Potential Flow and the Principle of Superposition:

Consider a two-dimensional, incompressible, irrotational flow, also referred to as a potential flow.

Then we know there is a potential ϕ such that:

$$\vec{V} = \vec{\nabla}\phi$$

$$\vec{\nabla}^2\phi = \vec{\nabla}\cdot\vec{V} = 0 \quad \text{Laplace's Equation}$$

and a stream function ψ such that:

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

$$-\vec{\nabla}^2\psi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{Laplace's Equation}$$

Now, suppose $\phi_1, \phi_2, \dots, \phi_n$ are solutions to Laplace's equation:

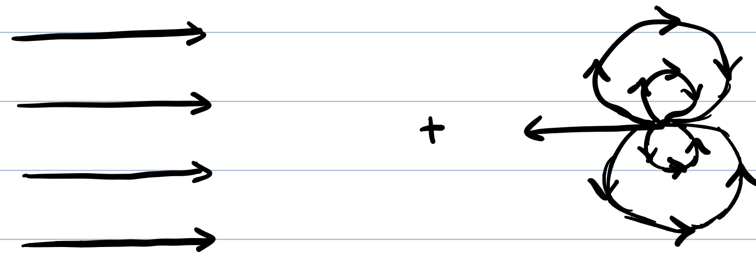
$$\vec{\nabla}^2\phi_1 = \vec{\nabla}^2\phi_2 = \dots = \vec{\nabla}^2\phi_n = 0$$

Then so is:

$$\phi = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n \; !$$

This is the principle of superposition. It allows us to build complex flow solutions from a family of elementary flows that when added together satisfy appropriate boundary conditions.

As an example, we can obtain non-lifting flow over a cylinder by adding a doublet to a uniform flow:



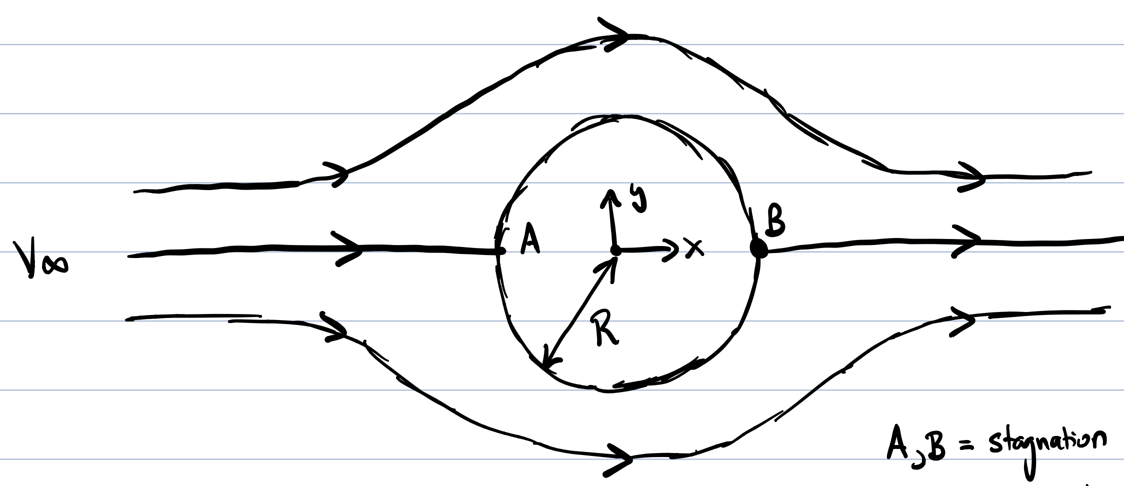
Uniform Flow: $\psi = V_\infty r \sin\theta$

Doublet: $\psi = -\frac{K}{2\pi}\frac{\sin\theta}{r}$

Defining $R^2 \equiv K/2\pi V_\infty$, the resulting stream function is:

