Roadmap: From Continued Fractions to Modular Forms and Elliptic Curves

Expanded Curriculum with Bridges, Applications, and Fluency Plan

Audience and tone. For a mathematically gifted 9-year-old, comfortable with arithmetic, introductory algebra, and eager to explore. Lessons remain visual, computational, and playful, with proofs postponed unless short and accessible.

Materials. Graph paper, straightedge, compass, colored pencils; calculator or plotting app capable of parametric plots and discrete sampling.

Unit A — Tiny Complex/Calculus Boosters

Goal: Comfort with rotations, complex numbers, series, and Fourier intuition; prepare for Möbius maps and q-series.

Lesson A1: Complex Numbers and Rotations

What we will build. Interpret $x + iy$ in polar form $re^{i\theta}$, and use multiplication by $e^{i\theta}$ to rotate points in $\mathbb{C}$. Learn to rotate about arbitrary centers a via $a + e^{i\theta}(z - a)$.

- **Activities:** Convert between rectangular and polar forms; compute $e^{i\pi/3}$ and $e^{i\pi/4}$; rotate $(2, -1)$ by 45° about origin and $(1, 2)$.
- **Visual/Computer:** Plot $z(\theta) = e^{i\theta}(2 - i)$ for $\theta \in [0, 2\pi]$ and see the circle.

Lesson A2: Power Series in One Page

What we will build. Approximate $e^x, \sin x, \cos x$ using Taylor series; apply Newton's binomial theorem $(1 + x)^\alpha = \sum \binom{\alpha}{n} x^n$.

- **Activities:** Expand $(1 + x)^{1/2}$ up to x^4; approximate $\sqrt{1.04}$ and compare.
- **Visual/Computer:** Overlay $y = \sin x$ with its cubic Taylor polynomial near 0.

Lesson A3: First Look at Fourier

What we will build. Express periodic functions as $f(\theta) = a_0 + \sum_{n\geq 1}(a_n \cos n\theta + b_n \sin n\theta)$; connect to beats identity.

- **Activities:** Sample an 8-point square wave; identify odd harmonics.
- **Visual/Computer:** Plot S_1, S_3, S_5 partial sums; observe Gibbs overshoot.

Lesson A4 (Bridge): Matrices & Simple Möbius Maps

What we will build. Read 2×2 integer matrices, determinant $ad - bc$; compose simple maps $x \mapsto \frac{ax+b}{cx+d}$.

- **Activities:** Multiply $M_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; interpret T, S.
- **Visual/Computer:** Apply S, T to points on real line; plot the motion.

Capstone A:

1. Rotate multiple complex points about origin and arbitrary centers; verify distances preserved.
2. Approximate $\sin(0.3)$, $e^{0.5}$, $(1.02)^{1/2}$ via Taylor/binomial series; estimate remainder.
3. Sample and Fourier-approximate a simple wave; compare to original.
4. Given simple Möbius maps, express as products in S, T; apply to a set of points.

Unit B — Continued Fractions to $\mathrm{SL}_2(\mathbb{Z})$

Lesson B1: CF as Matrix Products

What we will build. Compute convergents of $\alpha = [a_0; a_1, a_2, \dots]$ via $M(a) = \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$; product $M(a_0)\cdots M(a_k) = \left(\begin{smallmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{smallmatrix}\right)$.

- **Activities:** Compute $\sqrt{14}$ convergents by matrix and recurrence.
- **Visual:** Tree diagram of rationals with CF branches.

Lesson B2: Möbius Transformations and Fundamental Domain

What we will build. Action $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ on $\mathbb{H}$; reduce to $\mathcal{F} = \{|\Re\tau| \le \frac{1}{2}, |\tau| \ge 1\}$ using S, T.

- **Activities:** Reduce $\tau = 3 + 2i$ to $\mathcal{F}$; record S, T word.
- **Visual:** Modular tessellation; orbit of a point.

