lie on both sheets of the hyperboloid. One simply defines a virtual Descartes configuration to be the image on the two-sheeted hyperboloid resulting from stereographic projection through (-1,0,0,...,0) of any Descartes configuration on the unit sphere. The resulting hyperbolic coordinate matrix \mathbf{W}_{-} is to be defined by (6.8). One can define a "virtual (oriented) hyperbolic sphere", and define its oriented radius and center using the formulas following (6.3); the center and oriented radius of some "virtual hyperbolic spheres" may then be (non-real) complex numbers, although by definition their hyperbolic curvature-center coordinates will be real.

Theorem 6.2 is readily deducible from the spherical generalized Descartes theorem via stereographic projection through the "south pole" (-1,0,0,...0) in \mathbb{R}^{n+1} , mapping oriented Descartes configurations on the upper sheet H_+^n of the hyperboloid to spherical Descartes configurations which lie entirely on the upper hemisphere S_+^n of the unit sphere. See Figure 4 in §5.

In applying stereographic projection, the locus of a hyperbolic (n-1)-sphere on the hyperbolic is mapped to the locus of a spherical (n-1)-sphere on the unit n-sphere, and also to the locus of a Euclidean (n-1)-sphere in the plane $x_0 = 0$. Note however that the hyperbolic center, the spherical center of the associated spherical cap and the Euclidean center of the Euclidean sphere are typically all distinct in the sense that they usually lie on three different lines through the "south pole" (-1,0,...,0) in \mathbb{R}^{n+1} .

One can show that, if \mathbf{W}_{-} is defined as above, and if \mathbf{W}_{+} are the spherical Descartes coordinates associated to the resulting Descartes configuration on the sphere as in §4, then they are related by

$$\mathbf{W}_{-} = \mathbf{W}_{+} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{n} \end{bmatrix} . \tag{6.8}$$

We omit a proof of this fact, which can be carried out along the lines of §5. Given it, one immediately deduces Theorem 6.2 from Theorem 4.2, plus a converse if "virtual Descartes configurations" are included.

7. Apollonian Packings

Using stereographic projection we have a recipe to pass between Euclidean, spherical and hyperbolic Descartes configurations. It gives a one-to-one correspondence between configurations \mathbf{W}, \mathbf{W}_+ , and \mathbf{W}_- given by (5.1) and (6.8), namely

$$\mathbf{W} = \mathbf{W}_{+} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & I_{n} \end{bmatrix} = \mathbf{W}_{-} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n} \end{bmatrix}.$$
 (7.1)

Here we have extended the definition of Descartes configuration to "virtual Descartes configurations" in the hyperbolic case to be configurations on both sheets of the two-sheeted hyperboloid. This recipe clearly lifts to Apollonian packings.

Since the spherical and hyperbolic Soddy-Gossett theorems involve quadratic forms, an analogue of the relation (2.1) holds in spherical and hyperbolic geometry, permitting the easy calculation of the "curvatures" $\cot \alpha$ (resp. $\coth s$) of circles in spherical (resp. hyperbolic) packings. There is a notion of "integral Apollonian circle packing" for such "curvatures" which makes sense in spherical and hyperbolic geometry. Furthermore, analogues of the relation (2.4) hold in spherical and hyperbolic geometry as well, permitting the easy calculation of the centers in spherical (resp. hyperbolic) Apollonian packings.

Thus the standard Euclidean Apollonian packing pictured in Figure 3, with center at the origin, has a corresponding hyperbolic packing obtained by stereographic projection in which the bounding outer circle in the packing is the "absolute" in the unit ball model of the hyperbolic plane, and in which the $\coth(r)$'s are all integers, but *not* the same integers as in the Euclidean packing, calculated using \mathbf{W}_{-} in (7.1). See Figure 5.

Those circles that are tangent to the bounding circle are known as horocycles, and have infinite hyperbolic radius, so the corresponding value of $\coth(r)$ is 1. This explains the large number of circles assigned the value 1 in Figure 5, namely all those that touch the outer circle.

Similarly, in the spherical packing associated to the standard Euclidean packing in Figure 3, the cot α 's are all integers, different from both the Euclidean and hyperbolic cases, starting from (0,1,1,2). See Figure 6.