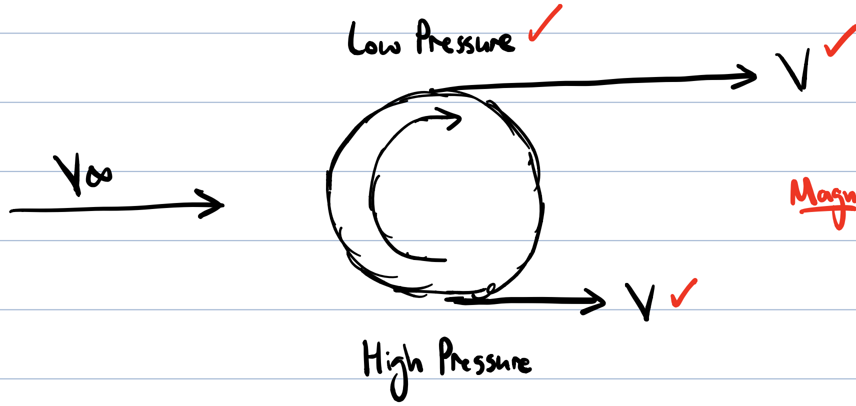
$$\boxed{\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right)}$$

The resulting streamlines and flow pattern is:



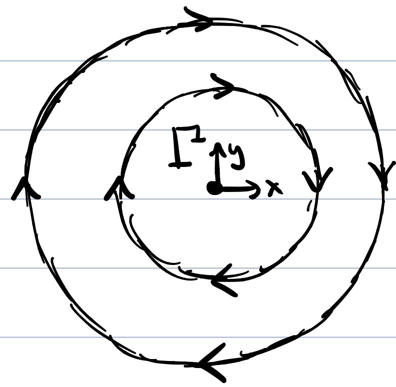
A, B = stagnation points

We can create lift by adding circulation to the cylinder:



Magnus Effect

Mathematically, we do this by adding an ideal vortex to the center of the cylinder:



Ideal Vortex Flow:

$$\Psi = \frac{\Gamma}{2\pi}\ln(r)$$

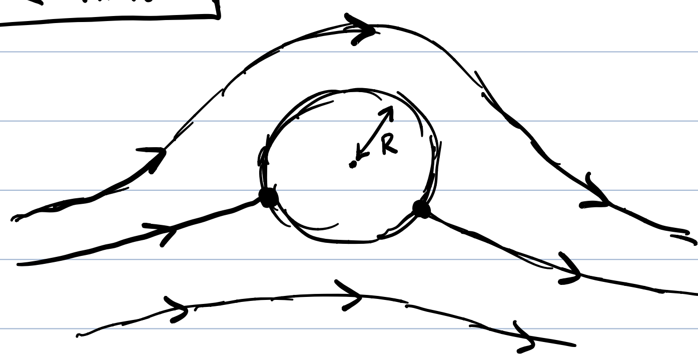
yielding the stream function:

Often taken as: $\frac{\Gamma}{2\pi}\ln\left(\frac{r}{R}\right)$

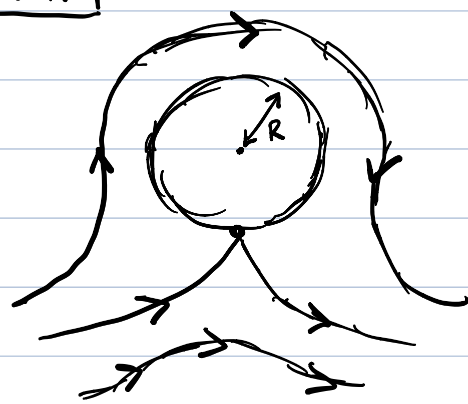
$$\boxed{\Psi = \left(V_\infty r \sin\theta\right)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln(r)}$$

Depending on the strength of Γ, we get different flow patterns.

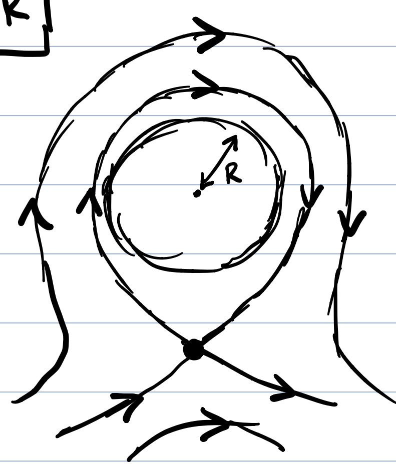
$\boxed{\Gamma < 4\pi V_\infty R}$



$\boxed{\Gamma = 4\pi V_\infty R}$



$\boxed{\Gamma > 4\pi V_\infty R}$



Not too surprisingly, we can model flow over airfoils with vortices too!