# Performance Analysis of Source-Seeking Algorithms with Integral Quadratic Constraints

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# Source-seeking Algorithms

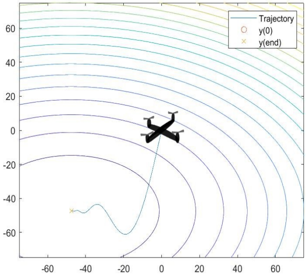



Figure 1: Single quadrotor

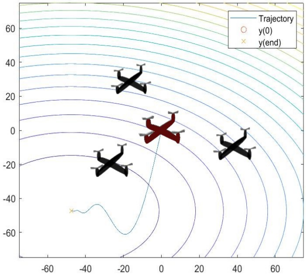



Figure 2: Multiple quadrotors

## Assumptions

- Field is differentiable (can be relaxed using sub-gradients) and convex
- Local gradients available at each agent (Replaced by gradient estimates in practise)

# Motivations and relevant works

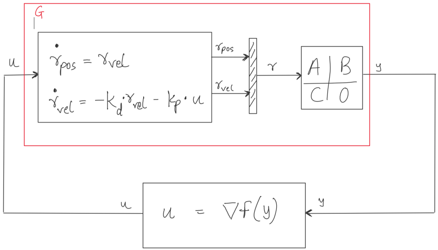
## Earlier work on analysis of source-seeking algorithms

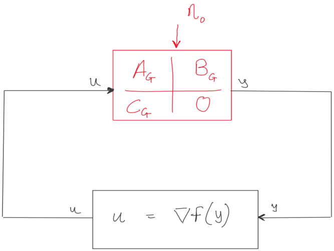
- Stability analysis but no performance guarantees [Attallah, 2020], [Datar, Paulsen and Werner, 2020]
- Lyapunov (storage) functions are constructed based on physically motivated energy like functions: Diagonal Lyapunov (storage) functions [Datar, Paulsen and Werner, 2020]
- Use small-gain arguments for general non-linear (LPV) agents [Attallah, Datar and Werner, 2020]

## Key contributions of this work

- Performance analysis with guaranteed rates of convergence
- Quantifiable robustness w.r.t underlying field
- Less conservative results with dynamic non-causal multipliers

# Control Architecture: Single Agent




$$
\begin{aligned}
\dot{\eta}(t) &= A_G \eta(t) + B_G u(t), \qquad \eta(0) = \eta_0 \\
y(t) &= C_G \eta(t) \\
u(t) &= \nabla f(y(t)),
\end{aligned}
\tag{1}
$$

# Decomposing flocking dynamics under quadratic fields

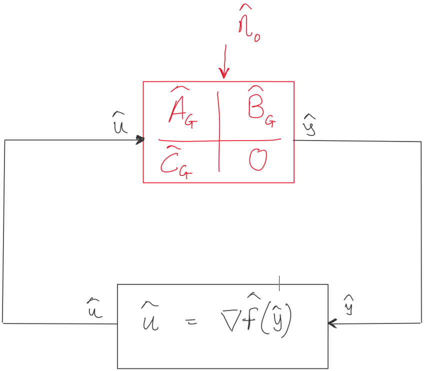



Figure 3: Block diagram

$$\begin{aligned}
\dot{\hat{\eta}}(t) &= (I_N \otimes A_G)\hat{\eta}(t) + (I_N \otimes B_G)\hat{u}(t), \\
\hat{y}(t) &= (I_N \otimes C_G)\hat{\eta}(t) \\
\hat{u}(t) &= \nabla \hat{f}(\hat{y}(t)) - \nabla \mathcal{V}(\hat{y}(t)) - (\mathcal{L} \otimes I_d)\dot{\hat{y}}(t).
\end{aligned} \tag{2}$$

# Decomposing flocking dynamics under quadratic fields

## Averaging out state, input and output:Center Of Mass(COM)

$\eta_c = \text{avg}(\hat{\eta})$, $y_c = \text{avg}(\hat{y})$, $u_c = \text{avg}(\hat{u})$

## Quadratic fields (Linear gradients)

- $f(y) = y^T Q y + c^T y + d \implies \nabla f(y) = 2Qy + c$
- $\text{avg}(\nabla \hat{f}(\hat{y})) = \nabla f(\text{avg}(\hat{y})) = \nabla f(y_c)$

