Ordered Sets for Data Analysis: An Algorithmic Approach

Lattices and Closures

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Bounds, joins, and meets

Let $(P, \leq)$ be a poset and $A \subseteq P$. The **upper bound** of a set $A \subseteq P$ is the set

$$A^u = \{b \in P \mid \forall a \in A\ (b \geq a)\}.$$

Supremum (least upper bound, join) of set $A \subseteq P$, denoted by $sup(A)$, is the least element (if it exists) of A^u, the upper bound of A:

$$\forall u \in A^u\ (sup(A) \leq u)$$

The **lower bound** and **infimum** (**greatest lower bound, meet**), denoted by $inf(A)$, are introduced dually:

$$A^\ell = \{b \in P \mid \forall a \in A\ (b \leq a)\}$$

and

$$\forall l \in A^\ell\ (inf(A) \geq l).$$

Bounds, joins, and meets (logically pedantic)

Let $(P, \leq)$ be a poset and $A \subseteq P$. The **upper bound** of a set $A \subseteq P$ is the set

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Supremum (least upper bound, join) of set $A \subseteq P$, denoted by $sup(A)$, is the least element of A^u, the upper bound of A (if it exists):

$$\forall u\, ((u \in A^u) \to (c \leq u))$$

The **lower bound** and **infimum** (**greatest lower bound, meet**), denoted by $inf(A)$, are defined dually:

$$A^\ell = \{b \in P \mid \forall\ a((a \in A) \to (b \leq a))\}$$

and

$$\forall l((l \in A^\ell) \to (inf(A) \geq l)).$$

Join and meet of the empty set

By the property of logical implication the upper bound and the lower bound of the empty set is the whole set P

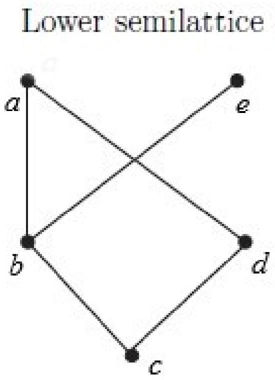
Hence, $sup(\emptyset)$ is the least element of the poset P if it exists, and $inf(\emptyset)$ is the largest element of the poset P if it exists.

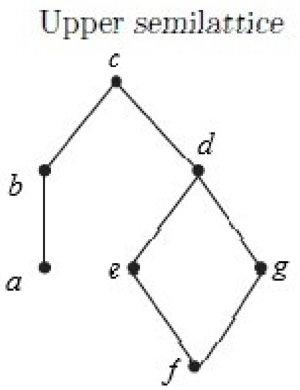
Semilattices

A partially ordered set (poset) $(SL, \leq)$ is an **upper semilattice** if any pair of elements $x, y \in SL$ has supremum $sup\{x, y\}$.
The definition of **lower semilattice** is dual. A partially ordered set $(SL, \leq)$ is a *lower semilattice* if any pair of element $x, y \in SL$ has infimum $inf\{x, y\}$.

Examples

Examples of semilattices given by diagrams.



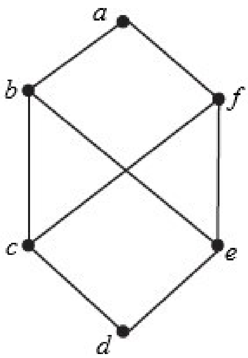


Lattice

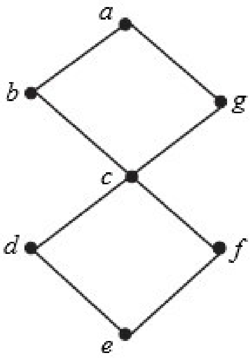
A partially ordered set $(L, \leq)$ is a **lattice** if any pair of elemenets $x, y \in L$ has supremum $sup\{x, y\}$ and infimum $inf\{x, y\}$.

Examples

A poset which is neither a lower, nor an upper semilattice



A poset, which is a lattice



Examples

- The set of natural numbers, partially ordered by "less than or equal" relation, is a lattice.
  The join of two numbers is their maximum and the meet is their minimum.
-
- The relation of **divisibility** defines a lattice on the set of natural numbers, where each pair of natural numbers has join, their **least common multiple**, and meet, their **greatest common divisor**.

