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FUZZY COMPLEX NUMBERS*

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Abstract: This paper first defines fuzzy complex numbers and shows closure under the basic arithmetic operations. It is shown that arithmetic of fuzzy complex numbers may be performed in terms of α-cuts. Next two special types of fuzzy complex numbers based on the forms $z = x + iy$ and $z = re^{i\theta}$ for regular complex numbers are investigated. Introducing a metric on the space of fuzzy complex numbers then allows us to discuss continuity and differentiability of fuzzy complex functions.

Keywords: Fuzzy sets; algebra; analysis; fuzzy numbers.

1. Introduction

We first need to present a definition of a fuzzy complex number. We will write regular complex numbers as $z = x + iy$, $w = x + iy$ and we use $\bar{Z}$, $\bar{W}$ for fuzzy complex numbers. A fuzzy complex number $\bar{Z}$ is defined by its membership function $\mu(z \mid \bar{Z})$ which is a mapping from the complex numbers C into $[0, 1]$. An α-cut of $\bar{Z}$ is

$$\bar{Z}^\alpha = \{z \mid \mu(z \mid \bar{Z}) > \alpha\}, \tag{1}$$

where $0 \leq \alpha < 1$, and we separately specify

$$\bar{Z}^1 = \{z \mid \mu(z \mid \bar{Z}) = 1\}. \tag{2}$$

The support of $\bar{Z}$, written $\mathrm{supp}(\bar{Z})$, is the closure of $\bar{Z}^0$.

Definition 1. $\bar{Z}$ is a fuzzy complex number if and only if:
(1) $\mu(z \mid \bar{Z})$ is continuous;
(2) $\bar{Z}^\alpha$, $0 \leq \alpha < 1$, is open, bounded, connected and simply connected; and
(3) $\bar{Z}^1$ is non-empty, compact, arcwise connected and simply connected.

We desire a very general definition for $\bar{Z}$ so that the regions $\bar{Z}^\alpha$, $0 \leq \alpha \leq 1$, are 'nice' connected subsets of C containing no holes. If $\bar{Z}^\alpha$, $0 \leq \alpha < 1$, is open and connected, then it is automatically arcwise connected. The simply connected assumption is to assure that $\bar{Z}^\alpha$, $0 \leq \alpha \leq 1$, will not contain any holes. $\bar{Z}^1$ being

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non-empty means that all fuzzy complex numbers are normalized ($\mu(z \mid \bar{Z}) = 1$ for some z). We purposely did not require $\bar{Z}^\alpha$, $0 \leq \alpha \leq 1$, to be convex because this would exclude one special type of fuzzy complex number based on the form $z = re^{i\theta}$ (see Section 4). Notice that the closure of $\bar{Z}^\alpha$, $0 \leq \alpha < 1$, is compact and connected, so in particular $\operatorname{supp}(\bar{Z})$ is compact and connected.

In the next section we show that this class of fuzzy complex numbers is closed under the basic arithmetic operations. We follow the historical development of real fuzzy numbers by first defining addition and multiplication from the extension principle and then show that these operations may be performed in terms of α-cuts. In the next two sections we investigate the basic properties of fuzzy complex numbers derived from the forms $z = x + iy$ and $z = re^{i\theta}$ for regular complex numbers. In the fifth section we introduce a metric on the set of fuzzy complex numbers $\bar{C}$ producing a complete metric space $\mathscr{C}$. Having $\mathscr{C}$ fuzzy complex analysis may begin with studying the continuity and differentiability of functions $h: \mathscr{C} \to \mathscr{C}$.

2. Closure

We first define all the basic arithmetic operations on $\bar{C}$. Let $f(z_1, z_2) = w$ be any mapping from $C \times C$ into C. We extend f to $\bar{C} \times \bar{C}$ into $\bar{C}$ using the extension principle. We write $f(\bar{Z}_1, \bar{Z}_2) = \bar{W}$ if

$$\mu(w \mid \bar{W}) = \sup\{\Pi(z_1, z_2) \mid f(z_1, z_2) = w\}, \tag{3}$$

where

$$\Pi(z_1, z_2) = \min\{\mu(z_1 \mid \bar{Z}_1), \mu(z_2 \mid \bar{Z}_2)\}. \tag{4}$$

One obtains $\bar{W} = \bar{Z}_1 + \bar{Z}_2$, or $\bar{W} = \bar{Z}_1\bar{Z}_2$, by using $f(z_1, z_2) = z_1 + z_2$, or $f(z_1, z_2) = z_1 z_2$, respectively. For subtraction we first define $-\bar{Z}$ as

$$\mu(z \mid -\bar{Z}) = \mu(-z \mid \bar{Z}), \tag{5}$$

and then set

$$\bar{Z}_1 - \bar{Z}_2 = \bar{Z}_1 + (-\bar{Z}_2). \tag{6}$$

For division we first define the reciprocal $\bar{Z}^{-1}$ of $\bar{Z}$ as

$$\mu(z \mid \bar{Z}^{-1}) = \mu(z^{-1} \mid \bar{Z}), \tag{7}$$

as long as there is some open disk centered at 0 $(0 = 0 + i0)$ disjoint from $\bar{Z}^0$. If $\bar{Z}^0$ is not bounded away from zero, then $\bar{Z}^{-1}$ remains undefined. When zero belongs to $\operatorname{supp}(\bar{Z})$, then $\operatorname{supp}(\bar{Z}^{-1})$ will not be bounded and by our definition of fuzzy complex numbers, $\bar{Z}^{-1}$ will not be a fuzzy complex number. We then set

$$\frac{\bar{Z}_1}{\bar{Z}_2} = \bar{Z}_1\bar{Z}_2^{-1}, \tag{8}$$

for the division of two fuzzy complex numbers.

The complex conjugate $\bar{Z}^*$ of $\bar{Z}$ is defined as

$$\mu(z \mid \bar{Z}^*) = \mu(\bar{z} \mid \bar{Z}), \tag{9}$$

where $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$. The modulus $|\bar{Z}|$ of $\bar{Z}$ is defined by

$$\mu(r \mid |\bar{Z}|) = \sup\{\mu(z \mid \bar{Z}) \mid |z| = r\}, \tag{10}$$

where r is the modulus of z.

