More Lagrangian Problems

Lagrange's Equation

Basically it is the 'next level' of Newton's laws of motion. It allows one to solve much more complicated dynamics systems.

The Lagranian is defined as:

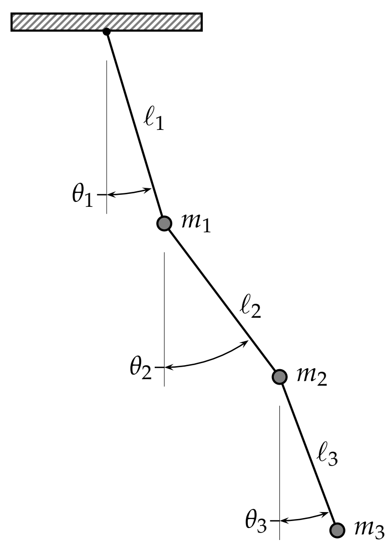
$$\mathsf{Lagrangian} = \mathsf{KineticEnergy} - \mathsf{PotentialEnergy} \Rightarrow \mathcal{L} = T - V$$

To obtain the equations of motion, we need to solve:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Triple Pendulum

Have already solutions for simple and double pendulums, so hence the triple pendulum is the next on the list. Perhaps the n-link pendulum will be next.



The positions of the masses can be calculated as:

$$\begin{aligned}
x_1 &= \ell_1 \sin\theta_1 \\
y_1 &= -\ell_1 \cos\theta_1 \\
x_2 &= \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 = x_1 + \ell_2 \sin\theta_2 \\
y_2 &= -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 = y_2 - \ell_2 \cos\theta_2 \\
x_3 &= \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 + \ell_3 \sin\theta_3 = x_2 + \ell_3 \sin\theta_3 \\
y_3 &= -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 - \ell_3 \cos\theta_3 = y_2 - \ell_3 \cos\theta_3
\end{aligned}$$

The speeds of the masses can be found by taking the derivatives:

$$\begin{aligned}
\dot{x}_1 &= \ell_1 \cos\theta_1 \cdot \dot{\theta}_1 \\
\dot{y}_1 &= \ell_1 \sin\theta_1 \cdot \dot{\theta}_1 \\
\dot{x}_2 &= \ell_1 \cos\theta_1 \cdot \dot{\theta}_1 + \ell_2 \cos\theta_2 \cdot \dot{\theta}_2 \\
\dot{y}_2 &= \ell_1 \sin\theta_1 \cdot \dot{\theta}_1 + \ell_2 \sin\theta_2 \cdot \dot{\theta}_2 \\
\dot{x}_3 &= \ell_1 \cos\theta_1 \cdot \dot{\theta}_1 + \ell_2 \cos\theta_2 \cdot \dot{\theta}_2 + \ell_3 \cos\theta_3 \cdot \dot{\theta}_3 \\
\dot{y}_3 &= \ell_1 \sin\theta_1 \cdot \dot{\theta}_1 + \ell_2 \sin\theta_2 \cdot \dot{\theta}_2 + \ell_3 \sin\theta_3 \cdot \dot{\theta}_3
\end{aligned}$$

Now solving for the *kinetic energy*:

$$\begin{aligned}
T = \frac{1}{2}mv^2 &= \frac{1}{2}m\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)^2 \\
&= \frac{1}{2}m_1\left(\dot{x}_1^2+\dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2+\dot{y}_2^2\right) + \frac{1}{2}m_3\left(\dot{x}_3^2+\dot{y}_3^2\right) \\
&= \frac{1}{2}m_1\left[\left(\ell_1\cos\theta_1\cdot\dot{\theta}_1\right)^2 + \left(\ell_1\sin\theta_1\cdot\dot{\theta}_1\right)^2\right] \\
&\quad + \frac{1}{2}m_2\left[\left(\ell_1\cos\theta_1\cdot\dot{\theta}_1 + \ell_2\cos\theta_2\cdot\dot{\theta}_2\right)^2 + \left(\ell_1\sin\theta_1\cdot\dot{\theta}_1 + \ell_2\sin\theta_2\cdot\dot{\theta}_2\right)^2\right] \\
&\quad + \frac{1}{2}m_3\left[\left(\ell_1\cos\theta_1\cdot\dot{\theta}_1 + \ell_2\cos\theta_2\cdot\dot{\theta}_2 + \ell_3\cos\theta_3\cdot\dot{\theta}_3\right)^2 + \left(\ell_1\sin\theta_1\cdot\dot{\theta}_1 + \ell_2\sin\theta_2\cdot\dot{\theta}_2 + \ell_3\sin\theta_3\cdot\dot{\theta}_3\right)^2\right] \\
&= \frac{1}{2}m_1\left[\ell_1^2\cos^2\theta_1\cdot\dot{\theta}_1^2 + \ell_1^2\sin^2\theta_1\cdot\dot{\theta}_1^2\right] \\
&\quad + \frac{1}{2}m_2\Big[\ell_1^2\cos^2\theta_1\cdot\dot{\theta}_1^2 + 2\ell_1\ell_2\cos\theta_1\cos\theta_2\cdot\dot{\theta}_1\dot{\theta}_2 + \ell_2^2\cos^2\theta_2\cdot\dot{\theta}_2^2 \\
&\qquad\qquad + \ell_1^2\sin^2\theta_1\cdot\dot{\theta}_1^2 + 2\ell_1\ell_2\sin\theta_1\sin\theta_2\cdot\dot{\theta}_1\dot{\theta}_2 + \ell_2^2\sin^2\theta_2\cdot\dot{\theta}_2^2\Big] \\
&\quad + \frac{1}{2}m_3\Big[\ell_1^2\cos^2\theta_1\cdot\dot{\theta}_1^2 + \ell_2^2\cos^2\theta_2\cdot\dot{\theta}_2^2 + \ell_3^2\cos^2\theta_3\cdot\dot{\theta}_3^2 + \ell_1^2\sin^2\theta_1\cdot\dot{\theta}_1^2 + \ell_2^2\sin^2\theta_2\cdot\dot{\theta}_2^2 + \ell_3^2\sin^2\theta_3\cdot\dot{\theta}_3^2 \\
&\qquad\qquad + 2\ell_1\ell_2\cos\theta_1\cos\theta_2\cdot\dot{\theta}_1\dot{\theta}_2 + 2\ell_1\ell_3\cos\theta_1\cos\theta_3\cdot\dot{\theta}_1\dot{\theta}_3 + 2\ell_2\ell_3\cos\theta_2\cos\theta_3\cdot\dot{\theta}_2\dot{\theta}_3 \\
&\qquad\qquad + 2\ell_1\ell_2\sin\theta_1\sin\theta_2\cdot\dot{\theta}_1\dot{\theta}_2 + 2\ell_1\ell_3\sin\theta_1\sin\theta_3\cdot\dot{\theta}_1\dot{\theta}_3 + 2\ell_2\ell_3\sin\theta_2\sin\theta_3\cdot\dot{\theta}_2\dot{\theta}_3\Big] \\
&= \frac{1}{2}m_1\left(\ell_1^2\dot{\theta}_1^2\right) \\
&\quad + \frac{1}{2}m_2\Big(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 \\
&\qquad\qquad + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\Big) \\
&\quad + \frac{1}{2}m_3\Big(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + \ell_3^2\dot{\theta}_3^2 \\
&\qquad\qquad + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) \\
&\qquad\qquad + 2\ell_1\ell_3\dot{\theta}_1\dot{\theta}_3\left(\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_3\right) \\
&\qquad\qquad + 2\ell_2\ell_3\dot{\theta}_2\dot{\theta}_3\left(\cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3\right)\Big) \\
&= \frac{1}{2}\left(m_1+m_2+m_3\right)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2+m_3\right)\ell_2^2\dot{\theta}_2^2 + \frac{1}{2}m_3\ell_3^2\dot{\theta}_3^2 \\
&\quad + \left(m_2+m_3\right)\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) \\
&\quad + m_3\ell_1\ell_3\dot{\theta}_1\dot{\theta}_3\left(\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_3\right) + m_3\ell_2\ell_3\dot{\theta}_2\dot{\theta}_3\left(\cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3\right) \\
&= \frac{1}{2}\left(m_1+m_2+m_3\right)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2+m_3\right)\ell_2^2\dot{\theta}_2^2 + \frac{1}{2}m_3\ell_3^2\dot{\theta}_3^2 \\
&\quad + \left(m_2+m_3\right)\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + m_3\ell_1\ell_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + m_3\ell_2\ell_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)
\end{aligned}$$