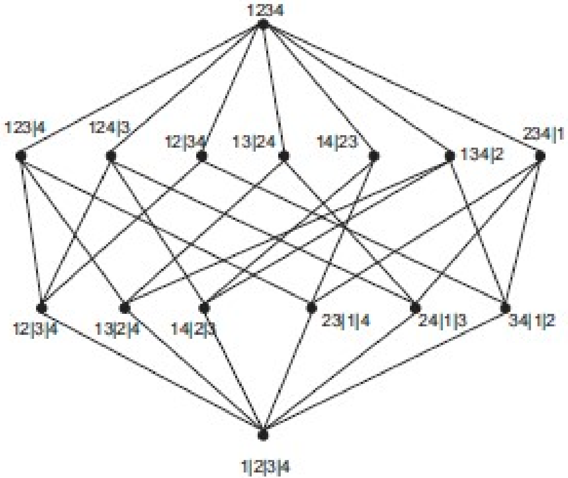
Examples

- The **powerset**, i.e., the set of all subsets of a set, ordered by containment relation, makes a lattice, where the *meet is the intersection* and the *join is the union of two sets*.
- The set of all multisets on a ground set U makes a lattice with meets and joins given by component-wise minimums and maximums of respective multiplicities.
- The set of partitions of set S ordered by the reflexive relation **"to be a rougher partition than"** makes a **partition lattice**. The join of two partitions A and B consists of the *unions* of all blocks from A and B that are *not disjoint* (i.e., having non-empty intersection).
  The meet of A and B consists of inclusion-maximal blocks, each of which is a non-empty intersection of blocks from A and B.

Example

Partition diagram of a four-element set.

$$A = \{1, 2, 3, 4\}$$



Lattice of order filters and ideals

For an arbitrary poset $(P, \leq)$ the set of order ideals $OI(P, \leq)$ is partially ordered by set-theoretic containment, which makes a *lattice*.

- Let I_1 and I_2 be two arbitrary ideals from $OI(P, \leq)$.
- Then $inf\{I_1, I_2\}$ is the set-theoretic intersection $I_1 \cap I_2$, and
- $sup\{I_1, I_2\}$ is the set-theoretic union $I_1 \cup I_2$.

Lattices as algebras

Theorem

An arbitrary set L is a lattice with respect to a partial order $\leq$ iff for any $x, y, z \in L$ one can define operations $\vee$ and $\wedge$ with the following properties:

L1 $x \vee x = x$, $x \wedge x = x$ *(idempotence)*

L2 $x \vee y = y \vee x$, $x \wedge y = y \wedge x$ *(commutativity)*

L3 $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *(associativity)*

L4 $x = x \wedge (x \vee y) = x \vee (x \wedge y)$ *(absorption)*

Theorem 1: Proof

Proof.

Necessity.

- Let L be a lattice w.r.t. a partial order $\leq$. We take *sup* as $\wedge$ and *inf* as $\vee$, and show that $\wedge$ and $\vee$ satisfy properties **L1-L4**.
- Note that if $x \leq y$ in a partial order $(P, \leq)$, then $\sup\{x, y\} = y$ and $\inf\{x, y\} = x$.
- Properties **L1, L2** hold due to the fact that an element can occur in a set only once and the order of elements in a set is inessential (*commutativity*).

Theorem 1: Proof (Continuation)

- By the definition of supremum

$$sup\{x, sup\{y, z\}\} \geq x, y, z,$$

hence, $sup\{x, sup\{y, z\}\} \geq sup\{x, y\}, sup\{x, sup\{y, z\}\} \geq z,$

and $sup\{x, sup\{y, z\}\} \geq sup\{sup\{x, y\}, z\}.$

Similarly, in the other direction

$$sup\{sup\{x, y\}, z\} \geq sup\{x, sup\{y, z\}\}.$$

Hence, $sup\{sup\{x, y\}, z\} = sup\{x, sup\{y, z\}\},$

thus, property **L3** holds.

Theorem 1: Proof (Continuation)

- By the definition of infimum, $x \geq inf\{x, y\}$, hence, by the definition of supremum,

$$sup\{x, inf\{x, y\}\} = x,$$

i.e., property **L4** also holds.

Theorem 1: Proof (Continuation)

Sufficiency.

