Homework 1
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Exercise 2. *Show that the number of monochromatic triangles in any* 2*-coloring of the edges of* K_n *is at least*

$$\frac{n(n-1)(n-5)}{24}$$

Proof. Let G be a K_n and Fix a vertex $v \in V(G)$. There are n such ways to fix a vertex v. Now color edges incident to v such that we maximize the number of triangles that are not monochromatic (i.e. contain edges of different colors). For each vertex $v \in V(G)$ color $\frac{n-1}{2}$ edges one color, and the remaining edges another color. This creates at most $\frac{n-1}{2}$ new triangles that are not monochromatic. For every vertex v we create at most $\frac{1}{2}\left(\frac{n-1}{2}\right)^2$ such triangles as we have double counted. The total number of triangles in G is $\binom{n}{3}$. Therefore, the total number of monochromatic triangles is

$$\geq \binom{n}{3} - \frac{n}{2}\left(\frac{n-1}{2}\right)^2 = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)^2}{8}$$

$$= \frac{n(n^2-6n+5)}{24} = \frac{n(n-1)(n-5)}{24}$$

□

Exercise 3. *Show that the number of triangles in a graph with n vertices and m edges is at least*

$$\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$$

Proof. Let G be a graph on n vertices and m edges, and consider $uv \in E(G)$. The $d(u) + d(v) > n$ if and only if $\exists$ a triangle containing the edge uv. Therefore, it follows that the number of common neighbors of u and v is at least $d(u) + d(v) - n$. Each common neighbor of u and v form a triangle in G and there are greater than $d(u) + d(v) - n$ such neighbors for each $uv \in E(G)$. This implies the number of triangles

$$\geq \frac{1}{3} \sum_{uv \in E(G)} (d(u) + d(v) - n) = \frac{1}{3} \sum_{v \in V(G)} \big(d(v)^2\big) - \frac{1}{3}mn$$

By Cauchy-Schwarz we have

$$\geq \frac{1}{3n}\left(\sum_{v \in V(G)} d(v)\right)^2 - \frac{1}{3}mn = \frac{1}{3n}\Big(2m\Big)^2 - \frac{1}{3}mn = \frac{4m^2}{3n} - \frac{1}{3}mn$$

□

Exercise 4. *Confirm that $T_k(n)$ is the largest (most edges) n-vertex complete multipartite graph without a (k + 1)-clique.*

Proof. Let G be a candidate graph for a complete multipartite graph without a $(k+1)$-clique. Observe that certainly an edge maximal graph without a $k+1$-clique requires the addition of only one edge to have a $k+1$-clique. This implies that such a graph must have a k_k embedded within it. Use a K_k as the starting point for G, certainly it has the most edges of any k-vertex multipartite graph. Now we will add vertices one by one, until we run out of vertices. We want to add a vertex in such a way that maximizes the number of edges we create. First add a vertex to any of the k-classes from the k-partite G for a new G. A new class wasn't created as it would have formed a $k+1$-clique with the other k classes. In general, each iteration we will add a vertex to a class of smallest size, which ensures that the number of edges that are created is maximal. If all classes at that current iteration have the same size, pick a class arbitrarily to add a vertex to. Repeat this procedure until we have used all of the vertices. Observe that each class differs by a size of at most 1, and more specifically the ending G will have class sizes $\lfloor \frac{n}{k} \rfloor$ and $\lceil \frac{n}{k} \rceil$. This is the Turán graph $T_k(n)$, so $G = T_k(n)$. $\square$

Exercise 6. *The maximum number of edges in a graph G on n vertices with no $(k+1)$-clique subgraph is at most*

$$\left(1-\frac{1}{k}\right)\frac{n^2}{2}$$

Proof. To each vertex $v \in V(G)$ assign a non-negative weight w(x) such that

$$\sum_{x \in V(G)} w(x) = 1$$

We would like to determine the max value of the following

$$S = \sum_{xy \in E(G)} w(y)w(x)$$

Assigning $1/n$ to each vertex gives the maximum of S is

$$S \geq \frac{|E(G)|}{n^2}$$

Showing S cannot exceed $\frac{1}{2}\left(1-\frac{1}{k}\right)$ will complete the proof. So we will employ the weight shifting technique to show this. Let x and y be non-adjacent vertices and let W_x and W_y be the sum of weights on vertices adjacent to x and y respectively. Assume $W_x \geq W_y$ and let $\epsilon > 0$, thus

$$\big(w(x)+\epsilon\big)W_x + \big(w(y)-\epsilon\big)W_y \geq w(x)W_x + w(y)W_y$$

