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Fuzzy complex analysis II: Integration

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Abstract: This paper continues the author's research in fuzzy complex analysis where we now define, and study the basic properties of, a fuzzy contour integral for fuzzy mappings which map a rectifiable curve in the complex plane into fuzzy complex numbers.

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1. Introduction

This paper continues the author's research in fuzzy complex analysis. In [3] we defined, and studied the basic properties of, (regular) fuzzy complex numbers which were then used in [6] to solve fuzzy quadratic equations. In defining our new solution concept for solving fuzzy equations we found that we needed a more general fuzzy complex number which was defined in [5] as a generalized fuzzy complex number and subsequently used in [4] where we defined the derivative of functions mapping a real interval into generalized fuzzy complex numbers.

Let $\mathscr{C}$ be the complex numbers, $\bar{\mathscr{C}}$ the generalized fuzzy complex numbers and let C be a rectifiable curve in $\mathscr{C}$. In this paper we are interested in defining, and deriving the basic properties of, an integral for a function $\bar{f}$ mapping C into $\bar{\mathscr{C}}$. This integral will be called a fuzzy line, or contour, integral. The definition, and basic properties through the Cauchy Integral Theorem, are presented in the next section. The fuzzy line integral is the integral of a fuzzy mapping and not the integral of a crisp mapping over a fuzzy set [21], or over a fuzzy curve or interval [8]. To define the fuzzy line integral of $\bar{f}$ we chose the definition of the integral of a fuzzy mapping from a closed interval $[a, b]$ into the set of (non-empty) fuzzy subsets of the reals in [9] to extend to the complex case. An alternate definition of a fuzzy contour integral, based on an integral of set-valued functions [2, 7] is also presented and we show that the two integrals are equal. This second procedure has been used in the literature [9, 11, 12, 14] to define an integral for functions mapping an interval $[a, b]$, or a probability space, into fuzzy subsets of $\mathbf{R}^n$.

The definitions of other fuzzy integrals [15, 17, 19] we believe will not easily generalize to the complex case. In [15] the author integrates an $\bar{f}$ mapping $[a, b]$ into real fuzzy numbers by integrating the end points of the α-cuts of the fuzzy numbers. In the complex case the boundary of the α-cuts of the fuzzy complex numbers will usually be simple closed curves in the complex plane. In [19] the author defines a Riemann–Stieltjes type integral for $\bar{f}$ mapping a subset the reals into the set of real fuzzy numbers using upper and lower Riemann sums. This requires an ordering for real fuzzy numbers which we do not have for fuzzy complex numbers. Also, our fuzzy integral is completely different from a Sugeno [17] type fuzzy integral which is an integral of a crisp mapping with respect to a fuzzy measure.

Let us now introduce some of the notation needed in the rest of this paper. We will always place a bar over a symbol if it represents a fuzzy set, or a collection of fuzzy sets. We write $z = x + iy$ (or w) for a regular complex number and let $\bar{W}$ be a fuzzy subset of $\mathscr{C}$ with membership function $y = \mu(z \mid \bar{W})$ with y in $[0, 1]$ and z in $\mathscr{C}$. The (weak) α-cut of $\bar{W}$ is

$$\bar{W}(\alpha) = \{z \mid \mu(z \mid \bar{W}) \geq \alpha\}, \tag{1}$$

for $0 < \alpha \leq 1$. We separately define $\bar{W}(0)$, the 0-cut of $\bar{W}$, as the closure of the union of the $\bar{W}(\alpha)$ for $0 < \alpha \leq 1$. $\bar{W}$ is a generalized fuzzy complex number if and only if: (1) $\mu(z \mid \bar{W})$ is upper semi-continuous; (2) $\bar{W}(\alpha)$ is compact, simply connected and arcwise connected for $0 \leq \alpha \leq 1$; and (3) $\bar{W}(1) \neq \emptyset$. However, in this paper we will make the stronger assumption that $\bar{W}(\alpha)$ is convex for $0 \leq \alpha \leq 1$. Then $\bar{\mathscr{C}}$ will be all generalized fuzzy complex numbers whose α-cuts are all convex.

Let C be a rectifiable curve in $\mathscr{C}$. Therefore, C may be described by a function $z = \phi(t) + \mathrm{i}\psi(t)$, $a \leqslant t \leqslant b$, where $\phi(t)$ and $\psi(t)$ are real-valued on $[a, b]$. We will also assume that $\phi(t)$ and $\psi(t)$ are continuously differentiable on $[a, b]$ so that C will be a smooth curve in $\mathscr{C}$. If $g: C \to \mathscr{C}$ we write

$$g(z) = u(x, y) + \mathrm{i}v(x, y)$$

where u and v are real-valued functions of $z = x + \mathrm{i}y$. Then the complex line integral of g on C is

$$\int_C g \, \mathrm{d}z = (\text{Real}) + \mathrm{i}(\text{Imaginary}), \tag{2}$$

where [1, p. 231; 10, p. 161]

$$\text{Real} = \int_a^b u(\phi, \psi)\phi' \, \mathrm{d}t - \int_a^b v(\phi, \psi)\psi' \, \mathrm{d}t, \tag{3}$$

$$\text{Imaginary} = \int_a^b v(\phi, \psi)\phi' \, \mathrm{d}t + \int_a^b u(\phi, \psi)\psi' \, \mathrm{d}t, \tag{4}$$

where ϕ' (ψ') denotes the derivative of ϕ (ψ), assuming all the Riemann integrals exist over $[a, b]$. A sufficient condition for all the Riemann integrals to exist is for g to be continuous on C. Let

$$\mathscr{I} = \{g: C \to \mathscr{C} \mid g \text{ is continuous on } C\}.$$

A fuzzy complex mapping on C will be $\bar{f}: C \to \bar{\mathscr{C}}$. Let $\bar{f}(z) = \bar{W}(z)$ for z in C. We assume that $\bar{W}(z) \neq \phi$ for each z in C. We write $y = \mu(w \mid \bar{W}(z))$ for the membership function of $\bar{W}(z)$ for w in $\mathscr{C}$ and $\bar{W}(z)(\alpha)$ for the α-cut of $\bar{W}(z)$ for $z \in C$, $0 \leqslant \alpha \leqslant 1$.