We first observe that addition and multiplication of fuzzy complex numbers may be performed using α-cuts. Let

$$S^\alpha = \{z_1 + z_2 \mid (z_1, z_2) \in \bar{Z}_1^\alpha \times \bar{Z}_2^\alpha\}, \tag{11}$$

and

$$P^\alpha = \{z_1 z_2 \mid (z_1, z_2) \in \bar{Z}_1^\alpha \times \bar{Z}_2^\alpha\}. \tag{12}$$

Theorem 1. *If $\bar{W} = \bar{Z}_1 + \bar{Z}_2$, then $\bar{W}^\alpha = S^\alpha$. If $\bar{W} = \bar{Z}_1\bar{Z}_2$, then $\bar{W}^\alpha = P^\alpha$.*

Proof. Both proofs are very similar so we will only argue that $\bar{W}^\alpha = S^\alpha$.

First assume that $0 \leq \alpha < 1$. If $w \in \bar{W}^\alpha$, then we may find z_1 and z_2 so that $z_1 + z_2 = w$ and $\Pi(z_1, z_2) > \alpha$ since $\mu(w \mid \bar{W}) > \alpha$. This means that $\mu(z_i \mid \bar{Z}_i) > \alpha$ for $i = 1, 2$, which implies $(z_1, z_2) \in \bar{Z}_1^\alpha \times \bar{Z}_2^\alpha$ and $w \in S^\alpha$. If $w \in S^\alpha$, then $w = z_1 + z_2$ with $\mu(z_i \mid \bar{Z}_i) > \alpha$ for $i = 1, 2$. This implies that $\Pi(z_1, z_2) > \alpha$, and it follows that $\mu(w \mid \bar{W})$ also exceeds α, and hence $w \in \bar{W}^\alpha$.

Now let $\alpha = 1$. If $w \in S^1$, then there are z_1 and z_2 so that $w = z_1 + z_2$ and $\Pi(z_1, z_2) = 1$. Hence $\mu(w \mid \bar{W}) = 1$ and $w \in \bar{W}^1$. Now suppose $w \in \bar{W}^1$. For each $n = 2, 3, \ldots$ we can find z_{1n} in $\operatorname{supp}(\bar{Z}_1)$ and z_{2n} in $\operatorname{supp}(\bar{Z}_2)$ so that $z_{1n} + z_{2n} = w$ and

$$\Pi(z_{1n}, z_{2n}) > 1 - \frac{1}{n}. \tag{13}$$

Since the supports are compact we may choose a subsequence $z_{1n_k} \to z_1$ and $z_{2n_k} \to z_2$ with $z_1 + z_2 = w$ and $\Pi(z_1, z_2) \geq 1$ because Π is continuous. This implies that $z_i \in Z_i^1$ for $i = 1, 2$ and hence $w \in S^1$.

The following lemmas are needed in the proof of Theorem 2.

Lemma 1. *If $\bar{W} = \bar{Z}_1 + \bar{Z}_2$, or $W = \bar{Z}_1\bar{Z}_2$, then $\bar{W}^\alpha$ is open for $0 \leq \alpha < 1$.*

Proof. Suppose $\bar{W} = \bar{Z}_1 + \bar{Z}_2$ and let $w \in \bar{W}^\alpha$ for some $0 \leq \alpha < 1$. From Theorem 1 there is a $(z_1, z_2) \in \bar{Z}_1^\alpha \times \bar{Z}_2^\alpha$ so that $z_1 + z_2 = w$. Since $\bar{Z}_2^\alpha$ is open choose an open disk $O(z_2, \epsilon)$, centered at z_2 with radius $\epsilon > 0$, contained in $\bar{Z}_2^\alpha$. Then $z_1 + O(z_2, \epsilon)$ is an open set, containing w, wholely inside $\bar{W}^\alpha$ since $\bar{W}^\alpha = S^\alpha$. Therefore $\bar{W}^\alpha$ is open.

If $\bar{W} = \bar{Z}_1\bar{Z}_2$, then with minor changes, like $z_1 z_2 = w$ and $z_1 \cdot O(z_2, \epsilon)$ in place of $z_1 + O(z_2, \epsilon)$, the above argument shows that $\bar{W}^\alpha$ is open.

Lemma 2. *Let $\bar{W}=\bar{Z}_1+\bar{Z}_2$, or $\bar{W}=\bar{Z}_1\bar{Z}_2$. Suppose $w_n \in \bar{W}^0$ converges to w and $\mu(w_n \mid \bar{W})$ converges to λ in $[0, 1]$. Then $\mu(w \mid \bar{W}) \geq \lambda$.*

Proof. First let $\bar{W}=\bar{Z}_1+\bar{Z}_2$. For every $\epsilon>0$ there is z_{1n} in $\bar{Z}_1^0$ and z_{2n} in $\bar{Z}_2^0$ so that $z_{1n}+z_{2n}=w_n$ and

$$\mu(w_n \mid \bar{W}) \geq \Pi(z_{1n}, z_{2n}) > \mu(w_n \mid \bar{W})-\epsilon. \tag{14}$$

Now all the z_{1n}, all the z_{2n}, and all the w_n belong to compact sets so we may choose a subsequence so that $z_{1n_k}\to z_1$, $z_{2n_k}\to z_2$, $w_{n_k}\to w$ and $z_1+z_2=w$ and

$$\lambda \geq \Pi(z_1, z_2) > \lambda-\epsilon, \tag{15}$$

because Π is continuous. Since ϵ was arbitrary we see that $\lambda=\Pi(z_1, z_2)$ which implies $\lambda \leq \mu(w \mid \bar{W})$.

The case $\bar{W}=\bar{Z}_1\bar{Z}_2$ is similar and hence is omitted.

Theorem 2. *If $\bar{Z}_1$ and $\bar{Z}_2$ are fuzzy complex numbers, then so are $\bar{Z}_1+\bar{Z}_2$, $\bar{Z}_1\bar{Z}_2$, $\bar{Z}_1-\bar{Z}_2$, and $\bar{Z}_1/\bar{Z}_2$.*

Proof. (a) First let $\bar{W}=\bar{Z}_1+\bar{Z}_2$. We show $\mu(w \mid \bar{W})$ is continuous by arguing that $w_n \to w$ implies $\mu(w_n \mid \bar{W}) \to \mu(w \mid \bar{W})$. It suffices to choose the w_n in $\bar{W}^0$.

