THE γ_5-PROBLEM AND ANOMALIES – A CLIFFORD ALGEBRA APPROACH

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It is shown that a strong correspondence between noncyclicity and anomalies exists. This allows, by fundamental properties of Clifford algebras, to build a simple and consistent scheme for treating γ_5 without using $(d-4)$-dimensional objects

Dimensional regularization has become the appropriate method regularizing divergent Feynman graphs. It respects gauge invariance and is self-consistent. The only subtlety involved is the γ_5 problem. In the following a method is presented which avoids $(n-4)$-dimensional objects, arising for example in the Breitenlohner–Maison scheme. The naive handling of γ_5 is justified and proven to be consistent with the prerequisites of dimensional regularization.

The notation is simple: F_d denotes a functional which is the continuation of ordinary integration to integration over a complex d-dimensional space in the sense of ref. [1], p denotes a loop momentum, q an exterior momentum, $\epsilon=d-4$ and $G^c(i,j)$ denotes a complexified Clifford algebra with i timelike and j spacelike components [2].

To introduce dimensional regularization as an integration over a space of complex dimension it is treated as a functional which fullfills some properties necessary for consistency and applicability in calculating diagrams:
linearity:

$$F_d\{ag(p)+bf(p)\}=aF_d\{g(p)\}+bF_d\{f(p)\}, \quad (1)$$

scaling:

$$F_d\{g(sp)\}=s^{-d}F_d\{g(p)\}, \quad (2)$$

translational invariance:

$$F_d\{g(p+q)\}=F_d\{g(p)\}. \quad (3)$$

Up to normalization this defines a unique functional [1]. Existence is proven by explicit construction in terms of ordinary Riemann integration. As a consequence, loop momenta appear as elements of infinite-dimensional vector space The d dependence is only in the functional, not in this underlying vector space.

Resulting from this infinity, the covariant tensor $g^{\mu\nu}$ is not the inverse of the metric tensor $g_{\mu\nu}$ – which may not exist – but is defined as a linear functional acting on the space of contravariant tensors in a well defined way: $g^{\mu\nu}g_{\mu\nu}=d$. This can be defined to commute with integration and to agree with the normal definition in the finite case [1].

With the help of this construction the Feynman integration can be carried out to reproduce the divergences of the perturbative theory as pole terms by analytical continuation of the resulting functions [1,3].

The need of an infinite dimensional vector space gives rise to the well known γ_5 problem. Trying to keep all the four-dimensional rules,

$$\mathrm{tr}(\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_4})=4\mathrm{i}\epsilon_{\mu_1\ldots\mu_4}, \quad \{\gamma_5,\gamma_\mu\}=0,$$

and cyclicity of the trace one ends with

$$(n-4)\,\mathrm{tr}(\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_4})=0, \quad (4)$$

which forbids an analytic continuation.

Note that to generate this inconsistency one starts with

$$\mathrm{tr}(\gamma_5\gamma_\alpha\gamma^\alpha\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}) \quad (5)$$

and uses the Clifford algebra relation as well as the anticommuting γ_5 and cyclicity of the trace. So to demonstrate the γ_5 problem one needs to trace of γ_5 with six γ matrices.

To get a consistent regularization, one of the above rules must be replaced in arbitrary complex dimensions. The Breitenlohner–Maison scheme keeps cyclicity but rejects the anticommuting γ_5:

$$\{\gamma_5, \gamma_\mu\} = 2\gamma_5\hat{\gamma}_\mu . \tag{6}$$

The $\hat{\gamma}$ objects are $(d-4)$-dimensional objects, so they disappear whenever the index is not an element of I/I_4, where I denotes the set of all values of the indices and I_4 the set of the four values representing Minkowski spacetime.

It is proven that the Breitenlohner–Maison scheme is the only consistent way of replacing the anticommuting γ_5 [4].

This method gives the correct results for the Adler–Bell–Jackiw anomaly and is also in agreement with Bardeen's theorem [5].

While producing the anomaly serves as a test for a valuable regularization, the appearing of so-called spurious anomalies in this scheme is problematic [6]. These terms generate from the possible contractions of the nonvanishing anticommutator

$$\{\gamma_5, \gamma_\mu\}\{\gamma_5, \gamma^\mu\} = 2\epsilon . \tag{7}$$

For example,

$$\begin{aligned}&(1+\gamma_5)\gamma_\mu(1+\gamma_5)\gamma_\alpha(1-\gamma_5)\gamma^\mu(1-\gamma_5)\\ &\quad = -8\epsilon(1+\gamma_5)\gamma_\alpha ,\end{aligned} \tag{8}$$

instead of being zero by chirality mismatch. This gives rise to finite counterterms to compensate such spurious anomalies. It is not clear a priori if these counterterms still respect Ward identities.

