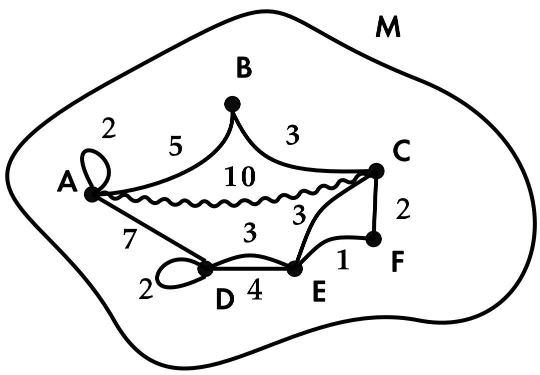
1. Metric axioms. Metric spaces. Norms. Normed linear spaces

Definition: (Metric space)

A metric space is a set X with a metric $\rho: X \times X \to [0, \infty)$ (or it can be valid notation d, in that way we can call it by 'distance') such that $\forall x, y, z \in X$, ρ satisfies the following properties:

1. Positive definite:
$$\rho(x,y) \geq 0, \quad \forall x \neq y$$
$$\rho(x,y) = 0 \iff x = y;\ \rho(x,x) = 0.$$
2. Symmetric:
$$\rho(x,y) = \rho(y,x).$$
3. Triangle Inequality:
$$\rho(x,z) \leq \rho(x,y) + \rho(y,z).$$

Example 1:



Let define some metric space with metric $\rho(x,y)$ is equal to the legth if the shortest path. Then $\dim(M) = 10$ and, e.g.:

$$\rho(A,A) = 0;$$
$$\rho(A,D) = 7;$$
$$\rho(A,C) = 8$$

Also we can show, for example, open ball on this example (we will define it little bit later):

$$B_3(A) = \{A\}$$
$$B_9(A) = \{A, B, C, D, E\}$$

Example 2: Given a set X:

- The discrete metric ρ on X is defined by:

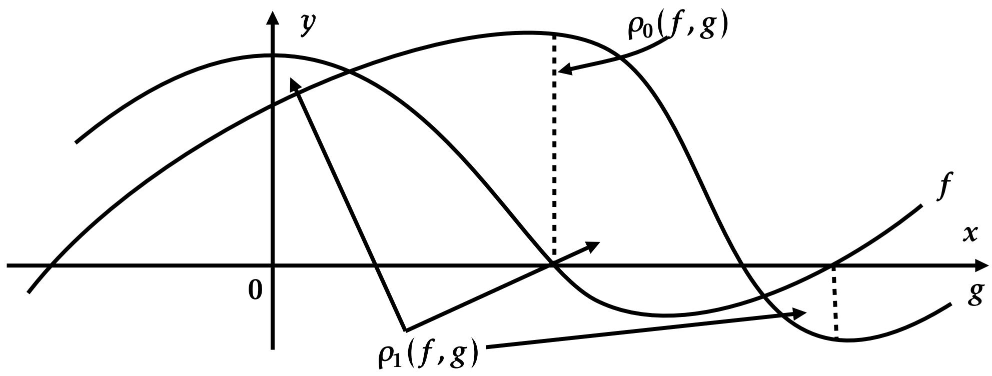
$$\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

- Metric on continuous functions:
  Let $X = C[0,1] = \{$continuous functions $f : [0,1] \to \mathbb{R}\}$. Then we can define metrics:

$$\rho_0(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$
$$\rho_1(f,g) = \int_0^1 |f(x) - g(x)| dx$$

Or:

$$\rho(f,g) = \rho_0(f,g) + |f(1) - g(1)|.$$



- Let $x = \mathbb{R}$. Possible metric:

$$\rho(x,y) = \left|e^x - e^y\right|.$$

- Another example of vector:

$$\rho(x,y) = \begin{cases} 1, & x - y \in \mathbb{Q} \\ 2, & x - y \notin \mathbb{Q} \\ 0, & x = y \end{cases}$$

Definition: (Continuous function)

The function is called continuous iff:

$$\lim_{x \to x_0} f(x) = y_0,$$

that is $\forall \varepsilon > 0 \, \exists \delta$:

$$f\left[B_\delta(x_0)\right] \subset B_\varepsilon\left[f(x_0)\right] \equiv B_\varepsilon(y_0)$$

Definition: (Open ball)

An open ball of radius $r > 0$ centered at the pont $x_0 \in X$ is the set:

$$B(x_0, r) = \{x \in X | \, \rho(x, x_0) < r\}$$

Definition: (Closed ball)

A closed ball of radius $r > 0$ centered at the point $x_0 \in X$ is the set:

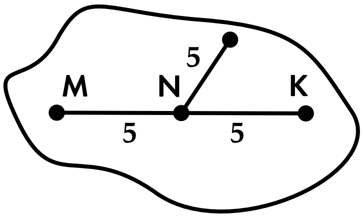
$$\overline{B}(x_0, r) = \{x \in X | \, f(x, x_0) \leq r\}.$$

Note

A ball centered at the point A of radius r in some metric space X''

$$B_r(A) = \{x \in X | \, f(A, x) \leq r\}.$$

Example 3: Let define space:



$$B_8(M) = M, N$$
$$B_6(N) = M, N, K$$

Definition: (Normed space)

A (complex or real) vector space V is called normed space if a function ('norm') $\nu : V \to \mathbb{R}$, denoted for $v \in V$ $||v||$, which satisfies the following axioms:

1. Positive definite property:
$$\nu\left(\vec{x}\right) > 0;$$
2. Homogeneity
$$\nu\left(\alpha\vec{x}\right) = |\alpha|\nu\left(\vec{x}\right);$$
3. Triangle inequality $\forall x, y \in V$:
$$\nu\left(\vec{x}+\vec{y}\right) \le \nu\left(\vec{x}\right) + \nu\left(\vec{y}\right).$$

Note

In the vector space $\mathbb{R}^n(\mathbb{C}^n)$ the following three norms are in common use:

- absolute norm:
$$||x||_1 = \sum_{i=1}^{n} |x_i|$$
- Euclidean norm:
$$|\vec{x}|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$
- maximum norm:
$$||u||_\infty = \max_{1\le i\le n} |u_i|.$$

Example 4: For the vector $x = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$ we have:

$$||x||_1 = 11;\ ||x||_2 = \sqrt{39};\ ||x||_\infty = 5,$$

whereas for the vector $x = \begin{bmatrix} 1+i \\ 2-3i \\ 4 \end{bmatrix}$,

$$||x||_1 = \sqrt{2} + \sqrt{13} + 4;\ ||u||_2 = \sqrt{31};\ ||u||_\infty = 4.$$

Prop

In a normed space V, the function $\rho\left(\vec{x},\vec{y}\right) = \nu\left(\vec{y} - \vec{x}\right)$ is a metric.

Proof: □

Note

Each normed space is a metric space.