Since $\mu(w_n \mid \bar{W})$ belongs to $[0, 1]$ there is a subsequence $\mu(w_{n_k} \mid \bar{W})$ converging to some λ in $[0, 1]$. We know that ([5], p. 31)

$$\liminf \mu(w_n \mid \bar{W}) \leq \lambda \leq \limsup \mu(w_n \mid \bar{W}). \tag{16}$$

Lemma 1 implies that

$$\{w \mid \mu(w \mid \bar{W}) \leq t\}, \tag{17}$$

is closed for all real t. Therefore, $\mu(w \mid \bar{W})$ is lower semicontinuous and it follows that ([6], p. 74)

$$\liminf \mu(w_n \mid \bar{W}) \geq \mu(w \mid \bar{W}). \tag{18}$$

However, from Lemma 2 we obtain

$$\mu(w \mid \bar{W}) \geq \lambda. \tag{19}$$

Hence

$$\liminf \mu(w_n \mid \bar{W})=\lambda=\mu(w \mid \bar{W}). \tag{20}$$

There is a subsequence $\mu(w_{n_j} \mid \bar{W})$ converging to $\limsup \mu(w_n \mid \bar{W})$ ([5], p. 32). Lemma 2 implies that

$$\mu(w \mid \bar{W}) \geq \limsup \mu(w_n \mid \bar{W}). \tag{21}$$

Therefore

$$\liminf \mu(w_n \mid \bar{W})=\mu(w \mid \bar{W})=\limsup \mu(w_n \mid \bar{W}), \tag{22}$$

so ([5], p. 32)

$$\lim \mu(w_n \mid \bar{W})=\mu(w \mid \bar{W}), \tag{23}$$

and $\mu(w \mid \bar{W})$ is continuous.

Theorem 1 shows that $\bar{W}^\alpha$, $0 \le \alpha \le 1$, is bounded because it is the sum of two bounded sets. Also, $\bar{W}^\alpha$ is open for all $0 \le \alpha < 1$ (also Lemma 1) and $\bar{W}^1$ is closed because $\mu(w \mid \bar{W})$ is continuous. Finally, we argue that $\bar{W}^\alpha$ is connected, arcwise connected, and simply connected for $0 \le \alpha \le 1$. Now $\bar{Z}_1^\alpha$ and $\bar{Z}_1^\alpha$ are connected, arcwise connected, and simply connected and therefore so is $\bar{Z}_1^\alpha \times \bar{Z}_2^\alpha$, for $0 \le \alpha \le 1$. Since S^α is the continuous image of $\bar{Z}_1^\alpha \times \bar{Z}_2^\alpha$ it follows that $\bar{W}^\alpha = S^\alpha$ is also connected, simply connected, and arcwise connected, for $0 \le \alpha \le 1$. We have shown that $\bar{W}$ satisfies all the conditions to be a fuzzy complex number.

(b) The proof for $\bar{W} = \bar{Z}_1\bar{Z}_2$ is similar to $\bar{W} = \bar{Z}_1 + \bar{Z}_2$ and is omitted.

(c) If $\bar{Z}$ is a complex fuzzy number, then so is $-\bar{Z}$, and therefore $\bar{Z}_1 - \bar{Z}_2$ is a fuzzy complex number.

(d) Since the mapping $z \to z^{-1}$, $z \ne 0$, is continuous and $(\bar{Z}^{-1})^\alpha = (\bar{Z}^\alpha)^{-1}$ we see that $\bar{Z}^{-1}$ is a fuzzy complex number if $\bar{Z}$ is a fuzzy complex number. Therefore $\bar{Z}_1/\bar{Z}_2$ is a fuzzy complex number.

Theorem 1 says

$$(\bar{Z}_1 + \bar{Z}_2)^\alpha = \bar{Z}_1^\alpha + \bar{Z}_2^\alpha \tag{24}$$

and

$$(\bar{Z}_1\bar{Z}_2)^\alpha = \bar{Z}_1^\alpha\bar{Z}_2^\alpha, \tag{25}$$

for $0 \le \alpha \le 1$. Subtraction and division may also be performed using α-cuts because

$$(\bar{Z}_1 - \bar{Z}_2)^\alpha = \bar{Z}_1^\alpha - \bar{Z}_2^\alpha \tag{26}$$

and

$$(\bar{Z}_1/\bar{Z}_2)^\alpha = \bar{Z}_1^\alpha(\bar{Z}_2^\alpha)^{-1}, \tag{27}$$

for $0 \le \alpha \le 1$. The complex conjugate $\bar{Z}^*$ of a fuzzy complex number $\bar{Z}$ is also a fuzzy complex number because the mapping $z = x + iy \to \bar{z} = x - iy$ is continuous and $(\bar{Z}^*)^\alpha = (\bar{Z}^\alpha)^*$, $0 \le \alpha \le 1$.

We now show that $|\bar{Z}|$ is a (truncated) real fuzzy number if $\bar{Z}$ is a fuzzy complex number. A real fuzzy number $\bar{N}$, with membership function $\mu(x \mid \bar{N})$, is specified by $(n_1/n_2, n_3/n_4)$ where: (1) $n_1 < n_2 \le n_3 < n_4$; (2) $y = \mu(x \mid \bar{N})$ is continuous and increasing from zero to one on $[n_1, n_2]$; (3) $\mu(x \mid \bar{N})$ is one on $[n_2, n_3]$; (4) $y = \mu(x \mid \bar{N})$ is continuous and decreasing from one to zero on $[n_3, n_4]$; and (5) $\mu(x \mid \bar{N}) = 0$ outside (n_1, n_4). Notice that we do not insist that $\mu(x \mid \bar{N})$ be strictly increasing on $[n_1, n_2]$ nor do we demand that $\mu(x \mid \bar{N})$ be strictly decreasing on $[n_3, n_4]$ [1, 2]. In the degenerate case where $n_1 = n_2 = n_3 = n_4 = n$, $\bar{N}$ is the real number n. $|\bar{Z}|$ can be a truncated real fuzzy number because $\mu(r \mid |\bar{Z}|)$ must be zero for $r < 0$. We have the following cases: (1) if 0 $(0 + i0)$ is not in $\bar{Z}^0$, then $|\bar{Z}|$ is a regular real fuzzy number with $n_1 \ge 0$; (2) if $0 \in \bar{Z}^0$ but $0 \notin \bar{Z}^1$, then $n_1 = 0$ and $\mu(0 \mid |\bar{Z}|) \in (0, 1)$; (3) if $0 \in \bar{Z}^1$ but $\bar{Z}^1 \ne \{0\}$, then $n_1 = n_2 = 0$ and $\mu(0 \mid |\bar{Z}|) = 1$; and (4) if $\bar{Z}^1 = \{0\}$, then $n_1 = n_2 = n_3 = 0$. In cases (2) through (4) $|\bar{Z}|$ will be a truncated real fuzzy number.

