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Fuzzy complex analysis I: Differentiation

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Abstract: We consider functions mapping real numbers into the set of generalized complex fuzzy numbers $\bar{C}^*$. We define two derivatives of such functions which are generalizations of the derivative of real fuzzy mappings presented by Dubois and Prade [4, 5]. Sufficient conditions on the function are given so that the derivative will be a function mapping real numbers into $\bar{C}^*$. If the functional value is always a rectangular complex fuzzy number, made up to LR real fuzzy numbers, then we show that the derivative is easily computed from the derivatives of the LR fuzzy numbers. Other properties of the derivatives are presented.

Keywords: Analysis; fuzzy numbers; fuzzy derivative.

1. Introduction

In this section we first review the definitions and basic properties of: (1) real (LR) fuzzy numbers; (2) complex, and generalized complex, fuzzy numbers; and (3) the derivative of functions mapping real numbers into the set of real fuzzy numbers. We will also present in this section the notation, and other definitions, needed in the rest of the paper. In the next section we give our definitions of the derivative, and discuss some of their elementary properties, of functions mapping real numbers into the set of generalized complex fuzzy numbers. The last section contains a brief summary and conclusions.

A real fuzzy number $\bar{N}$ is defined by its membership function $\mu(x \mid \bar{N})$ which is a mapping from the real numbers into $[0, 1]$. There are three numbers $-\infty < n_1 < n_2 < n_3 < +\infty$ so that: (1) $\mu(x \mid \bar{N}) = 0$ for $x \leqslant n_1$ and $x \geqslant n_3$; (2) $y = \mu(x \mid \bar{N})$ is continuous and montonically increasing from zero to one on $[n_1, n_2]$; (3) $\mu(n_2 \mid \bar{N}) = 1$; and (4) $y = \mu(x \mid \bar{N})$ is continuous and monotonically decreasing from one to zero on $[n_2, n_3]$. Now let $F:(a, b) \to$ the set of real fuzzy numbers so that $F(t) = \bar{N}(t)$ for $a < t < b$. We now have the membership function $y = \mu(x \mid \bar{N}(t))$ a function of t and the n_i become $n_i(t)$, $i = 1, 2, 3$, also functions of t. Of course, $-\infty < n_1(t) < n_2(t) < n_3(t) < +\infty$, $\mu(n_2(t) \mid \bar{N}(t)) = 1$, etc.

We will sometimes assume that $\bar{N}(t)$ is a LR fuzzy number [4]. Let L and R be continuous functions mapping $[0, +\infty)$ into $[0, 1]$, both are strictly decreasing with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. Then

$$\mu(x \mid \bar{N}(t)) = \begin{cases} L\left(\dfrac{n_2(t) - x}{n_2(t) - n_1(t)}\right), & n_1(t) \leqslant x \leqslant n_2(t), \\ R\left(\dfrac{x - n_2(t)}{n_3(t) - n_2(t)}\right), & n_2(t) \leqslant x \leqslant n_3(t), \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

is a LR fuzzy number for $a < t < b$. We will use the 'dot' notation for derivatives, or partial derivatives, with respect to t. Otherwise, we employ the 'prime' notation for the derivative of a function of one real variable. We therefore assume that $\dot{n}_i(t)$, the derivative, exists for $a < t < b$, $i = 1, 2, 3$. Notice that in our definition of LR fuzzy number $\bar{N}(t)$ the L and R do not change with t, only the n_i are functions of t.

The (weak) $\alpha -$ cut of a real fuzzy number $\bar{N}$ is

$$\bar{N}(\alpha) = \{x \mid \mu(x \mid \bar{N}) \geqslant \alpha\}, \tag{2}$$

for $0 < \alpha \leqslant 1$. We separately define the zero cut $\bar{N}(0)$ to be $[n_1, n_3]$. All the $\alpha -$ cuts will be closed intervals, so let

$$\bar{N}(\alpha) = [n_1(\alpha), n_2(\alpha)], \tag{3}$$

for $0 \leqslant \alpha \leqslant 1$. Of course $n_1(0) = n_1$, $n_1(1) = n_2(1) = n_2$ and $n_2(0) = n_3$. For $\bar{N}(t)$, an arbitrary real fuzzy number, the value of $F(t)$, we denote its α-cut as $\bar{N}(t)(\alpha)$ and it is equal to

$$[n_1(t, \alpha), n_2(t, \alpha)], \tag{4}$$

where the end points are now functions of t and α. We assume that $\dot{n}_i(t, \alpha)$, the partial with respect to t, exists for $a < t < b$ and $0 \leqslant \alpha \leqslant 1$, $i = 1, 2$.

The derivative $F'(t)$, of $F(t) = \bar{N}(t)$, is a fuzzy subset of the reals for each $a < t < b$, defined by its membership function (see [5, 6])

$$\mu(x \mid F'(t)) = \sup\{\alpha \mid x = \dot{n}_1(t, \alpha), x = \dot{n}_2(t, \alpha), 0 \leqslant \alpha \leqslant 1\}. \tag{5}$$

Let us adopt the convention that the supremum of the empty set is zero. Therefore, if $x \neq \dot{n}_1(t, \alpha)$ and if $x \neq \dot{n}_2(t, \alpha)$ for $0 < \alpha \leqslant 1$, then the membership value for x is zero. Examples of $F'(t)$ when $\bar{N}(t)$ is a LR fuzzy number may be found in [5]. For other approaches to defining $F'(t)$ see [3, 7, 8], and [6] compares the various approaches to $F'(t)$.

We will write $z = x + iy$ for regular complex fuzzy numbers and $\bar{Z}$ for (generalized) complex fuzzy numbers. $\bar{Z}$ is defined by its membership function $\mu(z \mid \bar{Z})$ which is a mapping from the complex numbers C into $[0, 1]$. A weak α-cut of $\bar{Z}$ is

$$\bar{Z}(\alpha) = \{z \mid \mu(z \mid \bar{Z}) \geqslant \alpha\}, \tag{6}$$

for $0<\alpha\leqslant 1$. We separately specify $\bar{Z}(0)$ to be the closure of the union of $\bar{Z}(\alpha)$ for $0<\alpha\leqslant 1$. A strong α-cut of $\bar{Z}$ is

$$\bar{Z}(\bar{\alpha})=\{z \mid \mu(z \mid \bar{Z})>\alpha\}, \tag{7}$$

for $0\leqslant\alpha<1$. We define $\bar{Z}(\bar{1})=\{z \mid \mu(z \mid \bar{Z})=1\}$.

Definition 1. $\bar{Z}$ is a complex fuzzy number if and only if [1]:
(a) $\mu(z \mid \bar{Z})$ is continuous;
(b) $\bar{Z}(\bar{\alpha})$ is open, bounded, connected and simply connected for $0\leqslant\alpha<1$; and
(c) $\bar{Z}(\bar{1})$ is non-empty, compact, arcwise and simply connected.

Definition 2. $\bar{Z}$ is a generalized complex fuzzy number if and only if [2]:
(a) $\mu(z \mid \bar{Z})$ is upper semi-continuous (u.s.c);
(b) $\bar{Z}(\alpha)$ is compact, arcwise and simply connected for $0\leqslant\alpha\leqslant 1$; and
(c) $\bar{Z}(1)$ is non-empty.