Keeping cyclicity seems arbitrary as for an infinite number of basic elements of the Clifford algebra one has automatically an infinite dimensional representation. But it is commonly believed that cyclicity is necessary for practical and conceptual reasons [6]. Rejecting it one has to answer questions like the following:
- How to read a Feynman diagram?
- How to do the algebra (traces and all that)?

In the following it is shown that one can handle all this by using some basic properties of Clifford algebras [2]. This gives a new consistent regularization scheme for γ_5.

The object under investigation is the infinite-dimensional algebra

$$G^c(1,\infty) = \lim_{k\to\infty} G^c(1, 2k) . \tag{9}$$

By induction in $G^c(1,\infty)$ exists a unique element τ with $\tau^2 = \mathbb{1}$ and $\{\gamma_\mu, \tau\} = 0$. $G^c(1,\infty)$ is realized through

$$G^c(1,\infty) = G^c(1,3) \otimes \lim_{k\to\infty} G^c(0, 2k) , \tag{10}$$

if four-dimensional Minkowski space is the base space. For two-dimensional field theory one may start with $G^c(1, 1)$ as well. So the Clifford algebra which had to be extended is embedded in the resulting infinite Clifford algebra via the tensor product. This is a direct consequence of the periodicity properties of these algebras. Emphasizing the fact that the only d dependence is in the functional F_d it follows that $G^c(1,\infty)$ is the appropriate algebraic background for the regulator F_d and the nature of the divergencies – infrared or ultraviolet – does not influence the algebraic sector, but is determined by the application of F_d.

To define a trace operation, note the trace operation in four dimensions. There the trace is a number times the trace of $\mathbb{1}$. This is a consequence of the fact that every element $\gamma_{\mu_1}...\gamma_{\mu_j}$ has a definite basis in the space of antisymmetric tensors. For example,

$$\gamma_\mu\gamma_\nu = \gamma_\mu \wedge \gamma_\nu + g_{\mu\nu}\mathbb{1} , \tag{11}$$

and so on for higher Clifford products. Only the element in Λ^0, the space of the 0-form is not traceless, so the trace acts as a projector on this subspace. Because of the above, the Clifford product has a direct expression in exterior products.

Recall the definition of $G^c(1,\infty)$,

$$\begin{aligned}&G^c(1,\infty) = G^c(1,3) \otimes \lim_{k\to\infty} G^c(0, 2k) ,\\ &\mathbb{1}_\infty = \mathbb{1}_4 \otimes \lim_{k\to\infty} \mathbb{1}_{2k} ,\\ &\tau = \tau_4 \hat{\otimes} \lim_{k\to\infty} \tau_{2k} .\end{aligned} \tag{12}$$

Now, trace can be defined to be the same as it is in four dimensions: a number, which is the coefficient of $(\mathbb{1}_4\otimes ...)$ times $\mathrm{tr}(\mathbb{1}_4)$. Here the induced volume element τ_4 is defined by $\tau_4 = \epsilon^{0123}_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\mu_1}\gamma_\mu\gamma_{\mu_3}\gamma_{\mu_4}$. The Levi-Civita tensor is the antisymmetric tensor belonging to the hyperplane which represents Minkowski spacetime embedded in the infinite dimensional vector space and vanishes on all other hyperplanes. This means, that for the trace one still looks for the identity in this four-dimensional subspace by using the embedding property (12). This gives a consistent definition of the trace operation which is in agreement with the ordinary four-dimensional results. For example, the trace of an odd number of gamma matrices is zero because $\mathbb{1}$ belongs to the even part of a Clifford algebra. $\mathrm{tr}(\tau\gamma_{\mu_1}\cdots\gamma_{\mu_4})=4\mathrm{i}\epsilon^{0123}_{\mu_1\ldots\mu_4}$ holds because only if all values of the indices $\mu_1,\ldots,\mu_4$ are elements of $(0, 1, 2, 3)$ one gets a nonzero coefficient for $\mathbb{1}_4=\tau_4^2$.

What happens to cyclicity? It follows from the above construction that the trace of an arbitrary string of γ matrices not containing τ does not fail to be cyclic because of the result of the trace operation which is zero for an odd number and a cyclic sum of products of metric tensors in the even case.

Regarding the traces involving τ, the first candidate is $\mathrm{tr}(\tau\gamma_{\mu_1}\cdots\gamma_{\mu_6})$ because all other traces with fewer γ matrices vanish or, in the case of four matrices, are cyclic for permutations of τ and the four γ matrices.

And in fact, calculating the difference

$$\mathrm{tr}(\tau\gamma_{\mu_1}\cdots\gamma_{\mu_6})-\mathrm{tr}(\gamma_{\mu_6}\tau\gamma_{\mu_1}\cdots\gamma_{\mu_5}) , \qquad (13)$$

one ends up with a tensor of sixth order which is antisymmetric in five indices. This should vanish identically in four dimensions but contracting it with $g^{\mu_1\mu_6}$ gives a result $\approx (d-4)\times\epsilon_{\mu_2\ldots\mu_5}$.