Theorem 3. $|\bar{Z}|^\alpha = |\bar{Z}^\alpha|$, $0 \le \alpha \le 1$. *$|\bar{Z}|$ is a truncated real fuzzy number.*

Proof. Let

$$n_1 = \inf\{|z| \mid z \in \bar{Z}^0\}, \tag{28}$$

$$n_2 = \inf\{|z| \mid z \in \bar{Z}^1\}, \tag{29}$$

$$n_3 = \sup\{|z| \mid z \in \bar{Z}^1\}, \tag{30}$$

$$n_4 = \sup\{|z| \mid z \in \bar{Z}^0\}. \tag{31}$$

Clearly, $\mu(r \mid |\bar{Z}|) = 1$ on $[n_2, n_3]$.

We first argue that $|\bar{Z}|^\alpha = |\bar{Z}^\alpha|$. Assume $0 \leq \alpha < 1$. If $x \in |\bar{Z}|^\alpha$, then there is a z so that $|z| = x$ and $\mu(z \mid \bar{Z}) > \alpha$. Hence $x \in |\bar{Z}^\alpha|$. Next let $x \in |\bar{Z}^\alpha|$. Then there is a z so that $x = |z|$ and $\mu(z \mid \bar{Z}) > \alpha$. This implies

$$\sup\{\mu(z \mid \bar{Z}) \mid |z| = x\} > \alpha, \tag{32}$$

and $x \in |\bar{Z}|^\alpha$

Next suppose $\alpha = 1$. If $x \in |\bar{Z}^1|$, then there is a z so that $x = |z|$ and $\mu(z \mid \bar{Z}) = 1$. Then the supremum of $\mu(z \mid \bar{Z})$ over all z such that $|z| = x$ is also one and $x \in |\bar{Z}|^1$. Now let $x \in |\bar{Z}|^1$. For each $n = 2, 3, \ldots$ there is a z_n in $\bar{Z}^0$ so that $|z_n| = x$ and

$$\mu(z_n \mid \bar{Z}) > 1 - \frac{1}{n}. \tag{33}$$

The z_n belong to compact $\text{supp}(\bar{Z})$ so there is a subsequence $z_{n_k} \to z$ with $|z| = x$ and

$$\mu(z \mid \bar{Z}) \geq 1. \tag{34}$$

Hence $x \in |\bar{Z}^1|$.

We now argue that $\mu(r \mid |\bar{Z}|)$ is continuous. The proof is similar to that in Theorem 2 showing $\mu(w \mid \bar{W})$ is continuous so we only sketch the details. Let $r_n \to r$ with $r_n \in |\bar{Z}|^0$. There is subsequence $\mu(r_{n_k} \mid |\bar{Z}|) \to \lambda$ in $[0, 1]$. Now $|\bar{Z}^\alpha|$ is open for $0 \leq \alpha \leq 1$, so $|\bar{Z}|^\alpha$ is open and

$$\{r \mid \mu(r \mid |\bar{Z}|) \leq t\}, \tag{35}$$

is closed for all t. Hence $\mu(r \mid |\bar{Z}|)$ is lower semicontinuous and

$$\liminf \mu(r_n \mid |\bar{Z}|) \geq \mu(r \mid |\bar{Z}|). \tag{36}$$

We now prove a lemma similar to Lemma 2 saying r_n in $|\bar{Z}|^0$ converging to r and $\mu(r_n \mid |\bar{Z}|)$ converging to λ in $[0,1]$ implies $\mu(r \mid |\bar{Z}|) \geq \lambda$. This implies $\liminf \mu(r_n \mid |\bar{Z}|) \geq \lambda$ and we obtain

$$\liminf \mu(r_n \mid |\bar{Z}|) = \lambda = \mu(r \mid |\bar{Z}|). \tag{37}$$

We also have a subsequence converging to the lim sup and the above lemma produces

$$\mu(r \mid |\bar{Z}|) \geq \limsup \mu(r_n \mid |\bar{Z}|). \tag{38}$$

Therefore, the lim inf equals the lim sup which equals $\mu(r \mid |\bar{Z}|)$ and this function is continuous.

Finally, we show $\mu(r \mid |\bar{Z}|)$ is increasing on $[n_1, n_2]$ (or $[0, n_2]$) and decreasing on $[n_3, n_4]$. We first argue that

$$\mu(r \mid |\bar{Z}|) = \sup\{\mu(z \mid \bar{Z}) \mid |z| \leq r\} \tag{39}$$

for $n_1 \leq r \leq n_2$ (or $0 \leq r \leq n_2$) and

$$\mu(r \mid |\bar{Z}|) = \sup\{\mu(z \mid \bar{Z}) \mid |z| \geq r\}, \tag{40}$$

for $n_3 \leq r \leq n_4$. The proofs of equations (39) and (40) are similar so we will only prove equation (39). Suppose for some fixed value of r there is a z_0 so that $|z_0| < r$ and $\mu(z_0 \mid \bar{Z})$ exceeds $\mu(r \mid |\bar{Z}|)$. We know that

$$\{z \mid |z| = r\} \cap \bar{Z}^\alpha = \emptyset, \tag{41}$$

for $\alpha > \mu(r \mid |\bar{Z}|)$. Also, $z_0 \in \bar{Z}^{\alpha_0}$ for some $\alpha_0 > \mu(r \mid |\bar{Z}|)$ so $\bar{Z}^\alpha$ is a subset of $\{z \mid |z| < r\}$, for $\alpha \geq \alpha_0$, since the $\bar{Z}^\alpha$ are connected. This implies $n_2 < n_1$, a contradiction. Now let $n_1 \leq x_1 < x_2 \leq n_2$ (or $0 \leq x_1 < x_2 \leq n_2$). We see

$$\mu(x_1 \mid |\bar{Z}|) \leq \mu(x_2 \mid |\bar{Z}|), \tag{42}$$

because

$$\{z \mid |z| \leq x_1\} \subset \{z \mid |z| \leq x_2\}. \tag{43}$$

If $n_3 \leq x_1 < x_2 \leq n_4$, then

$$\mu(x_1 \mid |\bar{Z}|) \geq \mu(x_2 \mid |\bar{Z}|), \tag{44}$$

since

$$\{z \mid |z| \geq x_1\} \supset \{z \mid |z| \geq x_2\}. \tag{45}$$

This completes the proof that $|\bar{Z}|$ is a (truncated) real fuzzy number.