Let g map C into the reals. We say g is u.s.c. if and only if $\{z \mid g(z)\geqslant t\}$ is closed for all real t. It is well known that g is u.s.c. if and only if $z_n\to z$ implies $\limsup g(z_n)\leqslant g(z)$. It was shown in [2] that if $\bar{Z}$ is a generalized complex fuzzy number and $\mu(z \mid \bar{Z})$ is continuous, then $\bar{Z}$ is a complex fuzzy number. Let $\bar{C}$ ($\bar{C}^*$) denote the set of complex (generalized complex) fuzzy numbers. It was shown in [1] that addition and multiplication in $\bar{C}$ can be done by strong α-cuts which means $(\bar{Z}_1+\bar{Z}_2)(\bar{\alpha})=\bar{Z}_1(\bar{\alpha})+\bar{Z}_2(\bar{\alpha})$ and $(\bar{Z}_1\bar{Z}_2)(\bar{\alpha})=\bar{Z}_1(\bar{\alpha})\bar{Z}_2(\bar{\alpha})$.

If $\bar{X}$ and $\bar{Y}$ are real fuzzy numbers with membership functions $\mu(x \mid \bar{X})$ and $\mu(y \mid \bar{Y})$, respectively, then $\bar{Z}=\bar{X}+\mathrm{i}\bar{Y}$ is a complex fuzzy number with membership function

$$\mu(z \mid \bar{Z})=\min(\mu(x \mid \bar{X}), \mu(y \mid \bar{Y})), \tag{8}$$

where $z=x+\mathrm{i}y$. $\bar{Z}$ was called a rectangular complex fuzzy number [1]. It was shown in [1] that $\bar{Z}(\bar{\alpha})=\bar{X}(\bar{\alpha})\times\bar{Y}(\bar{\alpha})$ and $(\bar{X}_1+\mathrm{i}\bar{Y}_1)+(\bar{X}_2+\mathrm{i}\bar{Y}_2)=(\bar{X}_1+\bar{X}_2)+\mathrm{i}(\bar{Y}_1+\bar{Y}_2)$.

Now let $F:(a, b)\to\bar{C}^*$ so that $F(t)=\bar{Z}(t)$ is a generalized complex fuzzy number for $a<t<b$. $\bar{Z}(t)(\alpha)$ is the weak α-cut of $\bar{Z}(t)$, $0<\alpha\leqslant 1$, which we also write as $\Omega_t(\alpha)$. $\bar{Z}(t)(0)$ is the closure of the union of $\Omega_t(\alpha)$ for $0<\alpha\leqslant 1$. We will assume that $\bar{Z}(t)(1)=\{z_1(t)\}$, a single point, for $a<t<b$ and also $z_1(t)$ belongs to the interior of $\Omega_t(\alpha)$ for $0\leqslant\alpha<1$. We assume that $\dot{z}_1(t)$ exists for all t.

Our last concept to define is for $F(t)$ to be 'star-like'. For any α-cut $\Omega_t(\alpha)$, $0\leqslant\alpha<1$, draw the ray $L(\beta)$, see Figure 1, from $z_1(t)$ making angle β $(0\leqslant\beta<2\pi)$ with the positive x-axis in the complex plane. Consider the set $L(\beta)\cap$ (boundary of $\Omega_t(\alpha)$) $=S(t, \alpha, \beta)$. We shall say $F(t)$ is star-like if and only if $S(t, \alpha, \beta)$ is a single point, say $z(t, \alpha, \beta)$, for all $0\leqslant\alpha<1$, and all $0\leqslant\beta<2\pi$. We now extend β to be $0\leqslant\beta\leqslant 2\pi$ knowing that $z(t, \alpha, 0)=z(t, \alpha, 2\pi)$ because we wish for β to range over a closed interval. We may write

$$z(t, \alpha, \beta)=x(t, \alpha, \beta)+\mathrm{i}y(t, \alpha, \beta), \tag{9}$$

for all t, α, β. We assume that $\dot{x}(t, \alpha, \beta)$ and $\dot{y}(t, \alpha, \beta)$ exist (these are partials on t) for all t, α, β. Let us agree that $z_1(t)=z(t, 1, \beta)$ for all β.

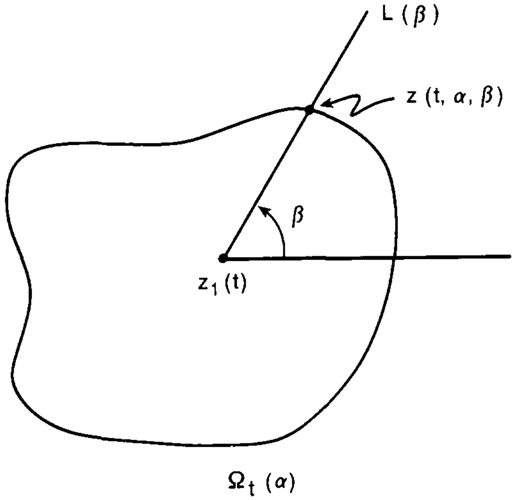

Fig. 1. Star-like α-cuts of $F(t)$.

Let us also agree that a closed, bounded, line segment, and a single point, are both star-like.

2. Differentiation

Let $F(t) = \bar{Z}(t) \in \bar{C}^*$ for $a < t < b$. We will assume throughout this section that $F(t)$ is star-like. $F'(t)$ will be a fuzzy subset of the complex numbers defined by its membership function $\mu(z \mid F'(t))$.

Definition 3.

$$\mu_1(z \mid F'(t)) = \sup\{\alpha \mid z = \dot{x}(t, \alpha, \beta) + i\dot{y}(t, \alpha, \beta),\ 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 2\pi\}. \tag{10}$$

We put a subscript of 'one' on μ in Definition 3 because we will present two ways to calculate $F'(t)$. For simplicity, we will write $\mu_1(z)$ for $\mu_1(z \mid F'(t))$. If $\bar{Z}(t)$ is a real fuzzy number for $a < t < b$, then Definition 3 reduces ($\beta = 0$ or π) to the definition of the derivative of function mapping the reals into the set of real fuzzy numbers presented in [5, 6]. Let $F'(t)(\alpha)$ denote a weak α-cut of $F'(t)$, $0 \leq \alpha \leq 1$.

Lemma 1. *Assume $\dot{x}(t, \alpha, \beta)$ and $\dot{y}(t, \alpha, \beta)$ are continuous functions of α and β. If $\mu_1(z) = \alpha$, $0 < \alpha < 1$, then there is a $\beta^* \in [0, 2\pi]$ so that*

$$z = \dot{x}(t, \alpha, \beta^*) + i\dot{y}(t, \alpha, \beta^*). \tag{11}$$

Proof. From the definition of $\mu_1(z)=\alpha$ there exist α_n, $\alpha \geq \alpha_n > \alpha - 1/n$, and $\beta_n \in [0, 2\pi]$ so that

$$z = \dot{x}(t, \alpha_n, \beta_n) + i\dot{y}(t, \alpha_n, \beta_n). \tag{12}$$

There is a convergent subsequence $\beta_{n_\kappa} \to \beta^*$ in $[0, 2\pi]$. By the continuity of $\dot{x}$ and $\dot{y}$ we obtain (since $\alpha_n \to \alpha$)

