Homework 2
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Exercise 21. *Prove that each graph G contains a bipartite graph with at least half as many edges.*

Proof. Consider the random graph G^* formed by assigning the vertices of G to one of two vertex partitions V_1 or V_2 with random and uniform probability $p = \frac{1}{2}$. An edge $xy \in G$ appears as a cross edge in G^* exactly when x and y are assigned to different vertex partitions. As the vertices of G are assigned independently, the probability an edge xy is a cross edge in G^* is also $p = \frac{1}{2}$.

Let X be the random variable that records the number of cross edges in $G*$. Consider the expected value of X and let m be the number of edges in $G*$

$$Ex[X] = \sum_{xy \in E(G^*)} p(xy) = \frac{m}{2}$$

By the probabilistic method $\exists$ a graph G with $X \geq E[X] = \frac{m}{2}$. Take such a graph and remove all edges between vertices in the same classes, and we have the desired bipartite graph with at least half as many edges. $\square$

Exercise 31. *Show that for any k-vertex tree T_k that ex(n,T_k) $\leq (k-1)n$.*

Assume to the contrary for the sake of a contradiction i.e. ex(n,T_k) $> (k-1)n$. Let G be an n-vtx graph with greater than $(k-1)n$ edges, and average degree d. It is clear that d is

$$> \frac{2|E(G)|}{n} = \frac{2(k-1)n}{n}$$

By exercise 30, G has a subgraph H with minimum degree

$$\delta(H) > \frac{2(k-1)n}{2n} = k-1$$

Claim: H contains T_k.

Proof. As H has $\delta(H) \geq k$ it follows that $|V(H)| \geq k$ as well.

Consider the following procedure.

1. Pick a vertex $v \in V(H)$
2. Start walking greedily to a neighbor of v such that the next step doesn't create a cycle. This vertex is the new v.
3. Repeat step (2) k-2 more times.

Observe:

1. We are able to execute step (2) successfully at each iteration as $\delta(H) \geq k$ and we can walk to a vertex that hasn't been visited yet, thus preventing the creation of a cycle.
2. As we iterated $k-1$ times our walk has $k-1$ edges, and from (1) is cycle-free.

This is the definition of a k-tree or T_k, which is a contradiction ※ .
Therefore our supposition was false and

$$ex(n, T_k) \leq (k-1)n$$

□

Exercise 34. *Show that there exists an n-vertex graph G with no cycle of length $\leq g$ and $\frac{1}{12}n^{1+1/g}$ edges.*

Proof. The construction is probabilistic with the deletion method. Consider the probability space of random graphs with edge probability

$$p = n^{1/g-1}$$

The expected number of edges is

$$\binom{n}{2}p \geq \frac{1}{4}n^{1+1/g}$$

The expected number of cycles $C_k, (k \leq g)$ is

$$\frac{n(n-1)(n-2)...(n-g+1)}{6g}p^g \leq \frac{1}{6g}n^g p^g \leq \frac{1}{12}n^{1+1/g}$$

Let X be the random variable defined by the number of edges minus the number of copies of $C_k, k \leq g$. Thus, by linearity of expectation we have

$$E[X] \geq \frac{1}{12}n^{1+1/g}$$

By the probabilistic method, there exists a graph G with $X \geq E[X]$ If we remove an edge from each $C_k, k \leq g$ we are left with an n-vertex C_k-free graph with $e(G) \geq \frac{1}{12}n^{1+1/g}$. $\square$

Exercise 40. *Show that the regularity lemma holds for graphs on $n < M$ vertices.*

Proof. The regularity lemma is trivially true when $n < M$, and does not reveal much useful information about the graph. Consider partitioning the vertices into singleton classes (i.e. $r = m$) to get Szemerédi partition P. Consider a pair of classes $(V_1, V_2) \in P$.

Claim: All pairs (V_1, V_2) are ϵ-regular

1. xy is an edge $x \in V_1, y \in V_2$.

$$d(V_1, V_2) = \frac{e(V_1, V_2)}{|V_1||V_2|} = \frac{1}{1} = 1$$
$$d(V_1', V_2') = \frac{e(V_1', V_2')}{|V_1'||V_2'|} = \frac{1}{1} = 1$$

$$|d(V_1, V_2) - d(V_1', V_2')| = |1 - 1| = 0 < \epsilon \checkmark$$

2. xy is not an edge $x \in V_1, y \in V_2$.

$$d(V_1, V_2) = \frac{e(V_1, V_2)}{|V_1||V_2|} = \frac{0}{1} = 0$$
$$d(V_1', V_2') = \frac{e(V_1', V_2')}{|V_1'||V_2'|} = \frac{0}{1} = 0$$

$$|d(V_1, V_2) - d(V_1', V_2')| = |0 - 0| = 0 < \epsilon \checkmark$$

All pairs of classes are ϵ-regular, and the regularity lemma holds for $n < M$.

