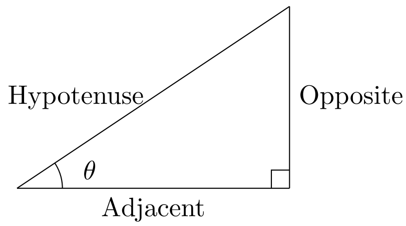
# Trigonometry with Triangles

## Right Triangle Definitions

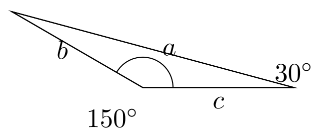



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

## Reference Angles

A reference angle is the acute angle ($< 90°$) formed by the terminal side of an angle and the horizontal axis.




In this triangle, angle $A$ is $150°$, and the reference angle $\theta_{\text{ref}}$ is:

$$\theta_{\text{ref}} = 180° - 150° = 30°$$

**Example:** Find the reference angle for $\theta = 150°$.

$$\theta_{\text{ref}} = 180° - 150° = 30°$$

So, the reference angle is $30°$.

When calculating trigonometric functions for obtuse angles, use the reference angle and adjust the sign based on the quadrant.

**Example:** Find $\sin 150°$.

$$\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$$

Since $150°$ is in Quadrant II, (sin+), $\sin 150° = \frac{1}{2}$.

## Signs of Trigonometric Functions

| Quadrant | Positive Functions |
|---|---|
| I | sin, cos, tan |
| II | sin |
| III | tan |
| IV | cos |

## Inverse Functions

The range of $\theta = \arccos(x)$ is given by $0 \le \theta \le \pi$.
The range of $\theta = \arcsin(x)$ is given by $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
The range of $\theta = \arctan(x)$ is given by $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

## Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

**Example:** If $\sin\theta = \frac{3}{5}$, then:

$$\cos\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

## Sum and Difference Formulas for Sine *and* Cosine

$$\sin(x+y) = \sin x\cos y + \cos x\sin y,$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y,$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y,$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y.$$

These identities allow us to break down more complicated trigonometric expressions into simpler combinations of sines and cosines of single angles.

**Example (using cosine):** Find $\cos(75°)$ using the sum formula.

Since $75° = 45° + 30°$, we write:

$$\cos(75°) = \cos\big(45° + 30°\big).$$

Using the sum formula for cosine:

$$\cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°.$$

Substitute known values:

$$\cos 45° = \frac{\sqrt{2}}{2}, \quad \sin 45° = \frac{\sqrt{2}}{2}, \quad \cos 30° = \frac{\sqrt{3}}{2}, \quad \sin 30° = \frac{1}{2}.$$

$$\cos(75°) = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right).$$

$$\cos(75°) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Hence,

$$\boxed{\cos(75°) = \frac{\sqrt{6}-\sqrt{2}}{4}.}$$

These sum and difference formulas are fundamental tools in trigonometry for simplifying expressions, evaluating trigonometric functions at specific angles, and deriving many other identities.

## Sum of Sine and Cosine in Amplitude-Phase Form

A common problem in trigonometry is to rewrite a linear combination of sine and cosine,

$$a\sin x + b\cos x,$$

in *amplitude-phase form* (also called *phase-shift form*), which typically looks like