$$z = \dot{x}(t, \alpha, \beta^*) + i\dot{y}(t, \alpha, \beta^*). \tag{13}$$

Define the functions

$$G_1(\alpha, \beta) = \dot{x}(t, \alpha, \beta), \tag{14}$$

$$G_2(\alpha, \beta) = \dot{y}(t, \alpha, \beta), \tag{15}$$

which map $[0, 1] \times [0, 2\pi]$ into the reals. Assume that both G_1 and G_2 are continuous. Let Re (Im) denote the range of G_1 (G_2). Since the G_i are continuous both Re and Im are closed, bounded, intervals. We note that $F'(t)(\alpha)$ is a subset of Re $\times$ Im for $0 < \alpha \leq 1$. Also, since Re $\times$ Im is closed, we obtain $F'(t)(0)$, the closure of the union of $F'(t)(\alpha)$ for $0 < \alpha \leq 1$, a subset of Re $\times$ Im. We also see that: (1) if $z \in$ Re $\times$ Im, then there exists a α and β so that $z = \dot{x}(t, \alpha, \beta) + i\dot{y}(t, \alpha, \beta)$ but $\mu_1(z)$ could still be zero; and (2) if $z \notin$ Re $\times$ Im, then $z \neq \dot{x}(t, \alpha, \beta) + i\dot{y}(t, \alpha, \beta)$ for all α and β so that $\mu_1(z) = 0$.

Lemma 2. *Assuming $\dot{x}(t, \alpha, \beta)$ and $\dot{y}(t, \alpha, \beta)$ are continuous in α and β, then $F'(t)(0) =$ Re $\times$ Im.*

Proof. (a) Let $z \in$ Re $\times$ Im. If $\mu_1(z) = \alpha > 0$, then $z \in F'(t)(\alpha)$ and $z \in F'(t)(0)$. So assume that $\mu_1(z) = 0$. Then there is a $\beta^* \in [0, 2\pi]$ so that

$$z = \dot{x}(t, 0, \beta^*) + i\dot{y}(t, 0, \beta^*). \tag{16}$$

Choose $\alpha_n \in (0, 1]$, $\alpha_n \downarrow 0$, and let

$$z_n = \dot{x}(t, \alpha_n, \beta^*) + i\dot{y}(t, \alpha_n, \beta^*). \tag{17}$$

By the continuity of $\dot{x}$ and $\dot{y}$ we see that $z_n \to z$, and z_n in the union of the $F'(t)(\alpha)$, $0 < \alpha \leq 1$. Hence z belongs to the closure of the union of the $F'(t)(\alpha)$, $0 < \alpha \leq 1$, and then belongs to $F'(t)(0)$.

(b) We have already established the fact that $F'(t)(0)$ is a subset of Re $\times$ Im.

Lemma 3. *If $\dot{x}$ and $\dot{y}$ are continuous functions of α and β, then $F'(t)(1) = \{\dot{z}_1(t)\}$.*

Proof. Clearly, $\dot{z}_1(t) \in F'(t)(1)$. Let z belong to $F'(t)(1)$ but $z \neq \dot{z}_1(t)$. From the definition of $\mu_1(z) = 1$, we may choose $\alpha_n \in [0, 1)$, $\alpha_n \uparrow 1$, and $\beta_n \in [0, 2\pi]$, so that

$$z = \dot{x}(t, \alpha_n, \beta_n) + i\dot{y}(t, \alpha_n, \beta_n). \tag{18}$$

There is a subsequence $\beta_{n_\kappa} \to \beta^* \in [0, 2\pi]$. By the continuity of $\dot{x}$ and $\dot{y}$ we obtain

$$z = \dot{x}(t, 1, \beta^*) + i\dot{y}(t, 1, \beta^*), \tag{19}$$

which is $\dot{z}_1(t)$, a contradiction.

For $0<\alpha<1$, define

$$\Gamma(\alpha)=\{\dot{x}(t,\gamma,\beta)+\mathrm{i}\dot{y}(t,\gamma,\beta)\mid \alpha\leqslant\gamma\leqslant 1, \qquad 0\leqslant\beta\leqslant 2\pi\}. \tag{20}$$

Lemma 4. *If $\dot{x}$ and $\dot{y}$ are continuous, then $F'(t)(\alpha)=\Gamma(\alpha)$, $0<\alpha<1$.*

Proof. (a) Let $z\in\Gamma(\alpha)$. Then

$$z=\dot{x}(t,\gamma,\beta)+\mathrm{i}\dot{y}(t,\gamma,\beta), \tag{21}$$

for some $\gamma\geqslant\alpha$ and some $\beta\in[0,2\pi]$. This implies that $\mu_1(z)\geqslant\alpha$ and then $z\in F'(t)(\alpha)$.

(b) Let $z\in F'(t)(\alpha)$ and suppose $\mu_1(z)=\gamma\geqslant\alpha$. If $\gamma=1$, then by Lemma 3 we have

$$z=\dot{z}_1(t)=\dot{x}(t,1,\beta)+\mathrm{i}\dot{y}(t,1,\beta), \tag{22}$$

for all β. Hence $z\in\Gamma(\alpha)$. Therefore, assume that $\gamma<1$. From Lemma 1 we have

$$z=\dot{x}(t,\gamma,\beta^*)+\mathrm{i}\dot{y}(t,\gamma,\beta^*), \tag{23}$$

for some $\beta^*\in[0,2\pi]$. Therefore $z\in\Gamma(\alpha)$.

Theorem 1. *If $\dot{x}$ and $\dot{y}$ are continuous in α and β, then $F'(t)\in\bar{C}^*$.*

Proof. All we need to do is show $\mu_1(z)$ is u.s.c. If $\mu_1(z)$ is u.s.c., then $F'(t)(\alpha)$ is closed for $0<\alpha\leqslant 1$. $F'(t)(0)$ is closed by definition. Lemma 2 implies $F'(t)(\alpha)$ is bounded, $0\leqslant\alpha\leqslant 1$. Hence $F'(t)(\alpha)$ is compact for $0\leqslant\alpha\leqslant 1$.

$F'(t)(\alpha)$, $0<\alpha<1$, is simply and arcwise connected because, by Lemma 4, it is the continuous image of a rectangle $[\alpha,1]\times[0,2\pi]$. $F'(t)(0)$ is a rectangle (Lemma 2) so it is simply and arcwise connected. Lemma 3 says that $F'(t)(1)$ is non-empty but also arcwise and simply connected since it is only a single point.

Let $z_n\to z$. We may assume that z is in $F'(t)(0)$ for otherwise $\mu_1(z_n)=\mu_1(z)=0$ for sufficiently large n. Let $\alpha_n=\mu_1(z_n)$ and $\alpha=\mu_1(z)$. We need to show $\limsup\alpha_n\leqslant\alpha$. So assume that $\limsup\alpha_n=\alpha^*>\alpha$.

There is a subsequence $\alpha_{n_\kappa}\to\alpha^*$. From Lemma 1 choose $\beta_{n_\kappa}\in[0,2\pi]$ so that

$$z_{n_\kappa}=\dot{x}(t,\alpha_{n_\kappa},\beta_{n_\kappa})+\mathrm{i}\dot{y}(t,\alpha_{n_\kappa},\beta_{n_\kappa}). \tag{24}$$

The β_{n_κ} in $[0,2\pi]$, have a convergent subsequence, which for notational simplicity we still write as β_{n_κ}, converging to β^* in $[0,2\pi]$. By the continuity of $\dot{x}$ and $\dot{y}$ we get

$$z=\dot{x}(t,\alpha^*,\beta^*)+\mathrm{i}\dot{y}(t,\alpha^*,\beta^*). \tag{25}$$

But this means $\mu_1(z)\geqslant\alpha^*>\alpha$, a contradiction.

