

Vector Notation and Representation

Unit Vector Notation

Vectors can be represented in different notations:

- **Angle Bracket Notation:**

A vector \vec{V} in two dimensions can be written as:

$$\vec{V} = \langle V_x, V_y \rangle$$

- **Column Vector Notation:**

$$\vec{V} = \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

- **Cartesian Coordinates:**

Using the unit vectors \hat{i} and \hat{j} :

$$\vec{V} = V_x \hat{i} + V_y \hat{j}$$

Where:

- V_x is the component of the vector along the x -axis. - V_y is the component of the vector along the y -axis. - \hat{i} and \hat{j} are unit vectors in the x and y directions, respectively.

Zero-Length and Parallel Vectors

- **Zero-Length Vector (Zero Vector):**

A vector with zero magnitude and no specific direction.

$$\vec{0} = \langle 0, 0 \rangle$$

Properties:

- Adding the zero vector to any vector leaves the original vector unchanged.

$$\vec{V} + \vec{0} = \vec{V}$$

- **Parallel Vectors:**

Two vectors are parallel if they have the same or opposite directions.

Conditions for Parallelism:

- Vectors \vec{A} and \vec{B} are parallel if:

$$\vec{A} = k\vec{B}$$

where k is a scalar constant.

- If $k > 0$, vectors are in the same direction. - If $k < 0$, vectors are in opposite directions.

- **Component Proportionality:**

$$\frac{A_x}{B_x} = \frac{A_y}{B_y}$$

If the ratios of the corresponding components are equal (and finite), the vectors are parallel.

Position Vectors and Vector Difference

If A and B are points in the Cartesian plane with position vectors \vec{a} and \vec{b} respectively, then the vector from point A to point B is given by:

$$\vec{AB} = \vec{b} - \vec{a}$$

This represents the displacement from point A to point B .

Example:

Let point A have coordinates $(2, 3)$ and point B have coordinates $(5, 7)$.

- Write the position vectors:

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

- Compute \vec{AB} :

$$\vec{AB} = \vec{b} - \vec{a} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5-2 \\ 7-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

- The magnitude of \vec{AB} is:

$$|\vec{AB}| = \sqrt{3^2 + 4^2} = 5$$

- The direction (angle) of \vec{AB} with respect to the horizontal axis is:

$$\theta = \arctan\left(\frac{4}{3}\right) \approx 53.13^\circ$$

Vector Decomposition and Addition

Decomposing Vectors into Components

Any vector \vec{V} can be decomposed into its horizontal and vertical components using trigonometric functions.

Given a vector of magnitude V at an angle θ with respect to the horizontal axis:

$$V_x = V \cos \theta$$

$$V_y = V \sin \theta$$

Adding Vectors Using Components

To add two vectors, decompose each vector into its components, add the corresponding components, and then recombine the resultant components to find the resultant vector.

Given two vectors \vec{A} and \vec{B} :

$$A_x = A \cos \alpha$$

$$A_y = A \sin \alpha$$

$$B_x = B \cos \beta$$

$$B_y = B \sin \beta$$

The resultant vector $\vec{R} = \vec{A} + \vec{B}$ has components:

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

The magnitude and direction of \vec{R} are:

$$R = \sqrt{R_x^2 + R_y^2}$$

$$\theta_R = \arctan\left(\frac{R_y}{R_x}\right)$$

Dot Product and Cosine Similarity

The **dot product** (or scalar product) of two vectors $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$ is defined as:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

The magnitudes of the vectors are $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ and $\|\vec{B}\| = \sqrt{B_x^2 + B_y^2 + B_z^2}$.

The **cosine similarity**, which gives the cosine of the angle θ between \vec{A} and \vec{B} , is:

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\|, \|\vec{B}\|}$$

Perpendicularity of Vectors

Two vectors \vec{A} and \vec{B} are **perpendicular** (orthogonal) if and only if their dot product is zero:

$$\vec{A} \cdot \vec{B} = 0 \iff \cos \theta = 0 \quad (\theta = 90^\circ)$$

This condition can be checked algebraically using the component form of the dot product or geometrically using the cosine of the angle.

Vector Equation of a Plane

A plane in three dimensions can be defined by a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector $\vec{n} = \langle a, b, c \rangle$ perpendicular to the plane.

Let $\vec{r} = \langle x, y, z \rangle$ be the position vector of an arbitrary point $P(x, y, z)$ on the plane, and let $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ be the position vector of P_0 .