The *potential energy* is:

$$
\begin{aligned}
V = mgh &= m_1 g y_1 + m_2 g y_2 + m_3 g y_3 \\
&= m_1 g\left(-\ell_1\cos\theta_1\right) + m_2 g\left(-\ell_1\cos\theta_1 - \ell_2\cos\theta_2\right) + m_3 g\left(-\ell_1\cos\theta_1 - \ell_2\cos\theta_2 - \ell_3\cos\theta_3\right) \\
&= -1\left[(m_1+m_2+m_3)g\ell_1\cos\theta_1 + (m_2+m_3)g\ell_2\cos\theta_2 + m_3 g\ell_3\cos\theta_3\right]
\end{aligned}
$$

Now computing the Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= T - V \\
&= \left[\frac{1}{2}\left(m_1+m_2+m_3\right)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2+m_3\right)\ell_2^2\dot{\theta}_2^2 + \frac{1}{2}m_3\ell_3^2\dot{\theta}_3^2\right. \\
&\qquad \left. + \left(m_2+m_3\right)\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + m_3\ell_1\ell_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + m_3\ell_2\ell_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\right] \\
&\qquad - (-1)\left[(m_1+m_2+m_3)g\ell_1\cos\theta_1 + (m_2+m_3)g\ell_2\cos\theta_2 + m_3 g\ell_3\cos\theta_3\right] \\
&= \frac{1}{2}\left(m_1+m_2+m_3\right)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2+m_3\right)\ell_2^2\dot{\theta}_2^2 + \frac{1}{2}m_3\ell_3^2\dot{\theta}_3^2 \\
&\qquad + \left(m_2+m_3\right)\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + m_3\ell_1\ell_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + m_3\ell_2\ell_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3) \\
&\qquad + (m_1+m_2+m_3)g\ell_1\cos\theta_1 + (m_2+m_3)g\ell_2\cos\theta_2 + m_3 g\ell_3\cos\theta_3
\end{aligned}
$$

There are a few degrees of freedom, that is θ_1, θ_2, and θ_3. So to determine the equations of motions, we need to compute:

$$
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0 \qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) - \frac{\partial\mathcal{L}}{\partial\theta_2} = 0 \qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_3}\right) - \frac{\partial\mathcal{L}}{\partial\theta_3} = 0
$$

Solving for θ_1:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} &= (m_1+m_2+m_3)\ell_1^2\dot{\theta}_1 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\dot{\theta}_2 + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\dot{\theta}_3 \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) &= (m_1+m_2+m_3)\ell_1^2\ddot{\theta}_1 \\
&\qquad + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 - (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2\left(\dot{\theta}_1-\dot{\theta}_2\right) \\
&\qquad + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot{\theta}_3 - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_3\left(\dot{\theta}_1-\dot{\theta}_3\right) \\
&= (m_1+m_2+m_3)\ell_1^2\ddot{\theta}_1 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot{\theta}_3 \\
&\qquad - (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + (m_1+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 \\
&\qquad - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_1\dot{\theta}_3 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_3^2 \\
\frac{\partial\mathcal{L}}{\partial\theta_1} &= -(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_1\dot{\theta}_3 - (m_1+m_2+m_3)g\ell_1\sin\theta_1 \\
&= -1\left[(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_1\dot{\theta}_3 + (m_1+m_2+m_3)g\ell_1\sin\theta_1\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} &= (m_1+m_2+m_3)\ell_1^2\ddot\theta_1 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_2 + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_3 \\
&\quad - (m_1+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 + (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_2^2 \\
&\quad - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1\dot\theta_3 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_3^2 \\
&\quad - (-1)\left[(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1\dot\theta_3 + (m_1+m_2+m_3)g\ell_1\sin\theta_1\right] \\
&= (m_1+m_2+m_3)\ell_1^2\ddot\theta_1 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_2 + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_3 \\
&\quad + (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_2^2 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_3^2 + (m_1+m_2+m_3)g\ell_1\sin\theta_1
\end{aligned}
$$