2. Fuzzy contour integral

Following [9] we first define a fuzzy set of mappings, a fuzzy subset of $\mathscr{I}$, called $\bar{\mathscr{F}}$, by its membership function

$$\mu(g \mid \bar{\mathscr{F}}) = \inf\{\mu(g(z) \mid \bar{W}(z)) \mid z \in C\}, \tag{5}$$

where $g \in \mathscr{I}$. Define $\bar{F}_1$, a fuzzy subset of $\mathscr{C}$, by its membership function

$$\mu(w \mid \bar{F}_1) = \sup\left\{\mu(g \mid \bar{\mathscr{F}}) \;\middle|\; \int_C g \, \mathrm{d}z = w\right\}, \tag{6}$$

for each w in $\mathscr{C}$. We adopt the convention that the supremum of the empty set is zero. Then $\bar{F}_1$ is our first definition of a fuzzy contour integral and we set

$$\int_C \bar{f} \, \mathrm{d}z = \bar{F}_1. \tag{7}$$

As pointed out in [9] the definition of $\bar{F}_1$ is just the classical extension principle applied to the mapping

$$\int_C (\cdot) \, \mathrm{d}z : \mathscr{I} \to \mathscr{C} \tag{8}$$

and the fuzzy subset $\bar{\mathscr{F}}$ of $\mathscr{I}$.

Our second definition of a fuzzy line integral follows our previous method of defining solutions to fuzzy equations [5] and the procedures in [9, 11, 12, 14]. Define

$$\Omega(\alpha) = \left\{\int_C g \, \mathrm{d}z \mid g \in \mathscr{I}, g(z) \in \bar{W}(z)(\alpha) \text{ on } C\right\}, \tag{9}$$

for $0 \leqslant \alpha \leqslant 1$. Define $\bar{F}_2$, a fuzzy subset of $\mathscr{C}$, by its membership function

$$\mu(w \mid \bar{F}_2) = \sup\{\alpha \mid w \in \Omega(\alpha)\}, \tag{10}$$

for w in $\mathscr{C}$. $\bar{F}_2$ is our second definition of a fuzzy contour integral and we have

$$\int_C \bar{f} \, \mathrm{d}z = \bar{F}_2. \tag{11}$$

Theorem 1. $\bar{F}_1 = \bar{F}_2$.

Proof. (a) We first argue that $\mu(w \mid \bar{F}_2) \leqslant \mu(w \mid \bar{F}_1)$ all w in $\mathscr{C}$. If $\bar{F}_2 = \emptyset$ the result is trivial so let $\mu(w \mid \bar{F}_2) = \lambda > 0$ for some w in $\mathscr{C}$. Given $\epsilon > 0$ there is an α, $0 < \alpha \leqslant 1$, so that $\lambda - \epsilon < \alpha \leqslant \lambda$ and $w \in \Omega(\alpha)$. Then there is a $g \in \mathscr{I}$ so that $g(z) \in \bar{W}(z)(\alpha)$ on C and $\int_C g \, \mathrm{d}z = w$. Hence $\mu(g \mid \bar{\mathscr{F}}) \geqslant \alpha$ so that

$$\mu(w \mid \bar{F}_1) \geqslant \alpha > \lambda - \epsilon.$$

So $\mu(w \mid \bar{F}_1) > \lambda - \epsilon$ and it follows that $\mu(w \mid \bar{F}_1) \geqslant \lambda$ since $\epsilon > 0$ was arbitrary.

(b) We show that $\mu(w \mid \bar{F}_1) \leqslant \mu(w \mid \bar{F}_2)$ for any $w \in \mathscr{C}$. If $\bar{F}_1 = \emptyset$ the result follows so let $\mu(w \mid \bar{F}_1) = \lambda > 0$ for some w in $\mathscr{C}$. Given $\epsilon > 0$ there is a $g \in \mathscr{I}$, $\int_C g \, \mathrm{d}z = w$, so that

$$\lambda - \epsilon < \mu(g \mid \bar{\mathscr{F}}) \leqslant \lambda.$$

Then $\lambda-\epsilon<\mu(g(z)\mid \bar{W}(z))$ on C so that $g(z)\in\bar{W}(z)(\lambda-\epsilon)$ on C. Hence $w\in\Omega(\lambda-\epsilon)$ and $\mu(w\mid\bar{F}_2)\geqslant\lambda-\epsilon$. It follows that $\mu(w\mid\bar{F}_2)\geqslant\lambda$ since $\epsilon>0$ was arbitrary.

Since $\bar{F}_1=\bar{F}_2$ let $\bar{F}$ be their common value and define the fuzzy contour integral to be

$$\int_C \bar{f}\,\mathrm{d}z=\bar{F}. \tag{12}$$

Of course, $\bar{F}$ may be empty.

Suppose that $\bar{f}(z)=A(z)$, a crisp subset of $\mathscr{C}$, for each z in $\mathscr{C}$. That is, $\mu(w\mid\bar{W}(z))=1$ for $w\in A(z)$ and is zero otherwise. Then it easily follows that $\bar{F}$ is a crisp subset of $\mathscr{C}$ given by

$$\left\{\int_C g\,\mathrm{d}z \mid g\in\mathscr{I},\, g(z)\in A(z) \text{ on } C\right\}, \tag{13}$$

which is an integral (the g must be continuous) for set-valued mappings [2, 7]. Further suppose that $A(z)=$ a single complex number $x(z)+\mathrm{i}y(z)$ on C, where $x(z)$ and $y(z)$ are real-valued functions of z. Then $\bar{F}$ reduces to the crisp complex number

$$\int_C [x(z)+\mathrm{i}y(z)]\,\mathrm{d}z, \tag{14}$$

when $x(z)+\mathrm{i}y(z)\in\mathscr{I}$, and $\bar{F}=\emptyset$ otherwise. That is, $\bar{F}$ reduces to an ordinary contour integral when $\bar{f}(z)=x(z)+\mathrm{i}y(z)\in\mathscr{I}$.

We next argue that $\bar{F}$ is contained in the fuzzy integrals presented in [9, 11, 12, 14] when we restrict everything to the real numbers. Let $C=[a,b]$ a closed interval of real numbers and suppose $\bar{f}(z)$ is a non-empty fuzzy subset of the reals for each z in $[a,b]$. Then

$$\mathscr{I}=\{g:[a,b]\to\mathbf{R}\mid g \text{ is continuous on } [a,b]\}.$$

We see that $\int_C\bar{f}\,\mathrm{d}z$ is contained in the fuzzy integral defined in [9, 11, 12, 14] because we require the g's to be continuous and the author's in [9, 11, 12, 14] only require the g's to be integrable on $[a,b]$. One usually assumes that g is continuous in the definition of the complex line integral [10, 13, 16].

