HA 1

Modern Method of Decision Making

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Task Description

Hedge algorithm for linear losses on the simplex. Consider the OCO problem with

Definition

Decision space: $K = \Delta_d$, where Δ_d is the d-dimensional probability simplex:

$$\Delta_d = \left\{ x = (x(1), \dots, x(d)) \in \mathbb{R}^d : \forall i,\ x(i) \in [0,1] \text{ and } \sum_{i=1}^{d} x(i) = 1 \right\}$$

Definition

Loss class:

$$\mathcal{F} = \left\{ x \in \Delta_d \to \langle x, l \rangle : \ l = (l(1), \dots, l(d)) \in [0,1]^d \right\}$$

The goal of this home assignment is to analyse a more general version of the Hedge algorithm that involves a time varying parameter "ε_t", independent of a prescribed time horizon, and guaranteed to have low regret at any time. The algorithm we want to analyse is as follows:

Algorithm 1 General Hedge

Require: $(\varepsilon_t)_{t \geq 1}$ ▷ Tuning parameters, $\varepsilon_t > 0$

Initialize:

$L_0 = (L_0(1), \dots, L_0(d)) = (0, \dots, 0)$ ▷ Cumulative loss function at time 0

for $t \geq 1$ **do**

- Play $x_t = (x_t(1), \dots, x_t(d)) \in \Delta_d$, where

$$x_t(i) := \frac{e^{-\varepsilon_t \mathcal{L}_{t-1}(i)}}{\sum_{j=1}^{d} e^{-\varepsilon_t \mathcal{L}_{t-1}(j)}}$$

- Receive loss $l_t = (l_t(1), \dots, l_t(d)) \in [0,1]^d$ and update:

$$L_t := L_{t-1} + l_t$$

end for

Note

To connect this version of the algorithm with the one discussed previously, note that the above algorithm can be equivalently described as follows:

Initialize:

$w_1 = (w_1(1), \dots, L_1(d)) = (1, \dots, 1)$

for $t \geq 1$ do

- Play $x_t = (x_t(1), \dots, x_t(d)) \in \Delta_d$, where

$$x_t(i) := \frac{w_t(i)}{\sum\limits_{j=1}^{d} w_t(j)}$$

- Receive loss $l_t = (l_t(1), \dots, l_t(d)) \in [0, 1]^d$ and update:

$$w_{t+1}(i) = (w_t(i))^{\frac{\varepsilon_{t+1}}{\varepsilon_t}} \cdot e^{-\varepsilon_{t+1} l_t(i)}$$

$$\triangleright\ w_t(i) = e^{-\varepsilon_t \mathcal{L}_{t-1}(i)}$$

end for

Note that we recover the simple Hedge algorithm whenever $\varepsilon_t = \varepsilon,\ \forall t \geq 1$.
The goal of the home assignment is to show the following result.

Solution

Theorem

i) Suppose that $0 < \varepsilon_{t+1} \leq \varepsilon_t,\ \forall t \geq 1$. Then, $\forall T \geq 1$, the Hedge algorithm with time varying parameter $(\varepsilon_t)_{t \geq 1}$ satisfies:

$$R_T \leq \frac{1}{8} \sum_{t=1}^{T} \varepsilon_t + \frac{\log d}{\varepsilon_{T+1}}$$

ii) In particular, chosing:

$$\varepsilon_t := \sqrt{\frac{8 \log d}{t}}$$

implies that, $\forall T \geq 1$,

$$R_T \leq \sqrt{2T \log d}$$

Proof: We divide the proof in 5 steps:

1. Define $W_t := \frac{1}{d} \sum\limits_{i=1}^{d} e^{-\varepsilon_t L_{t-1}(i)}$, $\forall t \geq 1$. Show that $\forall T \geq 1$:

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_1} \geq - \inf_{x \in \Delta_d} \sum_{t=1}^{T} \langle x, l_t \rangle - \frac{\log d}{\varepsilon_{T+1}};$$

Let us bound $\log\left(\frac{W_{T+1}}{W_1}\right)$ from below:

$$\log\left(\frac{W_{T+1}}{W_1}\right) = \log\left(\frac{1}{d}\sum_{i=1}^{d} e^{-\varepsilon_{T+1}L_T(i)}\right) - \log 1 = -\log d + \log\left(\sum_{i=1}^{d} e^{-\varepsilon_{T+1}L_T(i)}\right) \geq$$

$$\geq -\log d + \log\left(\max_i e^{-\varepsilon_{T+1}L_T(i)}\right) = -\varepsilon_{T+1}\min_i L_T(i) - \log d$$