Solving for θ_2:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot\theta_2} &= (m_2+m_3)\ell_2^2\dot\theta_2 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\dot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\dot\theta_3 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_2}\right) &= (m_2+m_3)\ell_2^2\ddot\theta_2 \\
&\quad + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_1 + (m_2+m_3)\ell_1\ell_2(-1)\sin(\theta_1-\theta_2)\dot\theta_1(\dot\theta_1-\dot\theta_2) \\
&\quad + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_3 + m_3\ell_2\ell_3(-1)\sin(\theta_2-\theta_3)\dot\theta_3(\dot\theta_2-\dot\theta_3) \\
&= (m_2+m_3)\ell_2^2\ddot\theta_2 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_3 \\
&\quad - (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1^2 + (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 \\
&\quad - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2\dot\theta_3 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_3^2 \\
\frac{\partial \mathcal{L}}{\partial \theta_2} &= (m_2+m_3)\,\ell_1\ell_2\dot\theta_1\dot\theta_2(-1)\sin(\theta_1-\theta_2)(-1) + m_3\ell_2\ell_3\dot\theta_2\dot\theta_3(-1)\sin(\theta_2-\theta_3) + (m_2+m_3)g\ell_2(-1)\sin\theta_2 \\
&= -1\left[-(m_2+m_3)\,\ell_1\ell_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) + m_3\ell_2\ell_3\dot\theta_2\dot\theta_3\sin(\theta_2-\theta_3) + (m_2+m_3)g\ell_2\sin\theta_2\right] \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} &= (m_2+m_3)\ell_2^2\ddot\theta_2 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_3 \\
&\quad - (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1^2 + (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 \\
&\quad - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2\dot\theta_3 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_3^2 \\
&\quad - (-1)\left[-(m_2+m_3)\,\ell_1\ell_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) + m_3\ell_2\ell_3\dot\theta_2\dot\theta_3\sin(\theta_2-\theta_3) + (m_2+m_3)g\ell_2\sin\theta_2\right] \\
&= (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_1 + (m_2+m_3)\ell_2^2\ddot\theta_2 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_3 \\
&\quad - (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1^2 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_3^2 + (m_2+m_3)g\ell_2\sin\theta_2
\end{aligned}
$$

Solving for θ_3:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot\theta_3} &= m_3\ell_3^2\dot\theta_3 + m_3\ell_1\ell_3\dot\theta_1\cos(\theta_1-\theta_3) + m_3\ell_2\ell_3\dot\theta_2\cos(\theta_2-\theta_3)\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_3}\right) &= m_3\ell_3^2\ddot\theta_3\\
&\quad + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_1 + m_3\ell_1\ell_3\dot\theta_1(-1)\sin(\theta_1-\theta_3)(\dot\theta_1-\dot\theta_3)\\
&\quad + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_2 + m_3\ell_2\ell_3\dot\theta_2(-1)\sin(\theta_2-\theta_3)(\dot\theta_2-\dot\theta_3)\\
&= m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_2 + m_3\ell_3^2\ddot\theta_3\\
&\quad - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1^2 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1\dot\theta_3\\
&\quad - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2^2 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2\dot\theta_3\\
\frac{\partial L}{\partial \theta_3} &= m_3\ell_1\ell_3\dot\theta_1\dot\theta_3(-1)\sin(\theta_1-\theta_3)(-1) + m_3\ell_2\ell_3\dot\theta_2\dot\theta_3(-1)\sin(\theta_2-\theta_3)(-1) + m_3g\ell_3(-1)\sin\theta_3\\
&= m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1\dot\theta_3 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2\dot\theta_3 - m_3g\ell_3\sin\theta_3\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_3}\right) - \frac{\partial L}{\partial \theta_3} &= m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_2 + m_3\ell_3^2\ddot\theta_3\\
&\quad - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1^2 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1\dot\theta_3\\
&\quad - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2^2 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2\dot\theta_3\\
&\quad - \Big[m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1\dot\theta_3 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2\dot\theta_3 - m_3g\ell_3\sin\theta_3\Big]\\
&= m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_2 + m_3\ell_3^2\ddot\theta_3\\
&\quad - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1^2 - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2^2\\
&\quad + m_3g\ell_3\sin\theta_3
\end{aligned}
$$

