Pseudoinverse를 사용하여 구한 least-squares solution $\hat{\mathbf{x}}$이 smallest length임을 증명

Q1 (교과서 6.3의 Exercises 23)

Let A be an $m \times n$ matrix. Prove that every vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written in the form $\mathbf{x} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p}$ is in Row A and $\mathbf{u}$ is in Nul A. Also, show that if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then there is a unique $\mathbf{p}$ in Row A such that $A\mathbf{p} = \mathbf{b}$.

Proof

Each $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{x} = \mathbf{p} + \mathbf{u}$ where $\mathbf{p}$ in Row A and $\mathbf{u}$ in (Row A)$^\perp$ by the Orthogonal Decomposition Theorem. (Row A)$^\perp$ = Nul A, so $\mathbf{u}$ is in Nul A. Suppose $A\mathbf{x} = \mathbf{b}$ is consistent. Let $\mathbf{x}$ be a solution and write $\mathbf{x} = \mathbf{p} + \mathbf{u}$ as above. Then $A\mathbf{p} = \mathbf{b}$ because $A\mathbf{x} = A\mathbf{p} + A\mathbf{u} = A\mathbf{p} + \mathbf{0} = \mathbf{b}$. Therefore, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{p}$ in Row A. Suppose there are $\mathbf{p}$ and $\mathbf{p}_1$ both in Row A satisfying $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{p} - \mathbf{p}_1$ in Nul A, since $A(\mathbf{p} - \mathbf{p}_1) = \mathbf{b} - \mathbf{b} = \mathbf{0}$. By the Orthogonal Decomposition Theorem, $\mathbf{p} = \mathbf{p}_1$ because $\mathbf{p} = \mathbf{p} + \mathbf{0}$ and $\mathbf{p} = \mathbf{p}_1 + (\mathbf{p} - \mathbf{p}_1)$ both decompose $\mathbf{p}$ as the sum of a vector in Row A and a vector in Nul A. Therefore, $\mathbf{p}$ is unique.

Q2 (교과서 7 Supplementary Excises 12)

Verify the properties of A^+:

a. For each $\mathbf{y}$ in $\mathbb{R}^m$, $AA^+\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto Col A.

b. For each $\mathbf{x}$ in $\mathbb{R}^n$, $A^+A\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto Row A.

c. $AA^+A = A$ and $A^+AA^+ = A^+$.

Proof

a. $AA^+\mathbf{y} = U_r D V_r^T V_r D^{-1} U_r^T \mathbf{y} = U_r U_r^T \mathbf{y}$

b. $A^+A\mathbf{x} = V_r D^{-1} U_r^T U_r D V_r^T \mathbf{x} = V_r V_r^T \mathbf{x}$

c. $AA^+A = U_r D V_r^T V_r D^{-1} U_r^T U_r D V_r^T = U_r D V_r^T = A$ and
$A^+AA^+ = V_r D^{-1} U_r^T U_r D V_r^T V_r D^{-1} U_r^T = V_r D^{-1} U_r^T = A^+$

Q3(교과서 7 Supplementary Excises 13)

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent, and let $\mathbf{x}^+ = A^+\mathbf{b}$. By the proof of Q1, there is exactly one vector $\mathbf{p}$ in Row A such that $A\mathbf{p} = \mathbf{b}$. The following steps prove that $\mathbf{x}^+ = \mathbf{p}$ and $\mathbf{x}^+$ is the minimum length solution of $A\mathbf{x} = \mathbf{b}$.

a. Show that $\mathbf{x}^+$ is in Row A.

b. Show that $\mathbf{x}^+$ is a solution of $A\mathbf{x} = \mathbf{b}$.

c. Show that if $\mathbf{u}$ is any solution of $A\mathbf{x} = \mathbf{b}$, then $\|\mathbf{x}^+\| \le \|\mathbf{u}\|$, with equality only if $\mathbf{u} = \mathbf{x}^+$.

Proof

a. $\mathbf{x}^+ = A^+\mathbf{b} = A^+A\mathbf{x} = V_rV_r^T\mathbf{x}$. Proven already by the proof of Q2. b.

b. $A\mathbf{x}^+ = AA^+A\mathbf{x} = A\mathbf{x} = \mathbf{b}$ by the proof of Q2. c.

c. Uniquely decompose $\mathbf{u}$ as $\mathbf{u} = \mathbf{x}^+ + (\mathbf{u} - \mathbf{x}^+)$ with $\mathbf{x}^+$ in Row A and $(\mathbf{u} - \mathbf{x}^+)$ in $(\text{Row } A)^\perp$. Then, $\|\mathbf{u}\|^2 = \|\mathbf{x}^+\|^2 + \|\mathbf{u} - \mathbf{x}^+\|^2 \ge \|\mathbf{x}^+\|^2$ with equality only if $\mathbf{u} = \mathbf{x}^+$.

Q4(교과서 7 Supplementary Excises 14)

Given any $\mathbf{b}$ in $\mathbb{R}^m$, show that $A^+\mathbf{b}$ is the least-squares solution of minimum length.

Proof

The least-squares solutions of $A\mathbf{x} = \mathbf{b}$ are the solutions of $A\mathbf{x} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}} = \text{proj}_{\text{Col} A}\mathbf{b}$. From the proof of Q3, the minimum length solution of $A\mathbf{x} = \hat{\mathbf{b}}$ is $A^+\hat{\mathbf{b}}$. Therefore, $A^+\hat{\mathbf{b}}$ is the minimum length least-squares solution of $A\mathbf{x} = \mathbf{b}$. From the proof of Q2. a, $\hat{\mathbf{b}} = AA^+\mathbf{b}$ so $A^+\hat{\mathbf{b}} = A^+AA^+\mathbf{b} = A^+\mathbf{b}$ by the proof of Q2. c. Thus $A^+\mathbf{b}$ is the least-squares solution of minimum length of $A\mathbf{x} = \mathbf{b}$.