The membership function $\mu_1(z)$ may, or may not, be continuous. In [5] there are examples of $F(t)=\bar{N}(t)$, $\bar{N}(t)$ is a LR real fuzzy number for all t, where $F'(t)$ has a discontinuous membership function.

We now introduce some special types of (generalized) complex fuzzy numbers which are the generalizations of LR real fuzzy numbers presented in [4]. Assume that

$$\dot{x}(t, \alpha, \beta) = a_1 f_1(\beta) g_1(\alpha) + b_1, \tag{26}$$

$$\dot{y}(t, \alpha, \beta) = a_2 f_2(\beta) g_2(\alpha) + b_2, \tag{27}$$

where a_1, b_1, a_2, b_2 are real constants. For example we could have

$$x(t, \alpha, \beta) = h_1(t) f_1(\beta) g_1(\alpha) + x_1(t), \tag{28}$$

$$y(t, \alpha, \beta) = h_2(t) f_2(\beta) g_2(\alpha) + y_1(t), \tag{29}$$

or

$$x(t, \alpha, \beta) = f_1(t, \beta) g_1(\alpha) + x_1(t), \tag{30}$$

$$y(t, \alpha, \beta) = f_2(\beta) g_2(t, \alpha) + y_1(t), \tag{31}$$

where $z_1(t) = x_1(t) + \mathrm{i}y_1(t)$. In the first case (equations (28) and (29)) we have $a_1 = \dot{h}_1(t)$, $b_1 = \dot{x}_1(t)$, $a_2 = \dot{h}_2(t)$, and $b_2 = \dot{y}_1(t)$. In the second case (equations (30) and (31)) we have $a_1 = 1$, $f_1(\beta) = \dot{f}_1(t, \beta)$, $b_1 = \dot{x}_1(t)$, $a_2 = 1$, $g_2(\alpha) = \dot{g}_2(t, \alpha)$, and $b_2 = \dot{y}_1(t)$.

We now need to make some assumptions about the f_i and g_i in equations (26) and (27). Assume $g_i(\alpha) > 0$ on $[0, 1)$, $g_i'(\alpha) < 0$ on $(0, 1)$, and $g_i(1) = 0$, for $i = 1, 2$. Also the $f_i(\beta)$ will be continuous for $0 \leq \beta \leq 2\pi$ with $f_1(\beta)$ 'cosine-like' and $f_2(\beta)$ 'sine-like'. Assume that $f_1(\beta)$ is strictly increasing (decreasing) on $(\pi, 2\pi)$ $((0, \pi))$ with $f_1(\frac{1}{2}\pi) = f_1(\frac{3}{2}\pi) = 0$. Next assume that $f_2(\beta)$ is strictly increasing (decreasing) on $(0, \frac{1}{2}\pi) \cup (\frac{3}{2}\pi, 2\pi)$ $((\frac{1}{2}\pi, \frac{3}{2}\pi))$ and $f_2(0) = f_2(\pi) = f_2(2\pi) = 0$. Our description of the boundary of $\Omega_t(\alpha)$, say equations (28) and (29), is: (1) $z_1(t) = x_1(t) + \mathrm{i}y_1(t)$ is the 'center' of $\Omega_t(\alpha)$ where we measure outward along $L(\beta)$ to the boundary; (2) the $f_i(\beta)$ move us around the boundary in a (counter) clockwise direction; (3) the $g_i(\alpha)$ move the boundary inward (outward) as α increases (decreases); and (4) the $h_i(t)$ and $z_i(t)$ move the region around the complex plane. We continue to assume that $\Omega_t(\alpha)$ is star-like for all t and α. If equations (26) and (27) hold for t in (a, b) we will say that $F(t)$ is LR on (a, b).

A generalized complex fuzzy number $\bar{Z}$ is degenerate when $\bar{Z}(\alpha)$ fails to have an interior for some $\alpha \in [0, 1)$. For $F(t) = \bar{Z}(t)$, the $\bar{Z}(t)$ can not be degenerate because we assumed that $z_1(t)$ is in the interior of $\bar{Z}(t)(\alpha)$ for $0 \leq \alpha < 1$. However, $F'(t)$ can be degenerate as when $a_1 = 0$, or $a_2 = 0$, in equations (26) and (27). If $a_1 = 0$, then only $z = b_1 + \mathrm{i}y$ can have $\mu_1(z) > 0$. If $a_2 = 0$, then only $z = x + \mathrm{i}b_2$ can have $\mu_1(z) > 0$. If both a_1 and a_2 are zero, then $F'(t)$ becomes the regular complex number $\dot{z}_1(t)$.

Theorem 2. *If $F(t)$ is* LR *on (a, b), then $F'(t)$ is star-like for t in (a, b).*

Proof. We first fix α, $0 < \alpha < 1$, and consider $F'(t)(\alpha)$. The special case of $\alpha = 0$ will be treated at the end of the proof. We next assume that $a_i \neq 0$ for $i = 1, 2$ and discuss a_1 zero or a_2 zero later on in the proof. Let $L(\alpha, \beta) = L(\beta) \cap F'(t)(\alpha)$.

Table 1. Determining if f_1 (f_2) must increase, or decrease, in equation (26) and (27), if $g_i(\alpha)$ increases. (Assumed $x \neq b_1$ and $y \neq b_2$, or $\beta^* \neq 0$, $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, 2π.)

$\text{sign}(x^* - b_1)$ $\text{sign}(y^* - b_2)$	$\text{sign}(a_1)$ $\text{sign}(a_2)$	$\text{sign}(f_1(\beta^*))$ $\text{sign}(f_2(\beta^*))$	$f_1(\beta)$ $f_2(\beta)$
+	+	+	dec[a]
+	−	−	inc
−	+	−	inc
−	−	+	dec

[a] dec = decreasing, inc = increasing.

Let $z^* = x^* + iy^*$ be the point on $L(\alpha, \beta)$ maximum distance from $b_1 + ib_2$, which exists since $L(\alpha, \beta)$ is compact. Next consider $z = (b_1 + ib_2)\lambda + z^*(1 - \lambda)$ for any $0 < \lambda < 1$. Let $\mu_1(z) = \alpha_0$. We claim that $\alpha_0 \geqslant \alpha$ so that $L(\alpha, \beta) \subset F'(t)(\alpha)$ and $F'(t)(\alpha)$ is star-like.

In order to prove our claim suppose that $\alpha_0 < \alpha$. From Lemma 1 there are β^*, β_0 in $[0, 2\pi]$ so that

$$z^* = (a_1 f_1(\beta^*) g_1(\alpha) + b_1) + i(a_2 f_2(\beta^*) g_2(\alpha) + b_2), \tag{32}$$

$$z = (a_1 f_1(\beta_0) g_1(\alpha_0) + b_1) + i(a_2 f_2(\beta_0) g_2(\alpha_0) + b_2). \tag{33}$$

Since $\alpha_0 < \alpha$ we must have $g_i(\alpha_0) > g_i(\alpha)$, $i = 1, 2$. That is, the g_i must have increased. Assuming $\beta^* \neq 0$, $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, 2π, Table 1 tells us what f_1 and f_2 must do for equations (32) and (33) to hold. We conclude for f_1 that $\beta^* \in (0, \frac{1}{2}\pi) \cup (\pi, \frac{3}{2}\pi)$ and for f_2 $\beta^* \in (\frac{1}{2}\pi, \pi) \cup (\frac{3}{2}\pi, 2\pi)$, a contradiction. Now consider the special cases $\beta^* = 0$, $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, 2π. Let us consider only $\beta^* = 0$ which implies $\beta_0 = 0$ also. Look at

$$\frac{x^* - b_1}{a_1} = f_1(0) g_1(\alpha), \tag{34}$$

as $x^* \to x$ and $\alpha \to \alpha_0$. The term $(x^* - b_1)/a_1$ is either positive and decreasing, or negative and increasing, both impossible with $f_1(\beta)$ fixed at $f_1(0)$.