## Averaging dynamics (2)

- $\text{avg}((I_N \otimes A_G)\hat{\eta}) = A_G \eta_c$ (Similary for $B_G$ and $C_G$)
- $\text{avg}(\nabla \mathcal{V}(\hat{y})) = \mathbf{0}$,$\text{avg}((\mathcal{L} \otimes I_d)) = \mathbf{0}$

$$\begin{aligned} \dot{\eta_c}(t) &= A_G \eta_c + B_G u_c, \\ \hat{y}_c(t) &= C_G \eta_c \\ \hat{u}_c(t) &= \nabla f(y_c). \end{aligned} \tag{3}$$

# Theory

## Relevant Literature

- Exponential IQCs introduced in [Lessard,Recht and Packard, 2016], [Hu and Seiler, 2016],
- Non-causal dynamic Zames Falb $\alpha$-IQCs [Freeman, 2018]
- Parameterization of dynamics Zames Falb IQCs [Veenman and Scherer, 2018]

## theoretical contribution of this work

- Derivation of a general non-causal dynamic $\alpha$-ZF IQCs non-causal dynamic multipliers with an LMI-able parameterization

# Loop in the deviation variables

## Equilibrium

$$\begin{aligned} 0 &= A_G \eta_* + B_G u_* = A_G \eta_* \\ y &= C_G \eta_* \\ u_* &= \nabla f(y_*) = 0 \end{aligned} \tag{4}$$

## Loop in the transformed variables

$$\begin{aligned} \dot{\tilde{\eta}}(t) &= A_G \tilde{\eta}(t) + B_G \tilde{u}(t), \qquad \tilde{\eta}(0) = \eta_0 - \eta_* \\ \tilde{y}(t) &= C_G \tilde{\eta}(t) \end{aligned} \tag{5}$$

and

$$\tilde{u}(t) = \nabla f(\tilde{y}(t) + y_*) \tag{6}$$

# Loop in the deviation variables

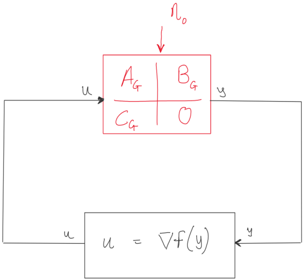



Figure 4: Original loop

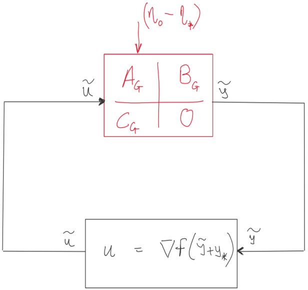


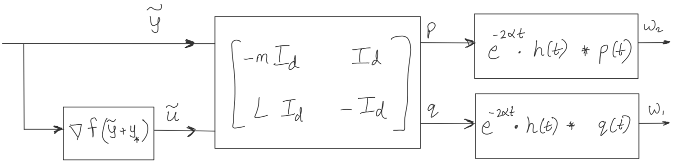

Figure 5: Loop in deviation variables

## Assumption 1

Let $f$ be a differentiable convex field with the minimum at $y_*$ satisfying

$$m||y_1 - y_2||^2 \leq (\nabla f(y_1) - \nabla f(y_2))^T (y_1 - y_2) \leq L||y_1 - y_2||^2 \quad \forall y_1, y_2.$$

# Zames Falb IQCs




**Theorem:** Let $h \in \mathcal{L}_1(-\infty, \infty)$ satisfy

$$h(s) \geq 0, \quad \forall s \in \mathbb{R} \text{ and } \int_{-\infty}^{\infty} h(s)ds \quad \leq H. \tag{7}$$

Then, under the assumptions on $f$, $\forall T \geq 0$ and $\forall \tilde{y} \in \mathcal{L}_{2e}$,

$$\int_0^T e^{2\alpha t}(Hp(t)^T q(t) - p(t)^T w_1(t) - q(t)^T w_2(t))dt \geq 0. \tag{8}$$