This implies we can shift all of the weight from one vertex y to some non-adjacent vertex x and not decrease S (assuming $W_y \leq W_x$). The graph has no K_{k+1} so we can shift all of the weight to the vertices of a K_k and not decrease S. Certainly S is maximized when each vertex has equal weight. A K_k has k vertices and the weights sum to 1 so each vertex will have weight equal to $(1-\frac{1}{k})$. Therefore, S is maximized at $\frac{1}{2}\left(1-\frac{1}{k}\right)$. Substitute this value for S and we get the following

$$\frac{1}{2}\left(1-\frac{1}{k}\right) \geq \frac{|E(G)|}{n^2}$$

Which is the theorem. □

Exercise 7. *Let $n \geq k+1$ and let G be an n-vertex graph. If $e(G) = ex(n, K_k) + 1$, then G contains the graph H formed by $K_k + 1$ minus an edge.*

Proof. Proceed by induction on n, the number of vertices. The base case is when $n = k+1$. Observe that G is simply the Turán graph with $k-1$ classes, and $k+1$ vertices. In other words

$$e(G) = ex(n, K_k) + 1 = e(T_{k-1}(n)) + 1$$

Therefore, it follows that $k-3$ of the $k-1$ classes contain only a single vertex. The remaining 2 classes have just 2 vertices each, for a total of $k+1 = n$. Let V_1 be the class with the added edge. Trivially $V1$ must be a class of size two. It follows by the definition of the Turán graph that this edge must also have both it's endpoints in V_1, as all other edges between vertices of different classes are already present. Let V_2 be the other class with 2 vertices and observe that all the vertices of G are connected to one another except for the vertices in $V2$. Therefore, G contains H a K_{k+1} minus the edge between vertices in $V2$.

Let $n > k+1$ and assume the statement above holds for graphs on fewer vertices.
Consider a vertex v of minimum degree in G.

Consider the following cases:

1. $k-1$ divides n.

   Once again from above we know that $e(G) = ex(n, k_k) + 1 = e(T_{k-1}(n)) + 1$. Therefore G is simply the Turán graph with $k-1$ classes each of size $\frac{n}{k-1}$, one with an edge between two of its vertices. Consider removing a vertex v of minimum degree to get G'. Observe that removing v doesn't decrease the number of classes, as each class has ≥ 2 vertices (assuming $k-1$ divides n). However G' is simply $T_{k-1}(n-1)$ plus an edge. By induction G' contains H, formed by K_{k+1} minus an edge, and G' is contained in G. Therefore, G also contains H!

2. $k-1$ doesn't divide n

   Once more from above we know that $e(G) = ex(n, k_k) + 1 = e(T_{k-1}(n)) + 1$. Therefore G is simply the Turán graph with $k-1$ classes each of size $\left\lfloor \frac{n}{k-1} \right\rfloor$ or $\left\lceil \frac{n}{k-1} \right\rceil$. Observe that there must be ≥ 3 classes of size at least 2. Let $V1$ be a larger class that we have added an edge between two of its vertices. Let $V2$ be a larger class that we remove a vertex v of minimum degree from, and let V_3 be one of the remaining largest classes. Observe that removing v doesn't decrease the number of classes, so we are left with $G' = T_{k-1}(n-1)$ plus an edge. This is a smaller graph so we apply induction to find H which is in G'. However, G contains G', so H is also in G!

□

Exercise 11. *Use Turáns theorem to show that among 3n points on the unit disk at least* $3\binom{n}{2}$ *of them are distance at most* $\sqrt{2}$

Proof. Build a graph G with $V(G)$ = points on the unit disk. Connect two points iff the sum of their euclidean distances is greater than $\sqrt{2}$.

<u>Claim:</u> G contains no K_4.

Proof. Pick a subset $S \subset V(G)$ of 4 distinct points. It will be sufficient to show that at least 1 edge between any of these points cannot be greater than $\sqrt{2}$. Thus this edge is not present in G and these 4 vertices cannot form a K_4.

Consider the following cases:

1. S contains a vertex at the center of the unit disk

   This case is easily resolved as this vertex is distance at most 1 from any other vertex, and cannot be contained in any K_4 (as its degree is 0 in G).

2. No vertex in S is the center of the unit disk

   Draw a line from the center of the disk through each individual point to the edge of the disk. This partitions the disk into potentially four parts P_1, P_2, P_3, and P_4. If there are less than 4 parts then ≥ 2 vertices lie on the same line and cannot possibly be distance greater than $\sqrt{2}$ from one another.