One may notice interesting numerical relations among the integers in these three packings. Consider a "loxodromic sequence" of spheres as studied in Coxeter [9], [11], where each sphere is produced by reflection in the largest sphere of the preceding Descartes configuration, For the "curvatures" one obtains for the Euclidean packing the infinite sequence E: (-1,2,2,3,15,38,...), for the spherical packing S: (0,1,1,2,8,21,...) and for the hyperbolic packing H: (-1,1,1,1,7,17,...). Note that S+H=E, since this is so for the initial values, and each sequence satisfies the same fourth order linear recurrence relation, which is $x_{n+1} = 2x_n + 2x_{n-1} + 2x_{n-2} - x_{n-3}$, by (2.1).

In the Euclidean case, there are infinitely many different kinds of Apollonian packings having integer curvatures for all circles, see [17]. The same occurs for both hyperbolic and

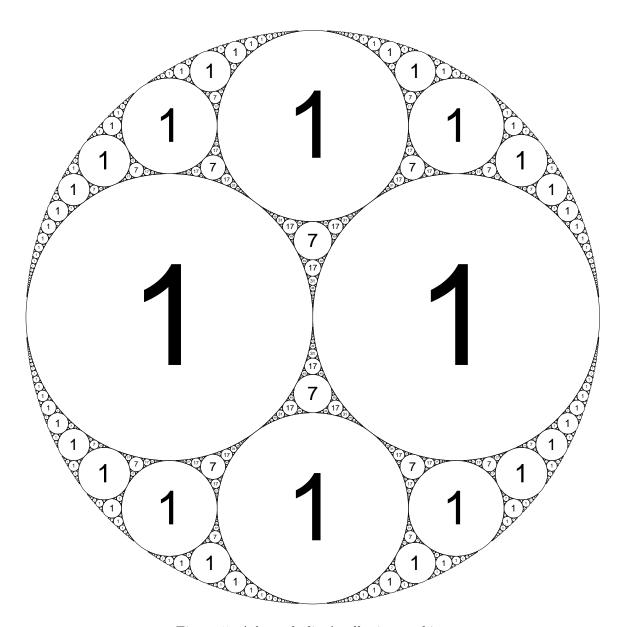


Figure 5: A hyperbolic Apollonian packing

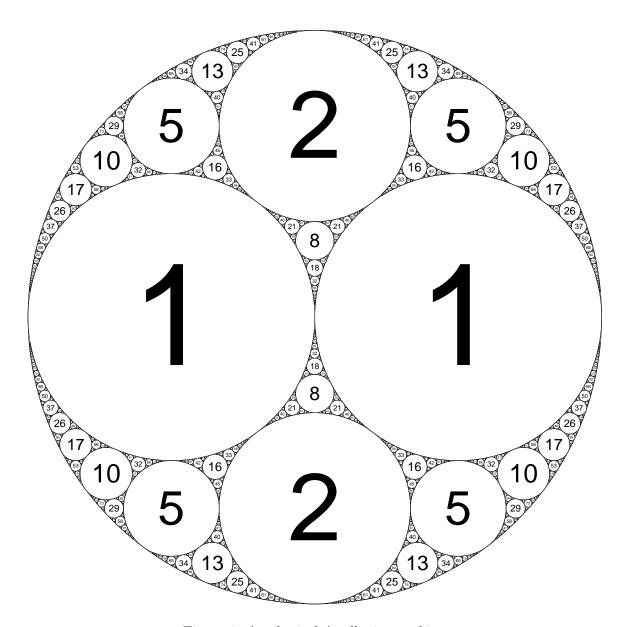


Figure 6: A spherical Apollonian packing

spherical Apollonian circle packings. In the hyperbolic case we also include among such integer hyperbolic circle packings some packings which are "virtual packings". Figure 5 is generated from the basic configuration having coth's (-1,1,1,1), and the next simplest hyperbolic case is (-2,3,5,6).

The Apollonian construction works also in higher dimensions, but gives sphere packings only in dimensions two and three; in dimensions four and higher we do not get proper packings; after several steps the spheres will overlap, see Boyd [6]. However "Apollonian sphere ensembles" continue to exist in all dimensions as collections of Descartes configurations, see [16].

There is a considerable amount of mathematics devoted to circle packings; Kenneth Stephenson's [30] bibliography of circle-packing papers lists over 90 papers since 1990. For further relations of Apollonian packings and the relation of integer Apollonian circle packings to the integer Lorentz group $O(1,3,\mathbb{Z})$, see our series of papers with Ron Graham and Catherine Yan [14],[15],[16], [17] and Söderberg [28].