Lesson B3: Farey Sequences and Ford Circles

What we will build. Best approximations; mediant $\frac{p+r}{q+s}$ between $\frac{p}{q}$ and $\frac{r}{s}$; tangency $ps - qr = 1$.

- **Activities:** Draw Ford circles with $q \le 5$; identify neighbors of 2/5.

Lesson B4: Quadratic Irrationals as Closed Geodesics

What we will build. Connect periodic CFs of $\sqrt{D}$ to closed loops on modular surface.

- **Activities:** Use $\sqrt{14}$ period to trace loop on tessellation.

Lesson B5: CFs in the Wild

What we will build. Apply CFs to best approximations; explain 355/113 for π.

- **Activities:** Approximate e and $\sqrt{2}$ to within 10^{-4} by CF.

Lesson B6: Nested Radicals & Quadratic Surprises

What we will build. Evaluate $\sqrt{k + \sqrt{k + \cdots}}$ as quadratic fixed points; trigonometric connection $\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} = 2\cos(\pi/2^{n+1})$; Viète's product for $\frac{2}{\pi}$.

- **Activities:** Solve $\sqrt{6 + \sqrt{6 + \cdots}}$; compare to CF expansion.
- **Visual:** Plot partial approximations.

Bridge to C: Fractions Modulo m

What we will build. Understand $\frac{a}{b}$ mod m when $\gcd(b, m) = 1$; invert small denominators.

Capstone B:

1. Reduce several τ to $\mathcal{F}$, recording S, T words.
2. Draw Ford circles up to $q = 7$; verify mediant property visually.
3. Match value of a nested radical to a quadratic irrational via CF expansion.

Unit C — Congruences and Quadratic Residues

Lesson C1: Modular Arithmetic and CRT

What we will build. Compute inverses mod m; solve $x \equiv a \pmod{m_1}$, $x \equiv b \pmod{m_2}$.

- **Activities:** Invert 7 mod 26; solve $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$.
- **Visual:** Wrap-around number lines.

Lesson C2: Quadratic Residues and Legendre Symbol

What we will build. List squares mod p; compute $\left(\frac{a}{p}\right)$ via Euler's criterion.

- **Activities:** Table squares mod 7, 11; evaluate $\left(\frac{2}{7}\right)$.
- **Visual:** Color quadratic residues on $p \times p$ grid.

Lesson C3: A Friendly Theta Function

What we will build. $\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}$; $\theta(q)^2$ counts $x^2 + y^2$.

- **Activities:** List coefficients to q^9; count integer solutions to $x^2 + y^2 = m$.
- **Visual:** Lattice points on circles.

Lesson C4: Fast Modular Exponentiation and Miller–Rabin

What we will build. Compute a^n mod m by square-and-multiply; run Miller–Rabin primality test.

- **Activities:** 3^{100} mod 13; test $n = 561$ with $a = 2, 3$.

Lesson C5: Square Roots mod p (Tonelli–Shanks)

What we will build. Solve $x^2 \equiv a \pmod p$ for odd prime p.

- **Activities:** $\sqrt{10}$ mod 13; verify by squaring.

Capstone C:

1. Solve 3 CRT problems quickly.
2. Invert several a mod m and check with multiplication.
3. Run Miller–Rabin on a 5-digit composite.
4. Find $\sqrt{a}$ mod p for two cases and verify.

Unit D — Elliptic Curves Two Ways

Pre-Bridge: From Grid to Donut

What we will build. Turn $\mathbb{R}^2/\mathbb{Z}^2$ into a paper torus; generalize to $\mathbb{C}/\Lambda$.

Lesson D1: Lattices and Complex Tori

What we will build. Lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$; fundamental parallelograms.

- **Activities:** Draw $\tau = i$ and $\tau = e^{i\pi/3}$ lattices.