$$R\sin\big(x+\phi\big),$$

or sometimes

$$R\cos\big(x-\phi\big).$$

**Key Steps**

1. **Identify the coefficients:**
$$a \sin x + b \cos x.$$
2. **Compute the amplitude $R$:** We define
$$R = \sqrt{a^2 + b^2}.$$
This follows from the Pythagorean identity once you set up the system:
$$\begin{cases} R\cos\phi = a, \\ R\sin\phi = b. \end{cases}$$
Squaring and adding both equations gives
$$(R\cos\phi)^2 + (R\sin\phi)^2 \;=\; a^2 + b^2,$$
and using $\cos^2\phi + \sin^2\phi = 1$ leads to
$$R^2 = a^2 + b^2.$$
3. **Solve for the phase $\phi$:** From the system
$$\begin{cases} R\cos\phi = a, \\ R\sin\phi = b, \end{cases}$$
you can divide one equation by the other to get
$$\tan\phi = \frac{b}{a} \quad (\text{assuming } a \neq 0).$$
Then
$$\phi = \arctan\!\Big(\frac{b}{a}\Big),$$
with possible adjustments depending on the signs of $a$ and $b$ (to get $\phi$ in the correct quadrant).
4. **Rewrite in the new form:** Once $R$ and $\phi$ are found, you have:
$$a \sin x + b \cos x = R \sin\big(x + \phi\big).$$
This identity can be verified using the sum formula for sine:
$$R\sin(x+\phi) = R\big[\sin x\cos\phi + \cos x\sin\phi\big] = R\cos\phi\,\sin x + R\sin\phi\,\cos x,$$
and noting that $R\cos\phi = a$ and $R\sin\phi = b$.

**Example**

Rewrite $3\sin x + \sqrt{3}\cos x$ in amplitude-phase form:

1. Identify $a = 3$ and $b = \sqrt{3}$.
2. Compute the amplitude:
$$R = \sqrt{a^2 + b^2} = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}.$$
3. Solve for $\phi$. We want:
$$2\sqrt{3}\,\cos\phi = 3 \quad\Longrightarrow\quad \cos\phi = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2},$$
$$2\sqrt{3}\,\sin\phi = \sqrt{3} \quad\Longrightarrow\quad \sin\phi = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}.$$
These correspond to $\phi = \frac{\pi}{6}$ (since $\cos\frac{\pi}{6} = \sqrt{3}/2$ and $\sin\frac{\pi}{6} = 1/2$).
4. Write the final expression:
$$3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\,\sin\!\Big(x + \tfrac{\pi}{6}\Big).$$

This method works in general for any linear combination $a\sin x + b\cos x$ and is especially helpful for analyzing maxima, minima, and phase shifts of trigonometric functions.

## Double Angle Formulas

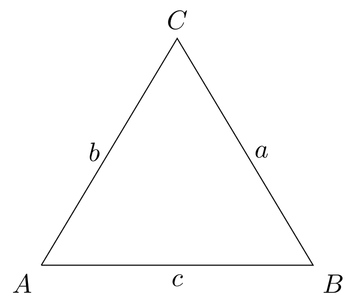
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta),$$

and using the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, we can derive alternate forms:

1. Replace $\sin^2(\theta)$ with $1 - \cos^2(\theta)$:
$$\cos(2\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1.$$
2. Replace $\cos^2(\theta)$ with $1 - \sin^2(\theta)$:
$$\cos(2\theta) = (1 - \sin^2(\theta)) - \sin^2(\theta) = 1 - 2\sin^2(\theta).$$

Thus, the double angle formula for cosine can be expressed in three equivalent ways:
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).$$

## Law of Sines




$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

**Example:** If $a = 7$, $A = 30°$, and $B = 45°$, find $b$:
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\begin{aligned} b &= a \times \frac{\sin B}{\sin A} \\ &= 7 \times \frac{\sin 45°}{\sin 30°} \\ &= 7 \times \frac{\frac{\sqrt{2}}{2}}{\frac{1}{2}} \\ &= 7 \times \sqrt{2} \approx 9.9 \end{aligned}$$

## Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

**Example:** Given $b = 5$, $c = 7$, and $A = 60°$:
$$\begin{aligned} a^2 &= 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 60° \\ &= 25 + 49 - 70 \times \left(\frac{1}{2}\right) \\ &= 74 - 35 \\ &= 39 \\ a &= \sqrt{39} \approx 6.24 \end{aligned}$$

