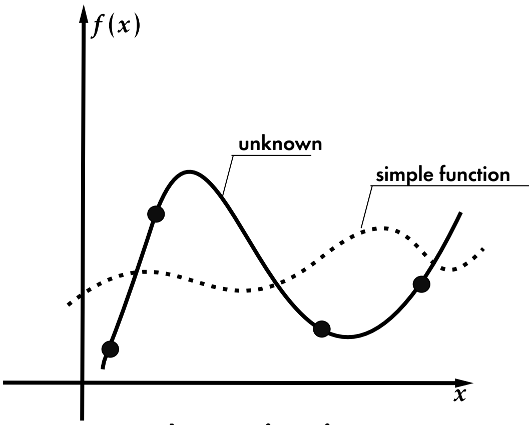
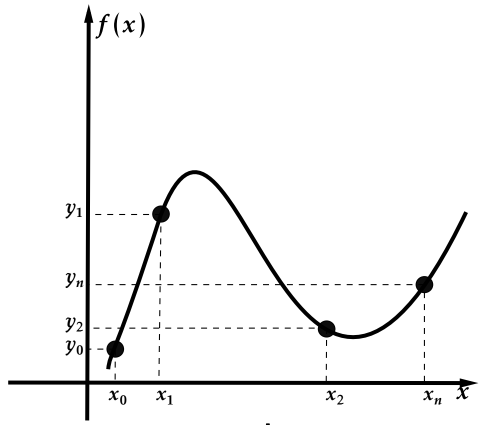
1. Approximation. Interpolation problem. Polynomial interpolation. Hermitian interpolation. Splines. Bézier curves and splines.



Approximation



Interpolation

Polynomial interpolation

Classical problem of polynomial interpolation

Problem: Suppose some function $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ is a polynomial of degree $\leq n$. Given the values of function $f(x)$:

$$\begin{cases} f(x_0) = y_0, \\ \ldots \\ f(x_n) = y_n \end{cases}$$

Recover $f(x)$ is our goal. That is the main problem to find a vector $\vec{a} = \begin{bmatrix} a_0 & \ldots & a_n \end{bmatrix}^\top = ?$

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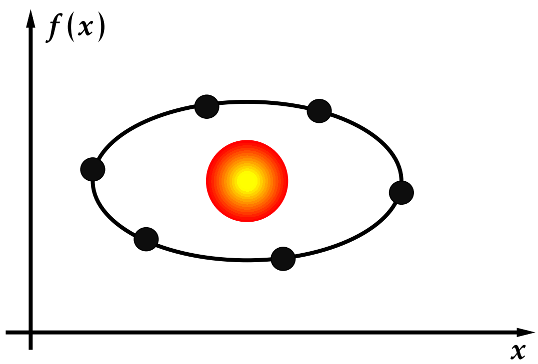


Some algebraic equations $f(x, y) = 0$. We need $\dfrac{n \cdot (n+3)}{2}$ observations to recover an orbit equation of degree equal to n (generally). From the system we have system of equations:

$$\begin{cases} a_0 + a_1x_0 + \ldots + a_nx_0^n = y_0 \\ a_0 + a_1x_1 + \ldots + a_nx_1^n = y_1 \\ \ldots \\ a_0 + a_1x_n + \ldots + a_nx_n^n = y_n \end{cases}$$

We can rewrite it by matrix product:

$$V\vec{a} = \vec{y},$$

where $a = \begin{bmatrix} a_0 & a_1 & \ldots & a_n \end{bmatrix}^\top$, $y = \begin{bmatrix} y_0 & y_1 & \ldots & y_n \end{bmatrix}^\top$ and V is a Vandermonde matrix:

$$V = \begin{bmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \ldots & x_n^n \end{bmatrix}$$

Note

Vendermonde determinant:

$$v = v(x_0, \dots, x_n) = \det V = (x_1 - x_0)(x_2 - x_0)\dots(x_2 - x_1)\dots(x_n - x_{n-1}) =$$
$$= \prod_{0 \le i < j \le n} (x_j - x_i).$$

Example 1: $\det V = v(x_0, x_1) = \begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix} = x_1 - x_0.$

Example 2: $\det V = v(x_0, x_1, x_2) = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x_2 - x_0)(x_1 - x_0)(x_2 - x_1).$

Corollary: if all $x_0, \dots, x_n$ are different $detV = v \neq 0$. Then $\vec{a} = V^{-1}\vec{y}$ is the unique solution.