After dividing the last inequality by the ε_{T+1} and using basic knowledge about properties of logarithms we can obtain:

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_{T+1}} \geq -\min_i \mathcal{L}_T(i) - \frac{\log d}{\varepsilon_{T+1}};$$

We can use the knowledge that $\min_i \mathcal{L}_T(i) = \inf_{x\in\Delta_d}\sum_{t=1}^{T}\langle x, l_t\rangle$:

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_{T+1}} \geq -\inf_{x\in\Delta d}\sum_{t=1}^{T}\langle x, l_t\rangle - \frac{\log d}{\varepsilon_{T+1}}.$$

Finally, as we know, that $\varepsilon_1 \geq \varepsilon_{T+1}$, we can obtain the desired inequality:

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_1} \geq -\inf_{x\in\Delta_d}\sum_{t=1}^{T}\langle x, l_t\rangle - \frac{\log d}{\varepsilon_{T+1}}.$$

2. Show that $\forall T \geq 1$:

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_1} = \sum_{t=1}^{T}(a_t + b_t)$$

where $\forall t \geq 1$:

$$\to a_t := \frac{1}{\varepsilon_t}\log\left(\frac{W_{t+1}^{\frac{\varepsilon_t}{\varepsilon_{t+1}}}}{\widetilde{W}_{t+1}}\right)$$

$$\to b_t := \frac{1}{\varepsilon_t}\log\left(\frac{\widetilde{W}_{t+1}}{W_t}\right),$$

$$\to \widetilde{W}_{t+1} := \frac{1}{d}\sum_{i=1}^{d} e^{-\varepsilon_t L_t(i)}.$$

Left side of the initial equality can be written in the following way:

$$\log\left(\frac{W_{T+1}^{\frac{1}{\varepsilon_{T+1}}}}{W_1^{\frac{1}{\varepsilon_1}}}\right) = \log\left(\prod_{t=1}^{T}\frac{W_{t+1}^{\frac{1}{\varepsilon_{t+1}}}}{W_t^{\frac{1}{\varepsilon_t}}}\right) = \sum_{t=1}^{T}\log\left(\frac{W_{t+1}^{\frac{1}{\varepsilon_t+1}}}{W_t^{\frac{1}{\varepsilon_t}}}\right)$$

At the same time:

$$\sum_{t=1}^{T}(a_t+b_t)=\sum_{t=1}^{T}\left(\log\left(\frac{W_{t+1}^{\frac{1}{\varepsilon_{t+1}}}}{\widetilde{W}_{t+1}^{\frac{1}{\varepsilon_t}}}\right)+\log\left(\frac{\widetilde{W}_{t+1}}{W_t}\right)^{\frac{1}{\varepsilon_t}}\right)=\sum_{t=1}^{T}\left(\log\left(\frac{W_{t+1}^{\frac{1}{\varepsilon_{t+1}}}}{W_t^{\frac{1}{\varepsilon_t}}}\right)\right)=$$

$$=\frac{\log W_{T+1}}{\varepsilon_{T+1}}-\frac{\log W_1}{\varepsilon_1}.$$

3. Show that, $\forall t\geq 1,\ a_t\leq 0$. Actually, we need to show:

$$\frac{1}{\varepsilon_{t+1}}\log W_{t+1}-\frac{1}{\varepsilon_t}\log\widetilde{W}_{t+1}\leq 0$$

or

$$\frac{\varepsilon_t}{\varepsilon_{t+1}}\log W_{t+1}\leq\log\widetilde{W}_{t+1}$$

Hence,

$$\log W_{t+1}^{\frac{\varepsilon_t}{\varepsilon_{t+1}}}\leq\log\widetilde{W}_{t+1}$$

and,

$$W_{t+1}^{\frac{\varepsilon_t}{\varepsilon_{t+1}}}\leq\widetilde{W}_{t+1}$$

$$\left(\frac{1}{d}\sum_{i=1}^{d}e^{-\varepsilon_{t+1}\mathcal{L}_t(i)}\right)^{\frac{\varepsilon_t}{\varepsilon_{t+1}}}\leq\frac{1}{d}\sum_{i=1}^{d}e^{-\varepsilon_t\mathcal{L}_t(i)}$$

Keeping in mind the Jensen's inequality, we can conclude that the last inequality is always true since

$$\frac{\varepsilon_t}{\varepsilon_{t+1}}\geq 1$$

.

