If M has a negative eigen value, there exist x1,x2 s.t f(x) < 0

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Is there an M, not +ve semidef s.t

$$[x_1 \quad x_2 \quad x_1x_2]\, M \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix} \geq 0 \quad \forall x \qquad ?$$

$\Rightarrow$

If $\exists \lambda > 0,\ v \neq 0 : \quad Mv = -\lambda v$

Then $\forall \alpha, \quad (\alpha v)^T M (\alpha v) = \alpha^2 v^T M v$

$$= -\alpha^2 \cdot \lambda \cdot \|v\|^2 < 0$$

If $\left\{ \alpha v \mid \alpha \in \mathbb{R},\ Mv = -\lambda v \right\} \cap \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix} \middle| \ x_1, x_2 \in \mathbb{R} \right\}$

is non-empty, then we have a contradiction.

Case 1] $v_1 \neq 0$, $v_2 \neq 0$, $v_3 \neq 0$ (No component of v is 0)

Let: $Mv = -\lambda v$

with, $x_1 = \dfrac{v_3}{v_2}$, $x_2 = \dfrac{v_3}{v_1}$, $[x_1 \quad x_2 \quad x_1x_2]\, M \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix}$

$$= \left[\frac{v_3}{v_2} \quad \frac{v_3}{v_1} \quad \frac{v_3^2}{v_1v_2}\right] M \begin{bmatrix} v_3/v_2 \\ v_3/v_1 \\ v_3^2/v_1v_2 \end{bmatrix}$$

$$= \left(\frac{v_3}{v_1v_2}\right)^2 [v_1 \quad v_2 \quad v_3]\, M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= -\lambda \cdot \left(\frac{v_3}{v_1v_2}\right)^2 \cdot \|v\|^2 < 0$$

Case 5] $\left(\begin{matrix}\text{One component of}\\ v \text{ is } 0\end{matrix}\right)$ $\nexists\, x_1, x_2 :$ $\alpha \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$$

$x_1 = v_3/v_2$	$x_1 = \alpha v_1$	$x_1 = \alpha v_1$
$x_2 = \alpha v_2$	$x_2 = v_3/v_1$	$x_2 = \alpha v_2$
$x_1 x_2 = \alpha v_3$	$x_1 x_2 = \alpha v_3$	$\underbrace{x_1 x_2 = \alpha^2 v_1 v_2}$
$\alpha \to \infty$	$\alpha \to \infty$	Consider $\alpha \to 0$
$\begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ \alpha v_2 \\ \alpha v_3 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{bmatrix} \to \begin{bmatrix} \alpha v_1 \\ 0 \\ \alpha v_3 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{bmatrix} \to \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ 0 \end{bmatrix}$

$\left(\begin{matrix}\text{One component}\\ \text{of}\\ v \text{ is } 0\end{matrix}\right)$

Case 2] $v_1 \neq 0$, $v_2 = 0$, $v_3 = 0$ $\Rightarrow$ $v_1 = x_1$, $v_2 = x_2 = 0$, $v_3 = x_1 x_2 = 0$ ✓

Case 3] $v_1 = 0$, $v_2 \neq 0$, $v_3 = 0$ $\Rightarrow$ — // —

Case 4] $v_1 = 0$, $v_2 = 0$, $v_3 \neq 0$ $\Rightarrow$ $m_{33} < 0$ $\Rightarrow$ contradiction can't happen

$\left(\begin{matrix}\text{Two}\\ \text{components of}\\ v = 0\end{matrix}\right)$