Algebra and Polynomials Cheat Sheet

Polynomial Basics

A polynomial in one variable x has the form:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with real or complex coefficients and $a_n \neq 0$.

Degree: The highest exponent n.
Leading Coefficient: The coefficient a_n.
Example:

$$4x^3 - 3x^2 + 2x - 1$$

has degree 3 and leading coefficient 4.

Factoring Techniques

Greatest Common Factor (GCF)

Factor out the largest common factor from all terms.
Example:

$$6x^3 + 9x^2 = 3x^2(2x + 3).$$

Difference of Squares

$$a^2 - b^2 = (a + b)(a - b).$$

Example:

$$x^2 - 9 = (x + 3)(x - 3).$$

Sum and Difference of Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Example:

$$8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9).$$

Factoring Quadratics (Trinomials)

For $ax^2 + bx + c$:
- If $a = 1$:

$$x^2 + bx + c = (x + m)(x + n), \quad m + n = b,\ mn = c.$$

Example:

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

- If $a \neq 1$ (AC Method): 1. Compute ac. 2. Find m, n such that $m + n = b$ and $mn = ac$. 3. Rewrite and factor by grouping.
Example:

$$2x^2 + 7x + 3 : \quad ac = 6,\ \text{find } (6, 1).$$

$$2x^2 + 7x + 3 = 2x^2 + 6x + x + 3 = 2x(x + 3) + (x + 3) = (2x + 1)(x + 3).$$

Perfect Square Trinomials

$$a^2 \pm 2ab + b^2 = (a \pm b)^2.$$

Example:

$$x^2 + 6x + 9 = (x + 3)^2.$$

Factoring by Grouping

Used for four-term polynomials by grouping terms and factoring out common factors.
Example:

$$x^3 + 3x^2 + 2x + 6.$$

Group:

$$(x^3 + 3x^2) + (2x + 6) = x^2(x + 3) + 2(x + 3) = (x^2 + 2)(x + 3).$$

Sum of Squares and Complex Factoring

a^2+b^2 does not factor over the reals, but $a^2+b^2 = (a+bi)(a-bi)$ over

Example:

$$x^2 + 1 = (x + i)(x - i).$$

Expanding $(a + b)^2$ with Imaginary Numbers:

$$(a + i)^2 = a^2 + 2ai + i^2 = a^2 + 2ai - 1.$$

Quadratic Equations

Standard Form

$$ax^2 + bx + c = 0.$$

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Discriminant: $\Delta = b^2 - 4ac$ - $\Delta > 0$: Two distinct real roots. - $\Delta = 0$: One real double root. - $\Delta < 0$: Two complex conjugate roots.
Example: Solve $x^2 - 4x + 3 = 0$:

$$\Delta = 16 - 12 = 4 > 0.$$

$$x = \frac{4 \pm 2}{2} = 3 \text{ or } 1.$$

Completing the Square

Convert $ax^2 + bx + c$ to vertex form: 1. If $a \neq 1$, divide through by a. 2. Move c/a to the other side. 3. Add $\left(\frac{b}{2a}\right)^2$ to both sides. 4. Factor the perfect square.
Example:

$$x^2 + 6x + 5 : \quad x^2 + 6x = -5.$$

Add $(6/2)^2 = 9$:

$$x^2 + 6x + 9 = 4 \implies (x + 3)^2 = 4.$$

Complex Roots and Conjugates

If $a + bi$ is a root, $a - bi$ is also a root.
Example: For $x^2 + 1 = 0$: Roots: i and $-i$.

Rational Roots and Rational Root Theorem

For a polynomial $a_n x^n + \cdots + a_0$, if there is a rational root $\frac{p}{q}$ (in lowest terms), then: - $p \mid a_0$ (p divides the constant term) - $q \mid a_n$ (q divides the leading coefficient)

Example: Consider $2x^3 - 3x^2 + 4x - 6$. Possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 6$ divided by the leading coeff factors (1,2). So test $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$, etc.