Theorem 3 may be used to deduce other results like the triangle inequality. We write (a, b) for an open interval of real numbers and $[a, b]$ for a closed interval of real number. We define: (1) $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$; and (2) $[a, b] \leq [c, d]$ if and only if $a \leq c$ and $b \leq d$. Let $\bar{M}$ and $\bar{N}$ be two real fuzzy numbers. We define α-cuts of real fuzzy numbers as in equations (1) and (2). We say $\bar{N} \geq 0$ if $n_1 \geq 0$. It is known that [3]

$$(\bar{M} + \bar{N})^\alpha = \bar{M}^\alpha + \bar{N}^\alpha, \tag{46}$$

and if $\bar{M} \geq 0$, $\bar{N} \geq 0$, then

$$(\bar{M}\bar{N})^\alpha = \bar{M}^\alpha \bar{N}^\alpha. \tag{47}$$

Theorem 4. (1) $|\bar{Z}_1 + \bar{Z}_2| \leq |\bar{Z}_1| + |\bar{Z}_2|$.
(2) $|\bar{Z}_1 \bar{Z}_2| = |\bar{Z}_1|\,|\bar{Z}_2|$.
(3) $|\bar{Z}_1 / \bar{Z}_2| = |\bar{Z}_1| / |\bar{Z}_2|$.

Proof. (1) The meaning of the first inequality is that the interval $|\bar{Z}_1 + \bar{Z}_2|^\alpha$ is less than or equal to the interval $(|\bar{Z}_1| + |\bar{Z}_2|)^\alpha$, for $0 \leq \alpha \leq 1$. From Theorem 1 and

Theorem 3 we see

$$|\bar{Z}_1+\bar{Z}_2|^\alpha=|(\bar{Z}_1+\bar{Z}_2)^\alpha|=|\bar{Z}_1^\alpha+\bar{Z}_2^\alpha| \\ =\{|z_1+z_2|\,|z_i\in\bar{Z}_i^\alpha,\ i=1,2\}. \tag{48}$$

From Theorem 3 and the results from real fuzzy numbers we have

$$(|\bar{Z}_1|+|\bar{Z}_2|)^\alpha=|\bar{Z}_1|^\alpha+|\bar{Z}_2|^\alpha=|\bar{Z}_1^\alpha|+|\bar{Z}_2^\alpha| \\ =\{|z_1|+|z_2|\mid z_i\in\bar{Z}_i^\alpha,\ i=1,2\}. \tag{49}$$

The result follows from $|z_1+z_2|\leq|z_1|+|z_2|$.

(2) We observe that

$$|\bar{Z}_1\bar{Z}_2|^\alpha=|(\bar{Z}_1\bar{Z}_2)^\alpha|=|\bar{Z}_1^\alpha\bar{Z}_2^\alpha|=\{|z_1z_2|\mid z_i\in\bar{Z}_i^\alpha,\ i=1,2\}. \tag{50}$$

We also deduce that

$$(|\bar{Z}_1|\,|\bar{Z}_2|)^\alpha=|\bar{Z}_1|^\alpha\,|\bar{Z}_2|^\alpha=|\bar{Z}_1^\alpha|\,|\bar{Z}_2^\alpha|=\{|z_1|\,|z_2|\,|z_i\in\bar{Z}_i^\alpha,\ i=1,2\}. \tag{51}$$

Hence, the α-cuts of $|\bar{Z}_1\bar{Z}_2|$ equal the corresponding α-cuts of $|\bar{Z}_1|\,|\bar{Z}_2|$ implying the two real fuzzy numbers are equal.

(3) We see that

$$|\bar{Z}_1/\bar{Z}_2|^\alpha=|\bar{Z}_1\bar{Z}_2^{-1}|^\alpha=|\bar{Z}_1^\alpha(\bar{Z}_2^{-1})^\alpha|=|\bar{Z}_1^\alpha(\bar{Z}_2^\alpha)^{-1}| \\ =\{|z_1/z_2|\mid z_i\in\bar{Z}_i^\alpha,\ i=1,2\}. \tag{52}$$

We also can write

$$(|\bar{Z}_1|/|\bar{Z}_2|)^\alpha=[|\bar{Z}_1|\,(|\bar{Z}_2|^{-1})]^\alpha=|\bar{Z}_1|^\alpha\,(|\bar{Z}_2|^{-1})^\alpha=|\bar{Z}_1^\alpha|\,|\bar{Z}_2^\alpha|^{-1} \\ =\{|z_1|/|z_2|\,|z_i\in\bar{Z}_i^\alpha,\ i=1,2\}. \tag{53}$$

Since the corresponding α-cuts are equal we obtain equality of the two real fuzzy numbers.

Since $\bar{Z}_1+\bar{Z}_2$ and $\bar{Z}_1\bar{Z}_2$ were defined from the extension principle the operations of addition and multiplication will enjoy the same basic properties they have when applied to real fuzzy numbers. Addition is associative, commutative, the complex number zero is the additive identity, and there is no additive inverse. Multiplication is associative, commutative, the complex number $1+i0$ is the multiplicitive identity, and there is no multiplicitive inverse. The distribution of multiplication over addition is sometimes true and sometimes false depending on the fuzzy complex numbers employed, just like for real fuzzy numbers.

Deducing the properties of fuzzy complex numbers under addition, multiplication, etc. is a lot like finding the basic properties of matrices. Some results from the algebra of real numbers do not carry over to matrices and the same is true for the algebra of complex numbers and fuzzy complex numbers. Each possible result on the algebra of fuzzy complex numbers must be looked at individually to see if it is true of false. For example,

$$|\bar{Z}|^2\neq\bar{Z}\bar{Z}^*, \tag{54}$$

and

$$\bar{Z}^{-1}\neq(|\bar{Z}|^2)^{-1}\bar{Z}^*. \tag{55}$$

In equation (54) the term on the left is a real fuzzy number but the item on the right is an fuzzy complex number. We leave it to the reader to find an example illustrating equation (55).

