Sets and Probability

Set Operations

Given a universal set U and subsets $A, B \subseteq U$:

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$$

Definition: The union of A and B, denoted $A \cup B$, is the set of all elements that are in A, in B, or in both.

$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$$

Definition: The intersection of A and B, denoted $A \cap B$, is the set of all elements that are in both A and B.

$$A^c = U \setminus A = \{x \in U : x \notin A\}$$

Definition: The complement of A, denoted A^c, is the set of all elements in the universal set U that are not in A.

$$A \setminus B = \{x \in A : x \notin B\}$$

Definition: The difference of A and B, denoted $A \setminus B$, is the set of all elements that are in A but not in B.

Probability Basics

For a probability space $(U, \mathcal{F}, P)$, where U is the sample space, $\mathcal{F}$ is the set of events (subsets of U), and P is a probability measure:
- $P(U) = 1$ - For mutually exclusive events A_i:

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

If $A \subseteq U$, then:

$$0 \leq P(A) \leq 1$$

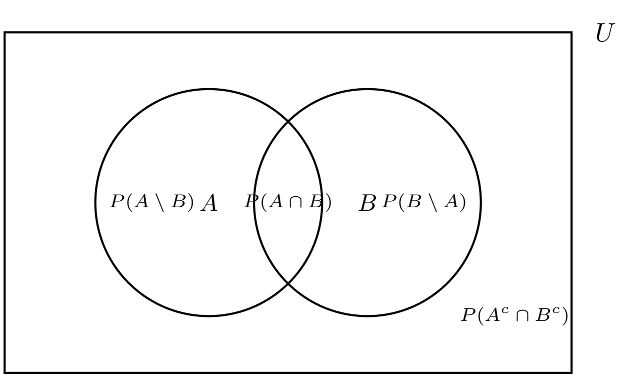
Addition Law of Probability

For any two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Explanation: The probability of $A \cup B$ (the union of A and B) is the sum of the probabilities of A and B, minus the probability of their intersection to avoid double-counting.

Venn Diagram of Two Events



Conditional Probability

For events A, B with $P(B) > 0$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Law

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

Law of Total Probability

$$P(A) = P(A \cap B) + P(A \cap B')$$

Stated in terms of conditional probability

$$P(A \cap B) = P(A \mid B)P(B)$$

$$P(A \cap B') = P(A \mid B')P(B')$$

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

Random Variables and Distributions

Probability Mass Function (PMF)

For a discrete random variable X, the probability mass function $P(X = x)$ is defined as:

$$P(X = x) = \begin{cases} p_x & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Definition: - X: A discrete random variable. - S: The support (set of possible values that X can take). - p_x: The probability assigned to the outcome x.

Bernoulli Distribution

A random variable X follows a Bernoulli distribution if:

$$P(X = x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0, \end{cases}$$

where $p \in [0, 1]$ is the probability of success.

Mean: $\mu = p$

Variance: $\sigma^2 = p(1 - p)$

Binomial Distribution

A random variable X follows a Binomial distribution $\text{Bin}(n, p)$ if it represents the number of successes in n independent Bernoulli trials:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

Parameters: - n: Number of trials. - p: Probability of success.

Mean: $\mu = np$

Variance: $\sigma^2 = np(1 - p)$

Negative Binomial Distribution

A random variable X follows a Negative Binomial distribution if it represents the number of trials required to achieve a fixed number of successes r in a sequence of independent Bernoulli trials with success probability p:

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, r+2, \dots$$

Parameters: - r: Number of successes. - p: Probability of success in each trial.

Mean: $\mu = \frac{r}{p}$

Variance: $\sigma^2 = \frac{r(1-p)}{p^2}$

Example: Suppose you are rolling a six-sided die, and you want to know the probability that the third "1" appears on the seventh roll. Here: - $r = 3$ (three successes, rolling a "1"). - $p = \frac{1}{6}$ (probability of rolling a "1" on a single trial). - $k = 7$ (the total number of trials).

$$P(X = 7) = \binom{6}{2} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^4 \approx 0.0335$$

Real-World Application The Negative Binomial distribution is commonly used to model scenarios where we are counting the number of trials needed to achieve a fixed number of successes. For example: - In customer service, modeling the number of calls needed to resolve r issues when the probability of resolving an issue in a single call is p. - In biology, determining the number of experiments required to observe r successful outcomes of a rare event.