Let $\bar{F}(\alpha)$ denote an α-cut of $\bar{F}$. We say that $\bar{f}:C\to\mathscr{C}$ is bounded if there is a bounded real-valued function h defined on C so that for each z in C we have $|w|\leqslant h(z)$ for all w in $\bar{W}(z)(0)$. Also, let $\Gamma(\alpha)$ be the set of all g in $\mathscr{I}$ so that $g(z)\in\bar{W}(z)(\alpha)$ for each z in C, $0\leqslant\alpha\leqslant1$.

Theorem 2. $\langle1\rangle$ *$\Omega(\alpha)\subset\bar{F}(\alpha)$ and*

$$\bar{F}(\alpha)=\bigcap\{\Omega(\beta)\mid 0<\beta<\alpha\},$$

when $\alpha>0$.

$\langle2\rangle$ *$\bar{F}(\alpha)$ is convex for $0\leqslant\alpha\leqslant1$. If $\bar{f}$ is bounded then $\bar{F}(\alpha)$ is bounded for $0\leqslant\alpha\leqslant1$ and $\bar{F}(0)$ is compact.*

Proof. $\langle1\rangle$(a) If $w\in\Omega(\alpha)$, with $\alpha>0$, then $\mu(w\mid\bar{F})\geqslant\alpha$ so that $w\in\bar{F}(\alpha)$.

(b) Let $w\in\bar{F}(\alpha)$ for $\alpha>0$. If $\mu(w\mid\bar{F})=\gamma$, then $\gamma\geqslant\alpha$. By the definition of supremum we see that $w\in\Omega(\beta)$ for $0<\beta<\gamma$. Hence w belongs to the intersection of the $\Omega(\beta)$ for $0<\beta<\alpha$.

Now choose w in the intersection of the $\Omega(\beta)$ for $0<\beta<\alpha$ with $\alpha>0$. Since w belongs to all the $\Omega(\beta)$ for $0<\beta<\alpha$ we have $\sup\{\gamma\mid w\in\Omega(\gamma)\}\geqslant\alpha$ and $\mu(w\mid\bar{F})\geqslant\alpha$. Therefore, w is in $\bar{F}(\alpha)$.

$\langle2\rangle$(a) For each z in C we have $\bar{W}(z)(\alpha)$ convex which implies that $\Gamma(\alpha)$ is also convex, $0\leqslant\alpha\leqslant1$. Hence $\Omega(\alpha)$ is convex, $0\leqslant\alpha\leqslant1$, because the complex line integral is a linear function on $\Gamma(\alpha)$ [10, p. 160]. So, for $\alpha>0$ we obtain $\bar{F}(\alpha)$ convex since it is, by $\langle1\rangle$ above, the intersection of convex sets.

$\bar{F}(0)$ is a special case. Let U be the union of the $\bar{F}(\alpha)$ for $0<\alpha\leqslant1$. U is convex since it is the union of a nested ($\alpha_1<\alpha_2$ implies $\bar{F}(\alpha_2)\subset\bar{F}(\alpha_1)$) collection of convex sets. The closure of U is also convex. But the closure of U is $\bar{F}(0)$.

(b) Assume $\bar{f}$ is bounded. We argue that $\Omega(0)$ is bounded which implies that $\bar{F}(\alpha)$, $\alpha>0$, is bounded since $\bar{F}(\alpha)$ is a subset of $\Omega(0)$, $\alpha>0$. In the definition of $\bar{f}$ bounded let $|h(z)|\leqslant M$ for z in C since h is a bounded real-valued function on C. Now $\Gamma(0)$ is bounded because if g is in $\Gamma(0)$, then $\max|g(z)|$, for z in C, is bounded by M. Then $\Omega(0)$ is bounded since $|\int_C g\,\mathrm{d}z|\leqslant M$ (length of C) for any g in $\Gamma(0)$.

U, the union of the $\bar{F}(\alpha)$ for $0<\alpha\leqslant1$, is a subset of $\Omega(0)$. So, the closure of U, which is $\bar{F}(0)$, is contained in the closure of $\Omega(0)$ which is compact. Hence, $\bar{F}(0)$ is compact.

However, $\bar{F}(\alpha)$ need not be compact for $\alpha>0$, $\mu(w\mid\bar{F})$ may not be upper semi-continuous, and $\bar{F}(\alpha)$ need not equal $\Omega(\alpha)$ for $\alpha > 0$. That is, $\bar{F}$ may not belong to $\mathscr{C}$. The following simple example illustrates these facts.

Example 1. Let $[a, b] = [0, 1]$ and

$$z = \phi(t) + i\psi(t) = t + i0, \quad 0 \leq t \leq 1,$$

so that the rectifiable curve C is the interval $[0, 1]$ on the x-axis in the complex plane. The fuzzy complex mapping $\bar{f}: C \to \mathscr{C}$ is

$$\bar{f}(t + i0) = \bar{W}(t + i0) = \bar{W}(t)$$

where $\bar{W}(t)$ will be defined by its α-cuts, $0 \leq t \leq 1$. The α-cuts of $\bar{W}(t)$ are:

(a) $0 \leq t < 0.5$:

$$\bar{W}(t)(\alpha) = \{t + iy \mid 2\alpha \leq y \leq 2 - 2\alpha\}, \quad 0 \leq \alpha \leq 0.25, \tag{15}$$

$$\bar{W}(t)(\alpha) = \{t + iy \mid 0.5 \leq y \leq 1.5\}, \quad 0.25 \leq \alpha \leq 0.50, \tag{16}$$

$$\bar{W}(t)(\alpha) = \{t + iy \mid \alpha \leq y \leq 2 - \alpha\}, \quad 0.50 \leq \alpha \leq 1, \tag{17}$$

(b) $0.5 \leq t \leq 1$:

$$\bar{W}(t)(\alpha) = \{t + iy \mid 2\alpha \leq y \leq 3 - 4\alpha\}, \quad 0 \leq \alpha \leq 0.25, \tag{18}$$

$$\bar{W}(t)(\alpha) = \{t + iy \mid 0.5 \leq y \leq 2.0\}, \quad 0.25 \leq \alpha \leq 0.50, \tag{19}$$

$$\bar{W}(t)(\alpha) = \{t + iy \mid \alpha \leq y \leq 3 - 2\alpha\}, \quad 0.50 \leq \alpha \leq 1. \tag{20}$$

One may check that $\bar{W}(t) \in \mathscr{C}$ for $0 \leq t \leq 1$. Actually, $\bar{W}(t)$ is a real fuzzy number placed in the complex plane, in the first quadrant, perpendicular to the real axis.

