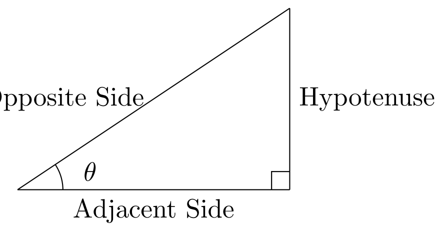
Trigonometry with Triangles

Right Triangle Definitions

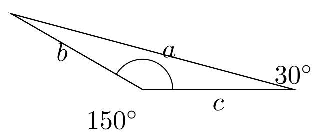


$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Reference Angles

A reference angle is the acute angle ($< 90°$) formed by the terminal side of an angle and the horizontal axis.



In this triangle, angle A is $150°$, and the reference angle θ_{ref} is:

$$\theta_{\text{ref}} = 180° - 150° = 30°$$

Example: Find the reference angle for $\theta = 150°$.

$$\theta_{\text{ref}} = 180° - 150° = 30°$$

So, the reference angle is $30°$.

When calculating trigonometric functions for obtuse angles, use the reference angle and adjust the sign based on the quadrant.

Example: Find $\sin 150°$.

$$\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$$

Since $150°$ is in Quadrant II, (sin+), $\sin 150° = \frac{1}{2}$.

Example: Find $\cos 150°$.

$$\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$$

Since $150°$ is in Quadrant II, (cos-), $\cos 150° = -\frac{\sqrt{3}}{2}$.

Signs of Trigonometric Functions

Quadrant	Positive Functions
I	sin, cos, tan
II	sin
III	tan
IV	cos

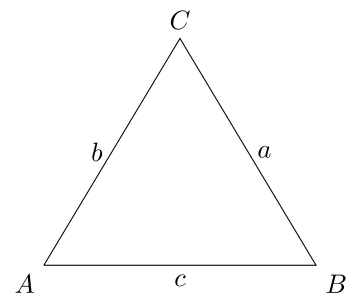
Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

Example: If $\sin\theta = \frac{3}{5}$, then:

$$\cos\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Law of Sines



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example: If $a = 7$, $A = 30°$, and $B = 45°$, find b:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$b = a \times \frac{\sin B}{\sin A}$$
$$= 7 \times \frac{\sin 45°}{\sin 30°}$$
$$= 7 \times \frac{\frac{\sqrt{2}}{2}}{\frac{1}{2}}$$
$$= 7 \times \sqrt{2} \approx 9.9$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Example: Given $b = 5$, $c = 7$, and $A = 60°$:

$$a^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 60°$$
$$= 25 + 49 - 70 \times \left(\frac{1}{2}\right)$$
$$= 74 - 35$$
$$= 39$$
$$a = \sqrt{39} \approx 6.24$$

Area of a Triangle

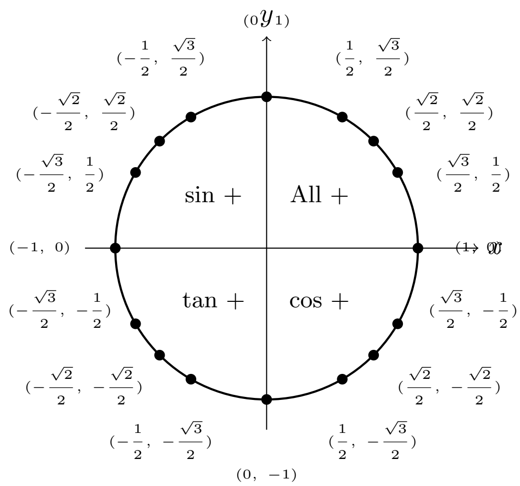
$$\text{Area} = \frac{1}{2}ab\sin C$$

Example: For $a = 5$, $b = 7$, and $C = 45°$:

$$\text{Area} = \frac{1}{2} \times 5 \times 7 \times \sin 45° = \frac{35}{2} \times \frac{\sqrt{2}}{2} = \frac{35\sqrt{2}}{4} \approx 12.37$$

Unit Circle and Angles

Unit Circle Diagram



The cosine is the x-value, the sine is the y-value.

Radians and Degrees

$$180^\circ = \pi \text{ radians}, \quad 1^\circ = \frac{\pi}{180} \text{ radians}$$

Example: Convert 135° to radians:

$$135^\circ \times \frac{\pi}{180^\circ} = \frac{3\pi}{4}$$

Alternate Formulation

For common angles, sine and cosine values can be calculated as:

$$\sin\theta = \frac{\sqrt{n}}{2}, \quad \cos\theta = \frac{\sqrt{4-n}}{2}$$

where n corresponds to the following angles:

$$\begin{aligned} n &= 0 \quad \text{for } \theta = 0^\circ, \\ n &= 1 \quad \text{for } \theta = 30^\circ, \\ n &= 2 \quad \text{for } \theta = 45^\circ, \\ n &= 3 \quad \text{for } \theta = 60^\circ, \\ n &= 4 \quad \text{for } \theta = 90^\circ. \end{aligned}$$

Reference Angles on the Unit Circle

To find the trigonometric functions of any angle, find its reference angle and determine the sign based on the quadrant.