If a certain rational root is known, you can use it to find missing coefficients. **Example:** Suppose a polynomial $2x^2 + mx + 3$ has a root $\frac{3}{2}$. Substitute $x = \frac{3}{2}$:

$$2\left(\frac{3}{2}\right)^2 + m\left(\frac{3}{2}\right) + 3 = 0.$$

$$2 \cdot \frac{9}{4} + \frac{3m}{2} + 3 = 0 \implies \frac{9}{2} + \frac{3m}{2} + 3 = 0.$$

Multiply by 2:

$$9 + 3m + 6 = 0 \implies 3m + 15 = 0 \implies m = -5.$$

Sum and Product of Roots

For $ax^2 + bx + c = 0$: - Sum of roots $= -\frac{b}{a}$ - Product of roots $= \frac{c}{a}$

Example:

$$x^2 - 5x + 6 = 0.$$

Sum of roots $= 5$, product of roots $= 6$. If you know sum $= 5$ and product $= 6$, you can reconstruct the quadratic: $x^2 - (sum)x + (product) = x^2 - 5x + 6$.

For a cubic $ax^3 + bx^2 + cx + d = 0$: - Sum of roots $= -b/a$ - Sum of product of roots taken two at a time $= c/a$ - Product of roots $= -d/a$

Dividing Polynomials

Long Division

Use polynomial long division if deg(dividend) $\geq$ deg(divisor).

Example: Divide $(2x^3 + 3x^2 - x + 5)$ by $(x + 2)$:

$$\begin{array}{r|rrr}
 & 2x^2 & -x & +1 \\
\hline
x+2 & 2x^3 + 3x^2 & -x & +5 \\
 & -2x^3 - 4x^2 & & \\
\hline
 & -x^2 & -x & \\
 & x^2 + 2x & & \\
\hline
 & & x & +5 \\
 & & -x & -2 \\
\hline
 & & & 3
\end{array}$$

Result: Quotient $= 2x^2 - x + 1$, Remainder $= 3$.

Synthetic Division

Faster method when dividing by $(x - r)$.

Example: Divide $2x^3 - 3x^2 + 4x - 5$ by $x - 2$:

$$\begin{array}{c|cccc}
2 & 2 & -3 & 4 & -5 \\
 & & 4 & 2 & 12 \\
\hline
 & 2 & 1 & 6 & 7
\end{array}$$

Quotient: $2x^2 + x + 6$, Remainder: 7.

Partial Fraction Decomposition

Express $\frac{P(x)}{Q(x)}$ as a sum of simpler fractions after factoring $Q(x)$.

Example with Repeated Factor:

$$\frac{2x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Multiply through by $(x-1)^2$:

$$2x + 3 = A(x-1) + B.$$

Set $x = 1$:

$$2(1) + 3 = 5 = B.$$

For $x = 0$:

$$3 = A(-1) + 5 \implies -A = -2 \implies A = 2.$$

Thus:

$$\frac{2x+3}{(x-1)^2} = \frac{2}{x-1} + \frac{5}{(x-1)^2}.$$

Asymptotes of Rational Functions

Vertical Asymptotes

Where denominator $= 0$ (and numerator $\neq 0$).

Example:

$$f(x) = \frac{x+2}{x^2-4}, \ x = \pm 2 \text{ are vertical asymptotes.}$$

Horizontal/Oblique Asymptotes

Compare degrees of numerator (N) and denominator (D): - $\deg(N) < \deg(D)$: $y = 0$. - $\deg(N) = \deg(D)$: $y = \frac{\text{leading coeff(N)}}{\text{leading coeff(D)}}$. - $\deg(N) > \deg(D)$: No horizontal asymptote; may have an oblique asymptote.

Example:

$$f(x) = \frac{2x^2 + 3}{x^2 - x}.$$

Degrees equal:

$$y = \frac{2}{1} = 2.$$

Graphing Polynomials and Roots

Multiplicity of Roots

- Odd multiplicity: graph crosses the x-axis. - Even multiplicity: graph touches and turns at the x-axis.

Example:

$$f(x) = (x + 2)(x - 1)^2$$

Root -2 (odd): crosses. Root 1 (even): touches and turns.

End Behavior

- Even degree, positive leading coeff: Up, Up. - Even degree, negative leading coeff: Down, Down. - Odd degree, positive leading coeff: Down, Up. - Odd degree, negative leading coeff: Up, Down.

Intermediate Value Theorem

If $f(a)$ and $f(b)$ have opposite signs, there's at least one root in (a, b).

Parabolas

A parabola is the graph of a quadratic function.

Standard Form:

$$y = ax^2 + bx + c.$$

Vertex Form:

$$y = a(x - h)^2 + k,$$

where (h, k) is the vertex and the axis of symmetry is $x = h$.