If $g \in \Gamma(\alpha)$, then $g(t + i0) = t + iv(t, 0)$ where $v(t, 0)$ must belong to the correct interval for y, given above, in the definition of the α-cuts of $\bar{W}(t)$. For example, if $0.5 \leq t \leq 1$ and $0 \leq \alpha \leq 0.25$, then $2\alpha \leq v(t, 0) \leq 3 - 4\alpha$ for t in $[0.5, 1]$. It follows that $\int_C g \, dz = 0.5 + i \int_0^1 v(t, 0) \, dt$ with $v(t, 0)$ continuous on $[0, 1]$. We may now determine $\Omega(\alpha)$ and $\bar{F}(\alpha)$, for $\alpha > 0$, from part $\langle 1 \rangle$ of Theorem 2. For example, if $0 < \alpha \leq 0.25$ we see that

$$\Omega(\alpha) = \{0.5 + iy \mid 2\alpha \leq y < 2.5 - 3\alpha\}$$

but $\bar{F}(\alpha)$ equals the closure of $\Omega(\alpha)$. Also, for $0.25 < \alpha \leq 0.50$ we obtain

$$\bar{F}(\alpha) = \Omega(\alpha) = \{0.5 + iy \mid 0.5 \leq y < 1.75\}.$$

But again, for $0.5 \leq \alpha < 1$, $\bar{F}(\alpha)$ is not equal to $\Omega(\alpha)$. Finally,

$$\bar{F}(1) = \Omega(1) = \{0.5 + i\}.$$

This example shows that $\bar{F}(\alpha)$ need not be compact because it is not closed in the interval $0.25 < \alpha \leq 0.5$. Hence, $\bar{F}$ is not a generalized fuzzy complex number.

These results are not completely satisfactory since we would like to have a theory, similar to the real case [9, 11, 12, 14], where, under very mild conditions on $\bar{f}$, we have $\bar{F}(\alpha) = \Omega(\alpha)$, $0 < \alpha \leq 1$, and $\bar{F} \in \mathscr{C}$. We see two things to do: (1) be more restrictive and use a smaller $\mathscr{I}$ to specify $\Gamma(\alpha)$ and $\Omega(\alpha)$; or (2) be less restrictive and use a larger $\mathscr{I}$ to define $\Gamma(\alpha)$ and $\Omega(\alpha)$. Both methods, as we shall see, will produce an integral $\int_C \bar{f} \, dz$ with the above mentioned properties[1].

2.1. More restrictive

In this subsection $\mathscr{I}_0$ will be a compact subset of $\mathscr{I}$. We assume that $\mathscr{I}$ has the supremum norm ($\|\cdot\|$) so that $\mathscr{I}$ is a complex Banach space [18, p. 148]. As before, define

$$\Gamma_0(\alpha) = \{g \in \mathscr{I}_0 \mid g(z) \in \bar{W}(z)(\alpha) \text{ on } C\}$$

and

$$\Omega_0(\alpha) = \{\textstyle\int_C g \, dz \mid g \in \Gamma_0(\alpha)\},$$

$0 \leq \alpha \leq 1$. $\bar{F}_0$, a fuzzy subset of the complex plane, is given by

$$\mu(w \mid \bar{F}_0) = \sup\{\alpha \mid w \in \Omega_0(\alpha)\},$$

and we set $\int_C \bar{f} \, dz = \bar{F}_0$. Of course, $\bar{F}_0$ will depend on which compact subset $\mathscr{I}_0$ we have chosen.

Since $\mathscr{I}_0$ need not be convex we need to drop the convexity condition on the α-cuts of $\bar{W}$ in $\mathscr{C}$. Let $\mathscr{E}$ be all fuzzy subsets $\bar{W}$ of the complex plane such that: (1) $\mu(w \mid \bar{W})$ is upper semi-continuous; (2) $\bar{W}(\alpha)$ is compact, $0 \leq \alpha \leq$

[1] Except that the fuzzy line integral will belong to $\bar{\mathscr{E}}$, where $\mathscr{E}$ is a subset of $\bar{\mathscr{E}}$, in Subsection 2.1. See that subsection for the details.

1; and (3) $\bar{W}(1) \neq \emptyset$. These are the type of fuzzy sets considered in [14] where as $\bar{W}$ in $\mathscr{C}$ are used in [11, 12].

Theorem 3. *$\bar{F}_0(\alpha) = \Omega_0(\alpha)$, $0 < \alpha \leq 1$. If $\Omega_0(1) \neq \emptyset$, then $\bar{F}_0 \in \bar{\mathscr{E}}$.*

Proof. 1. We first establish the following facts: (a) $\Gamma_0(\alpha)$ is a compact subset of $\mathscr{I}$; and (b) the mapping $Y: \Gamma_0(\alpha) \to \mathscr{C}$, given by $Y(g) = \int_C g\, dz$, is continuous.

(a) We argue that $\Gamma_0(\alpha)$ is a closed subset of compact $\mathscr{I}_0$ which proves that $\Gamma_0(\alpha)$ is compact. Let $\|g_n - g\| \to 0$ with $g_n \in \Gamma_0(\alpha)$. Then $g_n(z) \to g(z)$ for z in C. But $g_n(z)$ is in compact $\bar{W}(z)(\alpha)$ for all n, for each z in C. Hence $g(z) \in \bar{W}(z)(\alpha)$ for z in C and $g \in \Gamma_0(\alpha)$.

(b) If $\|g_n - g\| \to 0$ with $g_n, g \in \Gamma_0(\alpha)$, then [1, p. 399] $\int_C g_n\, dz \to \int_C g\, dz$ because g_n converges to g uniformly on C.

2. We now show that $\bar{F}_0(\alpha) = \Omega_0(\alpha)$, $1 < \alpha \leq 1$. We easily see, as in the proof of Theorem 2, that $\Omega_0(\alpha) \subset \bar{F}_0(\alpha)$, $0 < \alpha \leq 1$. So let $w \in \bar{F}_0(\alpha)$, $0 < \alpha \leq 1$, and assume $\mu(w \mid \bar{F}_0) = \beta \geq \alpha$. We argue that $w \in \Omega_0(\alpha)$ so that $\bar{F}_0(\alpha) \subset \Omega_0(\alpha)$. There are two cases to consider: (a) $\beta > \alpha$; and (b) $\beta = \alpha$.