8. Conclusion

We have extended the Descartes circle theorem, well known for n-dimensional Euclidean space, to n-dimensional spherical and hyperbolic space. We presented matrix generalizations of the Descartes circle theorem which characterize Descartes configurations in all three geometries, and which required for their formulation the use of a particular coordinate system in each of these geometries. Mauldon [22] generalized the Soddy-Gossett theorem in all three geometries to apply to sets of n+2 equally inclined spheres, as measured by an inclination parameter γ , with $\gamma=-1$ for touching spheres; our matrix theorems can be extended to the case of arbitrary γ as well.

Interestingly, there are one-dimensional analogues of all these theorems. For the Euclidean case in one dimension a "circle" consists of two points bounding an interval, and two "circles" are tangent if they have one point in common. The one-dimensional Descartes form is

$$\mathbf{Q}_1 := I_3 - \mathbf{1}_3 \mathbf{1}_3^T = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}. \tag{8.2}$$

A one-dimensional Euclidean Descartes configuration consists of two touching intervals, and a third "interval" which is the complement of their union, so that the three intervals cover the line \mathbb{R} . Call the third "interval" the *infinite interval*, and its "length" is defined to be the negative of the length of its complement, which is the union of the first two intervals. The radius is half the "length." The radii r_1, r_2, r_3 , of the three intervals then satisfy

$$r_1 + r_2 + r_3 = 0$$
,

which is equivalent to the Descartes relation

$$Q_1(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}) = -\frac{2}{r_1 r_2} - \frac{2}{r_1 r_3} - \frac{2}{r_2 r_3} = 0.$$
 (8.3)

The value of "curvature×center" of the infinite interval is defined as being equal to the "curvature×center" of the finite interval obtained by reflection sending $x \to \frac{1}{x}$. This describes

a positively oriented Descartes configuration; a negatively oriented one is obtained by reversing all signs. One can now define a 3×3 augmented matrix **W** exactly as in the augmented Euclidean Descartes theorem, and one finds that

$$\mathbf{W}^T \mathbf{Q}_1 \mathbf{W} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{8.4}$$

Conversely, every solution **W** to this equation corresponds to a one-dimensional Descartes configuration. The is even a notion of Apollonian packing in dimension n = 1, but it consists of a single Descartes configuration! This holds because there is only a single circle tangent to a pair of tangent one-dimensional circles. That is, the Descartes equation (8.3) is linear in each curvature variable $a_i = \frac{1}{r_i}$ separately, instead of quadratic, hence the reflection operation which generates new circles to add to the Apollonian packing in dimensions $n \geq 2$ does not exist. Finally, there are one-dimensional spherical and hyperbolic analogues of these results, defined via (7.1), taking n = 1. They can be established by stereographic projection.

The main results in this paper are theorems in *inversive geometry*, as described in Wilker [33], also Alexander [2] and Schwerdtfeger [26]. Inversive geometry is the geometry that preserves spheres and their incidences, which consists of the study of geometric properties preserved by the group M"ob(n) of conformal transformations of the space $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} \approx S^n$. The set of Descartes configurations form a single orbit under the action of the conformal group, and this group appears 5 in our results as the (real) automorphism group

$$Aut(Q_n) := \{ \mathbf{N} : \mathbf{N}^T \mathbf{Q}_n \mathbf{N} = \mathbf{Q}_n \}$$

of the Descartes quadratic form Q_n , which is a Lie group isomorphic to O(n+1,1), (see Wilker [33, Corollary p. 390] for the isomorphism), and the three generalized Descartes theorems given here are invariant under the action of $Aut(Q_n)$. Our results prompt the question: Is there a "natural" characterization of the global coordinate systems used in these geometries which yields the generalized Descartes circle theorems in the matrix form presented here?

⁵The conformal group is isomorphic to a subgroup of index 2 in $Aut(Q_n)$, and we introduced oriented Descartes configurations to keep track of the two cosets.