Poisson Distribution

A random variable X follows a Poisson distribution $\text{Poisson}(\lambda)$ if it represents the number of events occurring in a fixed interval:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \dots$$

Parameter: - λ: Average rate of occurrence.

Mean: $\mu = \lambda$

Variance: $\sigma^2 = \lambda$

Geometric Distribution

A random variable X follows a Geometric distribution if it represents the trial count until the first success in a series of independent Bernoulli trials with success probability p:

$$P(X = k) = (1-p)^{k-1} p, \quad k = 1, 2, 3, \dots$$

Parameters:

- p: Probability of success on each trial.

Mean: $\mu = \frac{1}{p}$

Variance: $\sigma^2 = \frac{1-p}{p^2}$

Standard Deviation: $\sigma = \sqrt{\frac{1-p}{p^2}}$

When to Use: Geometric distributions model the number of trials until the first success, common in reliability testing, queue theory, and repeated experiments until success.

Gaussian (Normal) Distribution

A random variable X follows a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ if its probability density function (PDF) is:

$$P(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Parameters: - μ: Mean (center of the distribution). - σ^2: Variance (spread of the distribution).

Properties: - Symmetric about μ. - $X \sim \mathcal{N}(\mu, \sigma^2)$ satisfies the Empirical Rule: - 68% of data within $[\mu - \sigma, \mu + \sigma]$. - 95% of data within $[\mu - 2\sigma, \mu + 2\sigma]$. - 99.7% of data within $[\mu - 3\sigma, \mu + 3\sigma]$.

Expected Value and Variance

Expected Value

For a discrete random variable X with possible values $x_1, x_2, \dots, x_n$ and corresponding probabilities $P(X = x_i)$:

$$E[X] = \sum_{i=1}^{n} x_i P(X = x_i)$$

Example: If X has the following probability mass function (PMF):

x	$P(X = x)$
1	0.2
2	0.5
3	0.3

The expected value is:

$$E[X] = (1 \times 0.2) + (2 \times 0.5) + (3 \times 0.3) = 2.1$$

Generalized Extreme Value (GEV) Distribution

A random variable X follows a GEV distribution if it models the maxima or minima of samples of various distributions:

$$f_X(x; \mu, \sigma, \xi) = \frac{1}{\sigma} \left[1 + \xi \left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \exp\left(-\left[1 + \xi \frac{x-\mu}{\sigma}\right]^{-\frac{1}{\xi}}\right),$$

where $\xi \neq 0$, $\sigma > 0$, and the domain of x depends on ξ.

Parameters: - Location: μ - Scale: σ - Shape: ξ

Mean ($\xi < 1$):

$$\mu + \frac{\sigma}{\xi}(\Gamma(1-\xi) - 1)$$

Variance:

$$\frac{\sigma^2}{\xi^2}[\Gamma(1-2\xi) - \Gamma^2(1-\xi)], \quad \xi < 0.5$$

Standard Deviation: Square root of variance.

Applications: - Extreme environmental events (floods, earthquakes). - Financial risk management.

Solving for the Mean of a Normal Distribution Using Symmetry

Given a normal random variable $X \sim N(\mu, 2^2)$, and the probability:

$$P(-4\mu < X < 6\mu) = 0.6528$$

we are asked to solve for the mean μ.

Step-by-Step Calculation:

1. Recognize the interval is symmetric about μ:
   $6\mu - \mu = \mu - (-4\mu) = 5\mu$.
2. Express the interval in terms of deviation from the mean:
   $P(\mu - 5\mu < X < \mu + 5\mu)$.
3. Rewrite as: $P(-5\mu < X - \mu < 5\mu)$.
4. Standardize using $Z = \frac{X-\mu}{2}$.
5. The probability becomes: $P\left(\frac{-5\mu}{2} < Z < \frac{5\mu}{2}\right)$.
6. Use symmetry: $P\left(0 < Z < \frac{5\mu}{2}\right) = \frac{0.6528}{2} = 0.3264$.
7. Total cumulative area: $\Phi\left(\frac{5\mu}{2}\right) = 0.5 + 0.3264 = 0.8264$.
8. Invert the standard normal CDF: $\Phi^{-1}(0.8264) = 0.94$.
9. Set up equation: $\frac{5\mu}{2} = 0.94$.
10. Solve for μ: $\mu = \frac{2 \cdot 0.94}{5} = 0.376$.
11. Round to two decimal places: $\mu \approx 0.38$.