Lagrange form of interpolation polynomial

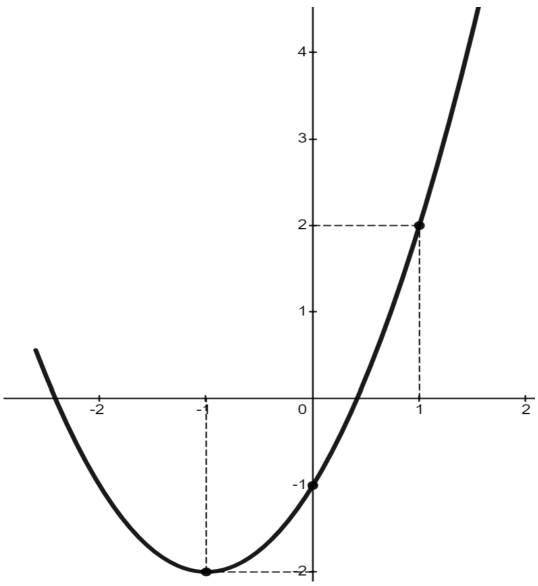
$$f(x) = \sum_{i=0}^{n} \frac{v(x_0, \dots, x, \dots, x_n)}{v(x_0, \dots, x_i, \dots x_n)} y_i = \sum_{i=0}^{n} y_i \frac{(x - x_0)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}.$$

Example 3: Let

$$x_0 = -1;\ y_0 = -2$$
$$x_1 = 0;\ y_1 = -1$$
$$x_2 = 1;\ y_2 = 2.$$

Let's apply Lagrange form of interpolation polynomial to calculate it:

$$f(x) = -2 \cdot \frac{(x-0)(x-1)}{(-1-0)(-1-1)} -$$
$$-1 \cdot \frac{(x+1)(x-1)}{(0+1)(0-1)} + 2 \cdot \frac{(x+1)(x-0)}{(1+1)(1-0)} =$$
$$= -x(x-1) + x^2 - 1 + x(x+1) = x^2 + 2x - 1.$$



Example of Lagrange form of interpolation polynomial.

Hermitian interpolation or interpolation with multiple knots

Definition

A number x_1 is a root of a polynomial $f(x)$ with multiplicity d if

$$f(x) = (x - x_1)^d \cdot g(x)$$

for some polynomial $g(x)$ such that $g(x_1) \neq 0$

Lemma

x_1 is a root of multiplicity d for a polynomial $f(x)$ if and only if:

$$\begin{cases} f(x_1) = 0 \\ f'(x_1) = 0 \\ \vdots \\ f^{(d-1)}(x_1) = 0 \\ f^{d}(x_1) \neq 0 \end{cases}$$

Proof: x_1 is a root of $f(x)$ of multiplicity $d \geq 1$, if and only if:

$$f'(x) = d \cdot (x - x_1)^{d-1} \cdot g(x) + (x - x_1)^d g'(x) = \\ = (x - x_1)^{d-1} \underbrace{(dg(x) + (x - x_1) g'(x))}_{h(x)},$$

where $h(x_1) = dg(x_1) \neq 0 \Leftrightarrow x_1$ is a root of $f'(x)$ with multiplicity $d - 1$. □

