Differentiation

Definition

The derivative of $f(x)$ at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Product Rule

If $f(x) = u(x)\, v(x)$, then

$$f'(x) = u'(x)\, v(x) + u(x)\, v'(x).$$

Quotient Rule

If $f(x) = \dfrac{u(x)}{v(x)}$, then

$$f'(x) = \frac{u'(x)\, v(x) - u(x)\, v'(x)}{[v(x)]^2}.$$

Chain Rule

If $f(x) = h(g(x))$, then

$$f'(x) = h'(g(x)) \;\cdot\; g'(x).$$

Implicit Differentiation

Given $F(x, y) = 0$, differentiate both sides w.r.t. x, treating $y = y(x)$:

$$F_x + F_y\, y' = 0 \quad\implies\quad y' = -\frac{F_x}{F_y}.$$

Example: From $x^2 + y^2 = 1$, $2x + 2y\, y' = 0 \Rightarrow y' = -\frac{x}{y}$.

Differentiation by Substitution

To differentiate $f(x) = H(u(x))$, set $u = u(x)$:

$$\frac{df}{dx} = H'(u)\, \frac{du}{dx}.$$

Example: If $f(x) = (2x^3+x)^4$, then $u = 2x^3+x$, $du = (6x^2+1)dx$, so

$$f' = 4u^3 \cdot (6x^2+1) = 4(2x^3+x)^3(6x^2+1).$$

Integration

Definition

Definite integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x, \quad \Delta x = \frac{b-a}{n}.$$

Indefinite integral:

$$\int f(x)dx = F(x) + C, \quad F'(x) = f(x).$$

Linearity Rules

$$\int \big[f(x) + g(x)\big] dx = \int f(x)dx + \int g(x)dx,$$

$$\int c\, f(x)dx = c \int f(x)dx.$$

Implicit Integration

For $y' = \frac{g(x)}{h(y)}$, separate:

$$h(y)\, dy = g(x)dx \quad\implies\quad \int h(y)\, dy = \int g(x)dx + C.$$

Integration by Substitution

Let $u = g(x)$ so $du = g'(x)\, dx$. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

Example: $\int xe^{x^2} dx$: with $u = x^2$, $du = 2xdx$, gives $\frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$.

Definite Integration by Substitution

To compute $\int_a^b f(g(x))g'(x)\, dx$, set $u = g(x)$ and update limits accordingly:

$$\int_{x=a}^{x=b} f(g(x))g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du.$$

Variable	Lower limit	Upper limit
x	a	b
$u = g(x)$	$g(a)$	$g(b)$

Example: Compute

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx.$$

Set $u = \tan x$, so $du = \sec^2 x\, dx$. Limits:

$$x = 0 \implies u = \tan 0 = 0, \quad x = \tfrac{\pi}{4} \implies u = \tan\tfrac{\pi}{4} = 1.$$

Table of limits:

Variable	Lower limit	Upper limit
x	0	$\frac{\pi}{4}$
u	0	1

Thus,

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx = \int_0^1 u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

Integration by Parts

Based on the product rule: $(uv)' = u'v + uv'$, we get the formula:

$$\int u\,dv = uv - \int v\,du.$$

Choose u and dv from the integrand so that du and v are easier to compute.

Example: $\int xe^x\,dx$: let $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, so:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Definite Integration by Parts

Apply the same rule with limits:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

Example: Compute $\int_0^1 x\ln(x+1)\,dx$.

Let $u = \ln(x+1)$, so $du = \frac{1}{x+1}\,dx$; let $dv = x\,dx$, so $v = \frac{x^2}{2}$. Then:

$$\int_0^1 x\ln(x+1)\,dx = \left[\frac{x^2}{2}\ln(x+1)\right]_0^1 - \int_0^1 \frac{x^2}{2(x+1)}\,dx.$$