Next we consider the case of a_1 or a_2 zero. Let $a_1 = 0$, but a_2 not zero. Then $F'(t)$ is degenerate. If $z \in F'(t)(\alpha)$, then $z = b_1 + iy$ and we only need to consider $\beta = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$. Let $\beta = \frac{1}{2}\pi$ and $L(\alpha, \frac{1}{2}\pi) = L(\frac{1}{2}\pi) \cap F'(t)(\alpha)$. As above let $z^* = b_1 + iy^*$ be the point on $L(\alpha, \frac{1}{2}\pi)$ farthest from $b_1 + ib_2$. Let $b_1 + iy$ be any point on $L(\alpha, \frac{1}{2}\pi)$ between $b_1 + ib_2$ and $b_1 + iy^*$. If $\mu_1(b_1 + iy) = \alpha_0$ assume $\alpha_0 < \alpha$. Consider

$$\frac{y^* - b_2}{a_2} = f_2(\tfrac{1}{2}\pi) g_2(\alpha), \tag{35}$$

as $y^* \to y$ and $\alpha \to \alpha_0$. The right side of equation (35) is positive and increasing. However, the left hand side is either positive and decreasing or negative and increasing. Hence $\alpha_0 \geqslant \alpha$ and $L(\alpha, \frac{1}{2}\pi)$ is contained in $F'(t)(\alpha)$. The case $\beta = \frac{3}{2}\pi$ is similar and omitted.

The case $a_2 = 0$, and $a_1 \neq 0$ is similar to $a_1 = 0$, $a_2 \neq 0$ and is omitted. Now let $a_1 = a_2 = 0$. We obtain $\mu_1(z) = 1$ for $z = b_1 + ib_2$ and $\mu_1(z) = 0$ otherwise. Here $F'(t)$ is the complex number $\dot{z}_1(t)$.

Lastly, assume that $\alpha = 0$. Lemma 2 says that $F'(t)(0)$ is a rectangle, or in the degenerate cases a bounded line segment or a single point, and hence is star-like.

We now define two fuzzy subsets of the real numbers, Re $F'(t)$ and Im $F'(t)$, to be used in our second definition of $F'(t)$. Let

$$\begin{aligned}\mu_r(x) &= \mu(x \mid \operatorname{Re} F'(t))\\ &= \sup\{\alpha \mid x = \dot{x}(t, \alpha, \beta),\ 0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant 2\pi\},\end{aligned} \tag{36}$$

$$\begin{aligned}\mu_i(y) &= \mu(y \mid \operatorname{Im} F'(t))\\ &= \sup\{\alpha \mid y = \dot{y}(t, \alpha, \beta),\ 0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant 2\pi\}.\end{aligned} \tag{37}$$

Definition 4. Define $F'(t)$, a fuzzy subset of C, by

$$\mu_2(z) = \mu_2(z \mid F'(t)) = \min(\mu_r(x), \mu_i(y)), \tag{38}$$

where $z = x + iy$.

Theorem 3. $\mu_1(z) \leqslant \mu_2(z)$.

Proof. Suppose $\mu_1(z) = \alpha > 0$ with $z = x + iy$. Let $\varepsilon > 0$. From the definition of $\mu_1(z) = \alpha$ there is an $1 \geqslant \alpha^* > \alpha - \varepsilon$ and a $\beta^* \in [0, 2\pi]$ so that

$$x = \dot{x}(t, \alpha^*, \beta^*), \tag{39}$$

$$y = \dot{y}(t, \alpha^*, \beta^*). \tag{40}$$

It follows that

$$\mu_r(x) \geqslant \alpha^*, \tag{41}$$

$$\mu_i(y) \geqslant \alpha^*. \tag{42}$$

Hence

$$\mu_2(z) \geqslant \alpha^* > \alpha - \varepsilon. \tag{43}$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

Theorem 4. *Assume $\dot{x}$ and $\dot{y}$ are continuous functions of α and β. Using Definition 4 we have $F'(t) \in \bar{C}^*$ and $F'(t)$ is star-like.*

Proof. (1) We first argue, as in the proof of Theorem 1, that μ_r and μ_i are u.s.c. The result that μ_2 is u.s.c. follows because the minimum of two u.s.c. functions is u.s.c.

(2) As in the Proof of Lemma 3 we can show that $F'(t)(1) = \{\dot{z}_1(t)\}$.

(3) Let, for $0 < \alpha < 1$,

$$\Gamma_r(\alpha) = \{\dot{x}(t, \gamma, \beta) \mid \alpha \leqslant \gamma \leqslant 1,\ 0 \leqslant \beta \leqslant 2\pi\}, \tag{44}$$

$$\Gamma_i(\alpha) = \{\dot{y}(t, \gamma, \beta) \mid \alpha \leqslant \gamma \leqslant 1,\ 0 \leqslant \beta \leqslant 2\pi\}. \tag{45}$$

As in the proof of Lemma 4 we may show

$$\Gamma_r(\alpha) = (\operatorname{Re} F'(t))(\alpha), \tag{46}$$

$$\Gamma_i(\alpha) = (\operatorname{Im} F'(t))(\alpha). \tag{47}$$

Also, as in Lemma 2 we may deduce that

$$F'(t)(0) = \operatorname{Re} \times \operatorname{Im}. \tag{48}$$

This proves the result that $F'(t) \in \bar{C}^*$ since for $0 < \alpha < 1$,

$$F'(t)(\alpha) = (\operatorname{Re} F'(t))(\alpha) \times (\operatorname{Im} F'(t))(\alpha). \tag{49}$$

(4) $F'(t)$ is star-like because its α-cuts, $0 \leqslant \alpha < 1$, are rectangles.

Theorem 5. *Suppose $\bar{X}(t)$ and $\bar{Y}(t)$ are* LR *real fuzzy numbers for all t. If $F(t) = \bar{X}(t) + \mathrm{i}\bar{Y}(t)$, then $F'(t) = \bar{X}'(t) + \mathrm{i}\bar{Y}'(t)$, when we use Definition* 4 *for $F'(t)$.*

Proof. We show $\operatorname{Re} F'(t) = \bar{X}'(t)$ and omit the details on $\operatorname{Im} F'(t) = \bar{Y}'(t)$.

