Stability of the damped pendulum with physically motivated energy and LaSalle

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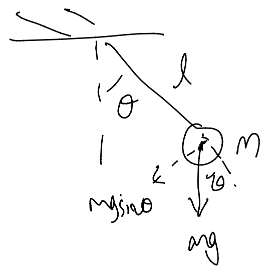
Pendulum:



$$I\ddot{\theta} = \tau$$

$$ml^2\ddot{\theta} = -b\dot{\theta} - (mg\sin\theta)\cdot l$$

$$\boxed{ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = 0}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad m = l = b = 1, \quad g = 1$$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 - \sin x_1 \end{bmatrix}}$$

Intuition: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is globally stable eq^n.

Proof:

$$V = \frac{1}{2}(ml^2)x_2^2 + mg(l - l\cos(x_1))$$

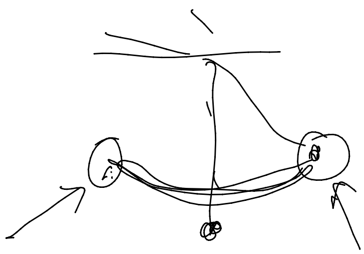
$$V = \frac{1}{2} x_2^2 + 1 - \cos(x_1)$$

$$\dot{V} = x_2 \dot{x}_2 + \sin(x_1)\, \dot{x}_1$$

$$= x_2\left[-x_2 - \sin x_1\right] + x_2 \sin x_1$$

$$= -x_2^2$$

$$\leq 0$$



$V(t) \leq V(0) \quad \forall t$ $\qquad \dot{V} = 0$

- only boundedness.
- Need La Salle's Invariance Principle to say that.

Traj converge to the longest invariant set in

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \dot{V}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\}$$

longest invariant set within this set is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Stability via LPV? Seems not possible

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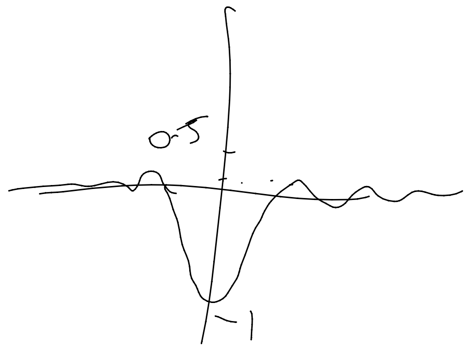
$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 - \sin x_1 \end{bmatrix}}$$

$$\left[\begin{array}{c|c} 0 & 1 \\ \hline \frac{\sin \rho}{\rho} & -1 \end{array}\right]$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-\sin x_1}{x_1} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rho = -\frac{\sin(x_1)}{x_1}$$



$$\boxed{\dot{x} = \begin{bmatrix} 0 & 1 \\ \rho & -1 \end{bmatrix} x}$$

$$\rho(t) \in [-1, 0.5]$$

- Since $\rho(t) \equiv 0.1$ is an acceptable trajectory.
- Cannot be done !

Stability via LPV 2

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta \cdot \dot{\theta} \\ -\sin\theta \cdot \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 x_3 \\ -x_1 x_3 \\ -x_3 - x_1 \end{bmatrix}$$

$$\rho = x_3$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & \rho & 0 \\ -\rho & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} x}_{\rho(t) \equiv 0 \;\Rightarrow\; \text{Infeasible}}$$

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & \rho_2 \\ 0 & 0 & -\rho_1 \\ -1 & 0 & -1 \end{bmatrix} x}_{\text{Always infeasible}}$$

Stability via SOS

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \\ \dot\theta \end{bmatrix} \qquad Eq^m \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1^2 + x_2^2 = 1$

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \end{bmatrix} = \begin{bmatrix} \cos\theta \cdot \dot\theta \\ -\sin\theta \cdot \dot\theta \\ \ddot\theta \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 x_3 \\ -x_1 x_3 \\ -x_3 - x_1 \end{bmatrix}}_{\text{polynomial system}}$$

$$V(x) = [\alpha_0\ \alpha_1\ \ldots\ \alpha_n] \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix}$$

$\boxed{V(x) \text{ is SOS}} \rightarrow V(x) \geq 0 \ \forall x$

$\downarrow$ SDP

$$\dot V(x) = [\alpha_0 \ldots \alpha_n] \begin{bmatrix} 0 \\ \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot x_1 x_2 + x_1 \dot x_2 \\ \dot x_1 x_3 + x_1 \dot x_3 \\ \dot x_2 x_3 + x_2 \dot x_3 \\ 2x_1 \dot x_1 \\ 2x_2 \dot x_2 \\ 2x_3 \dot x_3 \end{bmatrix}$$

$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix}\right) = m_1 x_1^2 + \ldots$ is SOS

$$\begin{bmatrix} m_1 & m_2 & m_3 \\ & m_4 & m_5 \\ & & m_6 \end{bmatrix}$$

$(*) M [0\ 1\ 1\ 1\ 0\ 0] \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$

$(*)^T M \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix}$ SOS $\rightarrow M \geq 0$

$V(x) = [\alpha_1 \ldots \alpha_n] \begin{bmatrix} x_1 \\ \vdots \\ x \end{bmatrix} \rightarrow V(x)$ is SOS

