Lesson A2: Power Series in One Page (Taylor & Binomial in Action)

Expanded, Step-by-Step Version for a Gifted Young Learner

What we will build

Tiny toolkits to approximate functions with *polynomials*:

- Memorize the core series (Maclaurin):
$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}, \quad \sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!}.$$
- Use the **generalized binomial series** for $|x| < 1$:
$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n, \qquad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$
- Build fast approximations with a *truncated polynomial* $T_N(x)$ and estimate the error by the size of the *next term*.

Audience note. Exact values at special angles are great; otherwise, short decimal work is fine. Everything stays computational and visual.

Vocabulary & Symbols

- $T_N(x) = \sum_{k=0}^{N} a_k x^k$: N-th Taylor polynomial at 0 (Maclaurin).
- $R_{N+1}(x) = f(x) - T_N(x)$: *remainder* (error). For alternating, decreasing term sizes: $|R_{N+1}(x)| \lesssim$ first omitted term.
- $n! = n(n-1)\cdots 1$: factorial.
- *Radius of convergence* (intuitive): how far the power series meaningfully "reaches." We use it qualitatively here.

Core Idea 1: Three series you can use on sight

$$\boxed{\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots && \text{(all } x\text{)} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \text{(all } x\text{)} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \text{(all } x\text{)} \end{aligned}}$$

Why the sin/cos signs? Use Euler: $e^{ix} = \cos x + i \sin x$ and equate real/imaginary parts.

Mini-check. For small x, $\sin x \approx x$, $\cos x \approx 1 - \frac{x^2}{2}$, $e^x \approx 1 + x$. These pass the "tiny x" sniff test.

Core Idea 2: Generalized binomial for tiny changes

For $|x| < 1$,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Special favorites:

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \cdots$$
$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

Trick: To approximate a^α, write $a = A(1+\delta)$ with A friendly and $|\delta| \ll 1$, so $a^\alpha = A^\alpha(1+\delta)^\alpha$.

Algorithm 1: Build a quick Taylor approximation

Input: function f (one of $e^x, \sin, \cos$), target x, degree N.
Output: $T_N(x)$ and an error estimate.

Step 1. Pick degree N (often $3, 4$, or 5 is enough for $|x| \le 0.5$).
Step 2. Write the first $N+1$ terms from the boxed series and plug x.
Step 3. **Error estimate:** use the absolute value of the next omitted term. For alternating series ($\sin, \cos$ at small x), this is a reliable bound.

Worked examples.

Ex. 1. $e^{0.1}$ with $N = 4$:

$$T_4 = 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} + \frac{0.1^4}{24} = 1.1051708333\ldots$$

True $e^{0.1} = 1.1051709180\ldots$ Error $\approx 8.47 \times 10^{-8}$. Next term $0.1^5/120 \approx 8.33 \times 10^{-8}$.

Ex. 2. $\sin(0.5)$ with $N = 5$:

$$T_5 = 0.5 - \frac{0.5^3}{6} + \frac{0.5^5}{120} = 0.4794270833\ldots$$

True $\sin(0.5) = 0.4794255386\ldots$ Error $\approx 1.54 \times 10^{-6}$. Next term $0.5^7/7! \approx 1.55 \times 10^{-6}$.

Ex. 3. $\cos(0.5)$ with $N = 4$:

$$T_4 = 1 - \frac{0.5^2}{2} + \frac{0.5^4}{24} = 0.8776041666\ldots$$

True $\cos(0.5) = 0.8775825619\ldots$ Error $\approx 2.16 \times 10^{-5}$. Next term $0.5^6/6! \approx 2.17 \times 10^{-5}$.

Algorithm 2: Binomial hacks for $\sqrt{}$ and friends

Goal: Approximate $(1+x)^\alpha$ for small $|x|$.

Step 1. Identify x so that your number looks like $1 + x$.
Step 2. Use 3–5 terms of the binomial series for α ($\frac{1}{2}$, $-\frac{1}{2}$, etc.).
Step 3. Multiply by any outer factor if you wrote $a = A(1+x)$.
Step 4. Use the next term's size as an error estimate.

Worked examples.