(a) There is a γ, $\alpha < \gamma < \beta$, so that w belongs to $\Omega_0(\gamma)$. It follows that $w \in \Gamma_0(\alpha)$ since $\Omega_0(\gamma) \subset \Omega_0(\alpha)$.

(b) Now $\beta = \alpha$. Choose $N \geq 1$ so that $\alpha - (1/n) > 0$ for $n \geq N$. Then $w \in \Omega_0(\alpha - (1/n))$ for $n \geq N$. It follows that there is a g_n in $\Gamma_0(\alpha - (1/n))$ so that $w = \int_C g_n\, dz$, $n \geq N$. Now the sequence g_n belongs to compact $\Gamma_0(\alpha - (1/N))$ so there is a subsequence g_{n_k} in $\Gamma_0(\alpha - (1/N))$ and a g in $\Gamma_0(\alpha - (1/N))$ so that $\|g_{n_k} - g\| \to 0$. Then, by upper semi-continuity,

$$\limsup \mu(g_{n_k}(z) \mid \bar{W}(z)) \leq \mu(g(z) \mid \bar{W}(z)), \quad (21)$$

for each z in C. But

$$\mu(g_{n_k}(z) \mid \bar{W}(z)) \geq \alpha - (1/n_k) \to \alpha, \quad (22)$$

for z in C. Hence

$$\mu(g(z) \mid \bar{W}(z)) \geq \alpha, \quad (23)$$

for $z \in C$ and g belongs to $\Gamma_0(\alpha)$. Also, from point 1(b) above

$$w = \int_C g_{n_k}\, dz \to \int_C g\, dz, \quad (24)$$

so $w = \int_C g\, dz$. That is, $w \in \Omega(\alpha)$.

3. We may now show that $\bar{F}(\alpha)$ is compact for $0 \leq \alpha \leq 1$. We first notice that $\Omega_0(\alpha)$ is compact for $0 \leq \alpha \leq 1$ since it is the continuous image of compact $\Gamma_0(\alpha)$ under $Y(\Gamma_0(\alpha)) = \Omega_0(\alpha)$. Since $\bar{F}(\alpha) = \Omega_0(\alpha)$, $0 < \alpha \leq 1$, from point 2 above, we have $\bar{F}_0(\alpha)$ compact, $0 < \alpha \leq 1$. As in the proof of Theorem 2 we see that $\bar{F}_0(0)$ is a closed subset of now compact $\Omega_0(0)$ and hence $\bar{F}_0(0)$ is also compact.

4. It is well-known that $\mu(w \mid \bar{F}_0)$ is upper semi-continuous if and only if $\{w \mid \mu(w \mid \bar{F}_0) \geq t\}$ is closed for all real numbers t. Since $\bar{F}_0(\alpha)$ is closed for $0 \leq \alpha \leq 1$ we know that $\mu(w \mid \bar{F}_0)$ is upper semi-continuous.

5. Finally, if $\Omega_0(1) \neq \emptyset$, then $\bar{F}_0(1) \neq \emptyset$ and $\bar{F}_0 \in \bar{\mathscr{E}}$.

So, if we restrict the g functions, which we are to integrate over C, to be is some compact subset of $\mathscr{I}$, we obtain the satisfactory result of the integral belonging to $\bar{\mathscr{E}}$ and α-cuts of the integral are equal to $\Omega_0(\alpha)$, $0 < \alpha \leq 1$. However, we have no preference as to which compact subset to use so we will next consider the union over all such compact subsets.

Theorem 4. (a) *For $0 \leq \alpha \leq 1$,*

$$\bigcup \{\Omega_0(\alpha) \mid \mathscr{I}_0 \text{ compact subset of } \mathscr{I}\} = \Omega(\alpha).$$

(b) *For $0 < \alpha \leq 1$.*

$$\bigcup \{\bar{F}_0(\alpha) \mid \mathscr{I}_0 \text{ compact subset of } \mathscr{I}\} \subset \bar{F}(\alpha).$$

Proof. (a) Since $\Omega_0(\alpha)$ is a subset of $\Omega(\alpha)$, we see that the union over all compact subsets is a subset of $\Omega(\alpha)$. To show the opposite inclusion let $w \in \Omega(\alpha)$ with $w = \int_C g\, dz$ and $g \in \Gamma(\alpha)$. Then $w \in \Omega_0(\alpha)$ with $\mathscr{I}_0 = \{g\}$.

(b) This follows from 1 above since $\bar{F}_0(\alpha) = \Omega_0(\alpha)$, $0 < \alpha \leq 1$, by Theorem 3 and $\Omega(\alpha) \subset \bar{F}(\alpha)$, $0 < \alpha \leq 1$, from Theorem 2.

We will proceed to develop further properties of $\int_C \bar{f}\, dz$ based on $\mathscr{I}$ and $\mathscr{I}_0$ but first let us consider a less restrictive approach to integration.

2.2. *Less restrictive*

In this subsection $\Gamma_0(\alpha)$ will be all $g: C \to \mathscr{C}$ so that: (1) $g(z) \in \bar{W}(z)(\alpha)$ on C; and (2)

$u(\phi(t), \psi(t))\phi'(t)$, $u(\phi(t), \psi(t))\psi'(t)$, $v(\phi(t), \psi(t))\phi'(t)$, and $v(\phi(t), \psi(t))\psi'(t)$ are all Lebseque integrable on $[a, b]$. If

$$\Omega_0(\alpha) = \left\{ \int_C g \, dz \mid g \in \Gamma_0(\alpha) \right\},$$

then define $\bar{F}_0$, a fuzzy subset of the complex plane, by its membership function

$$\mu(w \mid \bar{F}_0) = \sup\{\alpha \mid w \in \Omega_0(\alpha)\}$$

and set $\int_C \bar{f} \, dz = \bar{F}_0$. We will say that $\bar{f}$ is integrably bounded if and only if there is an integrable $h:[a, b] \to \mathbf{R}$ so that $|w| \leq h(t)$ for all w in $\bar{W}(z(t))(0)$, where $z(t) = \phi(t) + i\psi(t)$, for each t in $[a, b]$. Let

$$l(t) = \phi'(t) + i\psi'(t)$$

and

$$k(t) = (\phi'(t))^2 + (\psi'(t))^2,$$

$\alpha \leq t \leq b$. We will say that $k(t)$ is bounded away from zero on $[a, b]$ if there is an $\epsilon > 0$ so that $k(t) \geq \epsilon$ on $[a, b]$. If $\bar{f}: C \to \mathscr{C}$, then we define $\bar{r}$ on $[a, b]$ as $\bar{r}(t) = l(t)\bar{f}(z(t))$, or $\bar{r}(t) = l(t)\bar{W}(z(t))$, for t in $[a, b]$. It is not too difficult to see that $\bar{r}:[a, b] \to \mathscr{C}$.

