Lesson A1: Complex Numbers, Euler's Formula, and Rotations

Expanded, Step-by-Step Version for a Gifted Young Learner

What we will build

We link Euler's formula
$$e^{i\theta} = \cos\theta + i\sin\theta$$
to *rotating points in the plane.* By the end you can:

- move between rectangular form $x + iy$ and polar form $re^{i\theta}$,
- rotate about the origin by multiplying by $e^{i\theta}$,
- rotate about *any* center a using $a + e^{i\theta}(z - a)$,
- do the same rotation using the *rotation matrix* R_θ,
- run quick *checks* (distances preserved, special angles).

Audience note. We keep proofs tiny and hands-on. You will draw, compute, and check. (Parents/mentors: light calculator use is fine; exact values preferred at special angles.)

Vocabulary, Symbols, and the "sent to" notation

- i: the imaginary unit with $i^2 = -1$.
- $z = x+iy$: a complex number; $\Re z = x$, $\Im z = y$. The point (x, y) and the complex number $x + iy$ are two names for the same thing.
- $|z| = \sqrt{x^2 + y^2}$: *modulus* (distance from the origin).
- $\arg z$: *argument* (angle from the positive x-axis to z; measured in radians).
- $\overline{z} = x - iy$: *conjugate*; $z\overline{z} = |z|^2$.
- **"Sent to" notation.** When we write
$$(x, y) \ \mapsto\ (x', y'),$$
we mean: "our rule *sends* the point (x, y) to the new point (x', y')." Read $\mapsto$ as "goes to."

Units tip (radians vs degrees). We work in *radians.* To convert: $90° = \pi/2$, $60° = \pi/3$, $45° = \pi/4$, $30° = \pi/6$. In general, radians = $(\pi/180°) \times$ degrees.

Core idea: Multiplying by $e^{i\theta}$ rotates by θ

Let $z = x + iy$. Using Euler's formula,
$$e^{i\theta}z = (\cos\theta + i\sin\theta)(x + iy) = (x\cos\theta - y\sin\theta) + i\,(x\sin\theta + y\cos\theta).$$
So the rule is
$$(x, y) \ \mapsto\ \big(x\cos\theta - y\sin\theta,\quad x\sin\theta + y\cos\theta\big).$$

Rotation matrix form. Writing points as column vectors,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R_\theta} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Two quick checks (distance is preserved).

- *Complex check:* $|e^{i\theta}| = 1 \Rightarrow |e^{i\theta} z| = |z|$.
- *Coordinate check:* $x'^2 + y'^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = x^2 + y^2$.

Rotations are *rigid motions*: lengths and angles stay the same.

Algorithm 1: Rectangular $\leftrightarrow$ Polar (with quadrant adjustment)

Goal: Write $z = x + iy$ as $z = re^{i\theta}$ with $r \geq 0$, $\theta \in (-\pi, \pi]$.

Algorithm 1A: Rectangular $\to$ Polar

Step 1. Compute the modulus: $r = \sqrt{x^2 + y^2}$.
Step 2. Compute a *base angle*: $\theta_0 = \arctan\left(\frac{|y|}{|x|}\right)$ (in $(0, \pi/2]$ if both nonzero).
Step 3. **Adjust to the correct quadrant** using the signs of (x, y):

- Quadrant I ($x > 0, y > 0$): $\theta = \quad \theta_0$.
- Quadrant II ($x < 0, y > 0$): $\theta = \pi - \theta_0$.
- Quadrant III ($x < 0, y < 0$): $\theta = -(\pi - \theta_0) = \theta_0 - \pi$.
- Quadrant IV ($x > 0, y < 0$): $\theta = -\theta_0$.
- On axes: use $\theta = 0,\ \pm\frac{\pi}{2},\ \pi$ appropriately.

(On a calculator, `atan2(y,x)` does this automatically.)
Step 4. Report $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Sanity check: $r\cos\theta \stackrel{?}{=} x,\ r\sin\theta \stackrel{?}{=} y$.

Worked example (with quadrant adjust). Convert $z = -\sqrt{2} + i\sqrt{2}$.

$$r = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2, \quad \theta_0 = \arctan\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Since $x < 0, y > 0$ (Quadrant II), $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Thus $z = 2e^{i3\pi/4}$. *Check:* $2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = 2(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -\sqrt{2} + i\sqrt{2}$.

Algorithm 1B: Polar $\to$ Rectangular

Given $z = re^{i\theta}$, compute

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Worked example. $z = 3e^{i\pi/6} \Rightarrow x = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2},\ \ y = 3 \cdot \frac{1}{2} = \frac{3}{2}$.

Practice A1 (do, then check):

- $z = 1 + i\sqrt{3} \to re^{i\theta}$.
- $z = 5e^{-i\pi/3} \to x + iy$.
- $z = -3i \to re^{i\theta}$ *and* $x + iy$ (tiny!).

