## Linear regression problem

Let $y, x_1, x_2, \dots, x_n$ be variables. We believe that dependent variable $y$ could be approximated by linear combination of independent variables $x_1, x_2, \dots, x_n$. That is,

$$y \approx a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

The problem is to find coefficients $a_1, a_2, \dots, a_n$. In order to find them we conduct $m$ experiments (samples) and get following dataset of observations

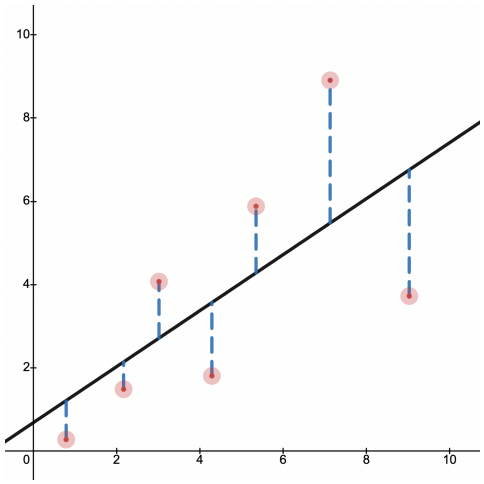
$$\begin{cases} y_1 \approx a_1 x_{11} + \dots + a_n x_{n1} \\ y_2 \approx a_2 x_{12} + \dots + a_n x_{n2} \\ \vdots \\ y_m \approx a_1 x_{1m} + \dots + a_n x_{nm} \end{cases}$$

where $x_{1i}, x_{2i}, \dots, x_{ni}, y_i$ is a result of $i$-th experiment. Clearly, the problem could is equivalent to matrix equation of the form

$$X\vec{a} \approx \vec{y},$$

where

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{nm} \end{bmatrix}, \quad \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$




In linear regression, the observations (red) are assumed to be the result of random deviations (blue) from an underlying relationship (black) between a dependent variable (y) and an independent variable (x).

The solution of the problem is least square solution (pseudosolution)

$$\vec{a} = X^{+}\vec{y}.$$

## Problem 1. Linear regression

Find linear approximation of temperature using following dataset of temperature observations. Find temperature on day 4.

| # | Day | Temperature |
|---|---|---|
| 1 | 1 | 19° |
| 2 | 2 | 18° |
| 3 | 2 | 20° |
| 4 | 3 | 15° |

Solution: Let us denote day and temperature of the row $i$ by $x_i$ and $y_i$ respectively. We assume that temperature ($y(x)$) could be approximated by linear model

$$y(x) = ax + b \in \mathbb{R},$$

where $x \in \mathbb{R}$ is number of the day and $a, b \in \mathbb{R}$ are unknown coefficients. This means that we need to find pseudosolution of the following system

$$X\vec{a} = \vec{y},$$

where

$$X = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 19 \\ 18 \\ 20 \\ 15 \end{bmatrix}, \quad \vec{a} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

So we need to evaluate the following $\vec{a} = X^+\vec{y}$. Since $X$ is full column rank, we get

$$\begin{aligned} X^+ = (X^*X)^{-1}X^* &= \left(2\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}\right)^{-1} X^* \\ &= \frac{1}{2}\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -2 & 0 & 0 & 2 \\ 5 & 1 & 1 & -3 \end{bmatrix} \end{aligned}$$

Hence, we get

$$\vec{a} = X^+\vec{y} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 0 & 2 \\ 5 & 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 19 \\ 18 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 22 \end{bmatrix}.$$

This means that linear model $y(x)$ is

$$y(x) = -2x + 22.$$

According to linear model $y(x)$, on day 4 temperature will be $y(4) = 14°$. ■