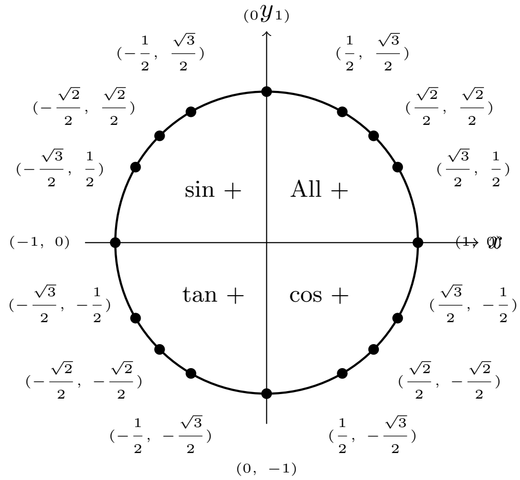
## Area of a Triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

**Example:** For $a = 5$, $b = 7$, and $C = 45°$:
$$\text{Area} = \frac{1}{2} \times 5 \times 7 \times \sin 45° = \frac{35}{2} \times \frac{\sqrt{2}}{2} = \frac{35\sqrt{2}}{4} \approx 12.37$$

# Unit Circle and Angles

## Unit Circle Diagram




The cosine is the x-value, the sine is the y-value.

## Radians and Degrees

$$180^\circ = \pi \text{ radians}, \quad 1^\circ = \frac{\pi}{180} \text{ radians}$$

**Example:** Convert 135° to radians:

$$135^\circ \times \frac{\pi}{180^\circ} = \frac{3\pi}{4}$$

## Alternate Formulation

For common angles, sine and cosine values can be calculated as:

$$\sin\theta = \frac{\sqrt{n}}{2}, \quad \cos\theta = \frac{\sqrt{4-n}}{2}$$

where $n$ corresponds to the following angles:

$$\begin{aligned} n &= 0 \quad \text{for } \theta = 0^\circ, \\ n &= 1 \quad \text{for } \theta = 30^\circ, \\ n &= 2 \quad \text{for } \theta = 45^\circ, \\ n &= 3 \quad \text{for } \theta = 60^\circ, \\ n &= 4 \quad \text{for } \theta = 90^\circ. \end{aligned}$$

## Reference Angles on the Unit Circle

To find the trigonometric functions of any angle, find its reference angle and determine the sign based on the quadrant.

**Example:** Find $\tan 225^\circ$.

$$\theta_{\text{ref}} = 225^\circ - 180^\circ = 45^\circ$$

$$\tan 225^\circ = \tan(180^\circ + 45^\circ) = \tan 45^\circ = 1$$

Since 225° is in Quadrant III, tan is positive.

**Example:** Find $\sin 210^\circ$.

$$\theta_{\text{ref}} = 210^\circ - 180^\circ = 30^\circ$$

$$\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$$

Since 210° is in Quadrant III, sin is negative.

**Example:** Find $\cos 300^\circ$.

$$\theta_{\text{ref}} = 360^\circ - 300^\circ = 60^\circ$$

$$\cos 300^\circ = \cos(360^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

Since 300° is in Quadrant IV, cos is positive.

## Coterminal Angles

Angles that differ by full rotations (360°) are coterminal.

**Example:** $\theta = -30^\circ$ is coterminal with 330° because:

$$-30^\circ + 360^\circ = 330^\circ$$

## Period of Stretched Trigonometric Functions

For trigonometric functions that are stretched or compressed horizontally, the period changes accordingly.

- For functions with a natural period of $2\pi$ (such as $\sin x$ and $\cos x$):

$$T = \frac{2\pi}{B}$$

- For functions with a natural period of $\pi$ (such as $\tan x$ and $\cot x$):

$$T = \frac{\pi}{B}$$

Where $B$ is the coefficient of $x$ in the function $f(x) = \sin(Bx)$ or $f(x) = \tan(Bx)$.

**Example:** Find the period of $f(x) = \sin(2x)$.

Since the natural period of $\sin x$ is $2\pi$:

$$T = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

So, the period of $\sin(2x)$ is $\pi$.

**Example:** Find the period of $f(x) = \tan\left(\frac{x}{3}\right)$.

Since the natural period of $\tan x$ is $\pi$:

$$T = \frac{\pi}{B} = \frac{\pi}{\frac{1}{3}} = 3\pi$$

So, the period of $\tan\left(\frac{x}{3}\right)$ is $3\pi$.