Variance of a Random Variable

The variance of a random variable X measures the spread of its values around the mean $\mu = E[X]$:

$$\text{Var}(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

Step-by-Step Calculation: 1. Compute $E[X]$. 2. Compute $E[X^2] = \sum_{i=1}^n x_i^2 P(X = x_i)$. 3. Subtract $(E[X])^2$ from $E[X^2]$.

Example: Using the PMF from above:

$$E[X^2] = (1^2 \times 0.2) + (2^2 \times 0.5) + (3^2 \times 0.3) = 0.2 + 2.0 + 2.7 = 4.9$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 4.9 - (2.1)^2 = 4.9 - 4.41 = 0.49$$

Properties of Expected Value and Variance

- For a constant c:

$$E[c] = c, \quad \text{Var}(c) = 0$$

- For a random variable X:

$$E[cX] = cE[X], \quad \text{Var}(cX) = c^2\text{Var}(X)$$

- For independent random variables X and Y:

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Application: Linear Combinations of Random Variables

For $Y = aX + b$:

$$E[Y] = aE[X] + b, \quad \text{Var}(Y) = a^2\text{Var}(X)$$

Statistics

Variance and Covariance

For a random variable X with mean $\mu_X = E[X]$:

$$\text{Var}(X) = E[(X-\mu_X)^2] = E[X^2] - (E[X])^2$$

For two random variables X and Y with means μ_X, μ_Y:

$$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$$

Sample-based formulas for n data points (x_i, y_i):

$$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$$

$$s_X^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2, \quad s_Y^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2$$

$$\text{Cov}(X,Y) = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

Linear Regression

For a simple linear regression with one predictor X and response Y, the model is:

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

The least-squares estimates for β_0 and β_1 using n observations (x_i, y_i) are:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\text{Cov}(X,Y)}{s_X^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

The fitted line is:

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

Combinatorics

Factorials

For a positive integer n:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

By definition, $0! = 1$

Permutations

The number of ways to arrange k distinct objects out of n distinct objects (order matters):

$$P(n,k) = \frac{n!}{(n-k)!}$$

Combinations

The number of ways to choose k elements out of n without regard to order (also called $\binom{n}{k}$):

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Common notation: - Combinations: $\binom{n}{k}$, $C(n,k)$ - Permutations: $P(n,k)$ or sometimes ${}_nP_k$

Examples

$$\binom{5}{2} = \frac{5!}{2! \cdot 3!} = \frac{120}{2 \cdot 6} = 10$$

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{120}{6} = 20$$

Gamma Function

For a complex number z with $\Re(z) > 0$:

$$\Gamma(z) \;=\; \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t$$

It satisfies the functional equation (recurrence relation):

$$\Gamma(z+1) \;=\; z\,\Gamma(z)$$

In particular, for a positive integer n:

$$\Gamma(n+1) \;=\; n!$$

A notable special value is

$$\Gamma\big(\tfrac{1}{2}\big) \;=\; \sqrt{\pi}$$

The reflection formula is given by

$$\Gamma(z)\,\Gamma(1-z) \;=\; \frac{\pi}{\sin(\pi z)}$$

Recursive computation for half-integers For any integer $m \geq 1$, one can compute

$$\Gamma\big(m+\tfrac{1}{2}\big) \;=\; \Big(m-\tfrac{1}{2}\Big)\,\Gamma\big(m-\tfrac{1}{2}\big),$$

with the base case $\Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}$. Hence

$$\Gamma\big(m+\tfrac{1}{2}\big) \;=\; \prod_{k=1}^{m} \Big(k-\tfrac{1}{2}\Big)\,\sqrt{\pi}.$$