Therefore, the equations of motion (or governing equations) are:

$$
\begin{aligned}
&(m_1+m_2+m_3)\ell_1^2\ddot\theta_1 + (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_2 + m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_3\\
&\qquad + (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_2^2 + m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_3^2 + (m_1+m_2+m_3)g\ell_1\sin\theta_1 = 0
\end{aligned}
$$

$$
\begin{aligned}
&(m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_1 + (m_2+m_3)\ell_2^2\ddot\theta_2 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_3\\
&\qquad - (m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1^2 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_3^2 + (m_2+m_3)g\ell_2\sin\theta_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&m_3\ell_1\ell_3\cos(\theta_1-\theta_3)\ddot\theta_1 + m_3\ell_2\ell_3\cos(\theta_2-\theta_3)\ddot\theta_2 + m_3\ell_3^2\ddot\theta_3\\
&\qquad - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot\theta_1^2 - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot\theta_2^2 + m_3g\ell_3\sin\theta_3 = 0
\end{aligned}
$$

Two quick checks, if m_2, m_3, ℓ_1, and ℓ_3 are all 0, then the equations should reduce to a single pendulum; and if m_3 and ℓ_3 are 0 then the system should reduce to a double pendulum.

Check 1, if m_2, m_3, ℓ_1, and ℓ_3 are 0, then we only have one equation:

$$m_1\ell_1^2\ddot\theta_1 + m_1g\ell_1\sin\theta_1 = 0 \quad \Rightarrow \quad \ddot\theta_1 + \frac{g}{\ell_1}\sin\theta_1 = 0$$

And this describes a simple pendulum.

Check 2, if m_3 and ℓ_3 are 0 then, we have two equations:

$$
\begin{aligned}
(m_1+m_2)\ell_1^2\ddot\theta_1 + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_2 + m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_2^2 + (m_1+m_2)g\ell_1\sin\theta_1 &= 0\\
m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot\theta_1 + m_2\ell_2^2\ddot\theta_2 - m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot\theta_1^2 + m_2g\ell_2\sin\theta_2 &= 0
\end{aligned}
$$

Divide equation 1 by ℓ_1 and equation 2 by m_2 & ℓ_2:

$$\begin{aligned}
(m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + m_2\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + (m_1+m_2)g\sin\theta_1 = 0 \\
\ell_1\cos(\theta_1-\theta_2)\ddot{\theta}_1 + \ell_2\ddot{\theta}_2 - \ell_1\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + g\sin\theta_2 = 0
\end{aligned}$$

And from past work, these are the equations for a double pendulum.

Simulation and solving numerically

The 'easy' solution is to use some linear alegbra:

$$[A]\left[\ddot{\theta}_i\right] = [B] \quad \Rightarrow \quad \left[\ddot{\theta}_i\right] = [A]^{-1}[B]$$

Arranging the solution equation into matrices we have:

$$\begin{bmatrix}
(m_1+m_2+m_3)\ell_1^2 & (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2) & m_3\ell_1\ell_3\cos(\theta_1-\theta_3) \\
(m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2) & (m_2+m_3)\ell_2^2 & m_3\ell_2\ell_3\cos(\theta_2-\theta_3) \\
m_3\ell_1\ell_3\cos(\theta_1-\theta_3) & m_3\ell_2\ell_3\cos(\theta_2-\theta_3) & m_3\ell_3^2
\end{bmatrix}
\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} =$$
$$\begin{bmatrix}
-(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_3^2 - (m_1+m_2+m_3)g\ell_1\sin\theta_1 \\
(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot{\theta}_3^2 - (m_2+m_3)g\ell_2\sin\theta_2 \\
m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_1^2 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot{\theta}_2^2 - m_3g\ell_3\sin\theta_3
\end{bmatrix}$$

