Problem 1. Pseudoinverse Matrix

Find

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}^{+}$$

Solution: From the lecture we know that $A^{+} = (A^{*}A)^{-1}A^{*}$.

$$A^{*} = A^{T} = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix};\ A^{*}A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = 0 + 1 + 4 + 9 = 14$$

$$(A^{*}A)^{-1} = \frac{1}{14} \Rightarrow A^{+} = (A^{*}A)^{-1}A^{*} = \frac{1}{14}\begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}.$$

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Problem 2. Skeletonization

Find a full rank decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Solution: We want such decomposition: $A_{m \times n} = F_{m \times r} \times G_{r \times n}$; m = 4, n = 3. Use Gauss transformation to find G:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \text{So, } G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

We see that first and second columns (a_1, a_2) are independent and $a_3 = -a_1 + 2a_2$, so matrix F will consist of a_1 and a_2:

$$F = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 1 & 2 \end{pmatrix}.\ \text{Let's check: } A = F \times G = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

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Problem 3. Pseudoinverse matrix. Skeletonization

Find

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{+}$$

<u>Solution:</u> Check $\operatorname{rank}(A)$: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = G \Rightarrow \operatorname{rank}(A) < \dim(A) \Rightarrow$ we need to use formula

$$A^+ = G^+F^+ = G^*(GG^*)^{-1}(F^*F)^{-1}F^*$$

$$G^* = G^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}; GG^* = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; (GG^*)^{-1} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \Longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

$$G^*(GG*)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Then

$$F = \begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, F^* = F^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \dots, (F^*F)^{-1}F* = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix}$$

All things considered,

$$A^+ = \frac{1}{3}\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & 2 & -2 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

To complete our solution we need to check all Moore-Penrose axioms, I leave it for the reader. ■

Problem 4. Proof

Prove that $Im(AA^+) = Im(AA^*) = Im(A)$.

<u>Solution:</u> to do ■