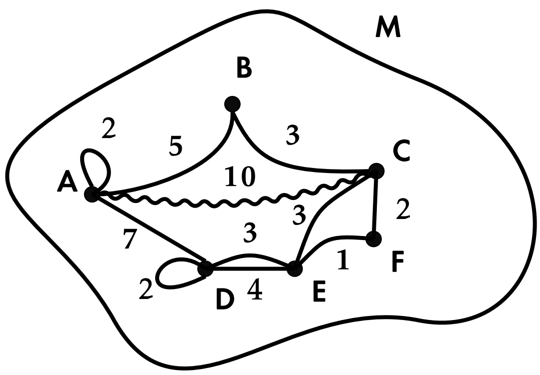
1. Metric axioms. Metric spaces. Norms. Normed linear spaces

Definition: (Metric space)

A metric space is a set X with a metric $\rho : X \times X \to [0, \infty)$ (or it can be valid notation d, in that way we can call it by 'distance') such that $\forall x, y, z \in X$, ρ satisfies the following properties:

1. Positive definite:
$$\rho(x,y) \geq 0, \quad \forall x \neq y$$
$$\rho(x,y) = 0 \Longleftrightarrow x = y; \ \rho(x,x) = 0.$$
2. Symmetric:
$$\rho(x,y) = \rho(y,x).$$
3. Triangle Inequality:
$$\rho(x,z) \leq \rho(x,y) + \rho(y,z).$$

Example 1:



Let define some metric space with metric $\rho(x,y)$ is equal to the legth if the shortest path. Then $\dim(M) = 10$ and, e.g.:

$$\rho(A,A) = 0;$$
$$\rho(A,D) = 7;$$
$$\rho(A,C) = 8$$

Also we can show, for example, open ball on this example (we will define it little bit later):

$$B_3(A) = \{A\}$$
$$B_9(A) = \{A, B, C, D, E\}$$

Example 2: Given a set X:

- The discrete metric ρ on X is defined by:

$$\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

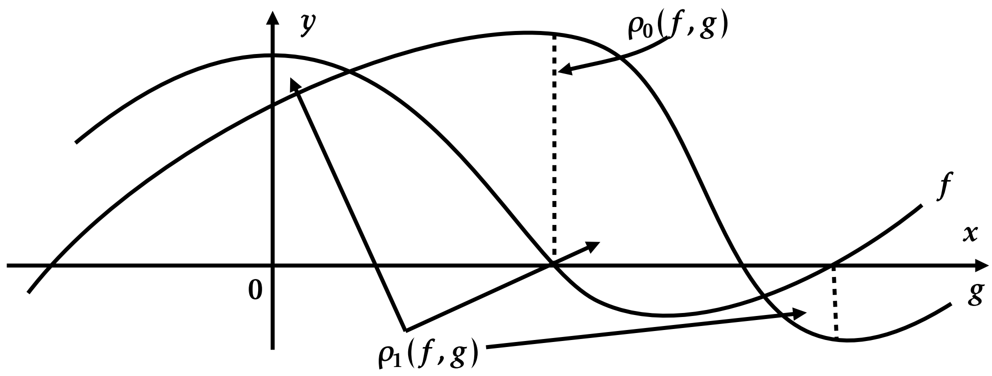
- Metric on continuous functions:

  Let $X = C[0,1] = \{$continuous functions $f : [0,1] \to \mathbb{R}\}$. Then we can define metrics:

$$\rho_0(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$
$$\rho_1(f,g) = \int_0^1 |f(x) - g(x)| dx$$

  Or:

$$\rho(f,g) = \rho_0(f,g) + |f(1) - g(1)|.$$



- Let $x = \mathbb{R}$. Possible metric:

$$\rho(x,y) = \left|e^x - e^y\right|.$$

- Another example of metric:

$$\rho(x,y) = \begin{cases} 1, & x-y \in \mathbb{Q} \\ 2, & x-y \notin \mathbb{Q} \\ 0, & x = y \end{cases}$$

Definition: (Continuous function)

The function is called continuous iff:

$$\lim_{x \to x_0} f(x) = y_0,$$

that is $\forall \varepsilon > 0 \, \exists \delta$:

$$f\left[B_\delta(x_0)\right] \subset B_\varepsilon\left[f(x_0)\right] \equiv B_\varepsilon(y_0)$$

Definition: (Open ball)

An open ball of radius $r > 0$ centered at the pont $x_0 \in X$ is the set:

$$B(x_0, r) = \{x \in X |\, \rho(x, x_0) < r\}$$

Definition: (Closed ball)

A closed ball of radius $r > 0$ centered at the point $x_0 \in X$ is the set:

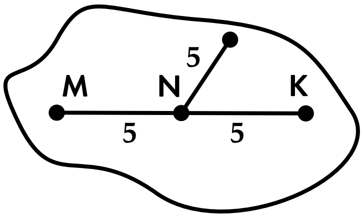
$$\overline{B}(x_0, r) = \{x \in X |\, f(x, x_0) \leq r\}.$$