   Observe that the interior angle of one of these partitions must be $\leq 90°$ as they all cannot have interior angle $\geq 90°$. Let x and y be vertices on the boundary of such a partition with interior angle $\leq 90°$. The distance between x and y is maximized when the angle of the partition is greatest ($90°$) and the points are pushed to the perimeter of the unit disk. Solving for the distance between them gives $d(x, y) = \sqrt{1^2 + 1^2} = \sqrt{2}$. Therefore, the edge xy is not present in G. Repeating this argument for all subsets S completes the claim.

□

Therefore G is K_4 free, and by Turáns theorem we get the following

$$e(G) \leq \left(1 - \frac{1}{4}\right)\frac{n^2}{2} = \frac{3}{8}n^2$$

The compliment of G has edges between points who's distance is at most $\sqrt{2}$. As we can have at most $\binom{n}{2}$ edges in any graph, the number of edges in $\overline{G}$ is

$$\geq \binom{n}{2} - \frac{3}{8}n^2 = \frac{4n^2 - 3n}{8}$$

Therefore, selecting $3n$ points guarantees the number of edges

$$\geq \frac{12n^2 - 9n}{8} \geq 3\binom{n}{2}$$

As edges are pairs with distance at most $\sqrt{2}$ this completes the proof. $\square$

Exercise 20. *For a fixed graph F, show that the following function is decreasing as n increases*

$$\frac{ex(n,F)}{\binom{n}{2}}$$

Proof. Let G be an extremal graph for F and count the number of pairs (e, v) where e is an edge of G and v is a vertex not incident to e. First pick an edge e, that edge is not incident to at most $n-2$ vertices. Summing over all edges of G gives the following lower bound on the number of pairs

$$\geq \sum_{xy \in E(G)} (n-2) = ex(n,F)(n-2)$$

Now instead fix a vertex $v \in V(G)$, there are n such ways to do this. Observe that once we have fixed a vertex the maximum number of edges that v is not incident to is the extremal number for a graph on $n-1$ vertices. This gives an upper bound on the number of pairs which is

$$\leq ex(n-1,F)n$$

Combining these bounds we get

$$ex(n,F)(n-2) \leq ex(n-1,F)n$$

WLOG multiply each side by $\frac{(n-1)}{2}$

$$\frac{1}{2}\big(ex(n,F)(n-1)(n-2)\big) \leq \frac{1}{2}\big(ex(n-1,F)n(n-1)\big)$$

Rearranging terms gives

$$\frac{ex(n,F)}{\frac{n(n-1)}{2}} \leq \frac{ex(n-1,F)}{\frac{(n-1)(n-2)}{2}}$$

$\Longrightarrow$

$$\frac{ex(n,F)}{\binom{n}{2}} \leq \frac{ex(n-1,F)}{\binom{n-1}{2}}$$

Therefore, the function is decreasing as n grows larger. $\square$

Exercise 24. *Prove Lemma 1.29. Let G be a $C4$-free bipartite graph with class sizes a and b, then*

$$e(G) \leq a\sqrt{b} + b$$

Proof. Let A be the class of size a and B the class of size b. Consider counting V structures centered on a vertex $v \in A$ in two different ways. First, fix a vertex $v \in A$ and count the number of V's centered at that vertex. There are at least $\binom{d(v)}{2}$ such V's. Summing over all vertices in A, we get the number of V's

$$\geq \sum_{v \in V(A)} \binom{d(v)}{2} \geq a\binom{\frac{1}{a}\sum d(v)}{2} = a\binom{e(G)/a}{2}$$

$$\geq a\left(\frac{(e(G)/(a-1))^2}{2}\right) \geq a\left(\frac{(\frac{e(G)}{a}-1)^2}{2}\right)$$

Now count V's coming from B to get a related upper bound. Observe that for every two vertices x and y in B they have at most 1 common neighbor (else we have a C_4). Therefore the number of V's is

$$\leq \binom{b}{2} \leq \frac{b^2}{2}$$

Combining these bounds we get

$$a\left(\frac{(\frac{e(G)}{a}-1)^2}{2}\right) \leq \frac{b^2}{2}$$

Solving for $e(G)$ gives the statement of lemma 1.29.

$$a\left(\frac{e(G)}{a}-1\right)^2 \leq b^2$$

$$\frac{e(G)}{\sqrt{a}} - \sqrt{a} \leq b$$

$$E(G) \leq b\sqrt{a} + a$$

Swapping the classes that we counted gives the other bound

$$E(G) \leq a\sqrt{b} + b$$

$\square$