The first graphs where this tensor enters the calculation is the VVA anomaly and it is already shown by Nicolai et al. [7] that this reproduces the anomaly correctly. Looking at Bardeen's theorem, it is easily seen that all the anomalous Ward identities can be generated by noncyclicity as well as by the BM scheme for which this is shown by Feikes.

Here the arbitrariness where to start reading the graph reflects the non locality of the anomaly. Graphs with the fermion loops not involving γ_5 remain unique and can be regularized by local counterterms. So the different possibilities to read a graph in the noncyclicity scheme give different results only for the anomaly graphs. But this ambiguity is a typical anomalous property manifest in all regularization schemes and is handles by installing some condition like vector current conservation. In the noncyclicity scheme this condition is simply implemented as a rule where to start reading the graph. This removes all ambiguities in calculating arbitrary many loops.

Also in this scheme the anomaly is determined by the one-loop graph as it should because divergent subgraphs can be compensated by local counterterms for the vertex function and spinor self-energy which have the same structure as the unrenormalized functions because dimensional regularization is compatible with renormalization.

So there is a strong correspondence between the rising of anomalous terms and a failure of cyclicity which is a failure of a typical property of finite dimensions. This can also be seen on a more fundamental level. Regarding scalar products like $\bar{\psi}(M)\psi$ where M is an arbitrary linear combination of elements of $\mathrm{G^c}(1,\infty)$ this is almost already defined by the trace definition. In finite dimensions this scalar product may be written as $\mathrm{tr}[(\psi\bar{\psi})(M)]$ where $(\psi\bar{\psi})$ is a matrix which also is a sum of basic elements of the Clifford algebra, the coefficients may be restricted by some laws like the Dirac equation. To see the effect of noncyclicity in infinite dimensions explicitly one can use the identity

$$AB=\tfrac{1}{2}\{A,B\}+\tfrac{1}{2}[A,B] , \qquad (14)$$

valid in arbitrary finite or infinite dimensions for matrices A, B. This gives

$$\bar{\psi}(M)\psi=\tfrac{1}{2}\,\mathrm{tr}(\{(\psi\bar{\psi}),(M)\}+[(\psi\bar{\psi}),(M)]), \qquad (15)$$

where tr is the trace defined before. The commutator term on the right vanishes if the trace is cyclic but is the origin of anomalies if not.

Neglecting this term the BM scheme installs a nonvanishing anticommutator for γ_5 with γ matrices to reproduce the anomaly with the consequence of spurious anomalies and very complicated algebraic rules. Also there are some problems with continuity in this scheme regarding the element τ.

The conclusion of all this is that the anomalies are the only manifest effects of the γ_5 problem and so, one has no problems in regularizing the perturbative expansion. Explicit noncyclicity arises only for anomalous graphs and in all other cases one can use even cyclicity.

So the naive treatment of γ_5 in the anomaly free graphs is justified by basic properties of Clifford algebras. Avoiding $(d-4)$-dimensional objects calculations are very much simplified without loosing consistency. There are no problems with Ward identities and the guessed problems with noncyclicity [6] disappear thanks to the Clifford algebra properties. So contrary to the conclusions in ref. [6], noncyclicity does not fail to be a consistent regularization scheme and, avoiding spurious terms, has even conceptual advantages.

Also $\bar{\psi}\tau\psi$ transforms as a pseudoscalar and P, C, T can be defined as usual.

Obviously there are connections between noncyclicity and anomaly. From the viewpoint of noncommutative geometry [8], this should be taken into future considerations.

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References

[1] J.C. Collins, Renormalization, Cambridge Monographs on Mathematical Physics (Cambridge U.P., Cambridge, 1984).

[2] R. Coqueraux, Spinors, reflections and Clifford algebras: a review, University of Texas preprint UTTG-21-85.

[3] G. 't Hooft and M. Veltman, Nucl. Phys. B 44 (1972) 189.

[4] G. Rufa, Mathematical foundations of the dimensional method, Institut für Physik, Mainz preprint Mz-TH/88-12 (1988).

[5] J. Feikes, Anomalien in dimensionaler regularisierung, Diplomarbeit, II. Institut für theoretische Physik, Universität Hamburg (October 1988), unpublished.

[6] G. Bonneau, Renormalization, regularization: a critical, but non-technical, review, Laboratoire de Physique Théorique et Hautes Energies, Paris preprint PAR LPTHE 88-52 (1988).

[7] H. Nicolai and P.K. Townsend, Phys. Lett. B 93 (1980) 111.

[8] R. Coqueraux, Noncommutative geometry and theoretical physics, CNRS, Centre de Physique Théorique, Luminy-Marseille, J. Geom. Phys., to appear.