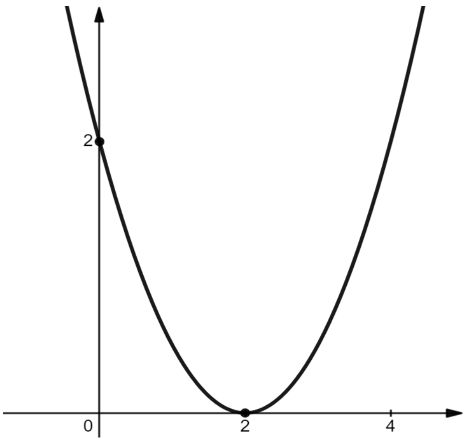
Problem (Brief): find $f(x)$ by m knots with multiplicities $h_1, h_2, \dots, h_m$.

Complete: to find a polynomial $f(x)$ of degree $\leq n - 1$, such that for some different $\underbrace{x_1, x_2, \dots, x_m}_{\text{knots}} \in \mathbb{R}$ and $\underbrace{h_1, \dots, h_m}_{\text{multiplicities}} \in \mathbb{N}$ with $h_1 + h_2 + \dots + h_m = n$, and $y_1, y_1^{(1)}, \dots, y_m^{h_m - 1}$;

$$f(x_1) = y_1;\ f'(x_1) = y_1^{(1)}, \dots, f(x_1)^{(h_1-1)} \\ \dots \\ f(x_m) = y_m, \dots, f^{(h_m-1)}(x_m) = y_m^{(h_m-1)}.$$

Prop

This problem always has a unique solution.



Example of interpolation with multiple knots

Example 4:

$$f(x) = ax^2 + bx + c; \\ f'(x) = 2ax + b \\ x_1 = 0;\quad f(0) = 2; x_2 = 2;\quad f(2) = 0;\ f'(2) = 0;$$

$$\begin{cases} f(0) = a0^2 + b0 + c = 2; & n = 3, \deg f \leq 2 \\ f(2) = 4a + 2b + c = 0; & x_1 = 0,\ h_1 = 1 \\ f'(2) = 4a + b = 0; & x_2 = 2,\ h_2 = 2. \end{cases}$$

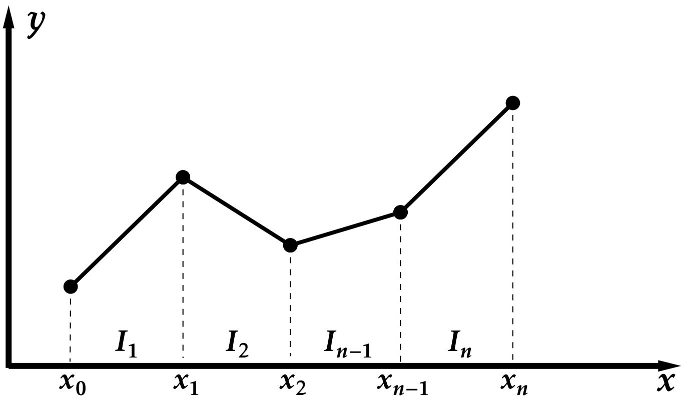
Solving equations we can obtain next results:

$$\begin{cases} c = 2 \\ b = -2 \\ a = \dfrac{1}{2} \end{cases} \Longrightarrow f(x) = \frac{1}{2}x^2 - 2x + 2.$$

Splines

An idea: to interpolate a "smooth" function $f(x)$ with knots $x_0, \dots, x_n$, on each $I_i = [x_{i-1}, x_i]$ put $f(x) = f_i(x)$.

