보충자료

Theorem 3 a 와 d의 증명 과정. 이를 통해 Theorem 2는 자동으로 증명됨

Q1

Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A.

Proof

Since $\mathbf{x}$ is an eigenvector of A, $\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T(\lambda\mathbf{x}) = \lambda\overline{\mathbf{x}} \cdot \mathbf{x}$. $\overline{\mathbf{x}} \cdot \mathbf{x}$ is real and positive because $\overline{z}z$ is nonnegative for every complex number z. Since $\overline{\mathbf{x}}^T A\mathbf{x}$ is real, so is λ. Now, let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Then $A\mathbf{x} = A\mathbf{u} + iA\mathbf{v}$ and $\lambda\mathbf{x} = \lambda\mathbf{u} + i\lambda\mathbf{v}$. The real part of $A\mathbf{x}$ is $A\mathbf{u}$ and the real part of $\lambda\mathbf{x}$ is $\lambda\mathbf{u}$ because the entries of A, $\mathbf{u}$, $\mathbf{v}$ and λ are real. Since $A\mathbf{x}$ and $\lambda\mathbf{x}$ are equal, their real parts are also equal. Thus $A\mathbf{u} = \lambda\mathbf{u}$, which shows that the real part of $\mathbf{x}$ is an eigenvector of A.

Proof of Theorem 3 a

By the proof of Q1, Theorem 3 a is proved.

Q2

Let block matrix $G = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$. Show that $\det G = (\det A)(\det B)$. From this, deduce that the characteristic polynomial of G is the product of the characteristic polynomials of A and B.

Proof

Let U and V be echelon forms of A and B, obtained with the number of r and s interchanges, respectively without scaling. Then $\det A = (-1)^r \det U$ and $\det B = (-1)^s \det V$ because row replacements do not change the determinants. When A is reduced to U, G is reduced in the form $G' = \begin{bmatrix} U & Y \\ 0 & B \end{bmatrix}$. Using the row operations that reduce B to V, G' can be further reduced to $G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. There were $r + s$ row interchanges when G is reduced to G'', so $\det G = (-1)^{r+s} \det \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. Since U and V are upper triangular, so is

$G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. Therefore, the determinant of G'' equals the product of the diagonal entries which equals $(\det U)(\det V)$. Thus $\det G = (-1)^{r+s}(\det U)(\det V) = (\det A)(\det B)$.

For any scalar λ, $G - \lambda I = \begin{bmatrix} A-\lambda I & X \\ 0 & B-\lambda I \end{bmatrix}$ where I represents various identity matrices of appropriate sizes. Since $\det G = (\det A)(\det B)$,
$\det(G-\lambda I) = \left(\det(A-\lambda I)\right)\left(\det(B-\lambda I)\right)$.

Q3

Schur factorization of an $n\times n$ matrix A is in the form of $A = URU^T$, where U is an orthogonal matrix and R is an $n\times n$ upper triangular matrix.
Let A be an $n\times n$ matrix with n real eigenvalues, counting multiplicities, denoted by $\lambda_1, \ldots, \lambda_n$. Show that A admits a Schur factorization.

Proof

Let $\mathbf{u}_1$ be a unit eigenvector corresponding to λ_1, let $\mathbf{u}_2,\ldots, \mathbf{u}_n$ be any other vectors such that $\{\mathbf{u}_1,\ldots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, and then let $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$. Then, $AU = [\lambda_1\mathbf{u}_1 \quad A\mathbf{u}_2 \quad \cdots \quad A\mathbf{u}_n]$. Since $\mathbf{u}_1$ is a unit vector and $\mathbf{u}_2,\ldots, \mathbf{u}_n$ are orthogonal to $\mathbf{u}_1$, the first column of $U^T AU$ is

$$U^T(\lambda_1\mathbf{u}_1) = \lambda_1 U^T\mathbf{u}_1 = \lambda_1\mathbf{e}_1 \tag{1.1}$$

Eq. (1.1) implies that $U^T AU$ has the form shown below.

$$U^T AU = \begin{bmatrix} \lambda_1 & * & * & * & * \\ 0 & & & & \\ \vdots & & A_1 & & \\ 0 & & & & \end{bmatrix} \tag{1.2}$$

Viewing $U^T AU$ as a 2×2 block upper triangular matrix with A_1 as the (2,2) block. Then from the proof of Q2,

$$\det(U^T AU - \lambda I) = \left(\det(\lambda_1 - \lambda)\right)\left(\det(A_1 - \lambda I_{n-1})\right) = (\lambda_1 - \lambda)\det(A_1 - \lambda I_{n-1}) \tag{1.3}$$

Eq. (1.3) shows that the eigenvalues of $U^T AU$, namely, $\lambda_1,\ldots, \lambda_n$, consist of λ_1 and the eigenvalues of A_1. Thus, the eigenvalues of A_1 are $\lambda_2, \ldots, \lambda_n$. Repeating Eqs.(1.1) to (1.3) for successively smaller matrices ($A_1, \ldots$) and then piecing together the results prove A admits a Schur factorization.

Q4

Suppose $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

Proof

$P^{-1}AP = R$. Since P is orthogonal, $R = P^T AP$. Hence $R^T = (P^T AP)^T = P^T A^T P^{TT} = P^T AP = R$, which shows that R is symmetric. Since R is upper triangular, its entries above the diagonal must be zeros to be symmetric. Thus R is a diagonal matrix.

Proof of Theorem 3 d

By Theorem 3 a, a symmetric matrix A has n real eigenvalues. By the proof of Q3, A can be factorized into the Schur form $A = PRP^{-1}$. By the proof of Q4, R is a diagonal matrix. Therefore, the symmetric matrix A is orthogonally diagonalizable.