3. Rectangular fuzzy complex numbers

If $\bar{X}$ and $\bar{Y}$ are real fuzzy numbers with membership functions $\mu(x \mid \bar{X})$ and $\mu(y \mid \bar{Y})$, respectively, then

$$\bar{Z} = \bar{X} + \mathrm{i}\bar{Y}, \tag{56}$$

is a fuzzy complex number with membership function

$$\mu(z \mid \bar{Z}) = \min(\mu(x \mid \bar{X}), \mu(y \mid \bar{Y})), \tag{57}$$

where $z = x + \mathrm{i}y$.

Theorem 5. *$\bar{Z}^\alpha = \bar{X}^\alpha \times \bar{Y}^\alpha$, for $0 \leq \alpha \leq 1$.*

Proof. Assume $0 \leq \alpha < 1$. If $z \in \bar{Z}^\alpha$, then $\min(\mu(x \mid \bar{X}), \mu(y \mid \bar{Y})) > \alpha$ implying that both membership functions exceed α and therefore $(x, y) \in \bar{X}^\alpha \times \bar{Y}^\alpha$ where $z = x + \mathrm{i}y$. Suppose $(x, y) \in \bar{X}^\alpha \times \bar{Y}^\alpha$, then the minimum of the membership functions at x and y, respectively, exceeds α so that $\mu(z \mid \bar{Z}) > \alpha$ and $z \in \bar{Z}^\alpha$, where $z = x + \mathrm{i}y$.

Now let $\alpha = 1$. If $(x, y) \in \bar{X}^1 \times \bar{Y}^1$, then we easily see that $\mu(z \mid \bar{Z}) = 1$ for $z = x + \mathrm{i}y$ and therefore $z \in \bar{Z}^1$. Next let $z \in \bar{Z}^1$. Then there is an x and y so that $z = x + \mathrm{i}y$ and $\mu(x \mid \bar{X}) = \mu(y \mid \bar{Y}) = 1$. Hence $(x, y) \in \bar{X}^1 \times \bar{Y}^1$.

Theorem 5 says that the α-cuts of $\bar{Z}$ are rectangles and this is why this type of $\bar{Z}$ is called a rectangular complex fuzzy number.

Theorem 6. (1) $\bar{Z}^* = \bar{X} + \mathrm{i}(-\bar{Y})$.
(2) *If $\bar{Z}_j = \bar{X}_j + \mathrm{i}\bar{Y}_j$, $j = 1, 2$, then $\bar{Z}_1 \pm \bar{Z}_2 = (\bar{X}_1 \pm \bar{X}_2) + \mathrm{i}(\bar{Y}_1 \pm \bar{Y}_2)$.*
(3) $(\bar{Z}_1 \pm \bar{Z}_2)^\alpha = (\bar{X}_1^\alpha \pm \bar{X}_2^\alpha) + \mathrm{i}(\bar{Y}_1^\alpha \pm \bar{Y}_2^\alpha)$, $0 \leq \alpha \leq 1$.
(4) $|\bar{Z}|^\alpha = [(\bar{X}^\alpha)^2 + (\bar{Y}^\alpha)^2]^{1/2}$, $0 \leq \alpha \leq 1$.

Proof. (1) Obvious.

(2) We show the addition formula is true. Let $\bar{W} = \bar{Z}_1 + \bar{Z}_2$, then

$$\mu(w \mid \bar{W}) = \sup\{\Pi(z_1, z_2) \mid z_1 + z_2 = w\}, \tag{58}$$

where Π is given in Eq. 4. Define $\Gamma(x_1, x_2, y_1, y_2)$ to be the minimum of $\mu(x_i \mid \bar{X}_i)$, $\mu(y_i \mid \bar{Y}_i)$, $i = 1, 2$. Then we see that

$$\Gamma(x_1, x_2, y_1, y_2) = \Pi(z_1, z_2), \tag{59}$$

if $x_1 + \mathrm{i}y_1 = z_1$, $x_2 + \mathrm{i}y_2 = z_2$.

Let $\bar{X} = \bar{X}_1 + \bar{X}_2$ and $\bar{Y} = \bar{Y}_1 + \bar{Y}_2$ so that

$$\mu(x \mid \bar{X}) = \sup\{\Pi(x_1, x_2) \mid x_1 + x_2 = x\}, \tag{60}$$

$$\mu(y \mid \bar{Y}) = \sup\{\Pi(y_1, y_2) \mid y_1 + y_2 = y\}, \tag{61}$$

where $\Pi(x_1, x_2)$ ($\Pi(y_1, y_2)$) is the minimum of $\mu(x_1 \mid \bar{X}_1)$ and $\mu(x_2 \mid \bar{X}_2)$ ($\mu(y_1 \mid \bar{Y}_1)$ and $\mu(y_2 \mid \bar{Y}_2)$). If $\bar{Z} = \bar{X} + \mathrm{i}\bar{Y}$, then $\mu(z \mid \bar{Z})$ is the minimum of $\mu(x \mid \bar{X})$, $\mu(y \mid \bar{Y})$, where $z = x + \mathrm{i}y$.

We first argue that $\mu(w \mid \bar{W}) \leq \mu(w \mid \bar{Z})$. Let $w = x + \mathrm{i}y$, $x_1 + x_2 = x$, and $y_1 + y_2 = y$. Now $\Gamma(x_1, x_2, y_1, y_2)$ is less than or equal to $\Pi(x_1, x_2)$ and $\Pi(y_1, y_2)$ implying it is also less than or equal to $\mu(x \mid \bar{X})$ and $\mu(y \mid \bar{Y})$. Hence $\Gamma \leq \mu(w \mid \bar{Z})$ which implies that $\mu(w \mid \bar{W}) \leq \mu(w \mid \bar{Z})$.

