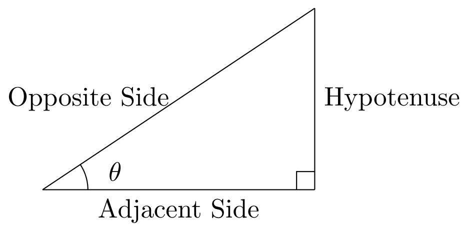
Trigonometry with Triangles

Right Triangle Definitions

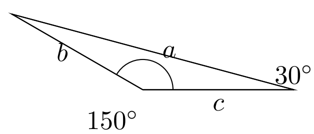


$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Reference Angles

A reference angle is the acute angle ($< 90^\circ$) formed by the terminal side of an angle and the horizontal axis.



In this triangle, angle A is 150°, and the reference angle θ_{ref} is:

$$\theta_{\text{ref}} = 180^\circ - 150^\circ = 30^\circ$$

Example: Find the reference angle for $\theta = 150^\circ$.

$$\theta_{\text{ref}} = 180^\circ - 150^\circ = 30^\circ$$

So, the reference angle is 30°.

When calculating trigonometric functions for obtuse angles, use the reference angle and adjust the sign based on the quadrant.

Example: Find $\sin 150^\circ$.

$$\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$$

Since 150° is in Quadrant II, (sin+), $\sin 150^\circ = \frac{1}{2}$.

Example: Find $\cos 150^\circ$.

$$\cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

Since 150° is in Quadrant II, (cos-), $\cos 150^\circ = -\frac{\sqrt{3}}{2}$.

Signs of Trigonometric Functions

Quadrant	Positive Functions
I	sin, cos, tan
II	sin
III	tan
IV	cos

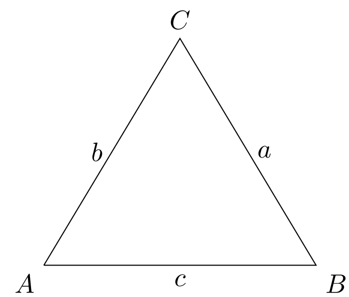
Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

Example: If $\sin\theta = \frac{3}{5}$, then:

$$\cos\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Law of Sines



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example: If $a = 7$, $A = 30^\circ$, and $B = 45^\circ$, find b:

$$\begin{aligned} \frac{a}{\sin A} &= \frac{b}{\sin B} \\ b &= a \times \frac{\sin B}{\sin A} \\ &= 7 \times \frac{\sin 45^\circ}{\sin 30^\circ} \\ &= 7 \times \frac{\frac{\sqrt{2}}{2}}{\frac{1}{2}} \\ &= 7 \times \sqrt{2} \approx 9.9 \end{aligned}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Example: Given $b = 5$, $c = 7$, and $A = 60^\circ$:

$$\begin{aligned} a^2 &= 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 60^\circ \\ &= 25 + 49 - 70 \times \left(\frac{1}{2}\right) \\ &= 74 - 35 \\ &= 39 \\ a &= \sqrt{39} \approx 6.24 \end{aligned}$$

Area of a Triangle

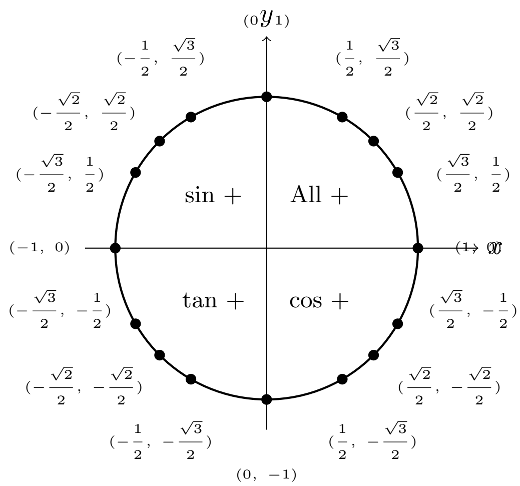
$$\text{Area} = \frac{1}{2}ab\sin C$$

Example: For $a = 5$, $b = 7$, and $C = 45^\circ$:

$$\text{Area} = \frac{1}{2} \times 5 \times 7 \times \sin 45^\circ = \frac{35}{2} \times \frac{\sqrt{2}}{2} = \frac{35\sqrt{2}}{4} \approx 12.37$$

Unit Circle and Angles

Unit Circle Diagram



The cosine is the x-value, the sine is the y-value.

Radians and Degrees

$$180^\circ = \pi \text{ radians}, \quad 1^\circ = \frac{\pi}{180} \text{ radians}$$

Example: Convert 135° to radians:

$$135^\circ \times \frac{\pi}{180^\circ} = \frac{3\pi}{4}$$

Alternate Formulation

For common angles, sine and cosine values can be calculated as:

$$\sin\theta = \frac{\sqrt{n}}{2}, \quad \cos\theta = \frac{\sqrt{4-n}}{2}$$

where n corresponds to the following angles:

$$\begin{aligned} n &= 0 \quad \text{for } \theta = 0^\circ, \\ n &= 1 \quad \text{for } \theta = 30^\circ, \\ n &= 2 \quad \text{for } \theta = 45^\circ, \\ n &= 3 \quad \text{for } \theta = 60^\circ, \\ n &= 4 \quad \text{for } \theta = 90^\circ. \end{aligned}$$

Reference Angles on the Unit Circle

To find the trigonometric functions of any angle, find its reference angle and determine the sign based on the quadrant.

Example: Find $\tan 225^\circ$.

$$\theta_{\text{ref}} = 225^\circ - 180^\circ = 45^\circ$$

$$\tan 225^\circ = \tan(180^\circ + 45^\circ) = \tan 45^\circ = 1$$

Since 225° is in Quadrant III, tan is positive.

Example: Find $\sin 210^\circ$.

$$\theta_{\text{ref}} = 210^\circ - 180^\circ = 30^\circ$$

$$\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$$

Since 210° is in Quadrant III, sin is negative.

Example: Find $\cos 300^\circ$.

$$\theta_{\text{ref}} = 360^\circ - 300^\circ = 60^\circ$$

$$\cos 300^\circ = \cos(360^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

Since 300° is in Quadrant IV, cos is positive.

Coterminal Angles

Angles that differ by full rotations (360°) are coterminal.

Example: $\theta = -30^\circ$ is coterminal with 330° because:

$$-30^\circ + 360^\circ = 330^\circ$$