Single-Qubit Introduction

1 What Is a Qubit?

A qubit is the *quantum analogue* of a classical bit. Mathematically it is a **unit vector** in a two–dimensional complex vector space. Dirac's *bra–ket* notation writes such vectors as

$$|\psi\rangle \;=\; \alpha\,|0\rangle + \beta\,|1\rangle\,, \qquad \alpha, \beta \in \mathbb{C},\; |\alpha|^2 + |\beta|^2 = 1.$$

The Greek letter ψ (psi) labels the state; the vertical bar and angle bracket form a *ket*.

2 Why Those Particular Column Vectors?

- $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are chosen as the *computational basis*. They are **orthonormal**—see §.
- Any qubit state can be written as a linear combination $\alpha\,|0\rangle + \beta\,|1\rangle$ because $\{|0\rangle\,, |1\rangle\}$ is a complete basis for $\mathbb{C}^2$.

(What does *orthonormal* mean?)

Let $|u\rangle$ and $|v\rangle$ be column vectors in a complex vector space.

1. Conjugate transpose (††). Given a column vector $|u\rangle = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, its **conjugate transpose** (also called the *Hermitian adjoint*) is

$$|u\rangle^\dagger \;=\; \begin{bmatrix} u_1^* & u_2^* & \dots & u_n^* \end{bmatrix},$$

where $*$ denotes complex conjugation. In words:

1. *Transpose*: turn the column into a row.
2. *Conjugate*: replace each entry $a + ib$ with $a - ib$.

For a matrix A, $A^\dagger = (A^T)^*$—transpose first, then conjugate every element.

2. Inner product. The *Dirac inner product* of $|u\rangle$ and $|v\rangle$ is

$$\langle u|v\rangle \;=\; |u\rangle^{\dagger}\,|v\rangle \;=\; \sum_{i=1}^{n} u_i^{*}\,v_i.$$

3. Orthogonality. Vectors are **orthogonal** when their inner product vanishes:

$$\langle u|v\rangle = 0.$$

4. Normalisation. A vector is **normalised** when its inner product with itself equals 1:

$$\langle u|u\rangle = 1.$$

Its **norm** (length) is $\||u\rangle\| = \sqrt{\langle u|u\rangle}$.

5. Orthonormality. A set of vectors is *orthonormal* if every pair is orthogonal and every vector is normalised:

$$\langle u|v\rangle = 0, \qquad \langle u|u\rangle = 1,\ \langle v|v\rangle = 1,\ \ldots$$

Example (computational basis).

$$\langle 0|1\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0, \qquad \langle 0|0\rangle = 1,\ \langle 1|1\rangle = 1.$$

Thus $\{|0\rangle, |1\rangle\}$ is orthonormal, and $\||0\rangle\| = \||1\rangle\| = 1$.

3 Probabilities from Amplitudes

If $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, measuring in the computational basis yields

$$P(0) = |\alpha|^2, \qquad P(1) = |\beta|^2.$$

The squared modulus converts a complex amplitude into a real, non-negative probability, and normalisation $|\alpha|^2 + |\beta|^2 = 1$ guarantees the outcomes exhaust all possibilities.

4 Two Fundamental Single-Qubit Gates

4.1 Pauli–X Gate (Bit Flip)

$$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \qquad X\,|0\rangle = |1\rangle\,,\ X\,|1\rangle = |0\rangle\,.$$

4.2 Hadamard Gate (Superposition Maker)

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ H|1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

H is its own inverse ($H^2 = I$) and converts definite states into equal superpositions (and vice versa).

5 Worked Examples

Example 1: Apply X to $|0\rangle$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

Example 2: Apply H to $|0\rangle$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Example 3: Apply H to $|1\rangle$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Example 4: Verify $H^2 = I$

$$H^2 = \left(\tfrac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^2 = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Guided exercise. Show that $HXH = Z$, where $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and evaluate $HXH|0\rangle$ and $XHX|0\rangle$, noting the phase difference.

6 Key Takeaways

1. A qubit is a length-1 complex vector.
2. **Orthonormal** basis vectors are perpendicular ($\langle u|v\rangle = 0$) *and* length-1 ($\langle u|u\rangle = 1$, $\||u\rangle\| = 1$).
3. Conjugate transpose (†) means transpose + complex conjugation.
4. Probabilities come from squared moduli of amplitudes.
5. Single-qubit gates are 2×2 unitary matrices.