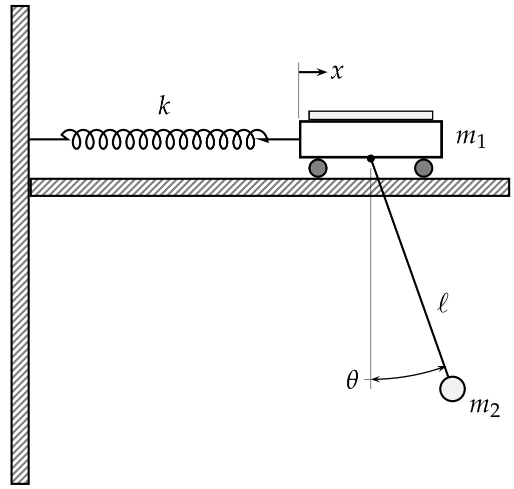
So then,

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} =
\begin{bmatrix}
(m_1+m_2+m_3)\ell_1^2 & (m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2) & m_3\ell_1\ell_3\cos(\theta_1-\theta_3) \\
(m_2+m_3)\ell_1\ell_2\cos(\theta_1-\theta_2) & (m_2+m_3)\ell_2^2 & m_3\ell_2\ell_3\cos(\theta_2-\theta_3) \\
m_3\ell_1\ell_3\cos(\theta_1-\theta_3) & m_3\ell_2\ell_3\cos(\theta_2-\theta_3) & m_3\ell_3^2
\end{bmatrix}^{-1}$$
$$\times \begin{bmatrix}
-(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 - m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_3^2 - (m_1+m_2+m_3)g\ell_1\sin\theta_1 \\
(m_2+m_3)\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 - m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot{\theta}_3^2 - (m_2+m_3)g\ell_2\sin\theta_2 \\
m_3\ell_1\ell_3\sin(\theta_1-\theta_3)\dot{\theta}_1^2 + m_3\ell_2\ell_3\sin(\theta_2-\theta_3)\dot{\theta}_2^2 - m_3g\ell_3\sin\theta_3
\end{bmatrix}$$

This is what will be used in the Runge Kutta solver

Spring-Cart-Pendulum System

Classic problem involving a spring, pendulum, and horizontal motion.



There are two things that are moving: the cart and the pendulum. For the cart we have:

$$\text{Cart Position} = x \qquad\qquad \text{Cart Velocity} = \dot{x}$$

For the pendulum we have:

$$\begin{aligned}
\text{Pendulum Position} : x &= \ell \sin\theta \\
y &= -\ell\cos\theta \\
\text{Pendulum Velocity} : \dot{x} &= \ell\cos\theta \cdot \dot{\theta} \\
\dot{y} &= \ell \sin\theta \cdot \dot{\theta}
\end{aligned}$$

Note that since the pendulum is attached to the cart it can move. Therefore, the total pendulum x velocity becomes:

$$\begin{aligned}
\dot{x}_{\text{total}} &= \dot{x}_{\text{cart}} + \dot{x}_{\text{pendulum}} \\
&= \dot{x} + \ell\cos\theta \cdot \dot{\theta}
\end{aligned}$$

The *kinetic energy* is then:

$$\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \left(\frac{1}{2}mv^2\right)_{\text{cart}} + \left(\frac{1}{2}mv^2\right)_{\text{pendulum}} \\
&= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\left(\dot{x} + \ell\cos\theta\cdot\dot{\theta}\right)^2 + \left(\ell\sin\theta\cdot\dot{\theta}\right)^2\right] \\
&= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\dot{x}^2 + 2\ell\cos\theta\dot{x}\dot{\theta} + \ell^2\cos^2\theta\dot{\theta}^2 + \ell^2\sin^2\theta\dot{\theta}^2\right] \\
&= \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_2\ell\cos\theta\dot{x}\dot{\theta} + \frac{1}{2}m_2\ell^2\dot{\theta}^2
\end{aligned}$$