- Let operations $\vee$, $\wedge$ with properties **L1**-**L4** be given on L.
- Define the following relation $\leq$ on elements of L: $x \leq y := x \wedge y = x$.
  Note: $x \leq y$ iff $x \vee y = y$.
- Let $x \leq y$, then $x \wedge y = x$ and $x \vee y = (x \wedge y) \vee y = y$ by **L4**.
- Inversely, if $x \vee y = y$, then $x \wedge y = x \wedge (x \vee y) = x$ by property **L4** and $x \leq y$ holds.

Theorem 1: Proof (Continuation)

- We show that L is a lattice w.r.t. $\leq$, where $sup\{x, y\}$ and $inf\{x, y\}$ are defined as $x \vee y$ and $x \wedge y$, respectively.
- First $x \vee y \geq y$, since $(x \vee y) \wedge y = y$.
- Second, let $z \geq x$ and $z \geq y$. Then $z \vee x = z$, $z \vee y = y$ and $z \vee (x \vee y) = (z \vee x) \vee y = z \vee y = z$,
- hence $z \geq (x \vee y)$ and $x \vee y$ is the join of x and y.
- Similarly, $x \wedge y$ is the meet of elements x and y w.r.t. partial order $\leq$.

$\square$

A lattice as an algebra $(L, \vee, \wedge)$

Theorem 1 allows one to consider a lattice as an algebra $(L, \vee, \wedge)$ with properties **L1**-**L4**. **Natural order** of a lattice given as algebra with properties **L1**-**L4** is the relation $\leq \subseteq L \times L$ defined as $x \leq y \overset{def}{=} x \wedge y = x$ (or, equivalently, as $x \leq y \overset{def}{=} x \vee y = y$).

Complete lattices

A lower semilattice is called **complete** if any subset X of it (e.g., the empty set) has infimum $\bigwedge X$. Complete upper semilattice is defined dually.

- An upper semilattice has the largest element (the unit of the upper semilattice) equal to the supremum of the set of all lattice elements.
- The lower semilattie has the least element (lattice zero) equal to the infimum of all elements of the semilattice.

Lattices

A lattice is called **complete** if every subset of it, including the empty one, has supremum and infimum.

Note: A complete semilattice is a complete lattice.

Proof.

Consider a complete semilattice w.r.t. operation $\bigwedge$, then we define operation $\bigvee$ as follows: $\bigvee X := \bigwedge\{z \mid z \geq x \text{ for all } x \in X\}$.

By definition, $\bigwedge\{z \mid z \geq x \text{ for all } x \in X\} \geq x$ for all $x \in X$. Then for all z such that $z \geq x$ for all $x \in X$ one has $\bigwedge X \geq z$ by the definition of infimum, hence $\bigvee X$ is the supremum of X. $\square$

Lattices

By the definition, *the empty subset of elements* of a complete lattice has infimum.
Supremum of the empty set is the smallest element (zero) of the lattice. Dually, infimum of the empty set is the largest element (unit) of the lattice.

$$\bigvee \emptyset = 0 \quad \bigwedge \emptyset = 1$$

Hence, a complete lattice has zero and unit, i.e., every complete lattice is bounded.

Lattices

Theorem

All finite lattices are complete.

Proof.

Supremum and infimum of the emptyset are equal to zero and unit of the lattice, resepectively, while supremum and infimum of any nonempty subset is defined by pairwise supremums and infimums: for $X = \{x_1, \ldots, x_n\}$ one can set $\bigwedge X = x1 \wedge x2 \wedge \cdots \wedge x_n$ independent of the order of elements and brackets, by associativity and commutativity of operation $\wedge$. $\square$

Closure system

If X is a set and $L \subseteq 2^X$ such that $X \in L$ and for any nonempty subset $A \subseteq L$ one has $\bigcap\{a \in A\} \in L$, then L is called **closure system (Moore family) over** X.

Closure system

Theorem

A closure system L over X is a complete lattice, where infimum and supremum are given in the following way:

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$$

$$\bigvee_{i \in I} A_i = \bigcap \{B \in L \mid \bigcup_{i \in I} A_i \subseteq B\}.$$

Closure system

- In Data Science an important example of a closure system is *a set of descriptions of some objects from a domain together with all possible intersections of such descriptions*.
- These intersections give *similarity of objects* in terms of common parts of descriptions.
- Such closure systems will be studied in detail within Formal Concept Analysis.