We need to specify $x(t, \alpha, \beta)$ as we go around the boundary of an α-cut of $F(t)$, $0 \leqslant \alpha < 1$. Consulting Figure 2 we see that

$$x(t, \alpha, \beta) = x_2(t, \alpha), \quad -\beta_1 \leqslant \beta \leqslant \beta_1, \tag{50}$$

$$x(t, \alpha, \beta) = x_2(t) + \cot \beta[y_2(t, \alpha) - y_2(t)], \quad \beta_1 \leqslant \beta \leqslant \beta_2, \tag{51}$$

$$x(t, \alpha, \beta) = x_1(t, \alpha), \quad \beta_2 \leqslant \beta \leqslant \beta_3, \tag{52}$$

$$x(t, \alpha, \beta) = x_2(t) - \cot \beta[y_2(t) - y_1(t, \alpha)], \quad \beta_3 \leqslant \beta \leqslant \beta_4, \tag{53}$$

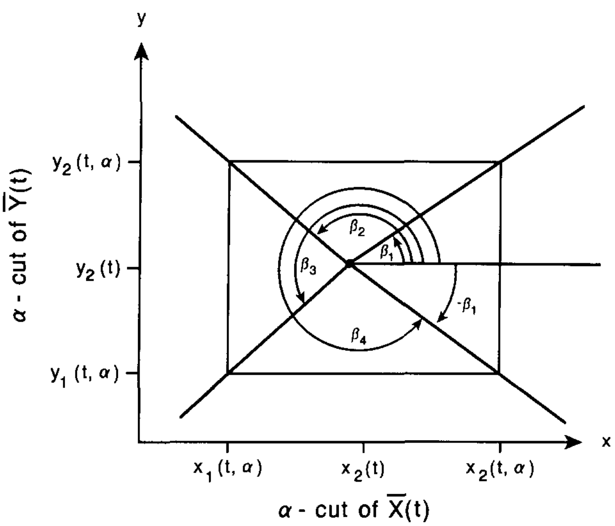

Fig. 2. An α-cut of $F(t) = \bar{X}(t) + \mathrm{i}\bar{Y}(t)$.

where

$$x_2(t, \alpha) = x_2(t) + (x_3(t) - x_2(t))R^{-1}(\alpha), \tag{54}$$

$$y_2(t, \alpha) = y_2(t) + (y_3(t) - y_2(t))R^{-1}(\alpha), \tag{55}$$

$$x_1(t, \alpha) = x_2(t) - (x_2(t) - x_1(t))L^{-1}(\alpha), \tag{56}$$

$$y_1(t, \alpha) = y_2(t) - (y_2(t) - y_1(t))L^{-1}(\alpha). \tag{57}$$

Now set $x = \dot{x}(t, \alpha, \beta)$, substitute for $y_2(t, \alpha)$ and $y_1(t, \alpha)$, then solve equations (51) and (53) for α, we get

$$\alpha = R\left(\frac{x - \dot{x}_2(t)}{\dot{y}_3(t) - \dot{y}_2(t)} \tan \beta\right), \tag{58}$$

$$\alpha = L\left(\frac{\dot{x}_2(t) - x}{\dot{y}_2(t) - \dot{y}_1(t)} \tan \beta\right), \tag{59}$$

assuming $\dot{y}_3(t) - \dot{y}_2(t)$ $(\dot{y}_2(t) - \dot{y}_1(t))$ are not zero and $\beta \neq \frac{1}{2}\pi$ $(\frac{3}{2}\pi)$. In equation (58) ((59)) we wish to find β, $\beta_1 \leqslant \beta \leqslant \beta_2$ $(\beta_3 \leqslant \beta \leqslant \beta_4)$ to maximize α. Since R and L are decreasing functions, the maximum value of α in equation (58) ((59)) is for $\beta = \beta_1$ or β_2 $(\beta = \beta_3$ or $\beta_4)$. This means that the maximum α on the top (bottom) of the rectangle in Figure 2 is at the corners where $\dot{x}(t, \alpha, \beta) = \dot{x}_1(t, \alpha)$ or $\dot{x}_2(t, \alpha)$. Hence

$$\mu_r(x) = \sup\{\alpha \mid x = \dot{x}_1(t, \alpha),\ x = \dot{x}_2(t, \alpha)\} \tag{60}$$

which is the definition of $\bar{X}'(t)$.

Now for the special cases where the expressions in equations (58) and (59) are undefined. The result follows because $x = \dot{x}_2(t)$ if and only if: (1) $\beta = \frac{1}{2}\pi$ or $\dot{y}_3(t) = \dot{y}_2(t)$ or $\alpha = 1$ for $\beta_1 \leqslant \beta \leqslant \beta_2$; and (2) $\beta = \frac{3}{2}\pi$ or $\dot{y}_2(t) = \dot{y}_1(t)$ or $\alpha = 1$ for $\beta_3 \leqslant \beta \leqslant \beta_4$. But in either case $\mu_r(x) = \mu(x \mid \bar{X}'(t)) = 1$ when $x = \dot{x}_2(t)$.

Let $G(t)$ be another function mapping (a, b) into $\bar{C}^*$. That is, $G(t) = \bar{W}(t) \in \bar{C}^*$ for $a < t < b$. We also assume that $G(t)$ is star-like. Let $\bar{W}(t)(1) = \{p_1(t)\}$, which is in the interior of $\bar{W}(t)(\alpha)$ for $0 \leqslant \alpha < 1$, and assume $\dot{p}_1(t)$ exists on (a, b). The point where the ray $L(\beta)$ (as in Figure 1) hits the boundary of $\bar{W}(t)(\alpha)$ is $w(t, \alpha, \beta) = u(t, \alpha, \beta) + \mathrm{i}v(t, \alpha, \beta)$, $0 \leqslant \alpha < 1$, $0 \leqslant \beta \leqslant 2\pi$. We assume that $\dot{u}(t, \alpha, \beta)$ and $\dot{v}(t, \alpha, \beta)$ exist for all t, α, β. Therefore, $G'(t)$, using either definition, exists for t in (a, b) and we may consider $F'(t) + G'(t)$.

Let $H(t) = F(t) + G(t)$ and $\bar{V}(t) = \bar{Z}(t) + \bar{W}(t)$. We first need to show that: (1) $\bar{V}(t) \in \bar{C}^*$; (2) $\bar{V}(t)(1) = \{z_1(t) + p_1(t)\}$ which is in the interior of $\bar{V}(t)(\alpha)$, $0 \leqslant \alpha < 1$ and $\dot{z}_1(t) + \dot{p}_1(t)$ exists; and (3) $\bar{V}(t)(\alpha)$ is star-like for $0 \leqslant \alpha < 1$. Then $H'(t) = (F(t) + G(t))'$ exists, by either definition, and we may compare $F'(t) + G'(t)$ and $(F(t) + G(t))'$. In [1] we showed that if $\bar{Z}(t)$ and $\bar{W}(t)$ are in $\bar{C}$, then $\bar{Z}(t) + \bar{W}(t) \in \bar{C}$. The proof that $\bar{C}^*$ is closed under addition is similar and omitted. The facts in (2) above are easily seen so let us now argue that $\bar{V}(t)(\alpha)$ is star-like, $0 \leqslant \alpha < 1$.