The *potential energy* is:

$$
\begin{aligned}
V &= V_{\mathsf{cart}} + V_{\mathsf{pendulum}} \\
&= \frac{1}{2}k(\text{Cart Position})^2 + mg(\text{Pendulum } y \text{ Position}) \\
&= \frac{1}{2}kx^2 + mg(-\ell\cos\theta) \\
&= \frac{1}{2}kx^2 - m_2 g\ell\cos\theta
\end{aligned}
$$

The *Lagrangian* becomes:

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_2\ell\cos\theta\dot{x}\dot{\theta} + \frac{1}{2}m_2\ell^2\dot{\theta}^2 - \frac{1}{2}kx^2 + m_2 g\ell\cos\theta
\end{aligned}
$$

Since we have two variables (x and θ), we will have two equations of motion:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{x}} &= (m_1+m_2)\dot{x} + m_2\ell\cos\theta\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) &= (m_1+m_2)\ddot{x} + m_2\ell\cos\theta\ddot{\theta} - m_2\ell\sin\theta\dot{\theta}^2 \\
\frac{\partial\mathcal{L}}{\partial x} &= -kx \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} &= (m_1+m_2)\ddot{x} + m_2\ell\cos\theta\ddot{\theta} - m_2\ell\sin\theta\dot{\theta}^2 + kx
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m_2\ell\cos\theta\dot{x} + m_2\ell^2\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= m_2\ell\cos\theta\ddot{x} - m_2\ell\sin\theta\dot{x}\dot{\theta} + m_2\ell^2\ddot{\theta} \\
\frac{\partial\mathcal{L}}{\partial\theta} &= -m_2\ell\sin\theta\dot{x}\dot{\theta} - m_2 g\ell\sin\theta \\
&= -\left[m_2\ell\sin\theta\dot{x}\dot{\theta} + m_2 g\ell\sin\theta\right] \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} &= m_2\ell\cos\theta\ddot{x} - m_2\ell\sin\theta\dot{x}\dot{\theta} + m_2\ell^2\ddot{\theta} + m_2\ell\sin\theta\dot{x}\dot{\theta} + m_2 g\ell\sin\theta \\
&= m_2\ell\cos\theta\ddot{x} + m_2\ell^2\ddot{\theta} + m_2 g\ell\sin\theta
\end{aligned}
$$

Therefore, the final equations of motion are:

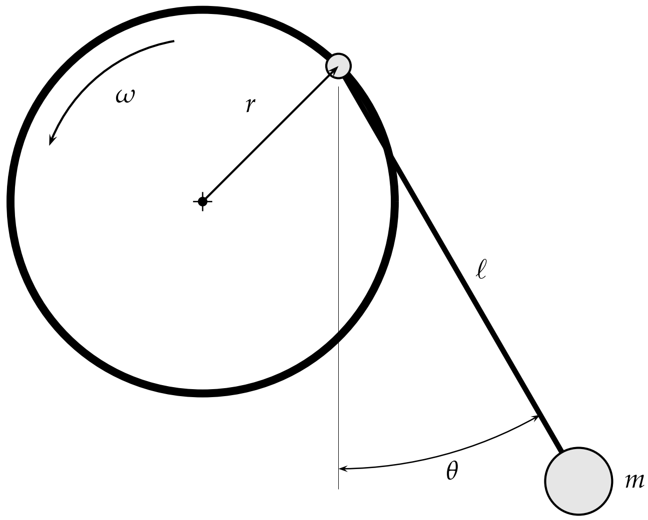
$$
\begin{aligned}
(m_1+m_2)\ddot{x} + m_2\ell\cos\theta\ddot{\theta} - m_2\ell\sin\theta\dot{\theta}^2 + kx &= 0 \\
m_2\ell\cos\theta\ddot{x} + m_2\ell^2\ddot{\theta} + m_2 g\ell\sin\theta &= 0
\end{aligned}
$$

Solving using some matrix magic:

$$
\begin{aligned}
\begin{bmatrix} m_1+m_2 & m_2\ell\cos\theta \\ m_2\ell\cos\theta & m_2\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} &= \begin{bmatrix} m_2\ell\sin\theta\dot{\theta}^2 - kx \\ -m_2 g\ell\sin\theta \end{bmatrix} \\
\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} &= \begin{bmatrix} m_1+m_2 & m_2\ell\cos\theta \\ m_2\ell\cos\theta & m_2\ell^2 \end{bmatrix}^{-1}\begin{bmatrix} m_2\ell\sin\theta\dot{\theta}^2 - kx \\ -m_2 g\ell\sin\theta \end{bmatrix}
\end{aligned}
$$

Simple Pendulum on a Wheel

This problem is what the title says, a simple pendulum that is attached to a rotating wheel. (that is rotating at a contant velocity). Note, to keep things easy, inertia is not considered in the wheels motion.



