Course 1

Linear Algebra for Data Science

Annotation

In the lecture course, we consider some topics of linear algebra beyond the standard first year course which are extremely important for applications. Mostly, these are applications to data analysis and machine learning, as well as to economics and statistics. We begin with inversions of rectangle matrices, that is, we discuss pseudo-inverse matrices (and their connections to the linear regression model). Among others, we discuss iteration methods (and their using in models of random walk on a graph applied to Internet search such as PageRank algorithm), matrix decompositions (such as SVD) and methods of dimension decreasing (with their connection to some image compression algorithms), and the theory of matrix norms and perturbation theory (for error estimates in matrix computations). The course includes also symbolic methods in systems of algebraic equations, approximation problems, Chebyshev polynomials, functions with matrices such as exponents etc. We plan to invite some external lecturers who successfully apply linear algebra in their work. The students are also be invited to give their own talks on additional topics of applied or theoretical linear algebra.

Final Grade

$$\text{GRADE} = \frac{\text{test1} + \text{test2}}{2} + \underbrace{\begin{matrix}\text{Bonus}\\ \text{for a talk}\end{matrix}}_{\leq 5} + \underbrace{\begin{matrix}\text{Bonus}\\ \text{for classes}\end{matrix}}_{\leq 1\ldots 2}.$$

1. Pseudoinverse matrices. Skeletonization. Singular value decomposition (SVD)

Problem 1. Skeletonization. Pseudoinverse matrix

Find the pseudoinverse matrix to matrix A using skeletonization:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution: Let's start with skeletonization and then find the pseudoinverse matrix by the formula:

$$A^+ = G^*(G, G^*)^{-1}(F^*, F)^{-1}F^*$$

$$A = \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{matrix}}}^{F} & \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{matrix}}}^{G} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{bmatrix}$$

Let's check:

$$F \cdot G = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Correct. Now we need to find the pseudoinverse matrix:

$$G^*(G, G^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6}$$

Note

Matrix (G, G^*) is called Gramm Matrix and contains results of scalar products.

$$(F^*, F)^{-1}F^* = \left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{6} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

All things considered, we can obtain pseudoinverse matrix A^+:

$$\frac{1}{36} \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 13 & -4 & 0 \\ -4 & 4 & 0 \\ 5 & 4 & 0 \end{bmatrix}$$

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Singular Value Decomposition of matrix $A \in M_{m\times n}(\mathbb{C})$ is a decomposition of a kind:

$$A = U\Sigma V^*,$$

where $U \in M_{m\times m}(\mathbb{C})$ and $V \in M_{n\times n}(\mathbb{C})$ are unitary matrices:

$$U^*U = UU^* = UU^{-1} = I$$
$$V^*V = VV^* = VV^{-1} = I$$

and $\Sigma \in M_{m\times n}(\mathbb{C})$ is a diagonal matrix of a kind:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_r \\ & & 0 & \end{bmatrix},$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ – singular values. ($r = \operatorname{rank} A$).

Problem 2. Singular Value Decomposition

Find singular value decomposition of a matrix:

$$A = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

Solution: Let's start with finding $V \in M_{2\times 2}(\mathbb{R})$ and $\Sigma \in M_{2\times 2}(\mathbb{R})$:

$$A^\top A = \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 25 & 25 \end{bmatrix}.$$

Next step is to find eigenvalues and eigenvectors for the obtained matrix.

$$\det\left(\begin{bmatrix} 25-\lambda & 25 \\ 25 & 25-\lambda \end{bmatrix}\right) = 0 \iff \lambda^2 - 50\lambda = 0.$$

We can obtain eigenvalues: $\lambda_1 = 50$; $\lambda_2 = 0$. Then singular values is equal to: $\sigma_1 = \sqrt{50}$, $\sigma_2 = 0$. Then:

$$\Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

After that we can find eigenvalues:

$$\begin{bmatrix} -25 & 25 \\ 25 & -25 \end{bmatrix} \vec{x} = 0$$

$\vec{x}_1$, for example, can be equal $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We need to normalize it, so $\vec{x}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\frac{\sqrt{2}}{2}$.

$$V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Let's find matrix U though:

$$AA^\top = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & 32 \end{bmatrix}$$

Repeating the same scenario:

$$\det\left(\begin{bmatrix} 18-\lambda & 0 & 24 \\ 0 & -\lambda & 0 \\ 24 & 0 & 32-\lambda \end{bmatrix}\right) \Longleftrightarrow \lambda^3 - 50\lambda^2 = 0.$$

Then we can obtain singular values $\sigma_1 = \sqrt{50}, \sigma_2 = 0, \sigma_3 = 0$.

$$\begin{bmatrix} -32 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & -18 \end{bmatrix} \vec{x} = 0$$

Solving by Gaussian elimination:

$$\left[\begin{array}{ccc|c} -4 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} \frac{3}{4} \\ 0 \\ 1 \end{bmatrix} \cdot \frac{4}{5}$. Similarly,

$$\left[\begin{array}{ccc|c} 18 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 32 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \cdot \frac{3}{5}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

All things considered, the result of singular value decomposition:

$$U = \frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

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