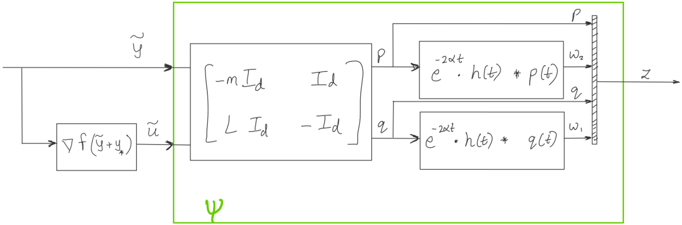
# Parameterize $h(t)$

- Fix $A_\nu = \begin{bmatrix} \rho & 0 & \dots & 0 \\ 1 & \rho & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & \rho \end{bmatrix}, B_\nu = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$
- Parameterize $h(t)$ as

$$h(t) = \begin{cases} P_1 e^{-A_\nu t} B_\nu, & \text{if } t < 0 \\ P_3 e^{A_\nu t} B_\nu, & \text{if } t \geq 0 \end{cases} \tag{9}$$

- If $\mathrm{LMI}(H, P_1, P_3) < 0$, conditions on $h(t)$ hold [Scherer, 2013].
- For searching over only causal(anti-causal) multipleirs, set $P_1 = 0$ ($P_3 = 0$)
- For searching over static multipliers(Circle criterion), set $P_1 = 0$ and $P_3 = 0$

# Parameterize $h(t)$




$$(Hp^T q - p^T w_1 - q^T w_2) = \tilde{z}^T \begin{bmatrix} \mathbf{0} & \begin{bmatrix} H & -P_3 \\ -P_1^T & \mathbf{0} \end{bmatrix} \\ * & \mathbf{0} \end{bmatrix} \tilde{z}$$

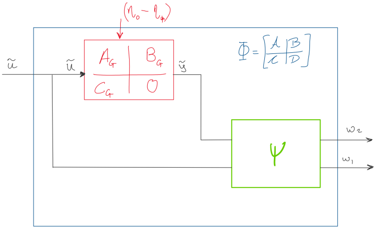
Define $\mathbb{P}$ such that every $P \in \mathbb{P}$ generates a valid $h(t)$.

$$\mathbb{P} = \left\{ \begin{bmatrix} \mathbf{0} & \begin{bmatrix} H & -P_3 \\ -P_1^T & \mathbf{0} \end{bmatrix} \\ * & \mathbf{0} \end{bmatrix} : \mathrm{LMI}(H, P_1, P_3) < 0 \right\}$$

# Main Results

**Theorem:** Under the assumptions on $f$, $\forall T \geq 0$ and $\forall \tilde{y} \in \mathcal{L}_{2e}$,

$$\int_0^T e^{2\alpha t} \tilde{z}^T(t)(P \otimes I_d)\tilde{z}(t)dt \geq 0 \quad \forall P \in \mathbb{P}, \forall T \geq 0.$$

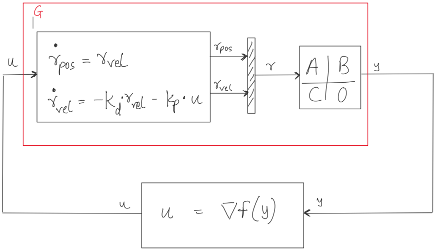



**Theorem:** Under the assumptions on $f$, if $\exists \mathcal{X} > 0, P \in \mathbb{P}$ such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} + 2\alpha\mathcal{X} & \mathcal{X}\mathcal{B} \\ \mathcal{B}^T\mathcal{X} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} (P \otimes I_d) \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} \leq 0,$$