The math is pretty easy, and it is just a single degree of freedom problem.

First, the angle (ϕ) of where the pendulum's pivot is on the wheel is at time t is:

$$\phi = \phi_0 + \omega t$$

Where ϕ_0 is the initial angle and ω is the angular speed of the rotating wheel.

The position of the pendulum's pivot is then:

$$x_{\text{pivot}} = r\cos\omega t \qquad y_{\text{pivot}} = r\sin\omega t$$

The pendulum's mass position is:

$$x_{\text{pend}} = \ell\sin\theta \qquad y_{\text{pend}} = \ell\cos\theta$$

Therefore the final/total position is:

$$x = x_{\text{pivot}} + x_{\text{pend}} = r\cos\omega t + \ell\sin\theta \qquad y = y_{\text{pivot}} + y_{\text{pend}} = r\sin\omega t + \ell\cos\theta$$

Now the velocities will be found to make the kinetic energy calculations a little bit easier to determine.

$$\dot{x} = -r\omega\sin\omega t + \ell\cos\theta\dot{\theta} \qquad \dot{y} = r\omega\cos\omega t + \ell\sin\theta\dot{\theta}$$

Now computing the *kinetic energy*:

$$\begin{aligned}
T &= \frac{1}{2}mv^2 = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2\right] \\
&= \frac{1}{2}m\left[(-r\omega\sin\omega t + \ell\cos\theta\dot{\theta})^2 + (r\omega\cos\omega t + \ell\sin\theta\dot{\theta})^2\right] \\
&= \frac{1}{2}m\left[r^2\omega^2\sin^2\omega t - 2r\ell\omega\cos\theta\sin\omega t\dot{\theta} + \ell^2\cos^2\theta\dot{\theta}^2 + r^2\omega^2\cos^2\omega t + 2r\ell\omega\sin\theta\cos\omega t\dot{\theta} + \ell^2\sin^2\theta\dot{\theta}^2\right] \\
&= \frac{1}{2}mr^2\omega^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + mr\ell\omega\sin(\theta - \omega t)\dot{\theta}
\end{aligned}$$

Next, the *potential energy*:

$$V = mgh = mgr\sin\omega t - mg\ell\cos\theta$$

So the Lagrangian becomes:

$$\mathcal{L} = T - V = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + mr\ell\omega\sin(\theta - \omega t)\dot{\theta} - mgr\sin\omega t + mg\ell\cos\theta$$

Now taking some derivatives so that the equation of motion can be found:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m\ell^2\dot{\theta} + mr\ell\omega\sin(\theta - \omega t) \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= m\ell^2\ddot{\theta} + mr\ell\omega\cos(\theta - \omega t)\cdot(\dot{\theta} - \omega) \\
&= m\ell^2\ddot{\theta} + mr\ell\omega\cos(\theta - \omega t)\dot{\theta} - mr\ell\omega^2\cos(\theta - \omega t) \\
\frac{\partial\mathcal{L}}{\partial\theta} &= mr\ell\omega\cos(\theta - \omega t)\dot{\theta} - mg\ell\sin\theta \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} &= m\ell^2\ddot{\theta} - mr\ell\omega^2\cos(\theta - \omega t) + mg\ell\sin\theta
\end{aligned}$$

Then after dividing everything by $m\ell^2$, the final equation of motion becomes:

$$\ddot{\theta} - \frac{r\omega^2}{\ell}\cos(\theta - \omega t) + \frac{g}{\ell}\sin\theta = 0$$

Two things to note: 1) the mass does not matter; 2) if r or ω are 0, then the system reduces to a simple pendulum.