Example 5: Given the values $f(x_i) = y_i$, let $f(x_i)$ be a linear function:



Example of a linear spline

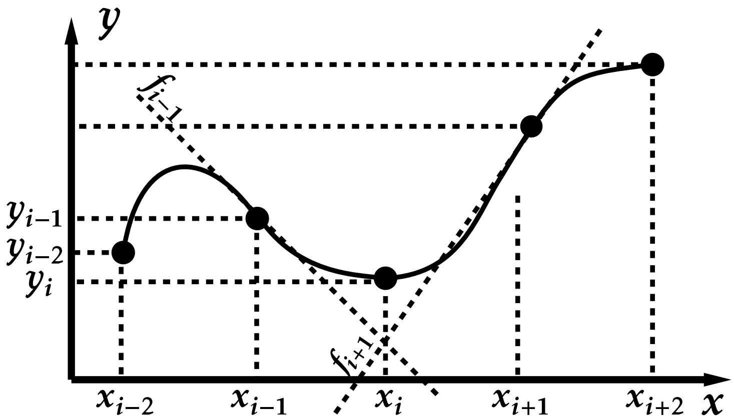
Quadratic Spline

Let $f(x)$ have a continuous derivative $f'(x)$:

$$f(x_i) = y_i; \quad i = 0, 1, \dots, n.$$

Then put $f_i(x) = a_i x^2 + b_i x + c_i$ and let

$$f(x) = \begin{cases} f_1(x);\ x \in I_1 \\ f_2(x);\ x \in I_2 \\ \vdots \\ f_n(x);\ x \in I_n. \end{cases}$$



For each f_i, $i > 1$:

$$\begin{cases} f_i(x_{i-1}) = y_{i-1} \\ f_i'(x_{i-1}) = f_{i-1}'(x_{i-1}) \\ f_i(x_i) = y_i \end{cases}$$

For $i = 1$, we add $f_1'(x_0) = 0$ or $f_1'(x_0) = x$ (if it is known) or $f_1'(x_n) = f_n'(x_n)$ (for periodic processing).

Cubic spline

$f(x)$ has continuous $f''(x)$: for $i = 1$, two initial conditions $f'(x_0) = f''(x_0) = 0$

$$\begin{cases} f_i(x_{i-1}) = y_{i-1} \\ f_i(x_{i-1}) = f_{i-1}(x_{i-1}) \\ f_i''(x_{i-1}) = f_{i-1}''(x_{i-1}) \\ f_i(x_i) = y_i \end{cases}$$

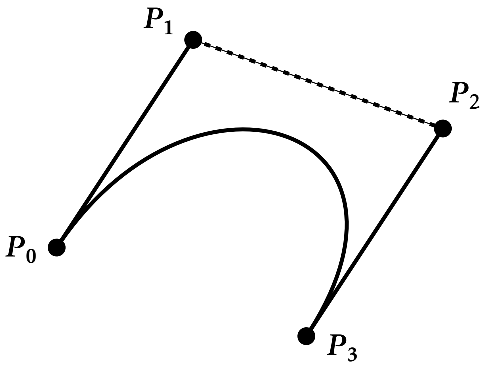
Bézier curves and splines

Problem: Given $A_0 A_1 \ldots A_n \in \mathbb{R}^m$ approximate the path

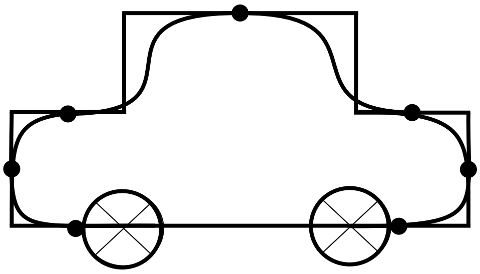
$$A_0 \to A_1 \to A_2 \to \ldots \to A_n$$

by a smooth curve.

Pierre Bézier
(Renault, 1962)



Paul de Faget
de Casteljau
(Citroën, 1959)



Applications of interpolation splines and curves

Explicit formula for Bézier curve:

$$B(t) = \sum_{i=0}^{n} A_i \cdot b_{n,i}(t),$$

where $b_{n,i}$ – Bernstein polynomials of a kind:

$$b_{n,i} = C_n^i (1-t)^{n-i} t^i$$