Define

$$J(\alpha) = \{f:[a, b] \to \mathbf{R}^2 \mid f(t) \in l(t)W(z(t))(\alpha) \text{ on } [a, b] \text{ and } f \text{ is integrable}\}.$$

Next set

$$\Phi(\alpha) = \left\{ \int_a^b f(t) \, dt \mid f \in J(\alpha) \right\}, \quad 0 \leq \alpha \leq 1.$$

Define $\bar{R}$, a fuzzy subset of $\mathbf{R}^2$, by its membership function

$$\mu(w \mid \bar{R}) = \sup\{\alpha \mid w \in \Phi(\alpha)\}$$

and set $\int_a^b \bar{r} \, dt = \bar{R}$. We see that $\Phi(\alpha)$ is the Aumann integral [2] of the set valued mapping $t \to l(t)\bar{W}(z(t))(\alpha)$ for t in $[a, b]$ and $\bar{R}$ is the fuzzy integral of $\bar{r}$ [9, 11, 12, 14].

Theorem 5. *If $k(t)$ is bounded away from zero on $[a, b]$, then $\Omega_0(\alpha) = \Phi(\alpha)$, $0 \leq \alpha \leq 1$, so $\bar{F}_0 = \bar{R}$.*

Proof. 1. We first argue that $\Omega_0(\alpha) \subset \Phi(\alpha)$. Let $g \in \Gamma_0(\alpha)$ and define

$$f_1(t) = u(\phi(t), \psi(t))\phi'(t) - v(\phi(t), \psi(t))\psi'(t)$$

and

$$f_2(t) = v(\phi(t), \psi(t))\phi'(t) + u(\phi(t), \psi(t))\psi'(t),$$

for t in $[a, b]$. Then $f = (f_1, f_2) \in J(\alpha)$. We will identify a point (x, y) in the complex plane with the complex number $x + iy$. So we see that

$$\int_C g \, dz = \left(\int_a^b f_1(t) \, dt, \int_a^b f_2(t) \, dt \right) = \int_a^b f(t) \, dt$$

and it follows that $\Omega_0(\alpha)$ is a subset of $\Phi(\alpha)$.

2. We show that $\Phi(\alpha)$ is a subset of $\Omega_0(\alpha)$. Let $f = (f_1, f_2)$ be in $J(\alpha)$. Define $u(x, y)$ at $x = \phi(t)$, $y = \psi(t)$ for t in $[a, b]$, by

$$u(x, y) = (f_1(t)\phi'(t) + f_2(t)\psi'(t))/k(t)$$

and define $v(x, y)$ at $x = \phi(t)$, $y = \psi(t)$ for t in $[a, b]$, by

$$v(x, y) = (f_2(t)\phi'(t) - f_1(t)\psi'(t))/k(t).$$

Then $g \in \Gamma_0(\alpha)$ and $\int_C g \, dz = \int_a^b f(t) \, dt$ so that $\Phi(\alpha)$ is a subset of $\Omega_0(\alpha)$.

Now we may adopt the results on $\int_a^b \bar{r} \, dt$, from the fuzzy real variable case in [11], [12], [14] to $\int_C \bar{f} \, dz$ since the integrals are equal when $k(t)$ is bounded away from zero. For example, if $\bar{f}$ is integrably bounded (and therefore so is $\bar{r}$) and if $\bar{r}$ is measurable, then $\bar{R} = \bar{F}_0$ is in $\mathscr{C}$ and α-cuts of $\bar{F}_0$ are equal to $\Omega_0(\alpha)$, $0 < \alpha \leq 1$ (see [11, 14]). We shall not proceed in this direction any further in this paper because it only involves a direct translation of the known results on $\int_a^b \bar{r} \, dt$ to $\int_C \bar{f} \, dz$.

The theory of $\int_C f \, dz$ based on the definition of $\Gamma_0(\alpha)$ in this subsection appears very satisfactory until one realizes that it is rare (if ever) that anyone bases the complex line integral on the Lebesgue integral. The most general integrals used to define a complex line integral are the Riemann–Stieltjes integrals. Before we proceed on to developing some elementary properties of $\int_C \bar{f} \, dz$ based on $\mathscr{I}$ and compact subsets of $\mathscr{I}$, let us return to Example 1 and show, using the $\Gamma_0(\alpha)$ of this subsection, that $\bar{F}_0$ belongs to $\mathscr{C}$.

Example 2. This is a continuation of Example 1 using the same C and $\bar{f}$. The graph of α-cuts of $\bar{f}$ is

$$E_\alpha = \{(t, w) \mid a \leq t \leq b, w \in \bar{W}(z(t))(\alpha)\}, \tag{25}$$

for $0 \le \alpha \le 1$. Each E_α, $0 \le \alpha < 1$, is the union of two rectangles and E_1 is a straight line segment. So, E_α belongs to $\mathscr{A} \times \mathscr{B}$ where $\mathscr{A}$ is the Lebesgue measurable subsets of $[a, b]$ and $\mathscr{B}$ is the Borel subsets of $\mathbf{R}^2$. This means that $\bar{f}$ is measurable [11, 14]. Clearly, $\bar{r}$ is also measururable because $\bar{r} = \bar{f}$ since $l(t) = 1$ on $[a, b]$, Also, $\bar{f}$ is integrably bounded. Hence, using the result cited above $\int_C \bar{f}\, dz$ is a generalized fuzzy complex number and its α-cuts equal $\Omega_0(\alpha)$, $0 < \alpha \le 1$.