Next, we show that $\mu(w \mid \bar{Z}) \leq \mu(w \mid \bar{W})$. Let $w = x + \mathrm{i}y$. For any $\epsilon > 0$ there are x_i^*, y_i^*, $i = 1, 2$, so that $x = x_1^* + x_2^*$, $y = y_1^* + y_2^*$, and

$$\Pi(x_1^*, x_2^*) > \mu(x \mid \bar{X}) - \epsilon, \tag{62}$$

$$\Pi(y_1^*, y_2^*) > \mu(y \mid \bar{Y}) - \epsilon. \tag{63}$$

It follows that

$$\Gamma(x_1^*, x_2^*, y_1^*, y_2^*) > \mu(w \mid \bar{Z}) - \epsilon. \tag{64}$$

This implies $\mu(w \mid \bar{W}) > \mu(w \mid \bar{Z}) - \epsilon$. Since $\epsilon > 0$ was arbitrary we have $\mu(w \mid \bar{W}) \geq \mu(w \mid \bar{Z})$.

The two inequalities say $\mu(w \mid \bar{W}) = \mu(w \mid \bar{Z})$ and we have the result for equality.

(3) This follows from Theorems 1 and 5.

(4) We see that

$$|\bar{Z}|^\alpha = |\bar{Z}^\alpha| = \{(x^2 + y^2)^{1/2} \mid x \in \bar{X}^\alpha, y \in \bar{Y}^\alpha\}. \tag{65}$$

But this set is $[(\bar{X}^\alpha)^2 + (\bar{Y}^\alpha)^2]^{1/2}$.

We do not obtain such nice results for the multiplication of rectangular fuzzy complex numbers because $(\bar{Z}_1\bar{Z}_2)^\alpha$ is not necessarily a rectangle. Therefore, $\bar{Z}_1\bar{Z}_2$ will not necessarily equal

$$(\bar{X}_1\bar{X}_2 - \bar{Y}_1\bar{Y}_2) + \mathrm{i}(\bar{X}_1\bar{Y}_2 + \bar{X}_2\bar{Y}_1), \tag{66}$$

when $\bar{Z}_j = \bar{X}_j + \mathrm{i}\bar{Y}_j$, $j = 1, 2$, because the fuzzy complex number in equation (66) is a rectangular fuzzy complex number. However, α-cuts of $\bar{Z}_1\bar{Z}_2$ will be contained in the corresponding α-cuts of the fuzzy complex number in equation (66).

4. $\bar{Z}$ based on $re^{\mathrm{i}\theta}$

If $\bar{R} \geq 0$ is a real fuzzy number and $\bar{\theta}$ is another real fuzzy number so that the diameter of $\mathrm{supp}(\bar{\theta}) < 2\pi$, then

$$\bar{Z} = \bar{R}\exp(\mathrm{i}\bar{\theta}), \tag{67}$$

is a fuzzy complex number with

$$\mu(z \mid \bar{Z}) = \min(\mu(r \mid \bar{R}), \mu(\theta \mid \bar{\theta})),$$

where $z = re^{\mathrm{i}\theta}$. The algebra of these types of fuzzy complex numbers, defined in terms of their α-cuts, has been investigated in [3].

Theorem 7. $\bar{Z}^\alpha = \bar{R}^\alpha \exp(\mathrm{i}\bar{\theta}^\alpha)$, $0 \leq \alpha \leq 1$.

Proof. We take the set

$$\{re^{\mathrm{i}\theta} \mid r \in \bar{R}^\alpha,\ \theta \in \bar{\theta}^\alpha\}, \tag{68}$$

to be $\bar{R}^\alpha \exp(\mathrm{i}\bar{\theta}^\alpha)$. The proof is similar to the proof of Theorem 5 and is omitted.

Notice that $\bar{Z}^\alpha$ is not a convex subset of C. For this reason we did not demand the α-cuts of $\bar{Z}$ to be convex in Definition 1.

Theorem 8. (1) *If* $\bar{Z}_j = \bar{R}_j \exp(\mathrm{i}\bar{\theta}_j)$, $j = 1, 2$, *then*: (a) $\bar{Z}_1\bar{Z}_2 = \bar{R}_1\bar{R}_2 \exp(\mathrm{i}[\bar{\theta}_1 + \bar{\theta}_2])$; (b) $|\bar{Z}_1\bar{Z}_2| = \bar{R}_1\bar{R}_2$; *and* (c) $\bar{Z}_1/\bar{Z}_2 = \bar{R}_1\bar{R}_2^{-1} \exp(\mathrm{i}[\bar{\theta}_1 - \bar{\theta}_2])$.

(2) $\bar{Z}^* = \bar{R} \exp(\mathrm{i}[-\bar{\theta}])$.

(3) $|\bar{Z}| = \bar{R}$.

(4) $\bar{Z}^{-1} = \bar{R}^{-1} \exp(\mathrm{i}[-\bar{\theta}])$.

(5) $(\bar{Z}_1\bar{Z}_2)^\alpha = \bar{R}_1^\alpha\bar{R}_2^\alpha \exp(\mathrm{i}[\bar{\theta}_1^\alpha + \bar{\theta}_2^\alpha])$, $0 \leq \alpha \leq 1$.

Proof. (1)(a). The proof is similar to the proof of (1) in Theorem 6 and is omitted. (3) We show $|\bar{Z}| = \bar{R}$ first and then return to (1)(b). Now

$$\mu(r \mid |\bar{Z}|) = \sup\{\mu(z \mid \bar{Z}) \mid |z| = r\}, \tag{69}$$

and

$$\mu(z \mid \bar{Z}) = \min(\mu(r \mid \bar{R}), \mu(\theta \mid \bar{\theta})). \tag{70}$$

All our fuzzy numbers, real or complex, are normalized so choose θ so that $\mu(\theta \mid \bar{\theta})$ is one and it follows that

$$\mu(r \mid |\bar{Z}|) = \mu(r \mid \bar{R}). \tag{71}$$

(1)(b). This now follows from (1)(a) and (3).

(4) We show this first and then return to (1)(c). This result follows immediately from

$$\mu(z \mid \bar{Z}^{-1}) = \mu(z^{-1} \mid \bar{Z}), \tag{72}$$

and $z^{-1} = r^{-1}e^{-\mathrm{i}\theta}$ when $z = re^{\mathrm{i}\theta}$.

(1)(c). This follows from (1)(a) and (4).

(2) This follows from

$$\mu(z \mid \bar{Z}^*) = \mu(\bar{z} \mid \bar{Z}), \tag{73}$$

and $\bar{z} = re^{-\mathrm{i}\theta}$ when $z = re^{\mathrm{i}\theta}$.