References

- [1] D. Aharonov and K. Stephenson, Geometric sequences of discs in the Apollonian packing, Algebra i Analiz 9 (1997), No. 3, 104–140. [English version: St. Petersburg Math. J. 9 (1998), 509–545.]
- [2] H. W. Alexander, Vectorial inversive and non-Euclidean geometry, Amer. Math. Monthly **74** (1967), 128–140.
- [3] A. F. Beardon, The Geometry of Discrete Groups, Springer-Verlag: New York 1983.
- [4] H. Beecroft, Properties of Circles in Mutual Contact, Lady's and Gentleman's Diary **139** (1842), 91–96, (1846), p. 51.
- [5] M. Berger, Geometry II, Springer-Verlag: Berlin 1987.
- [6] D. W. Boyd, The osculatory packing of a three-dimensional sphere. Canadian J. Math. **25** (1973), 303–322.
- [7] J. L. Coolidge, A treatise on the circle and the sphere, Clarendon Press: Oxford 1916.
- [8] H. S. M. Coxeter, The problem of Apollonius, Amer. Math. Monthly 75 (1968), 5–15.
- [9] H. S. M. Coxeter, Loxodromic sequences of tangent spheres, Aequationes Mathematicae 1 (1968), 104–121.
- [10] H. S. M. Coxeter, Introduction to Geometry, Second Edition, John Wiley and Sons, New York, 1969.
- [11] H. S. M. Coxeter, Numerical distances among the spheres in a loxodromic sequence, The Mathematical Intelligencer 19 No. 4, (1997), 41–47.
- [12] R. Descartes. *Oeuvres de Descartes, Correspondance IV*, (C. Adam and P. Tannery, Eds.), Paris: Leopold Cerf 1901.
- [13] T. Gossett, The Hexlet, Nature **139**(1937), 62.
- [14] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Packings: Geometry and Group Theory I. The Apollonian Group, eprint: arXiv math.MG/0010298
- [15] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Packings: Geometry and Group Theory II. Super- Apollonian Group and Integral Packings, eprint: arXiv math.MG/0010302
- [16] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Packings: Geometry and Group Theory III. Higher Dimensions, eprint: arXiv math.MG/0010324
- [17] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. Wilks and C. Yan, Apollonian Circle Packings: Number Theory, eprint: arXiv math.NT/0009113
- [18] T. E. Heath, A History of Greek Mathematics, Volume II. From Aristarchus to Diophantus, Dover, New York 1981. (Original: Clarendon Press, Oxford 1921).

- [19] K.E. Hirst, The Apollonian packing of circles, J. Lond. Math. Soc., 42 (1967), 281–291.
- [20] E. Kasner and F. Supnick, The Apollonian packing of circles, Proc. Nat. Acad. Sci. USA 29 (1943), 378–384.
- [21] R. Lachlan, On systems of circles and spheres, Phil. Trans. Roy. Soc. London, Ser. A 177 (1886), 481–625.
- [22] J. G. Mauldon, Sets of equally inclined spheres, Canadian J. Math. 14 (1962), 509–516.
- [23] D. Pedoe, On a theorem in geometry, Amer. Math. Monthly 74 (1967), 627–640.
- [24] W. F. Reynolds, Hyperbolic geometry on a hyperboloid, Amer. Math. Monthly **100** (1993), 442–455.
- [25] P. J. Ryan, Euclidean and non-Euclidean Geometry, Cambridge Univ. Press: Cambridge-New York 1986.
- [26] H. Schwerdtfeger, Geometry of Complex Numbers. Circle Geometry, Moebius Transformation, Non-Euclidean Geometry. Dover Publications: New York 1979.
- [27] F. Soddy, The Kiss Precise. Nature (June 20, 1936), p. 1021.
- [28] B. Söderberg, Apollonian tiling, the Lorentz group, and regular trees, Phys. Rev. A 46 (1992), No. 4, 1859–1866.
- [29] J. Steiner, Einige geometrische Betrachtungen, J. reine Angew. Math. 1 (1826), 161–184 and 252–288. (Also: J. Steiner, *Gesammelte Werke*, Vol. I, Reimer: Berlin 1881, pp. 17–76.)
- [30] K. Stephenson, Circle packing bibliography as of April 1999, preprint at: http://www.math.utk.edu/~kens.
- [31] S. M. Stigler, Stigler's Law of Eponymy, Transactions of the New York Academy of Sciences Ser. 2, **39** (1980), 147–158. (Merton Festschift Volume, Ed T. Gieryn).
- [32] J. B. Wilker, Four proofs of a generalization of the Descartes circle theorem, Amer. Math. Monthly **76** (1969), 278–282.
- [33] J. B. Wilker, Inversive Geometry, in: *The Geometric Vein*, (C. Davis, B. Grünbaum, F. A. Sherk, Eds.), Springer-Verlag: New York 1981, pp. 379–442.