Closure operator

Closure systems are tightly related to **closure operators**. A closure operator on set G is a mapping $\mathrm{cl} : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$ that takes every subset $X \subseteq G$ to its **closure** $\mathrm{cl}X \subseteq G$ and has the following properties:

1. $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$ *(idempotence)*
2. $X \subseteq \mathrm{cl}(X)$ *(extensivity)*
3. $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$ *(monotonicity)*

Subset $X \subseteq G$ is called closed if $\mathrm{cl}(X) = X$.

Connection between closure operators and closure systems

Let a set X be given, then:

1. A set of subsets of X closed w.r.t. some closure operator $\mathrm{cl}(X)$ makes a closure system over X. A closure system defines a closure operator where all elements of the closure system are closed w.r.t. the closure operator.

Proof. 1.

Let A and B be closed sets, i.e., $\mathrm{cl}(A) = A$, $\mathrm{cl}(B) = B$.
Then, by the monotonicity of closure operator,
$\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) = A$, $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(B) = B$, hence,
$\mathrm{cl}(A \cap B) \subseteq A \cap B$. By the extensivity of a closure operator,
$A \cap B \subseteq \mathrm{cl}(A \cap B)$, hence $\mathrm{cl}(A \cap B) = A \cap B$, i.e., the intersection of closed sets is a closed set. $\square$

Connection between closure operators and closure systems

2. Let L be a closure system over set X. Define operator $\varphi(A) = \bigcap_{A \subseteq X, X \in L} X$.
   $\varphi(A)$ is monotonic and extensive.
   Let's show that it is also *idempotent*.

Proof.

- Consider $\varphi\varphi(A) = \bigcap_{\varphi(A) \subseteq X, X \in L} X$.
- Due to the extensivity, one has $A \subseteq \varphi(A)$, hence, $\varphi(A) \subseteq X \to A \subseteq X$.
- Hence, $A \subseteq X \varphi(A) \subseteq X \Leftrightarrow \varphi\varphi(A) = \varphi(A)$, i.e., the idempotence is proved.
- Thus, $\varphi(A)$ is a closure operator such that all sets from the closure system L are closed with respect to it.

□

Examples

- Concise representation of **association rules**, an important tool of Data Mining, is based on so-called closed itemsets (closed sets of attributes).

Suppose that we have a dataset given by sets of objects described by sets of attributes (itemsets in terms of Data Mining).

Support of an itemset B is the set of objects with descriptions containing B.

B is **a closed itemset** if $B \cup \{a\}$ has smaller support than B for any $a \notin B$. A mapping taking an itemset A to the smallest closed itemset B containing A is a closure operator.

Duality principle for lattices

A statement **dual to a statement** on lattices can be obtained by replacing symbols $\leq, \vee, \wedge, 0, 1$ with symbols $\geq, \wedge, \vee, 1, 0$, respectively.

If poset $(V, \leq)$ is a (complete) lattice, then a dual partially ordered set $(V, \leq)^d = (V, \geq)$ is also a (complete) lattice called dual to the initial one.

A diagram of a dual lattice can be obtained from a diagram of the initial lattice by "turning it upside down."

Sublattices

A triple $\mathcal{K} = (K, \wedge, \vee)$ is a **sublattice** of lattice $\mathcal{L} = (L, \wedge, \vee)$ if $K \subseteq L$ is a nonempty subset of set L with the following property: $a, b \in K$ implies that $a \wedge b \in K$, $a \vee b \in K$, where $\wedge$ and $\vee$ are taken in L, i.e., they are restrictions of operations of $\mathcal{L}$ to $\mathcal{K}$.

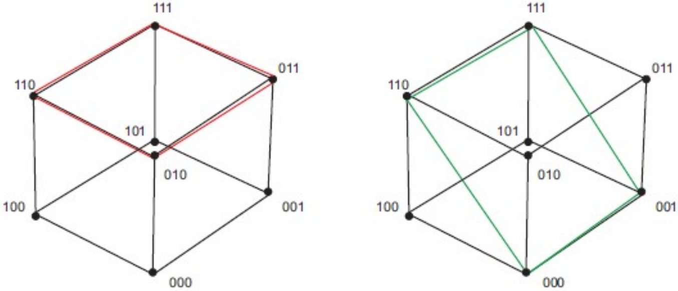
Example

The lattice of binary vectors of dimension 3 and its sublattice. Binary vectors of a fixed dimension make a lattice w.r.t. componentwise disjunctions and conjunctions.