Note

A ball centered at the point A of radius r in some metric space X"

$$B_r(A) = \{x \in X |\, f(A, x) \leq r\}.$$

Example 3: Let define space:



$$B_8(M) = \{M, N\}$$
$$B_6(N) = \{M, N, K\}$$

Definition: (Normed space)

A (complex or real) vector space V is called normed space if a function ('norm') $\nu : V \to \mathbb{R}$, denoted for $v \in V$ $||v||$, which satisfies the following axioms:

1. Positive definite property:
$$\nu(\vec{x}) > 0;$$
2. Homogeneity
$$\nu(\alpha\vec{x}) = |\alpha|\nu(\vec{x});$$
3. Triangle inequality $\forall x, y \in V$:
$$\nu(\vec{x} + \vec{y}) \leq \nu(\vec{x}) + \nu(\vec{y}).$$

Example 4: Euclidean norm:

$$|\vec{x}|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Prop

In a normed space V, the function $\rho(\vec{x}, \vec{y}) = \nu(\vec{y} - \vec{x})$ is a metric.

Proof: For proving positive definition of function $\nu(\vec{x}, \vec{y})$ we need a lemma:

Lemma

$\nu(\vec{0}) = 0$

Proof: $\nu(0 \cdot \vec{0}) = 0 \cdot \nu(\vec{0}) = 0.$ □

1. Positive definition:
$$\rho(\vec{x}, \vec{y}) = \nu(\vec{y} - \vec{x}) > 0$$
$$\rho(\vec{x}, \vec{x}) = \nu(\vec{x} - \vec{x}) = \nu(\vec{0}) = 0$$
2. Symmetric:
$$\rho(\vec{x}, \vec{y}) = \nu(\vec{y} - \vec{x}) = |-1|\nu(\vec{x} - \vec{y}) = \nu(\vec{x} - \vec{y}) = \rho(\vec{y}, \vec{x})$$
3. Triangle inequality:
$$\rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z}) = \nu(\vec{y} - \vec{x}) + \nu(\vec{z} - \vec{y}) \geq$$
$$\geq \nu(\vec{y} - \vec{x} + \vec{z} - \vec{y}) = \nu(\vec{z} - \vec{x}) = \rho(\vec{x}, \vec{z})$$

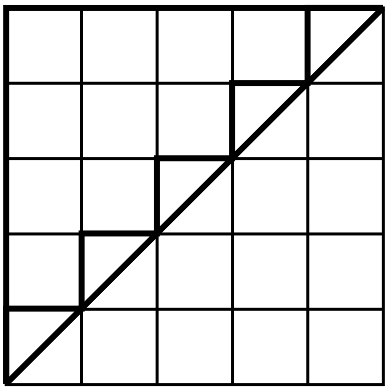
□

Note

Each normed space is a metric space.

Example 5: In the vector space $\mathbb{R}^n(\mathbb{C}^n)$ the following three norms are in common use:

- Manhattan norm (Taxicab norm):



This is the norm of the following kind:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Let's prove that this function is a norm.

1. Positive definite property: Let $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$. Obviously $\|x\|_1 \geq 0$. Also $\|x\|_1 = 0$ iff $x = 0$.

2. Homogeneity property:

$$\forall c \in \mathbb{R}: \; \|c \cdot x\|_1 = \sum_{i=1}^{n} |c \cdot x_i| = |c| \cdot \sum_{i=1}^{n} |x_i| = |c| \cdot \|x\|_1.$$

3. Triangle inequality $\forall x, y \in \mathbb{R}^n$:

$$\|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) =$$
$$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\|_1 + \|y\|_1.$$

- Maximum norm (Infinity norm):

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

1. The function $\|x\|_\infty$ is positive since it is the maximum over a set of positive terms $|x_i|$.
2. Homogeneity property:

$$\|\alpha \cdot x\|_\infty = \max_{1 \leq i \leq n} |\alpha \cdot x_i| = \max_{1 \leq i \leq n} |\alpha| \cdot |x_i| = |\alpha| \cdot \max_{1 \leq i \leq n} = |\alpha| \cdot \|x\|_\infty.$$

3. Triangle inequality:

$$\|x + y\|_\infty = \max_{1 \leq i \leq n} |x_i + y_i| \leq \max_{1 \leq i \leq n} (|x_i| + |y_i|) \leq \max_{1 \leq i \leq n} |x_i| + \max_{1 \leq i \leq n} |y_i| = \|x\|_\infty + \|y\|_\infty.$$

Example 6: For the vector $x = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$ we have:

$$\|x\|_1 = 11; \; \|x\|_2 = \sqrt{39}; \; \|x\|_\infty = 5,$$

whereas for the vector $x = \begin{bmatrix} 1+i \\ 2-3i \\ 4 \end{bmatrix}$,

$$||x||_1 = \sqrt{2} + \sqrt{13} + 4; \ ||u||_2 = \sqrt{31}; \ ||u||_\infty = 4.$$