$\square$

Exercise 41. *Show that the regularity lemma holds for n-vertex graphs with $o(n^2)$ many edges when n is large.*

Proof. Consider what happens to the density between classes as n tends towards infinity. Let A, B be pairs of classes in the Szemerédi partition P created by partitioning the vertex set of an n-vertex graph G into r parts.

$$d(A,B) = \frac{e(A,B)}{|A||B|}$$

Observe the following

$$e(A,B) = o(n^2)$$

and

$$|A||B| = \frac{n^2}{r^2}$$

As r is fixed, when n get's large the bottom term r^2 becomes negligible and n^2 dominates. Therefore, the density

$$d(A,B) = \frac{e(A,B)}{|A||B|} = \frac{o(n^2)}{O(n^2)} \to 0$$

and the number of ϵ-regular pairs tends towards $\binom{r}{2}$. Therefore, when n is large enough there are eventually $\leq \epsilon r^2$ pairs that are not ϵ-regular, and the regularity lemma holds for n-vertex graphs with $o(n^2)$ many edges (for n large enough).

□

Exercise 42. *Suppose G is a bipartite graph with classes A and B each of size n. Show that if the maximum degree of G is at most $\epsilon^2 n$ then the pair (A,B) is ϵ-regular.*

Proof. In order for the pair (A, B) to satisfy epsilon regularity the following must be true

$$|d(A, B) - d(A', B')| < \epsilon$$

$\forall A' \subset A, B' \subset B$ such that $|A'| > \epsilon|A|, |B'| > \epsilon|B|$.

It will suffice to show that neither $d(A, B)$ nor $d(A', B')$ exceed epsilon.

1. $$d(A, B) = \frac{e(A, B)}{|A||B|} = \frac{e(A, B)}{n^2}$$

   As the maximum degree of a vertex in G is $\epsilon^2 n$

   $$e(A, B) \leq \epsilon^2 n^2$$

   and we get that

   $$d(A, B) \leq \frac{\epsilon^2 n^2}{n^2} = \epsilon^2 < \epsilon$$

2. $$d(A', B') = \frac{e(A', B')}{|A'||B'|}$$

   As the maximum degree of a vertex in G is ϵn^2

   $$e(A', B') \leq \epsilon^2 n|A'|$$

   This gives

   $$d(A', B') \leq \frac{\epsilon^2 n|A'|}{|A'||B'|} = \frac{\epsilon^2 n}{|B'|}$$

   By epsilon regularity

   $$|B'| > \epsilon|B| = \epsilon n$$

   Therefore,

   $$d(A', B') < \frac{\epsilon^2 n}{\epsilon n} = \epsilon$$

   Combining bounds 1 and 2 we get the following

   $$|d(A, B) - d(A', B')| < \epsilon$$

   which is the theorem!

$\square$

Exercise 46. *Suppose G is a graph where each edge is in exactly one triangle. Use the triangle removal lemma to show that the number of edges in G is $o(n^2)$.*

Proof. First, observe that as each edge is in exactly one triangle the number of triangles in G is exactly $\frac{1}{3}m$, where $m = |E(G)|$. The triangle removal lemma states that if a graph G contains fewer than βn^3 triangles it can be made triangle free with the removal of relatively few edges (fewer than αn^2 edges to be exact).

Consider removing edges from G to make it triangle free. It can be made triangle free by removing one edge from each triangle. As each edge is apart of exactly 1 triangle, we remove exactly $\frac{m}{3}$ edges. The triangle removal lemma tells us that there were fewer than βn^3 triangles, and therefore m is $o(n^2)$. $\square$

Exercise 53. *Show that if a bipartite graph is ϵ-regular, then the complement (of the edges between the classes) is also ϵ-regular.*

Proof. Let G be such a graph, and observe the relationship between the densities of class $A, B \in G$ and their respective complements $\overline{A}, \overline{B} \in \overline{G}$

$$d\big(A, B\big) + d\big(\overline{A}, \overline{B}\big) = 1$$

rearranging terms gives

$$d\big(A, B\big) = 1 - d\big(\overline{A}, \overline{B}\big)$$

By epsilon regularity we know that for the two classes $A, B \in G$ we have have

$$|d(A, B) - d(A', B')| < \epsilon$$

$\forall A' \subset A, B' \subset B$ such that $|A'| > \epsilon|A|$ and $|B'| > \epsilon|B|$.

We would like to show the same for the compliments of A, B and we know

$$|d(A, B) - d(A', B')| < \epsilon$$

Substituting $1 - d\big(\overline{A}, \overline{B}\big)$ for $d(A, B)$, and $1 - d\big(\overline{A'}, \overline{B'}\big)$ for $d(A', B')|$ gives

$$|1 - d\big(\overline{A}, \overline{B}\big) - (1 - d\big(\overline{A'}, \overline{B'}\big))| > \epsilon$$

Simplifying gives

$$|d\big(\overline{A'}, \overline{B'}\big) - d\big(\overline{A}, \overline{B}\big)| < \epsilon$$

Which gives the theorem. $\square$