Ex. 5. $\sqrt{1.04} = (1+0.04)^{1/2}$. Using terms through x^4:

$$1 + \tfrac{1}{2}(0.04) - \tfrac{1}{8}(0.04)^2 + \tfrac{1}{16}(0.04)^3 - \tfrac{5}{128}(0.04)^4 = 1.0198039000\ldots$$

True $\sqrt{1.04} = 1.0198039027\ldots$ Error $\approx 2.7 \times 10^{-9}$ (tiny because $x = 0.04$ is small).

Ex. 6. $\dfrac{1}{\sqrt{1.1}} = (1+0.1)^{-1/2}$ with terms through x^3:

$$1 - \tfrac{1}{2}(0.1) + \tfrac{3}{8}(0.1)^2 - \tfrac{5}{16}(0.1)^3 = 0.9534375.$$

True $0.9534625892\ldots$ Error $\approx 2.51 \times 10^{-5}$. Next term size $\approx 1.72 \times 10^{-5}$.

Mini-Labs (paper or computer)

- **Overlay plots.** On $[-0.5, 0.5]$, draw $y = \sin x$ and its cubic; draw $y = \cos x$ and its quartic.
- **Convergence feel.** Fix $x = 0.3$. Compute T_1, T_3, T_5 for $\sin x$; watch the error shrink like the next term.
- **Radius intuition.** Try $(1+x)^{1/2}$ at $x = 0.9$ vs $x = 1.1$. See why $|x| < 1$ matters.

Practice (do, then check)

- **P1.** $e^{-0.2}$ with terms through x^4.
- **P2.** $\sin(18°)$ using radians $x = \pi/10$ with terms through x^5.
- **P3.** $\cos(15°)$ with terms through x^4 (use $x = \pi/12$).
- **P4.** Expand $(1+x)^{-1/2}$ to x^3. Evaluate at $x = 0.1$ and compare to a calculator.
- **P5.** Expand $(1+x)^{1/3}$ to x^3. Evaluate at $x = -0.06$ to approximate $\sqrt[3]{0.94}$.
- **P6.** (Geometric series warm-up) Compute $1 + 0.2 + 0.2^2 + \cdots + 0.2^5$ and compare to $\frac{1-0.2^6}{1-0.2}$.

Capstone A2: Series Approximation Bake-off

1. Approximate $\sin(0.3)$, $e^{0.5}$, and $(1.02)^{1/2}$ using 3, 4, and 5 terms. For each: record the first omitted term's size and compare to the true error.
2. Bonus: $\cos(15°)$ with quartic. True value $= \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.9659258263$. How close are you?

Bridge to Lesson A3 (First Look at Fourier)

Polynomials approximate *local* behavior (near a point). Fourier series approximate *periodic* behavior:

$$f(\theta) = a_0 + \sum_{n\geq 1} \big(a_n \cos n\theta + b_n \sin n\theta\big).$$

Think "Taylor near 0" vs "Fourier around the circle." In A3 we'll sample a periodic signal and build its first harmonics.

Selected Answers (numerical)

- **From examples:** $e^{0.1} \approx 1.1051708333$ (true 1.1051709180); $\sin(0.5) \approx 0.4794270833$ (true 0.4794255386); $\cos(0.5) \approx 0.8776041667$ (true 0.8775825619); $\sqrt{1.04} \approx 1.0198039000$ (true 1.0198039027); $1/\sqrt{1.1} \approx 0.9534375$ (true 0.9534625892).
- **P3 hint-check:** With $x = \pi/12 \approx 0.261799$, $1 - \frac{x^2}{2} + \frac{x^4}{24} \approx 0.9659262729$, true 0.9659258263. Error $\approx 4.47 \times 10^{-7}$.

Challenge (optional)

- Differentiate the geometric series $\sum_{n\geq 0} x^n = \frac{1}{1-x}$ (for $|x| < 1$) to get a series for $\frac{1}{(1-x)^2}$. Integrate to get $\ln(1+x)$ (with care about constants).
- Show that for $|x| \leq 0.5$, the alternating series error for $\sin x$ after the $x^5/5!$ term is at most $x^7/7!$.