4. Show that, $\forall t\geq 1$:

$$b_t\leq\frac{\varepsilon_t}{8}-\langle x_t,l_t\rangle.$$

Let us do it iteratively:

$$b_t=\frac{1}{\varepsilon_t}\log\left(\frac{\widetilde{W}_{t+1}}{W_t}\right)=\frac{1}{\varepsilon_t}\log\left(\frac{\sum_{i=1}^{d}e^{-\varepsilon_t\mathcal{L}_t(i)}}{\sum_{i=1}^{d}e^{-\varepsilon_t\mathcal{L}_{t-1}(i)}}\right)=$$

$$=\frac{1}{\varepsilon_t}\log\left(\frac{\sum_{i=1}^{d}e^{-\varepsilon_t l_t(i)}\cdot e^{-\varepsilon_t\mathcal{L}_{t-1}(i)}}{\sum_{i=1}^{d}e^{-\varepsilon_t\mathcal{L}_{t-1}(i)}}\right)=$$

$$=\frac{1}{\varepsilon_t}\log\left(\sum_{i=1}^{d}x_t(i)e^{-\varepsilon_t l_t(i)}\right)=\frac{1}{\varepsilon_t}\log\mathbb{E}\left[e^{-\varepsilon_t x_t}\right]\leq$$

Using Hoeffding's inequality:

$$\leq \frac{1}{\varepsilon_t}\left(\frac{\varepsilon_t^2}{8} - \varepsilon_t \mathbb{E}[x_t]\right) = \frac{\varepsilon_t}{8} - \langle x_t, l_t \rangle.$$

5. Show that statements i) and ii) hold true by combining the 4 previous results.

   i) Let us use 2nd and 4th results. Since $\langle x_t, l_t \rangle = \mathbb{E}[\hat{l}_t]$:

   $$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_1} = \sum_{t=1}^{T}(a_t + b_t) \leq \sum_{t=1}^{T}\left(\frac{\varepsilon_t}{8} - \langle x_t, l_t \rangle\right) \leq \frac{\sum_{t=1}^{T} \varepsilon_t}{8} - \mathbb{E}[\hat{l}_t].$$

   Now let us use the 1st result:

   $$-\inf_{x \in \Delta d} \sum_{t=1}^{T} \langle x, l_t \rangle - \frac{\log d}{\varepsilon_{T+1}} \leq \frac{\sum_{t=1}^{T} \varepsilon_t}{8} - \mathbb{E}[\hat{l}_t]$$

   Finally,

   $$R_t = \sum_{t=1}^{T} \mathbb{E}[\hat{l}_t] - \inf_{x \in \Delta d} \sum_{t=1}^{T} \langle x, l_t, \rangle \leq \frac{\sum_{t=1}^{T} \varepsilon_t}{8} + \frac{\log d}{\varepsilon_{T+1}}.$$

   ii) Let us substitute ε_t with $\sqrt{\frac{8 \log d}{t}}$. Then the inequality becomes

   $$R_t \leq \frac{\sqrt{8 \log d}}{8} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} + \sqrt{\log d (T+1)} \leq \frac{\log d}{8}(\sqrt{T} + \sqrt{T+1})$$

   It is known that

   $$T \geq 1 \Rightarrow \frac{1}{16 + 4\sqrt{8}} < T \Rightarrow T + 1 < (17 + 4\sqrt{8})T \Rightarrow \sqrt{T+1} < \sqrt{(17 - 4\sqrt{8})T} \Longrightarrow$$

   $$\Rightarrow \sqrt{T+1} < (2\sqrt{8} - 1)\sqrt{T} \Rightarrow \frac{\sqrt{T} + \sqrt{T+1}}{\sqrt{8}} < 2\sqrt{T}$$

   Then we can say that $\frac{\log d}{8}(\sqrt{T} + \sqrt{T+1}) \leq \sqrt{\log d}(\sqrt{T} + \sqrt{T}) = \sqrt{\log d 2T}$

□