- All sum of squares f are positive
- $\exists f$ is positive ; f not SOS

$$= [\alpha_0 \ldots \alpha_n] \begin{bmatrix} 0 \\ x_2x_3 \\ -x_1x_3 \\ -x_3 - x_1 \\ x_2^2 x_3 - x_1^2 x_3 \\ x_2 x_3^2 - x_3 x_1 - x_1^2 \\ -x_1 x_3^2 - x_2 x_3 - x_1 x_2 \\ 2x_1x_2x_3 \\ -2x_1x_2x_3 \\ -2x_3^2 - 2x_1x_3 \end{bmatrix}$$

$\boxed{V(x) \geq 0}$ want

$\boxed{V(x) \text{ is SOS}} \rightarrow$ LMI

$\boxed{-\dot V(x) \text{ is SOS}} \rightarrow$ LMI

$\Rightarrow$

$\boxed{\dot V(x) \leq 0}$ (want)

$\boxed{x_1^2 + x_2^2 = 1}$ not used yet

Q. Is $m : \mathbb{R}^2 \rightarrow \mathbb{R}$ sum of squares?

$$m(x) = x_1^2 + 2x_1x_2 + x_2^2$$

$$= \underbrace{(x_1\ x_2)}\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\geq 0}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

S

$S \geq 0 \Rightarrow S = DD^T$

$m(x) = x^T D D^T x = \| D^T x \|^2$

$\lambda(x)$

$V(x) \geq 0$

$-\dot V(x) + \lambda(x) [\underbrace{x_1^2 + x_2^2}_{1} - 1]$ is SOS

Manual SOS decomposition for an example

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- $m(x) = [x_1 \;\; x_2 \;\; x_1x_2] \; M \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix}$

- Look for $z(x)$ and Q s.t. $z(x)^T Q z(x) = m(x)$ & $Q \succeq 0$.

- $$[x_1 \;\; x_2 \;\; x_1x_2] \cdot M \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \end{bmatrix} = M_{11} \cdot x_1^2 + 2M_{12} \cdot x_1x_2 + 2M_{13}x_1^2x_2 + M_{22}x_2^2 + 2M_{23}x_2^2x_1 + M_{33}x_1^2x_2^2$$

- $$[1 \;\; x_1 \;\; x_2 \;\; x_1x_2 \;\; x_1^2 \;\; x_2^2] \; Q \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix} = Q_{00} + 2Q_{01}x_1 + 2Q_{02}x_2 + 2Q_{03}x_1x_2 + 2Q_{04}x_1^2 + 2Q_{05}x_2^2$$

$$+ Q_{11}x_1^2 + 2Q_{12}x_1x_2 + 2Q_{13}x_1^2x_2 + 2Q_{14}x_1^3 + 2Q_{15}x_1x_2^2$$

$$+ Q_{22}x_2^2 + 2Q_{23}x_1x_2^2 + 2Q_{24}x_2x_1^2 + 2Q_{25}x_2^3$$

$$+ Q_{33}x_1^2x_2^2 + 2Q_{34}x_1^3x_2 + 2Q_{35}x_1x_2^3$$

$$+ Q_{44}x_1^4 + 2Q_{45}x_1^2x_2^2$$

$$+ Q_{55}x_2^4$$

$$= \overbrace{Q_{00}}^{0} + \overbrace{2Q_{01}}^{0}x_1 + \overbrace{2Q_{02}}^{0}x_2 + \overbrace{(2Q_{03} + 2Q_{12})}^{2M_{12}}x_1x_2$$

$$+ \overbrace{(2Q_{04} + Q_{11})}^{M_{11}}x_1^2 + \overbrace{(2Q_{05} + Q_{22})}^{M_{22}}x_2^2$$

$$+ \overbrace{(2Q_{13} + 2Q_{24})}^{2M_{13}}x_1^2x_2 + \overbrace{(2Q_{14})}^{0}x_1^3 +$$

$$+ \overbrace{(2Q_{15} + 2Q_{23})}^{2M_{23}}x_1x_2^2 + \overbrace{(2Q_{25})}^{0}x_2^3$$

$$+ \overbrace{(Q_{33} + 2Q_{45})}^{M_{33}}x_1^2x_2^2 + \overbrace{(2Q_{34})}^{0}x_1^3x_2 + \overbrace{(2Q_{35})}^{0}x_1x_2^3$$

$$+ \overbrace{Q_{44}}^{0}x_1^4 + \overbrace{Q_{55}}^{0}x_2^4$$

$$+\underbrace{Q_{44}}_{}\overset{0}{\tilde{}}\, x_1^4 \quad + Q_{55}\, x_2^4$$

Find $Q \succeq 0$:

$$\begin{bmatrix} 0 & 0 & 0 & Q_{03} & Q_{04} & Q_{05} \\ & Q_{11} & Q_{12} & Q_{13} & 0 & Q_{15} \\ & & Q_{22} & Q_{23} & Q_{24} & 0 \\ & & & Q_{33} & 0 & 0 \\ & & & & 0 & Q_{45} \\ & & & & & 0 \end{bmatrix} \succeq 0$$

and

$$\left.\begin{aligned} (Q_{03}+Q_{12}) &= M_{12} \\ (2Q_{04}+Q_{11}) &= M_{11} \\ (2Q_{05}+Q_{22}) &= M_{22} \\ Q_{13}+Q_{24} &= M_{13} \\ Q_{15}+Q_{23} &= M_{23} \\ Q_{33}+2Q_{45} &= M_{33} \end{aligned}\right\} = \left\{ \begin{bmatrix} Q_{11}+2Q_{04} & Q_{03}+Q_{12} & Q_{13}+Q_{24} \\ * & Q_{22}+2Q_{05} & Q_{15}+Q_{23} \\ * & * & Q_{33}+2Q_{45} \end{bmatrix} \right. = M$$