The **vector equation of the plane** is:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

Equivalently, the **scalar equation** of the plane is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Matrices

Matrix Multiplication Algorithm

To multiply two matrices A and B :

- Let A be an $m \times n$ matrix and B an $n \times p$ matrix. - The product $C = AB$ is an $m \times p$ matrix where:

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}.$$

Matrices and Vectors

- A vector $\vec{v} \in \mathbb{R}^n$ can be represented as a column matrix:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

- To transform \vec{v} using a matrix A :

$$\vec{v}' = A\vec{v}.$$

Example: Let

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$A\vec{v} = \begin{bmatrix} 2x + 0 \cdot y \\ -1 \cdot x + 3y \end{bmatrix}.$$

Determinant of a Matrix

For a square matrix A :

- 2×2 :

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Example: For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $\det(A) = 2 \cdot 4 - 3 \cdot 1 = 5$.

- 3×3 :

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Example: For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\det(A) = 0$ (rows are linearly dependent).

Inverse of a 2×2 Matrix Using the Determinant

Given a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

if $\det(A) = ad - bc \neq 0$, then the inverse of A is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. We already found that $\det(A) = 5$, so:

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}.$$

Matrix Minor and Cofactor

- The *minor* M_{ij} of element A_{ij} is the determinant of the matrix formed by deleting the i -th row and j -th column. - The *cofactor* is $C_{ij} = (-1)^{i+j} M_{ij}$.

Example: Let

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$M_{1,3} = \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2, \quad C_{1,3} = (-1)^{1+3} \cdot 2 = 2.$$

Cross Product via Cofactor Expansion

Given two 3D vectors

$\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \quad \mathbf{b} = \langle b_1, b_2, b_3 \rangle,$

the cross product $\mathbf{a} \times \mathbf{b}$ is defined as the determinant of the following 3×3 matrix:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Example: Let

$\mathbf{a} = \langle 2, -1, 5 \rangle, \quad \mathbf{b} = \langle -1, 4, 2 \rangle.$

Then:

$$\begin{aligned} \mathbf{i} : \quad & \begin{vmatrix} -1 & 5 \\ 4 & 2 \end{vmatrix} = (-1)(2) - (5)(4) = -2 - 20 = -22, \\ \mathbf{j} : \quad & \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = (2)(2) - (5)(-1) = 4 + 5 = 9, \\ \mathbf{k} : \quad & \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} = (2)(4) - (-1)(-1) = 8 - 1 = 7. \end{aligned}$$

So,

$\mathbf{a} \times \mathbf{b} = -22\mathbf{i} - 9\mathbf{j} + 7\mathbf{k} = \langle -22, -9, 7 \rangle.$

Identity Matrix

The identity matrix I_n is the $n \times n$ matrix with ones on the diagonal and zeros elsewhere.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad AI_n = I_nA = A.$$

Augmented Matrices

An augmented matrix $[A|\vec{b}]$ represents a system $A\vec{x} = \vec{b}$.

Example: The system

$$\begin{cases} x - y + 2z = 3 \\ 2x + 3y + z = 7 \\ -x + 4y + 5z = -2 \end{cases}$$

has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 3 & 1 & 7 \\ -1 & 4 & 5 & -2 \end{array} \right].$$

Easy-to-Invert Matrices

Certain matrix classes have straightforward inverses:

Diagonal Matrices A diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ is invertible iff $d_i \neq 0$ for all i , with

$D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1}).$

Example: If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then $D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}.$

Permutation Matrices A permutation matrix P is obtained by permuting rows of I_n . It satisfies $P^{-1} = P^T$. Example: Swapping

rows 1 and 2 in I_3 gives $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $P^{-1} = P^T = P.$

Frobenius (Shear) Matrices A Frobenius (or elementary shear) matrix is of the form

$F = I + U,$

where all nonzero off-diagonal entries of U lie in a single column (or row) and $U^2 = 0$. Hence

$F^{-1} = I - U.$

Example: Let

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \quad F = I_4 + U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$F^{-1} = I_4 - U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Row Echelon Form and Gaussian Elimination

Row Echelon Form (REF): A matrix is in REF if:

- All nonzero rows are above rows of all zeros.
- The leading entry (pivot) of each nonzero row is to the right of the pivot above.
- All entries below a pivot are zero.

Gaussian Elimination Algorithm: To solve $A\vec{x} = \vec{b}$:

1. Form $[A|\vec{b}]$.
2. Apply row operations (swap, scale, add multiples) to reach REF.
3. (Optional) Continue to RREF for immediate solutions.
4. Back-substitute if in REF.

Example: Solve

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 6 \end{cases}$$

Augmented:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & -1 & 6 \end{array} \right].$$

Steps lead to RREF and the solution $(x, y) = (19/7, 4/7).$

Reduced Row Echelon Form (RREF)

In addition to REF:

- Each pivot = 1.
- Each pivot is the only nonzero in its column.

Example: For

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix},$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\div(-2)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{textsubtract}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$