In [1] we showed that addition in $\bar{C}$ may be done by strong α-cuts. It is also true that addition in $\bar{C}^*$ may be accomplished by weak α-cuts. This means that $\bar{Z}(t)(\alpha) + \bar{W}(t)(\alpha) = \bar{V}(t)(\alpha)$, $0 \leqslant \alpha \leqslant 1$. We claim that the point where the ray

$L(\beta)$ hits the boundary of $\bar{V}(t)(\alpha)$, $0 \leq \alpha < 1$, is $s(t, \alpha, \beta) = s_1(t, \alpha, \beta) + \mathrm{i}s_2(t, \alpha, \beta)$ where $s_1(t, \alpha, \beta) = x(t, \alpha, \beta) + u(t, \alpha, \beta)$ and $s_2(t, \alpha, \beta) = y(t, \alpha, \beta) + v(t, \alpha, \beta)$. Addition is continuous so a point on the boundary of $\bar{Z}(t)(\alpha)$ plus a point on the boundary of $\bar{W}(t)(\alpha)$ gives a point on the boundary of $\bar{V}(t)(\alpha)$. Also, a point on $L(\beta) \cap (\bar{Z}(t)(\alpha))$ plus a point on $L(\beta) \cap (\bar{W}(t)(\alpha))$ is a point on $L(\beta) \cap (\bar{V}(t)(\alpha))$. Hence $\bar{V}(t)(\alpha)$ is star-like for $0 \leq \alpha < 1$.

Theorem 6. *$(F(t)+G(t))' \subset F'(t)+G'(t)$, using either definition of the derivative.*

Proof. We show $\mu_a(s \mid (F(t)+G(t))') \leq \mu_a(s \mid F'(t)+G'(t))$ for all complex s, for $a = 1, 2$. We may assume that s belongs to an α-cut of $(F(t)+G(t))'$ for $0 < \alpha \leq 1$. We treat the special case of $\alpha = 1$ at the end of the proof so assume that $0 < \alpha < 1$.

(a) *Definition* 3: Let $\alpha_3 = \mu_1(s \mid (F(t)+G(t))')$ so that

$$\alpha_3 = \sup\{\alpha \mid s = \dot{s}_1(t, \alpha, \beta) + \mathrm{i}\dot{s}_2(t, \alpha, \beta),\ 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 2\pi\}. \tag{61}$$

If $\varepsilon > 0$ there is a α_3^*, $1 \geq \alpha_3^* > \alpha_3 - \varepsilon$, and a β_3 so that

$$s = \dot{s}_1(t, \alpha_3^*, \beta_3) + \mathrm{i}\dot{s}_2(t, \alpha_3^*, \beta_3). \tag{62}$$

We see that (discussion preceding the theorem)

$$s_1(t, \alpha_3^*, \beta_3) = x(t, \alpha_3^*, \beta_3) + u(t, \alpha_3^*, \beta_3), \tag{63}$$

$$s_2(t, \alpha_3^*, \beta_3) = y(t, \alpha_3^*, \beta_3) + v(t, \alpha_3^*, \beta_3). \tag{64}$$

Let $z + w = s$ where

$$z = x(t, \alpha_3^*, \beta_3) + \mathrm{i}y(t, \alpha_3^*, \beta_3), \tag{65}$$

$$w = u(t, \alpha_3^*, \beta_3) + \mathrm{i}v(t, \alpha_3^*, \beta_3). \tag{66}$$

If $\mu_1(z \mid F'(t)) = \alpha_1$ and $\mu_1(z \mid G'(t)) = \alpha_2$, then it follows that $\alpha_1 \geq \alpha_3^*$ and $\alpha_2 \geq \alpha_3^*$. Now

$$\mu_1(s \mid F'(t)+G'(t)) = \sup_{w+z=s} (\min(\mu_1(z \mid F'(t)), \mu_1(z \mid G'(t))). \tag{67}$$

But we have just shown that we can find a z and w so that $z + w = s$ and the minimum in equation (67) is at least α_3^*. Hence the supremum is at least $\alpha_3^* > \alpha_3 - \varepsilon$. The desired result follows since $\varepsilon > 0$ was arbitrary.

(b) *Definition* 4: Now $\alpha_3 = \mu(s \mid (F(t)+G(t))')$ is the minimum of $\alpha_{31} = \mu(s_1 \mid \mathrm{Re}(F(t)+G(t)))$ and $\alpha_{32} = \mu(s_2 \mid \mathrm{Im}(F(t)+G(t)))$ for $s_1 + \mathrm{i}s_2 = s$. Given $\varepsilon > 0$ there is a α_{31}^*, $1 \geq \alpha_{31}^* > \alpha_{31} - \varepsilon$, and a β_{31} so that

$$s_1 = \dot{x}(t, \alpha_{31}^*, \beta_{31}) + \dot{u}(t, \alpha_{31}^*, \beta_{31}), \tag{68}$$

and there is a α_{32}^*, $1 \geq \alpha_{32}^* > \alpha_{32} - \varepsilon$, and a β_{32} so that

$$s_2 = \dot{y}(t, \alpha_{32}^*, \beta_{32}) + \dot{v}(t, \alpha_{32}^*, \beta_{32}). \tag{69}$$

Now let

$$x = \dot{x}(t, \alpha_{31}^*, \beta_{31}), \tag{70}$$

$$y = \dot{y}(t, \alpha_{32}^*, \beta_{32}), \tag{71}$$

$$u = \dot{u}(t, \alpha_{31}^*, \beta_{31}), \tag{72}$$

$$v = \dot{v}(t, \alpha_{32}^*, \beta_{32}), \tag{73}$$

and let $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$, so that $w + z = s$.

Now $\mu_2(z \mid F'(t))$ is the minimum of $\mu(x \mid \operatorname{Re} F'(t))$ and $\mu(y \mid \operatorname{Im} F'(t))$. Therefore $\mu_2(z \mid F'(t)) \geqslant \min(\alpha_{31}^*, \alpha_{32}^*)$. Similarly we see that $\mu_2(w \mid G'(t)) \geqslant \min(\alpha_{31}^*, \alpha_{32}^*)$. Hence

$$\mu_2(s \mid F'(t) + G'(t)) \geqslant \min(\alpha_{31}^*, \alpha_{32}^*). \tag{74}$$

But $\min(\alpha_{31}^*, \alpha_{32}^*) > \min(\alpha_{31} - \varepsilon, \alpha_{32} - \varepsilon) = \min(\alpha_{31}, \alpha_{32}) - \varepsilon = \alpha_3 - \varepsilon$. The result follows since $\varepsilon > 0$ was arbitrary.

Lastly we consider the $\alpha = 1$ case. Using either definition the $\alpha = 1$ cut of $(F(t) + G(t))'$ contains only $\dot{z}_1(t) + \dot{p}_1(t)$ and the $\alpha = 1$ cut of $F'(t) + G'(t)$ contains the single point $\dot{z}_1(t) + \dot{p}_1(t)$.

Let $F(t) = \bar{X}(t) + \mathrm{i}\bar{Y}(t)$ and $G(t) = \bar{U}(t) + \mathrm{i}\bar{V}(t)$ where $\bar{X}(t), \ldots, \bar{V}(t)$ are all LR real fuzzy numbers for $a < t < b$. Then $F(t) + G(t) = (\bar{X}(t) + \bar{U}(t)) + \mathrm{i}(\bar{Y}(t) + \bar{V}(t))$ and from Theorem 5 we get (using Definition 4)

$$(F(t) + G(t))' = (\bar{X}(t) + \bar{U}(t))' + \mathrm{i}(\bar{Y}(t) + \bar{V}(t))'. \tag{75}$$

However (see [5]),

$$(\bar{X}(t) + \bar{U}(t))' \subset \bar{X}'(t) + \bar{U}'(t), \tag{76}$$

$$(\bar{Y}(t) + \bar{V}(t))' \subset \bar{Y}'(t) + \bar{V}'(t). \tag{77}$$

Therefore

$$\begin{aligned}(F(t) + G(t))' &\subset (\bar{X}'(t) + \bar{U}'(t)) + \mathrm{i}(\bar{Y}'(t) + \bar{V}'(t)) \\ &= F'(t) + G'(t).\end{aligned} \tag{78}$$

So, we cannot improve on Theorem 6 even in the simple case of $F(t)$ and $G(t)$ rectangular complex fuzzy numbers built from LR real fuzzy numbers.