2.3. Properties based on $\mathscr{I}$ and compact $\mathscr{I}_0$

We will return to the development in Section 2.1 where $\mathscr{I}_0$ was some compact subset of $\mathscr{I}$. $\Gamma_0(\alpha)$, $\Omega_0(\alpha)$, $\bar{F}_0$ and $\bar{\mathscr{C}}$ are as defined in there. $\mathscr{I}$ stands for all continuous $g: C \to \mathscr{C}$ with $\Gamma(\alpha)$, $\Omega(\alpha)$ and $\bar{F}$ defined at the beginning of this section.

In the next theorem let $a < c < b$ and let C_i be the smooth rectifiable curve produced by $z = \phi(t) + i\psi(t)$ restricted to $[a, c]$ for $i = 1$ and to $[c, b]$ for $i = 2$. $\mathscr{I}_i$ will represent those g in $\mathscr{I}$ restricted to C_i, and $\Gamma_i(\alpha)$ is all g in $\mathscr{I}_i$ so that $g(z) \in \bar{W}(z)(\alpha)$ on C_i, $0 \le \alpha \le 1$, $i = 1, 2$. Then $\Omega_i(\alpha)$ is the set of all integrals over C_i of g in $\Gamma_i(\alpha)$, $0 \le \alpha \le 1$, $i = 1, 2$. We define $\bar{H}_i$ by its membership function

$$\mu(w \mid \bar{H}_i) = \sup\{\alpha \mid w \in \Omega_i(\alpha)\}, \quad i = 1, 2.$$

Similarly, if we are working with $\mathscr{I}_0$, we define $\mathscr{I}_{0i}$, $\Gamma_{0i}(\alpha)$, $\Omega_{0i}(\alpha)$, and $\bar{H}_{0i}$, $i = 1, 2$, as above.

Theorem 6. (a) *Using $\mathscr{I}$, $\bar{F} \subset \bar{H}_1 + \bar{H}_2$.*
(b) *Using $\mathscr{I}_0$, $\bar{F}_0 \subset \bar{H}_{01} + \bar{H}_{02}$.*

Proof. (a) If $\mu(w \mid \bar{F}) = \alpha$, then we argue that $\mu(w \mid \bar{H}_1 + \bar{H}_2) \ge \alpha$, for any w in $\mathscr{C}$. Assume $\alpha > 0$ for otherwise the result is true. Choose a sequence of $\alpha_n \uparrow \alpha$ with $0 < \alpha_n < \alpha$. Now $w \in \Omega(\alpha_n)$, for all n, so there is a $g_n \in \Gamma(\alpha_n)$ so that $w = \int_C g_n\, dz$. Restrict g_n to C_i producting $g_{ni} \in \Gamma_i(\alpha_n)$, $i = 1, 2$. Let $w_{ni} = \int_{C_i} g_{ni}\, dz$, $i = 1, 2$. Then $w_{ni} \in \Omega_i(\alpha_n)$, $i = 1, 2$, so that $\mu(w_{ni} \mid \bar{H}_i) \ge \alpha_n$, $i = 1, 2$. Then, using the extension principle for addition, we see that

$$\mu(w_n \mid \bar{H}_1 + \bar{H}_2) \ge \alpha_n,$$

where $w_n = w_{n1} + w_{n2}$. However $w_{n1} + w_{n2} = w$ for all n so that $\mu(w \mid \bar{H}_1 + \bar{H}_2) \ge \alpha_n \uparrow \alpha$. Hence $\mu(w \mid \bar{H}_1 + \bar{H}_2) \ge \alpha$ also.

(b) If $\mu(w \mid \bar{F}_0) = \alpha > 0$, then we show that $\mu(w \mid \bar{H}_{01} + \bar{H}_{02}) \ge \alpha$. By Theorem 3 we have $w \in \Omega_0(\alpha)$. So there is a $g \in \Gamma_0(\alpha)$ with $w = \int_C g\, dz$. Restrict g to C_i giving $g_i \in \Gamma_{0i}(\alpha)$, $i = 1, 2$. If $w_i = \int_{C_i} g_i\, dz$, then $w_1 + w_2 = w$. Now $\mu(w_i \mid \bar{H}_{0i}) \ge \alpha$, $i = 1, 2$, so from the extension principle, we obtain $\mu(w \mid \bar{H}_{01} + \bar{H}_{02}) \ge \alpha$ also.

For the next theorem let $\bar{f}_i: C \to \bar{\mathscr{C}}$ with $\bar{f}_i(z) = \bar{W}_i(z)$ on C, $i = 1, 2$. Let $\bar{f} = \bar{f}_1 + \bar{f}_2$ so that $\bar{f}(z) = \bar{W}_1(z) + \bar{W}_2(z)$ for z in C. $\bar{f}$ also maps C into $\bar{\mathscr{C}}$. Using $\mathscr{I}$ we define $\Gamma_i(\alpha)$ and $\Omega_i(\alpha)$ for $\bar{f}_i$ and its integral $\bar{F}_i = \int_C \bar{f}_i\, dz$. Also, let $\bar{F} = \int_C \bar{f}\, dz$.

Theorem 7. $\bar{F}_1 + \bar{F}_2 \subset \bar{F}$.

Proof. Let $\mu(w \mid \bar{F}_1 + \bar{F}_2) = \alpha > 0$. We show that $\mu(w \mid \bar{F}) \ge \alpha$. From the extension principle used to define $\bar{F}_1 + \bar{F}_2$ we may find w_1 and w_2 so that $\mu(w_i \mid \bar{F}) \ge \alpha - \epsilon$, for any given $\epsilon > 0$, $i = 1, 2$, with $w_1 + w_2 = w$. Therefore, $w_i \in \Omega_i(\alpha - \epsilon)$ so that there is a $g_i \in \Gamma_i(\alpha - \epsilon)$ with $\int_C g_i\, dz = w_i$, $i = 1, 2$. Let $g = g_1 + g_2$ and $\bar{W}(z) = \bar{W}_1(z) + \bar{W}_2(z)$ for z in C. We see that: (1) $g \in \mathscr{I}$; (2) $w = \int_C g\, dz$; and (3) $g(z) \in \bar{W}(z)(\alpha - \epsilon)$ on C because

$$\bar{W}(z)(\alpha - \epsilon) = \bar{W}_1(z)(\alpha - \epsilon) + \bar{W}_2(z)(\alpha - \epsilon).$$

Hence $w \in \Omega(\alpha - \epsilon)$ and $\mu(w \mid \bar{F}) \ge \alpha - \epsilon$. The result follows since $\epsilon > 0$ was arbitrary.