then, $||\tilde{y}(t)|| \leq \kappa e^{-\alpha t}$ holds for all $t \geq 0$,

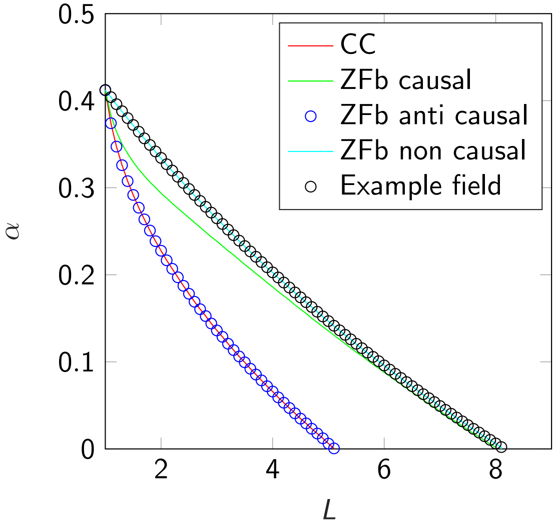
# Numerical results: Quadrotor



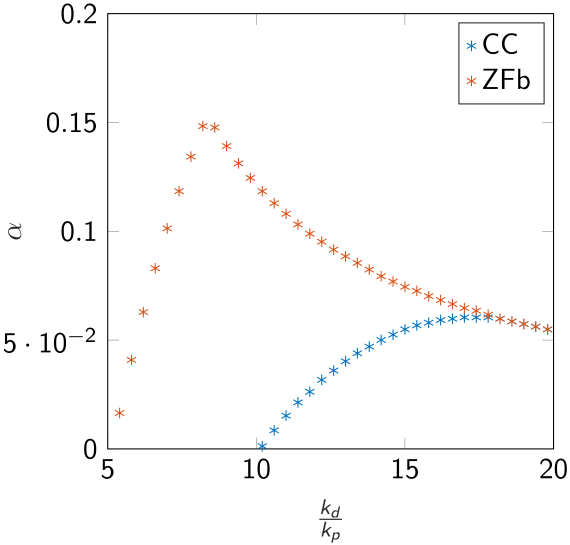

## Setup

- Linearized quadrotor model + LQR local controller
- Questions:
  - How robust is the given controller for different fields?
  - How do we select the PD gains for the pre-filter?
  - How conservative are the estimates on the rates of convergence?

# Numerical results: Robustness with respect to field



# Numerical results: Performance curves for kp,kd



# Numerical results: Quadrotor Trajectory

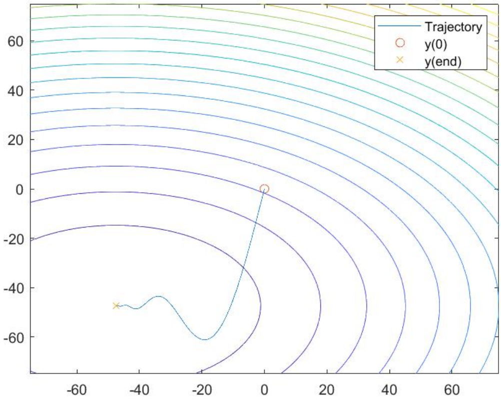



Figure 8: Robustness for Quadrotor and $m = 1$

# A toy example motivating non-causal multipliers

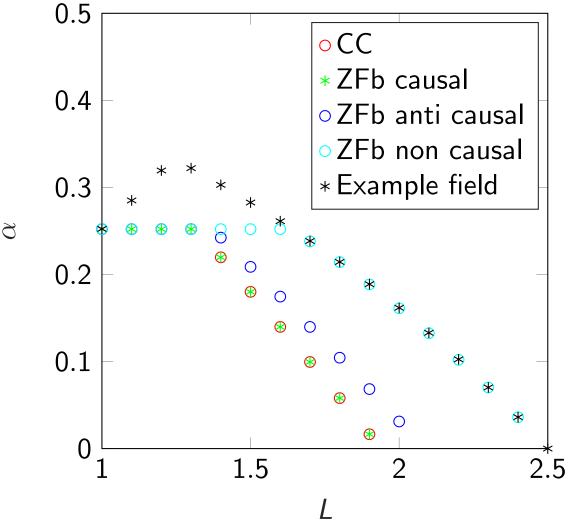



Figure 9: Robustness for $G(s) = 5\frac{(s-1)}{s(s^2+s+25)}$ and $m = 1$

# Ongoing work and next steps

## Forseeable extensions

- Source-seeking with formation control with a leader (Grounded Laplacians) (**Tested and seems to work** )
- LPV vehicle models
- Extend the full block ZF multipliers to the $\alpha$- IQC case
- Controller synthesis with DK-iteration?
- Mean squared stability analysis with MJLS

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Thank you