Differentiation

Definition

The derivative of a function $f(x)$ at a point x is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Power Rule

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$$

Example:

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

Constant Rule

$$f(x) = c \Rightarrow f'(x) = 0$$

Example:

$$f(x) = 2 \Rightarrow f'(x) = 0$$

Basic Derivatives

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \text{for } x > 0$$

Derivative of Exponential Functions

For $f(x) = a^x$, where $a > 0$ and $a \neq 1$:

$$\frac{d}{dx} a^x = a^x \ln a$$

Example:

$$\frac{d}{dx} 2^x = 2^x \ln 2$$

Derivative of Logarithmic Functions

For $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$:

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

Example:

$$\frac{d}{dx} \log_2 x = \frac{1}{x \ln 2}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{arccot} x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Product Rule

If $f(x) = u(x) \cdot v(x)$:

$$f'(x) = u'v + uv'$$

Quotient Rule

If $f(x) = \frac{u(x)}{v(x)}$:

$$f'(x) = \frac{u'v - uv'}{v^2}$$

Chain Rule

If $f(x) = h(g(x))$, then:

$$f'(x) = h'(g(x)) \cdot g'(x)$$

Example:

$$f(x) = (2x^3 + x)^4$$

Then:

$$f'(x) = 4(2x^3 + x)^3 \cdot (6x^2 + 1)$$

Implicit Differentiation

Implicit differentiation is used to find the derivative of a function when it is not explicitly solved for y in terms of x. For example, consider the equation:

$$x^2 + y^2 = 1$$

To differentiate implicitly:

1. Differentiate both sides of the equation with respect to x, treating y as a function of x ($y = y(x)$).
2. Apply the chain rule to terms involving y.

Steps:

- Differentiate $x^2 + y^2 = 1$:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

- Simplify:

$$2x + 2y\frac{dy}{dx} = 0$$

- Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example: Find $\frac{dy}{dx}$ if $x^2 + xy + y^2 = 7$.

- Differentiate both sides:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

- Apply product rule and chain rule:

$$2x + \left(\frac{d}{dx}(x \cdot y)\right) + 2y\frac{dy}{dx} = 0$$

$$2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

- Combine terms:

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

- Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Integration

Definition of Integration

The definite integral of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where: - $\Delta x = \dfrac{b-a}{n}$ is the width of each subinterval, - x_i^* is a sample point in the i-th subinterval $[x_{i-1}, x_i]$.

The integral represents the accumulation of the quantity $f(x)$ over the interval $[a, b]$.

Alternatively, the indefinite integral (antiderivative) of a function $f(x)$ is a function $F(x)$ such that:

$$\int f(x)\,dx = F(x) + C$$

where: - $F'(x) = f(x)$, - C is the constant of integration.

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1)$$

Constant Rule

$$\int c\,dx = cx + C$$

Example:

$$\int 5x^2\,dx = 5\int x^2\,dx = 5 \cdot \frac{x^3}{3} = \frac{5x^3}{3} + C$$

Sum Rule

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Basic Integrals

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (\text{for } a > 0,\ a \neq 1)$$

Example:

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int \log_a x\,dx = x\log_a x - \frac{x}{\ln a} + C$$

Example:

$$\int \log_2 x\,dx = x\log_2 x - \frac{x}{\ln 2} + C$$

Integrals of Trigonometric Functions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = -\ln |\cos x| + C$$

$$\int \cot x \, dx = \ln |\sin x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

Example Calculations

1.
$$\int x^2 \, dx = \frac{x^3}{3} + C$$

2.
$$\int 3^x \, dx = \frac{3^x}{\ln 3} + C$$

3.
$$\int \ln x \, dx = x \ln x - x + C$$

4.
$$\int \frac{1}{x \ln a} \, dx = \log_a x + C$$

5.
$$\int \sec^2 x \, dx = \tan x + C$$

6.
$$\int \tan x \, dx = -\ln |\cos x| + C$$

Substitution Method

The substitution method is used to simplify an integral by making a substitution to reduce it to a standard form. It is particularly useful when the integrand contains a composite function.

Steps for Substitution

1. Identify a substitution: Let $u = g(x)$, where $g(x)$ is part of the integrand. Compute $\frac{du}{dx} = g'(x)$ or equivalently $du = g'(x)\,dx$.
2. Rewrite the integral in terms of u: Substitute $g(x)$ with u and dx with $du/g'(x)$.
3. Perform the integration: Solve the integral in terms of u.
4. Back-substitute: Replace u with $g(x)$ to express the answer in terms of the original variable x.

General Formula

If $\int (ax+b)^n\,dx$, then:

$$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C, \quad n \neq -1.$$

Example 1

Evaluate $\int xe^{x^2}\,dx$.

Solution:

- Let $u = x^2$, so $\frac{du}{dx} = 2x$ or $du = 2x\,dx$.
- Rewrite the integral:
$$\int xe^{x^2}\,dx = \int e^u \cdot \frac{du}{2}.$$
- Simplify and integrate:
$$\int e^u \cdot \frac{1}{2}\,du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C.$$
- Back-substitute:
$$\frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Example 2

Evaluate $\int \frac{\ln x}{x}\,dx$.

Solution:

- Let $u = \ln x$, so $\frac{du}{dx} = \frac{1}{x}$ or $du = \frac{1}{x}\,dx$.
- Rewrite the integral:
$$\int \frac{\ln x}{x}\,dx = \int u\,du.$$
- Integrate:
$$\int u\,du = \frac{u^2}{2} + C.$$
- Back-substitute:
$$\frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

Example 3

Evaluate $\int \cos(3x)\,dx$.

Solution:

- Let $u = 3x$, so $\frac{du}{dx} = 3$ or $du = 3\,dx$, hence $dx = \frac{du}{3}$.
- Rewrite the integral:
$$\int \cos(3x)\,dx = \int \cos(u) \cdot \frac{du}{3}.$$
- Simplify and integrate:
$$\int \cos(u) \cdot \frac{1}{3}\,du = \frac{1}{3}\int \cos(u)\,du = \frac{1}{3}\sin(u) + C.$$
- Back-substitute:
$$\frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(3x) + C.$$