For the next theorem we will assume that $\Omega(1)$ (or $\Omega_0(1)$) are non-empty and $\mathscr{I}_0$ is closed under non-zero scalar multiplication. This last assumption is not too strict because if $\mathscr{I}_0$ is compact, then so is $\{\Delta g \mid g \in \mathscr{I}_0,\ \Delta \text{ complex},\ \Delta \ne 0 + i0\}$.

Theorem 8. *Using either $\mathscr{I}$ or $\mathscr{I}_0$ we have $\int_C (\Delta\bar{f})\, dz = \Delta \int_C \bar{f}\, dz$ for any complex Δ.*

Proof. The proof is the same for $\mathscr{I}$ or $\mathscr{I}_0$ so we will present it only for $\mathscr{I}$.

First assume that $\Delta = 0 + i0$. Then $\int_C (\Delta\bar{f})\, dz$ and $\Delta \int_C \bar{f}\, dz$ both equal the crisp complex number $0 + i0$. So now assume that $\Delta \ne 0 + i0$.

Let $\bar{h} = \Delta\bar{f}$ and $\bar{H} = \int_C \bar{h}\, dz$, $\bar{h}(z) = \Delta\bar{W}(z)$ for z in C. To show that $\Delta\bar{F} = \bar{H}$ we argue that $(\Delta\bar{F})(\alpha) = \bar{H}(\alpha)$ for $0 < \alpha \le 1$. But since $(\Delta\bar{F})(\alpha) = \Delta(\bar{F}(\alpha))$ we show $\Delta(\bar{F}(\alpha)) = \bar{H}(\alpha)$.

(a) Let $v \in \Delta(\bar{F}(\alpha))$. Then there is a $w \in \bar{F}(\alpha)$ so that $\Delta w = v$. If $\mu(w \mid \bar{F}) = \beta$, then $\beta \geqslant \alpha$. Given $\epsilon > 0$ there is a $\beta - \epsilon < \gamma \leqslant \beta$ so that $w \in \Omega(\gamma)$. That is $w = \int_C g \, dz$, $g \in \mathscr{I}$, $g(z) \in \bar{W}(z)(\gamma)$ on C. Let $g^* = \Delta g$. Then $v = \int_C g^* \, dz$ and

$$g^*(z) \in \Delta(\bar{W}(z)(\gamma)) = \Delta\bar{W}(z)(\gamma)$$

on C. Hence g^* belongs to $\Omega(\gamma)$ for $\bar{H}$ so $\mu(v \mid \bar{H}) \geqslant \gamma$. Therefore, $\mu(v \mid \bar{H}) > \beta - \epsilon$ and the result $v \in \bar{H}(\alpha)$ follows since $\epsilon > 0$ was arbitrary.

(b) Let $w \in \bar{H}(\alpha)$ and let $\mu(w \mid \bar{H}) = \beta \geqslant \alpha$. Given $\epsilon > 0$ there is a $\beta - \epsilon < \gamma \leqslant \beta$ so that $w \in \Omega(\gamma)$ for $\bar{H}$. There is a $g \in \mathscr{I}$ so that $\int_C g \, dz = w$,

$$g(z) \in \Delta\bar{W}(z)(\alpha) = \Delta(\bar{W}(z)(\alpha))$$

on C. Let $v = w/\Delta$. Then v is in $\Omega(\gamma)$ for $\bar{F}$. So $\mu(v \mid \bar{F}) > \beta - \epsilon$ and it follows that $v \in \bar{F}(\alpha)$ since $\epsilon > 0$ was arbitrary. Then $\Delta v = w \in \Delta(\bar{F}(\alpha))$.

A fuzzy subset $\bar{O}$ of the complex plane will be called a fuzzy zero if $\mu(0 + \mathrm{i}0 \mid \bar{O}) = 1$. If C is a simple closed curve, then we will say that $\Gamma(1)$ (or $\Gamma_0(1)$) is analytic if it contains a function analytic on, and inside C.

Theorem 9. *Suppose C is a simple closed curve and $\Gamma(1)$ (or $\Gamma_0(1)$) is analytic. Then using $\mathscr{I}$ (or $\mathscr{I}_0$) we obtain $\int_C \bar{f} \, dz = \bar{O}$.*

Proof. If $\Gamma(1)$ (or $\Gamma_0(1)$) is analytic, it follows from Cauchy's Integral Theorem [1, p. 510; 10, p. 163] that $0 + \mathrm{i}0 \in \Omega(1)$ (or $\Omega_0(1)$).

We should mention that Theorems 6 and 7 are the best possible. For Theorem 6: (1) using $\mathscr{I}$ one can find an example of a C and a $\bar{f}$ so that $\bar{H}_1 + \bar{H}_2$ is not contained in $\bar{F}$; and (2) using $\mathscr{I}_0$ one can find an example of an $\mathscr{I}_0$, C, and $\bar{f}$ so that $\bar{H}_{01} + \bar{H}_{02}$ is not contained in $\bar{F}_0$. Recall that $\mathscr{I}_0$ is any compact subset of $\mathscr{I}$ so $\mathscr{I}_0$ may be finite, countable, etc. For Theorem 7 one may find a C and $\bar{f}_i$ so that $\bar{F}$ is not contained in $\bar{F}_1 + \bar{F}_2$. Also, there is no Theorem 7 for $\mathscr{I}_0$ because one can find an $\mathscr{I}_0$, C, and $\bar{f}_i$ so that (1) $\bar{F}_{01} + \bar{F}_{02}$ is not contained in $\bar{F}_0$; and (2) $\bar{F}_0$ is not contained in $\bar{F}_{01} + \bar{F}_{02}$.

3. Summary and conclusions

In this paper we defined, and studied a few basic properties of, a fuzzy contour integral of a fuzzy complex mapping. A fuzzy complex mapping $\bar{f}$ maps a smooth rectifiable curve in the complex plane into (generalized) fuzzy complex numbers. We obtained fundamental results about α-cuts of the integral, when the integral is a fuzzy complex number, additivity, and the Cauchy Integral Theorem.

We studied three ways to define the fuzzy line integral: (1) using all the continuous functions which map the smooth rectifiable curve into the complex numbers; (2) using a compact subset of (1) above; and (3) using Lebesgue integrable functions. The most satisfactory theory appears to be the one based on Lebesgue integration because then, as shown in Subsection 2.2, one may directly translate the results from fuzzy real analysis [11, 14] to the fuzzy contour integral.

Future research will be concerned with determining connections between differentiation [4] and the fuzzy contour integral.

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