Example: Find $\tan 225^\circ$.

$$\theta_{\text{ref}} = 225^\circ - 180^\circ = 45^\circ$$

$$\tan 225^\circ = \tan(180^\circ + 45^\circ) = \tan 45^\circ = 1$$

Since 225° is in Quadrant III, tan is positive.

Example: Find $\sin 210^\circ$.

$$\theta_{\text{ref}} = 210^\circ - 180^\circ = 30^\circ$$

$$\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$$

Since 210° is in Quadrant III, sin is negative.

Example: Find $\cos 300^\circ$.

$$\theta_{\text{ref}} = 360^\circ - 300^\circ = 60^\circ$$

$$\cos 300^\circ = \cos(360^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

Since 300° is in Quadrant IV, cos is positive.

Coterminal Angles

Angles that differ by full rotations (360°) are coterminal.

Example: $\theta = -30^\circ$ is coterminal with 330° because:

$$-30^\circ + 360^\circ = 330^\circ$$

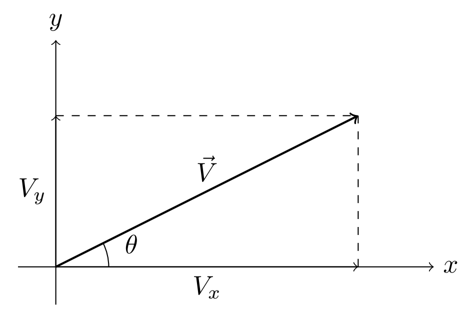
Vector Decomposition and Addition

Decomposing Vectors into Components

Any vector $\vec{V}$ can be decomposed into its horizontal and vertical components using trigonometric functions.

Given a vector of magnitude V at an angle θ with respect to the horizontal axis:

$$V_x = V\cos\theta$$
$$V_y = V\sin\theta$$



Example: A vector has a magnitude of $V = 5$ units and makes an angle of $\theta = 30^\circ$ with the horizontal axis. Find its components.

$$V_x = V\cos\theta = 5\cos 30^\circ = 5 \times \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2} \approx 4.33$$
$$V_y = V\sin\theta = 5\sin 30^\circ = 5 \times \frac{1}{2} = \frac{5}{2} = 2.5$$

Adding Vectors Using Components

To add two vectors, decompose each vector into its components, add the corresponding components, and then recombine the resultant components to find the resultant vector.

Given two vectors $\vec{A}$ and $\vec{B}$:

$$A_x = A\cos\alpha$$
$$A_y = A\sin\alpha$$
$$B_x = B\cos\beta$$
$$B_y = B\sin\beta$$

The resultant vector $\vec{R} = \vec{A} + \vec{B}$ has components:

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$

The magnitude and direction of $\vec{R}$ are:

$$R = \sqrt{R_x^2 + R_y^2}$$

$$\theta_R = \arctan\left(\frac{R_y}{R_x}\right)$$

Example: Add vector $\vec{A}$ of magnitude $A = 10$ units at $\alpha = 45^\circ$ and vector $\vec{B}$ of magnitude $B = 8$ units at $\beta = 120^\circ$.

Step 1: Decompose vectors into components.

$$A_x = 10\cos 45^\circ = 10 \times \frac{\sqrt{2}}{2} = 5\sqrt{2} \approx 7.07$$

$$A_y = 10\sin 45^\circ = 10 \times \frac{\sqrt{2}}{2} = 5\sqrt{2} \approx 7.07$$

$$B_x = 8\cos 120^\circ = 8 \times \left(-\frac{1}{2}\right) = -4$$

$$B_y = 8\sin 120^\circ = 8 \times \frac{\sqrt{3}}{2} = 4\sqrt{3} \approx 6.93$$

Step 2: Add components.

$$R_x = A_x + B_x = 7.07 + (-4) = 3.07$$

$$R_y = A_y + B_y = 7.07 + 6.93 = 14$$

Step 3: Calculate resultant magnitude and direction.

$$R = \sqrt{(R_x)^2 + (R_y)^2} = \sqrt{(3.07)^2 + (14)^2} \approx \sqrt{9.42 + 196} \approx \sqrt{205.42}$$

$$\theta_R = \arctan\left(\frac{R_y}{R_x}\right) = \arctan\left(\frac{14}{3.07}\right) \approx 77.5^\circ$$

So, the resultant vector has a magnitude of approximately 14.33 units and makes an angle of 77.5° with the horizontal axis.

Summary

- Decompose vectors into their x and y components using cos and sin functions. - Add the corresponding components to find the resultant components. - Calculate the magnitude and direction of the resultant vector from its components.