Lattice morphisms

A mapping φ between two complete lattices **respects supremums** (is **a join-morphism**) if

$$\varphi \bigvee X = \bigvee \varphi(X)$$



The mappings between the big lattice and its projection to "red" and "green" sublattices are examples of join-morphism.

$\wedge-$morphism is defined dually.

Lattice morphisms

A complete homomorphism (**homomorphism of complete lattices**) is a mapping between two complete lattices that is supremum and infimum morphism. **Isomorphism of complete lattices** is a bijective complete homomorphism.

Distributivity and modularrity

A lattice whith the following properties

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

is called *distributive*.

Example

Ring of sets over a set I is a set F of subsets of I that, together with two sets S and T contains their set-theoretic intersection $S \cap T$ and union $S \cup T$.

Modularrity

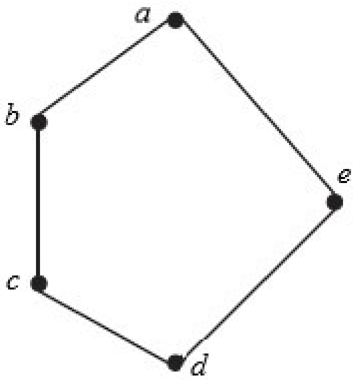
A lattice with the property

$$\text{if } x \leq z, \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z$$

is called **modular**.

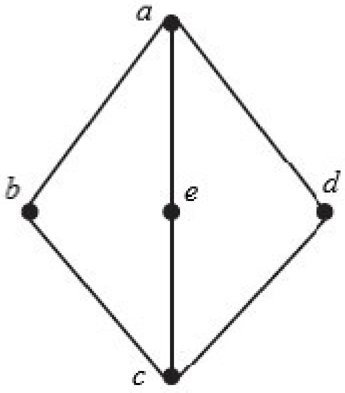
Lattices

Pentagon is a lattice with the following diagram:



Lattices

Diamond is a lattice with the following diagram:



Theorem

Lattice L **contains pentagon** (**diamond**) if it contains a sublattice which is a pentagon (diamond). The following theorem was proved by G. Birkhoff.

Theorem

1. *A lattice is distributive iff it contains neither a pentagon, nor a diamond.*
2. *A lattice is modular iff it does not contain a pentagon.*

Boolean lattice

A **Boolean lattice**, besides supremums and infimums (with distributivity property), has operation called complement.

Formally, lattice L is called Boolean if it is

1. distributive,
2. bounded, i.e., if it has the least element 0 and the largest element 1,
3. every element $a \in L$ has a unique **complement** $\overline{a} \in L$, i.e. $a \wedge \overline{a} = 0$ and $a \vee \overline{a} = 1$.

Boolean lattice

Statement

If L is a Boolean lattice, then the following relations hold:

1. $\overline{0} = 1$, $\overline{1} = 0$
2. $\overline{\overline{a}} = a$
3. $\overline{(a \vee b)} = \overline{a} \wedge \overline{b}$, $\overline{(a \wedge b)} = \overline{a} \vee \overline{b}$
4. $a \wedge \overline{b} = 0$ iff $a \leq b$ $a, b \in L$.

Irreducible elements of a lattice

Irreducible elements for lattices are similar to prime numbers for natural numbers.
Let L be a lattice. Element $x \in L$ is called $\vee-$**irreducible** (pronounced *supremum- (or join-) irreducible)* if

1. $x \neq 0$ (in case where L has zero, the least element),
2. If $x = a \vee b$, then $x = a$ or $x = b$.

$\wedge-$**irreducible** (infimum- or meet-irreducible) elements are defined dually. In infinite lattices irreducible elements might not exist.

Irreducible elements of a lattice

In diagrams of finite lattices:

- a $\vee-$irreducible element has only one neighbor from below,
- a $\wedge-$irreducible element has only one neighbor from above.

A join (meet) reducible element is an element that is not join (meet-) irreducible, so it can be represented as a join (meet) of all elements below (above) it.

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