To find the vertex from the standard form, use completing the square or $-\frac{b}{2a}$:

$$h = -\frac{b}{2a}, \quad k = f(h).$$

Example:

$$y = 2x^2 + 4x + 1.$$

Find vertex:

$$h = -\frac{4}{2 \cdot 2} = -1;\ k = 2(-1)^2 + 4(-1) + 1 = 2 - 4 + 1 = -1.$$

Vertex: $(-1, -1)$, so

$$y = 2(x + 1)^2 - 1.$$

Parabolas open upwards if $a > 0$ and downwards if $a < 0$.

Hyperbolas

A hyperbola is the set of all points (x, y) in the plane such that the *difference* of their distances to two fixed points (the foci) is constant.

Standard Forms

Depending on whether the transverse axis is horizontal or vertical, the standard form of a hyperbola centered at (h, k) can be written as:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{(horizontal transverse axis)},$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \quad \text{(vertical transverse axis)}.$$

Here: - (h, k) is the center of the hyperbola. - a is the semi-major axis (distance from center to each vertex). - b relates to the "co-vertices" and the asymptotes. - The asymptotes for the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ are given by

$$y - k \;=\; \pm\frac{b}{a}\,(x - h).$$

- The foci are located c units from the center, where $c^2 = a^2 + b^2$.

Getting a Hyperbola into Standard Form from Polynomial Form

Often, a hyperbola starts in the form of a general quadratic equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

with certain conditions on A, B, and C (specifically, for a hyperbola you typically have $B^2 - 4AC > 0$ when $B = 0$; or more generally, the discriminant indicates a hyperbola).

Steps to rewrite into standard form: 1. **Group the x-terms and y-terms**: Separate the equation into something like

$$Ax^2 + Dx \;+\; Cy^2 + Ey \;=\; -F.$$

2. **Factor out the coefficients of the squared terms**:

$$A\bigl(x^2 + \tfrac{D}{A}x\bigr) \;+\; C\bigl(y^2 + \tfrac{E}{C}y\bigr) = -F.$$

3. **Complete the square** for the x-group and the y-group separately:

$$A\Bigl(x^2 + \tfrac{D}{A}x + \dots\Bigr) - A(\dots) + C\Bigl(y^2 + \tfrac{E}{C}y + \dots\Bigr) - C(\dots) = -F.$$

The "..." refers to the term you add (and subtract) to complete the square. 4. **Rearrange and combine constants** to isolate perfect square expressions in x and y. 5. **Divide through** by whatever constant makes the right side $= 1$ (or -1, then multiply by -1 if needed). You'll end up with one positive term and one negative term (equal to 1), matching one of the standard forms for a hyperbola.

Example: Rewrite $4x^2 - 9y^2 + 8x + 18y = 77$ in standard form.

1. Group x-terms and y-terms:

$$4x^2 + 8x \;-\; 9y^2 + 18y = 77.$$

2. Factor out the coefficients of squared terms:

$$4(x^2 + 2x) \;-\; 9(y^2 - 2y) = 77.$$

3. Complete the square for each group:

   For $x^2 + 2x$:
$$x^2 + 2x + 1 - 1 = (x+1)^2 - 1.$$
   For $y^2 - 2y$:
$$y^2 - 2y + 1 - 1 = (y-1)^2 - 1.$$
   Substituting these back,
$$4\big[(x+1)^2 - 1\big] \; - \; 9\big[(y-1)^2 - 1\big] = 77.$$
   Distribute:
$$4(x+1)^2 - 4 \; - \; 9(y-1)^2 + 9 = 77.$$
   Combine constants:
$$4(x+1)^2 - 9(y-1)^2 + (9-4) = 77 \quad \Longrightarrow \quad 4(x+1)^2 - 9(y-1)^2 + 5 = 77.$$
$$4(x+1)^2 - 9(y-1)^2 = 72.$$

4. Divide through by 72:
$$\frac{4(x+1)^2}{72} \; - \; \frac{9(y-1)^2}{72} = 1,$$
   Simplify:
$$\frac{(x+1)^2}{18} \; - \; \frac{(y-1)^2}{8} = 1.$$

5. This is in the standard hyperbola form:
$$\frac{(x-(-1))^2}{(\sqrt{18})^2} \; - \; \frac{(y-1)^2}{(\sqrt{8})^2} = 1.$$