Lesson D2: Tori to Weierstrass Cubics (Peek)

What we will build. Statement $\mathbb{C}/\Lambda \leftrightarrow y^2 = 4x^3 - g_2x - g_3$.

Lesson D3: Group Law on a Cubic

What we will build. Chord-tangent addition; reflection in x-axis.

- **Activities:** On $y^2 = x^3 - x$, compute $(0,0) + (1,0)$.

Lesson D4: Counting Points mod p; a_p

What we will build. Count $\#E(\mathbb{F}_p)$; define $a_p = p + 1 - \#E(\mathbb{F}_p)$.

- **Activities:** Count for $p = 5, 7, 11$; make table.

Lesson D5: Scalar Multiplication (Double-and-Add)

What we will build. Compute nP efficiently from binary n.

Lesson D6: Toy ECC

What we will build. Simulate ECDH over $\mathbb{F}_p$ on a toy curve.

- **Activities:** Pick base point G; exchange public keys; compute shared secret.

Capstone D:

1. Add, double, and scalar multiply points on a curve.
2. Count $\#E(\mathbb{F}_p)$ for $p \leq 19$; plot a_p.
3. Run a toy ECC key exchange end-to-end.

Unit E — Modular Forms and the Bridge

Pre-Bridge Recap: Why q-series? Why S, T?

What we will build. Periodicity $\tau \mapsto \tau + 1 \Rightarrow$ Fourier in $q = e^{2\pi i\tau}$; $\tau \mapsto -1/\tau$ symmetry in $\mathbb{H}$.

Lesson E1: What is a Modular Form?

What we will build. $f : \mathbb{H} \to \mathbb{C}$ with $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f(\tau)$; q-expansion.

Lesson E2: Eisenstein Series

What we will build. $E_4(\tau) = 1 + 240 \sum \sigma_3(n)q^n$, $E_6(\tau) = 1 - 504 \sum \sigma_5(n)q^n$.

- **Activities:** Make divisor-sum tables $n \leq 10$; compute coefficients.

Lesson E3: Δ and j-Invariant

What we will build. $\Delta = q \prod(1 - q^n)^{24}$; $j(\tau) = 1728\, \frac{E_4^3}{E_4^3 - E_6^2}$.

Lesson E4: Hecke Operators (Taste)

What we will build. $(T_p f)(q) = \sum(a_{pn} + p^{k-1} a_{n/p}) q^n$.

Lesson E5: Bridge: Elliptic Curves ↔ Modular Forms

What we will build. Modularity theorem: $E/\mathbb{Q}$ matches weight-2 newform with coefficients a_p.

Lesson E6: Ramanujan and Chudnovsky π Formulas

What we will build. Rapidly converging π series from modular equations; e.g.,

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

Capstone E:

1. Compute q-expansion coefficients for E_4, E_6, Δ to $n = 8$.
2. Match a_p table of a curve to weight-2 modular form.
3. Compute π to 6+ digits with first few terms of Ramanujan's series.

Unit F (Optional Story): Fermat's Last Theorem Pipeline

Fermat → Frey curve → Ribet → Wiles. Map of the territory.

Fluency Plan

Daily skill ladders:

- **CF/matrices:** 3 CF expansions to depth 8; 3 matrix products; one τ to $\mathcal{F}$.
- **Mod arithmetic:** 5 inverses mod m; 2 CRT problems; one Tonelli–Shanks.
- **Elliptic:** Add/double 2 points mod p; count $\#E(\mathbb{F}_p)$.
- **q-series:** Generate coefficients for E_4, E_6, Δ up to $n = 10$.

Milestones

- **End A:** Rotate, expand series, do simple Fourier.
- **End B:** CF mastery, Ford circles, nested radicals.
- **End C:** CRT fluency, Miller–Rabin, Tonelli–Shanks.
- **End D:** ECC simulation, a_p computation.
- **End E:** q-expansion fluency, Ramanujan π computation, modular form–elliptic curve match.