(5) This follows from Theorem 7 and (1)(a).

The results are not as nice for the addition and subtraction of fuzzy complex numbers based on the form $re^{\mathrm{i}\theta}$. We also note that $\bar{R} \exp(\mathrm{i}\bar{\theta})$ is not equal to

$$\bar{R} \cos \bar{\theta} + \mathrm{i}\bar{R} \sin \bar{\theta}, \tag{74}$$

since this is a rectangular fuzzy complex number.

5. Fuzzy complex analysis

There appear to be two approaches to defining a metric on $\bar{C}$: (1) take a metric on the set of real fuzzy numbers and extend it to $\bar{C}$; and (2) define directly the distance between $\bar{Z}_1$ and $\bar{Z}_2$. In the first method we will need to define the real part of $\bar{Z}$ (Re $\bar{Z}$) and the imaginary part of $\bar{Z}$ (Im $\bar{Z}$).

Definition 2. $\operatorname{Re}\bar{Z}$ ($\operatorname{Im}\bar{Z}$) is the projection of the surface $\alpha = \mu(z \mid \bar{Z})$, $z = x + \mathrm{i}y$, onto the αx (αy) plane.

Since $\mu(z \mid \bar{Z})$ is continuous we see that

$$\mu(x \mid \operatorname{Re}\bar{Z}) = \max_{y} \{\mu(z \mid \bar{Z}) \mid x + \mathrm{i}y = z\}. \tag{75}$$

A similar expression holds for $\operatorname{Im}\bar{Z}$. $\bar{Z}$ will not equal $\operatorname{Re}\bar{Z} + \mathrm{i}\operatorname{Im}\bar{Z}$ unless it is a rectangular fuzzy complex number.

Theorem 9. $\operatorname{Re}\bar{Z}$ *and* $\operatorname{Im}\bar{Z}$ *are real fuzzy numbers.*

Proof. We show that $\operatorname{Re}\bar{Z}$ is a real fuzzy number. Let $[x_1, x_4]$ be the projection of $\operatorname{supp}(\bar{Z})$ onto the x-axis and let $[x_2, x_3]$ be the projection of $\bar{Z}^1$ onto the x-axis. Since $\bar{Z}^1$ is a subset of $\bar{Z}^0$ we see $x_1 < x_2$ and $x_3 < x_4$. Clearly $\mu(x \mid \operatorname{Re}\bar{Z}) = 1$ on $[x_2, x_3]$.

Let $0 \leq \alpha_1 < \alpha_2 < 1$, let the projection of $\bar{Z}^{\alpha_1}$ be (x_{l1}, x_{u1}), and let the projection of $\bar{Z}^{\alpha_2}$ be (x_{l2}, x_{u2}). Since $\bar{Z}^{\alpha_2}$ is a subset of $\bar{Z}^{\alpha_1}$ we have $x_{l1} \leq x_{l2}$ and $x_{u1} \geq x_{u2}$. It follows that $\mu(x \mid \operatorname{Re}\bar{Z})$ is increasing on $[x_1, x_2]$ and decreasing on $[x_3, x_4]$ because the α_1-cut of $\operatorname{Re}\bar{Z}$ is (x_{l1}, x_{u1}) and the α_2-cut of $\operatorname{Re}\bar{Z}$ is (x_{l2}, x_{u2}).

It remains to show $G(x) = \mu(x \mid \operatorname{Re}\bar{Z})$ is continuous. It suffices to show G^{-1} of $[0, t)$ and $(t, 1]$ is open for $0 < t < 1$. But G^{-1} of $[0, t)$ is the projection of open $\{z \mid \mu(z \mid \bar{Z}) < t\}$ and G^{-1} of $(t, 1]$ is the projection of open $\{z \mid \mu(z \mid \bar{Z}) > t\}$. Therefore, these inverse image sets are open and $G(x)$ is continuous.

Let D be any complete metric (see for example [4]) defined on the space of real fuzzy numbers. We may extend D to $\bar{C}$ in many ways and one procedure is to define

$$\bar{F}(\bar{Z}_1, \bar{Z}_2) = \max[D(\operatorname{Re}\bar{Z}_1, \operatorname{Re}\bar{Z}_2), D(\operatorname{Im}\bar{Z}_1, \operatorname{Im}\bar{Z}_2)]. \tag{76}$$

$\bar{F}$ is a metric on $\bar{C}$ producing a complete metric space.

The second method would be to employ the Hausdorff metric H on the sets

$$\{(z, t) \mid z \in \operatorname{supp}(\bar{Z}),\ 0 \leq t \leq \mu(z \mid \bar{Z})\}, \tag{77}$$

which are compact subsets of R^3. However, $\bar{C}$ with H is not complete because the real fuzzy numbers with H is not complete [1]. As in [1] we could consider a subset $\bar{C}_m$ of $\bar{C}$ and restrict H to $\bar{C}_m$ producing a complete metric space.

Let $\mathscr{C}$ be $\bar{C}$, or $\bar{C}_m$, with a metric giving a complete metric space. Now we may define, and study, continuity and differentiability of functions mapping $\mathscr{C}$ into $\mathscr{C}$. This will be investigated in a future paper.

6. Summary and conclusions

We first presented a very general definition of a fuzzy complex number and then showed that this class of fuzzy complex numbers is closed under the basic operations of arithmetic. We followed the historical development of real fuzzy numbers in first defining addition and multiplication by the extension principle and then showed that these operations may be performed in terms of α-cuts. We also proved that the modulus of a fuzzy complex number is a (truncated) real fuzzy number and showed that the triangle inequality holds for fuzzy complex numbers.

There are two special types of fuzzy complex numbers: (1) rectangular fuzzy complex numbers based on the form $z = x + iy$ of regular complex numbers; and (2) polar fuzzy complex fuzzy numbers based on the form $z = re^{i\theta}$ of regular complex numbers. We showed that addition and subtraction are easily performed for rectangular fuzzy complex numbers whereas multiplication and division is easily done for polar fuzzy complex numbers.

We briefly discussed two methods of defining a metric on the space of fuzzy complex numbers in order to get a complete metric space $\mathscr{C}$. Future research is needed to deduce properties of functions mapping $\mathscr{C}$ into $\mathscr{C}$. For example, one wonders what form the Cauchy–Riemann equations will take for differentiable $h: \mathscr{C} \to \mathscr{C}$.

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