Let us close this section with two examples.

Example 1. Let $F(t) = \bar{Z}(t)$ where the boundary of $\bar{Z}(t)(\alpha)$, $0 \leqslant \alpha < 1$, is described by (see Figure 1)

$$\begin{aligned}z(t, \alpha, \beta) = {} & [(\sin t + 2)(\cos \beta)(1 - \alpha)] + \mathrm{i}[(\cos t + 2)(\sin \beta)(1 - \alpha)] \\ & + [(2t - 1) + \mathrm{i}(3 - t)].\end{aligned} \tag{79}$$

$\bar{Z}(t)$ is an oscillating cone moving along the line $(2t - 1) + \mathrm{i}(3 - t)$. The boundary

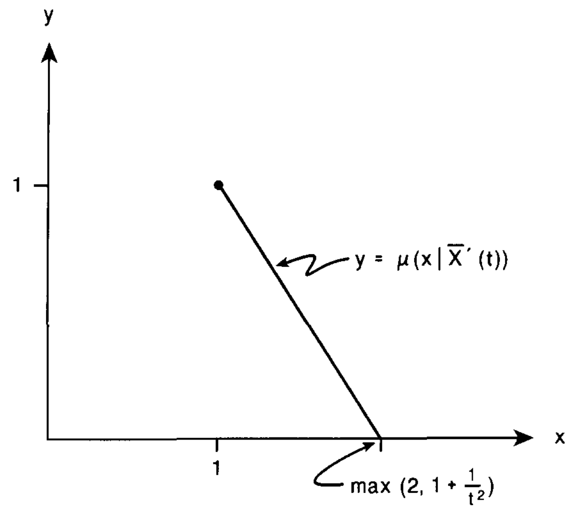

Fig. 3. $\bar{X}'(t)$ in Example 2.

of $F'(t)(\alpha)$, using Definition 3, for $0 \leqslant \alpha < 1$, is described by

$$z(t, \alpha, \beta) = [(\cos t + 2)(\cos \beta)(1 - \alpha)] + \mathrm{i}[(2 - \sin t)(\sin \beta)(1 - \alpha)] + (2 - \mathrm{i}), \tag{80}$$

which is also an oscillating, but stationary, cone.

Example 2. Let $F(t) = \bar{X}(t) + \mathrm{i}\bar{Y}(t)$ with $\bar{X}(t)$, $\bar{Y}(t)$ LR real fuzzy numbers and $L(x) = R(x) = 1 - x$. That is, $\bar{X}(t)$ and $\bar{Y}(t)$ are triangular real fuzzy numbers. Any triangular real fuzzy number $\bar{N}$ may be described by three numbers $(n_1/n_2/n_3)$ where $n_1 < n_2 < n_3$, (n_1, n_3) is the support of the triangle and the membership function is one at n_2. So let $\bar{X}(t) = (t - t^{-1}/t/2t)$ and let $\bar{Y}(t) = (t - t^2/t/t + t^2)$, for $t > 0$. Using Definition 4 and Theorem 5 we see that $F'(t) = \bar{X}'(t) + \mathrm{i}\bar{Y}'(t)$ where (see [5]), $\bar{Y}'(t) = (1 - 2t/1/1 + 2t)$ but $\bar{X}'(t)$ is not a fuzzy number. The graph of $\bar{X}'(t)$ is shown in Figure 3. So $\mu_2(z)$ will be discontinuous along the line

$$\{(1, y) \mid 1 - 2t < y < 1 + 2t\}. \tag{81}$$

3. Summary and conclusions

We presented two definitions of the derivative of a function mapping real numbers into the set of generalized complex fuzzy numbers ($F(t) = \bar{Z}(t) \in \bar{C}^*$, $a < t < b$). We showed that $F'(t)$, using the first definition, is always a subset of $F'(t)$, using the second definition. After placing some continuity assumptions on the functions which describe the boundary of α-cuts of $F(t)$, we showed that: (1) $F'(t)$, using either definition, is also a generalized complex fuzzy number ($F'(t) \in \bar{C}^*$ all t); and (2) $F'(t) = \bar{X}'(t) + \mathrm{i}\bar{Y}'(t)$ using the second definition, if

$F(t)=\bar{X}(t)+\mathrm{i}\bar{Y}(t)$ a rectangular complex fuzzy number and $\bar{X}(t)$, $\bar{Y}(t)$ are LR real fuzzy numbers. We also presented a definition of LR complex fuzzy numbers and showed that if $F(t)$ is LR, then $F'(t)$ is star-like (the α-cuts of $F'(t)$ may be described as rays, of varying length, coming out from some central point). Lastly, we showed that $(F(t)+G(t))'$ is always a subset of $F'(t)+G'(t)$, if either definition of the derivative is used.

We would like to generalize these results to finding $F'(\bar{Z})$ for $F:\bar{C}^*\to\bar{C}^*$. However, this depends on $F'(\bar{N})$ existing where F maps real fuzzy numbers into real fuzzy numbers. Since $F'(\bar{N})$ has not been defined [5] and studied, we are unable to define $F'(\bar{Z})$.

In this paper we always used the $z=x+\mathrm{i}y$ form of a complex number and not the polar form $z=re^{\mathrm{i}\theta}$. We did, however, consider the form $z=re^{\mathrm{i}\theta}$ but the results would be far more complicated than those using $z=x+\mathrm{i}y$. Consider equation (9) which gives $z=x+\mathrm{i}y$ description of the boundary of an α-cut of $F(t)$. Now translate this into polar form

$$z(t,\alpha,\beta)=r(t,\alpha,\beta)\exp(\mathrm{i}\theta(t,\alpha,\beta)). \tag{82}$$

In the definitions of derivative we must now take the t derivative of $z(t,\alpha,\beta)$ which will give a more complicated expression, if the polar form is employed, because of the product of two functions of t.

However, we may rewrite equation (82) as

$$z(t,\alpha,\beta)=z_1(t)+r(t,\alpha,\beta)\exp(\mathrm{i}\beta). \tag{83}$$

Then

$$\dot{x}(t,\alpha,\beta)+\mathrm{i}\dot{y}(t,\alpha,\beta)=\dot{z}_1(t)+\dot{r}(t,\alpha,\beta)\exp(\mathrm{i}\beta), \tag{84}$$

and we will obtain the same $\mu_1(z \mid F'(t))$ as in Definition 3 if we employ the polar form of $z(t,\alpha,\beta)$ given in equation (83). That is, we do not obtain a new derivative of $F(t)=\bar{Z}(t)\in\bar{C}^*$ which we could decompose into components as in equations (36) and (37).

Future research will be concerned with: (1) finding the differentiation formulas for the elementary functions such as $F(t)=\bar{Z}_1t+\bar{Z}_0$, $F(t)=\exp(t\bar{Z})$, $F(t)=\ln(\bar{Z}_1t+\bar{Z}_0)$, etc.; and (2) defining an integral for $F:(a,b)\to\bar{C}^*$.

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