Derivative & Integral Pairs

$$\frac{d}{dx}C = 0 \qquad \int 0\,dx = C$$

$$\frac{d}{dx}(ax+b)^n = n\,a\,(ax+b)^{n-1} \qquad \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\frac{d}{dx}e^{u(x)} = u'(x)\,e^{u(x)} \qquad \int u'(x)\,e^{u(x)}\,dx = e^{u(x)} + C$$

$$\frac{d}{dx}a^{u(x)} = u'(x)\,a^{u(x)}\ln a \qquad \int u'(x)\,a^{u(x)}\,dx = \frac{a^{u(x)}}{\ln a} + C$$

$$\frac{d}{dx}\ln|u(x)| = \frac{u'(x)}{u(x)} \qquad \int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

$$\frac{d}{dx}\log_a|u(x)| = \frac{u'(x)}{u(x)\,\ln a} \qquad \int \frac{u'(x)}{u(x)\,\ln a}\,dx = \log_a|u(x)| + C$$

$$\frac{d}{dx}x^n = n\,x^{n-1} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\frac{d}{dx}e^x = e^x \qquad \int e^x\,dx = e^x + C$$

$$\frac{d}{dx}a^x = a^x\ln a \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\frac{d}{dx}e^{px+q} = p\,e^{px+q} \qquad \int e^{px+q}\,dx = \frac{1}{p}e^{px+q} + C$$

$$\frac{d}{dx}a^{px+q} = p\,a^{px+q}\ln a \qquad \int a^{px+q}\,dx = \frac{1}{p\ln a}\,a^{px+q} + C$$

$$\frac{d}{dx}\ln x = \tfrac{1}{x} \qquad \int \tfrac{1}{x}\,dx = \ln|x| + C$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a} \qquad \int \frac{1}{x\ln a}\,dx = \log_a|x| + C$$

$$\frac{d}{dx}\ln|ax+b| = \frac{a}{ax+b} \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ \ln|ax+b| + C$$

Trigonometric Derivative & Integral Pairs

$$\frac{d}{dx}\sin x = \cos x \qquad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \int \tan x dx = -\ln\lvert\cos x\rvert + C$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \int \cot x dx = \ln\lvert\sin x\rvert + C$$

$$\frac{d}{dx}\sec x = \sec x, \tan x \qquad \int \sec x dx = \ln\big\lvert\sec x + \tan x\big\rvert + C$$

$$\frac{d}{dx}\csc x = -\csc x, \cot x \qquad \int \csc x dx = -\ln\big\lvert\csc x + \cot x\big\rvert + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \int \sec^2 x dx = \tan x + C$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \int \csc^2 x dx = -\cot x + C$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \int \sec x \tan x dx = \sec x + C$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \qquad \int \csc x \cot x dx = -\csc x + C$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}, dx = \arcsin x + C$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \int -\frac{1}{\sqrt{1-x^2}}, dx = \arccos x + C$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}, dx = \arctan x + C$$

$$\frac{d}{dx}\frac{1}{k}\arctan kx = \frac{1}{1+(kx)^2} \qquad \int \frac{1}{1+(kx)^2} dx = \frac{1}{k}\arctan kx + C$$

$$\frac{d}{dx}\frac{1}{k}\arctan \frac{x}{k} = \frac{1}{k^2+(x)^2} \qquad \int \frac{1}{k^2+(x)^2} dx = \frac{1}{k}\arctan \frac{x}{k} + C$$

$$\frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2} \qquad \int -\frac{1}{1+x^2}, dx = \operatorname{arccot} x + C$$

$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \int \frac{1}{|x|\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$$

$$\frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}} \qquad \int -\frac{1}{|x|\sqrt{x^2-1}} dx = \operatorname{arccsc} x + C$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \int \cosh x dx = \sinh x + C$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \int \sinh x dx = \cosh x + C$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \int \operatorname{sech}^2 x dx = \tanh x + C$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x \qquad \int \operatorname{csch}^2 x dx = -\coth x + C$$

$$\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{1+x^2}} \qquad \int \frac{1}{\sqrt{1+x^2}} dx = \operatorname{arsinh} x + C$$

$$\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}} \qquad \int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arcosh} x + C$$

$$\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2} \qquad \int \frac{1}{1-x^2} dx = \operatorname{artanh} x + C$$

$$\frac{d}{dx}\operatorname{arsech} x = -\frac{1}{x\sqrt{1-x^2}} \qquad \int -\frac{1}{x\sqrt{1-x^2}} dx = \operatorname{arsech} x + C$$

$$\frac{d}{dx}\operatorname{arccsch} x = -\frac{1}{|x|\sqrt{1+x^2}} \qquad \int -\frac{1}{|x|\sqrt{1+x^2}} dx = \operatorname{arccsch} x + C$$