Algorithm 2: Rotate about the origin by θ (complex multiplication)

Input: $z = x + iy$, angle θ. **Output:** $z' = e^{i\theta} z$.

Step 1. (Rectangular method) Compute

$$x' = x\cos\theta - y\sin\theta, \qquad y' = x\sin\theta + y\cos\theta.$$

Step 2. (Optional polar method) If $z = re^{i\phi}$, then $z' = re^{i(\phi+\theta)}$.
Step 3. **Checks:** $|z'| = |z|$ and special-angle sanity (e.g. $\theta = \pi$ flips signs).

Worked example (rectangular). Rotate $z = 2 - i$ by $\theta = \pi/2$.

$$x' = 2 \cdot 0 - (-1) \cdot 1 = 1, \qquad y' = 2 \cdot 1 + (-1) \cdot 0 = 2.$$

So $z' = 1 + 2i$. $|z| = |z'| = \sqrt{5}$.

Workable example (you try). Rotate $z = -2 + 2i$ by $\theta = \pi/4$. *Hint:* $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. *Answer (see Selected Answers):* page **??**.

Draw (graph paper). Plot z and z'. Use a right angle or rotate the paper to eyeball the turn. Distances to the origin should match.

Algorithm 2M: Rotate using the rotation matrix R_θ

Same rotation, but do it by a clear matrix-times-vector computation.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step-by-step dot products.

Step 1. **Row 1 $\cdot$ column:** $x' = (\cos\theta) \cdot x + (-\sin\theta) \cdot y = x\cos\theta - y\sin\theta$.
Step 2. **Row 2 $\cdot$ column:** $y' = (\sin\theta) \cdot x + (\cos\theta) \cdot y = x\sin\theta + y\cos\theta$.
Step 3. **Check orthogonality:** $R_\theta^\top R_\theta = I$ (explains length preservation).

Worked example (matrix). Rotate $z = (x, y) = (2, -1)$ by $\theta = \pi/3$.

$$R_{\pi/3} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

$$x' = \tfrac{1}{2} \cdot 2 + \left(-\tfrac{\sqrt{3}}{2}\right) \cdot (-1) = 1 + \tfrac{\sqrt{3}}{2}, \quad y' = \tfrac{\sqrt{3}}{2} \cdot 2 + \tfrac{1}{2} \cdot (-1) = \sqrt{3} - \tfrac{1}{2}.$$

So $z' = \left(1 + \frac{\sqrt{3}}{2}\right) + i\left(\sqrt{3} - \frac{1}{2}\right)$. *Distance check:* $x'^2 + y'^2 = 2^2 + (-1)^2 = 5$.

Practice A2M.

- Use $R_{\pi/6}$ to rotate $(3, 0)$. Compare with complex multiplication.
- Use R_π to rotate (a, b). What do you notice?

Algorithm 3: Rotate about a point a (not the origin)

Formula: $z_{\text{new}} = a + e^{i\theta}(z - a)$.

Step 1. Translate to the origin: $w = z - a$.
Step 2. Rotate about the origin: $w' = e^{i\theta} w$ (or $w' = R_\theta w$).
Step 3. Translate back: $z_{\text{new}} = a + w'$.
Step 4. **Check:** $|z_{\text{new}} - a| = |w'| = |w| = |z - a|$ (distance to center preserved).

Worked example (complex and compass). Rotate $z = 3 + i$ by $\theta = \pi/3$ about $a = 1 + i$.

$$w = z - a = (3 + i) - (1 + i) = 2, \quad w' = 2e^{i\pi/3} = 2\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3},$$

$$z_{\text{new}} = a + w' = (1 + i) + (1 + i\sqrt{3}) = 2 + i(1 + \sqrt{3}).$$

Distance check: $|z - a| = 2 = |z_{\text{new}} - a|$.

Draw (compass). Plot $a = (1, 1)$, $z = (3, 1)$. With center a and radius $|z - a| = 2$, draw the circle. Mark a 60° arc (equilateral-triangle construction). The intersection is z_{new}.

Practice A3.

- Rotate $z = 0$ by 120° about $a = -1 + i$.
- Rotate $z = (2, 0)$ by 90° about $a = (1, 1)$ (compute and construct).

Mini-labs and sanity checks

Compose rotations (angles add).

$$e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}.$$

Test: Start with $z = 1 + i$. Rotate by 30°, then 60°. Compare with one rotation by 90°.