The notations $||\cdot||_1, ||\cdot||_2$ and $||\cdot||_\infty$ are justified because of the fact that all these norms are special cases of the general Minkovskiy $p-$norm:

$$||x||_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}, \ p \geq 1$$

Similarly, in the vector space of real-valued continuous functions $\mathcal{C}[a, b]$, the following three norms are frequently used:

$$||f||_1 = \int_a^b |f(x)|dx; \ ||f||_2 = \sqrt{\int_a^b f^2(x)dx}; \ ||f||_\infty = \max_{x\in[a,b]} |f(x)|$$

And Minkovskiy norm:

$$||f||_p = \sqrt[p]{\int_a^b |f(x)|^p dx}$$

For example, for $V = \mathcal{C}[0, 1]$:

$$||f||_p = \sqrt[p]{\int_0^1 |f(x)|^p dx}$$

$$||f||_1 = \int_0^1 |f(x)|^p dx$$

$$||f||_\infty = ||f||_0 = \max_{x\in[0,1]} |f(x)|.$$

Some weighted norms:
Let $V = \mathcal{C}[0, 1]; \omega \geq 0$:

$$||f||_p^\omega = \left(\int_0^1 |f(x)|^p \cdot \omega(x)dx \right)^{\frac{1}{p}}$$

$$||f||_\infty^\omega = ||f||_0^\omega = \max_{x\in[0,1]} |f(x) \cdot \omega(x)|.$$

Balls in normed space

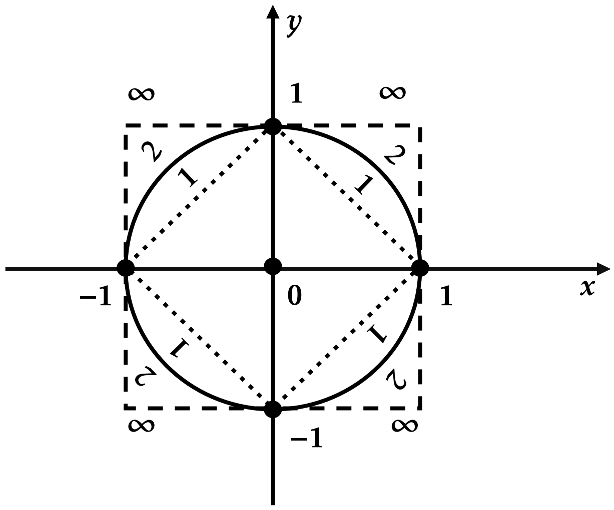
Prop

Let V be a normed space

- All balls of the same radius R are equal geometrically: a parallel translation of $B_R(\vec{x})$ by a vector $\vec{v} = \vec{y} - \vec{x}$ makes $B_R(\vec{y})$.
- For any two balls: $B = B_r(0)$ and $B' = B_R(0)$, there is a homotety $x \longmapsto \lambda\vec{x}$, which transfers B

onto B', where $\lambda = \dfrac{R}{r}$. Also $B_1^p(0)$ = unit ball for 0-norm.

Proof:



1.
$$B_R(\vec{x}) + V = \left\{\vec{a} \mid \nu\left(\vec{a}-\vec{x}\right) \le R\right\} + \left(\vec{y}-\vec{x}\right) =$$
$$= \left\{\vec{a}+\vec{y}-\vec{x} \mid \nu\left(\vec{a}-\vec{x}\right) \le R\right\} = \left\{\vec{b} \mid \nu\left(\vec{b}-\vec{y}\right) \le R\right\} =$$
$= B_R(\vec{y})$, where: $\vec{a}+\vec{y}-\vec{x} = \vec{b};\ \vec{a} = \vec{b}-\vec{y}+\vec{x}$.

2. $\dfrac{R}{r}\cdot\vec{a} = \vec{b};\ \vec{a} = \dfrac{r}{R}\cdot\vec{b}$.

$$\lambda \cdot B = \frac{R}{r}\cdot\left\{\vec{a} \mid \nu\left(\vec{a}\right) \le r\right\} = \left\{\frac{R}{r}\cdot\vec{a} \mid \nu\left(\vec{a}\right) \le r\right\} =$$
$$= \left\{\vec{b} \mid \nu\left(\frac{r}{R}\cdot\vec{b}\right) \le r\right\} = \left\{\vec{b} \mid \frac{r}{R}\cdot\nu\left(\vec{b}\right) \le r\right\} =$$
$$= \left\{\vec{b} \mid \nu\left(\vec{b}\right) \le \frac{r\cdot R}{r}\right\} = \left\{\vec{b} \mid \nu\left(\vec{b}\right) \le R\right\} = B'.$$

$\square$