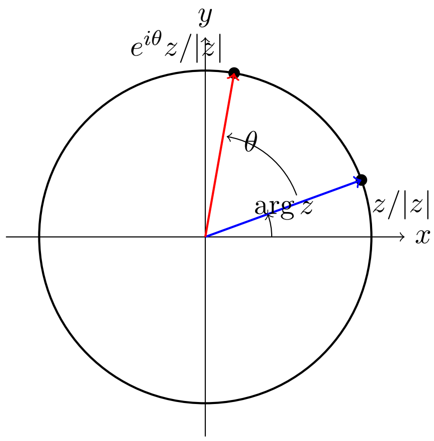
Genuine always-checks.

- **Distance preserved:** $|z| = |e^{i\theta} z|$ and $|z - a| = |z_{\text{new}} - a|$.
- **Special angles:** $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ land on obvious points.
- **Right angles stay right:** grid squares stay squares.

Common slips (and fixes).

- Mixing degrees and radians $\Rightarrow$ convert first.
- Forgetting quadrant adjust $\Rightarrow$ use a sign chart or `atan2(y,x)`.
- Dropping a minus sign in $x\cos\theta - \ y\sin\theta$ $\Rightarrow$ re-derive from the matrix once to reset.

Optional visual: unit circle with a rotated point



Challenge (optional)

- Prove $e^{i\theta}\overline{z} = \overline{e^{-i\theta}z}$. Geometric meaning?
- Find all θ for which rotation by θ keeps the square with vertices $(\pm 1, \pm 1)$ unchanged *as a set*.

Summary and next step

You can now (1) switch forms, (2) rotate about the origin by complex multiplication or by a matrix, and (3) rotate about any center. Next (Lesson A2) we build tiny Taylor/binomial toolkits for $\sin, \cos, e^x$ to prepare for Fourier and q-series.

Selected Answers

Practice A1

- $z = 1 + i\sqrt{3} \Rightarrow r = \sqrt{1+3} = 2,\ \theta = \arctan(\sqrt{3}/1) = \pi/3,\ z = 2e^{i\pi/3}$.
- $z = 5e^{-i\pi/3} \Rightarrow x = 5\cos(-\pi/3) = \frac{5}{2},\ y = 5\sin(-\pi/3) = -\frac{5\sqrt{3}}{2}$.
- $z = -3i \Rightarrow r = 3,\ \theta = -\pi/2$ (or $3\pi/2$ if you prefer $[0, 2\pi)$); as rectangular: $0 - 3i$.

Practice A2 & A2M

- Rotate $z = 3$ by $60°$: $x = 3, y = 0 \Rightarrow x' = 3 \cdot \frac{1}{2} - 0 = \frac{3}{2},\ y' = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \Rightarrow \frac{3}{2} + \frac{3\sqrt{3}}{2}i$.
- Rotate $z = -2 + 2i$ by $45°$:

$$x' = -2 \cdot \frac{\sqrt{2}}{2} - 2 \cdot \frac{\sqrt{2}}{2} = -2\sqrt{2}, \quad y' = -2 \cdot \frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2} = 0,$$

  so $z' = -2\sqrt{2} + 0\,i$. Distance check: $|z| = \sqrt{8} = 2\sqrt{2} = |z'|$.
- Using $R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, any $(a, b) \mapsto (-a, -b)$ (a $180°$ half-turn).

Practice A3

- Rotate $z = 0$ by $120°$ about $a = -1 + i$:

$$z_{\text{new}} = a + e^{i2\pi/3}(0 - a) = (-1 + i) + e^{i2\pi/3}(1 - i).$$

  Compute $e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

  $(1 - i) \cdot \Big(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \Big) = -\frac{1}{2} - i\Big(-\frac{\sqrt{3}}{2} \Big) + i\Big(-\frac{1}{2} \Big) + i^2\Big(-\frac{\sqrt{3}}{2} \Big) = -\frac{1}{2} + i\Big(\frac{\sqrt{3}}{2} - \frac{1}{2} \Big) + \frac{\sqrt{3}}{2}.$

  So

$$z_{\text{new}} = \Big(-1 - \frac{1}{2} + \frac{\sqrt{3}}{2} \Big) + i\Big(1 + \frac{\sqrt{3}}{2} - \frac{1}{2} \Big) = \Big(-\frac{3}{2} + \frac{\sqrt{3}}{2} \Big) + i\Big(\frac{1}{2} + \frac{\sqrt{3}}{2} \Big).$$

  (Any equivalent algebra earns full credit; numeric form also fine.)
- Rotate $z = (2, 0)$ by $90°$ about $a = (1, 1)$:

  $w = z - a = (1, -1), \quad R_{\pi/2} = \big(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\big), \quad w' = (1, -1) \mapsto (1 \cdot 0 - (-1) \cdot 1,\ 1 \cdot 1 + (-1) \cdot 0) = (1, 1).$

  Translate back: $z_{\text{new}} = a + w' = (2, 2)$.