Natural Numbers and Induction

A

In this appendix, we discuss one powerful and general method to deduce theorems and formulas depending on a natural parameter. This method, called *induction*, could be illustrated by the domino effect: if dominoes are stood on end one slightly behind the other, a slight push on the first will topple the others one by one.

As a toy example, let us prove that the area of the rectangle of the size $5 \times n$ is equal to $5n$. For $n = 1$, the rectangle can obviously be cut up into 5 unit squares, so its area is equal to 5 (the first domino falls down). Now, for $n = 2$ we explore the domino effect and cut up the rectangle into two ones of size 5×1. This gives the area $5 + 5 = 5 \cdot 2$. For $n = 3$, we cut up the rectangle into the rectangles 5×2 and 5×1: this gives the area $5 \cdot 2 + 5 \cdot 1 = 5 \cdot 3$. By the same way, we cut up any large rectangle of the size $5 \times n$ into two ones, that is, the rectangle $5 \times (n-1)$ and the 'ribbon' 5×1 (see Fig. A.1). At some moment, we can assume that the area of the first one is known to be $S_{n-1} = 5 \cdot (n-1)$ (the $(n-1)$-th domino has fallen down), so, we calculate the area as $S_n = 5 \cdot (n-1) + 5 \cdot 1 = 5 \cdot n$.

Similar (and slightly more complicated) methods can be applied in many problems. In order to give their formal description, we first discuss a formal introduction to natural numbers.

A.1 Natural Numbers: Axiomatic Definition

Natural numbers are known as the main and the basic objects in mathematics. Many complicated things such as rational and real numbers, vectors, and matrices can be defined via the natural numbers. "God made the integers; all else is the work of man", said Leopold Kronecker, one of the most significant algebraists of nineteenth century.

What are the natural numbers? This question admits a lot of answers, all in different levels of abstraction. The naïve definition says that these are just the numbers used in counting, that is, $0, 1, 2, \ldots$ In the geometry of the real line, the

F. Aleskerov et al., *Linear Algebra for Economists*, Springer Texts in Business and Economics, DOI 10.1007/978-3-642-20570-5,

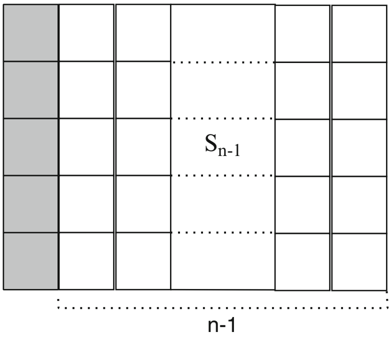

Fig. A.1 The area of a rectangle: proof by induction

set of natural numbers $\mathbb{N}$ is defined as the set of the following points in the line: an initial point (say, O); the right-hand end of the unit segment whose left-hand end is O; the right-hand end of the unit segment whose left-hand end is the just defined point; and so on. In the set theory, every natural number n is considered as the simplest set of n elements: $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}$ etc. These set-theoretical numbers play the role of etalons for counting all other finite sets.

Let us give the most general definition, that covers all previous ones. It is based on the axioms due to Peano[1] (1889). Assume the $\mathcal{N}$ is a set, 0 is an element of the set $\mathcal{N}$, and $s : \mathcal{N} \to \mathcal{N}$ is a function (called 'successor'). The set $\mathcal{N}$ is called *a set of natural numbers*, if the following three *Peano axioms* are satisfied.

1. If $s(m) = s(n)$, then $m = n$. (This means that the function s is injective).
2. There is no such n that $s(n) = 0$.
3. (Induction axiom). Suppose that there is a subset $A \subset \mathcal{N}$ such that (1) $0 \in A$ and (2) for every $n \in A$ we have $s(n) \in A$. Then $A = \mathcal{N}$.

For example, one can define an operation $n \mapsto n + 1$ as $n + 1 := s(n)$.

Exercise A.1. Define an operation s for the above versions of natural numbers, that is, for the collection N_1 of points in a real line and for the sequence of sets: $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\} \ldots$

Let us define also an operation $n \mapsto n + m$ via the two rules: $n + 0 := n$ and $n + (m + 1) := s(n + m)$. Let A be the set of the numbers $m \in \mathcal{N}$ for which we can calculate the sum $n + m$ using this rule. Then $0 \in A$ (by the first rule) and for every $m \in A$ we have $m + 1 \in A$ (by the second rule). It follows from the induction axiom that the sum $n + m$ is defined for all $m, n \in \mathcal{N}$.

[1] Giuseppe Peano (1858–1932) was a famous Italian mathematician. Being one of the founders of mathematical logic and set theory, he was also an author of many analytical discoveries including a continuous mapping of a line onto every point of a square. Another his discovery was *Latino sine Flexione* (or *Interlingua*), an artificial language based on Latin with simpler grammar.

Exercise A.2. Give definitions (in a similar way as the above definition of the addition) of the following operations with natural number m and n:
1. $m \cdot n$.
2. m^n.

This way to define an operation (from each natural number m to the next number $m + 1$ an so on) is called *recursion*. For example, a recursive formula gives a definition of the determinant of a matrix of order n, see the formula (3.6). The recursion definitions are appropriate for using in the induction reasonings, as described below.

A.2 Induction Principle

In order to deduce any significant property of natural numbers from the above axioms, one should use a special kind of reasoning, called *the induction principle*, or *mathematical induction*.

Let $P(n)$ be an arbitrary statement concerning a natural number n (like, for example, “n is equal to 5”, or $n + 2 = 2 + n$, or “either $n \leq 2$ or $x^n + y^n \neq z^n$ for any natural x, y, z”).

Theorem A.1. *Let $P(n)$ be a statement*[2] *depending on element n of a set of natural numbers $\mathcal{N}$. Suppose that the following two assumptions hold:*
1. (The basis, or The initial step) $P(0)$ is true.
2. (The inductive step) $P(n + 1)$ is true provided that $P(n)$ is true.
Then $P(n)$ is true for every $n \in \mathcal{N}$.

Note that the element n here is called *induction variable*, and the assumption that $P(n)$ holds in the inductive step is called *induction assumption*.

Proof. Let A be a set consisting of all natural numbers $n \in \mathcal{N}$ such that $P(n)$ is true. According to the basis of induction, we have $0 \in A$. By the induction step, for every $n \in A$ we have also $n + 1 \in A$. Thus, we can apply the induction axiom and conclude that $A = \mathcal{N}$. □

Example A.1. Let us prove the formula

$$0 + 1 + 2 + \cdots + n = \frac{n(n + 1)}{2}. \tag{A.1}$$

Let $P(n)$ be the above equality. For $n = 0$ it is obviously true: $0 = \frac{0\cdot 1}{2}$. This gives the basis of the induction. To prove the induction step, let us assume that the statement $P(n)$ is true for some n, that is, the equality (A.1) holds. We have to

[2]Note that we do not give here a strong mathematical definition of a term ‘statement’. At least, all statements consisting of arithmetical formulas with additions like “for every natural n” or “there exists natural x such that” are admissible.

deduce the statement $P(n+1)$, that is, the same formula with n replaced by $n+1$. Using the statement $P(n)$, we re-write the left hand side of the equality $P(n+1)$ as follows:

$$0+1+2+\cdots+n+(n+1)=\frac{n(n+1)}{2}+(n+1)=\frac{(n+1)(n+2)}{2}.$$

This equality is equivalent to $P(n+1)$, so, the induction step is complete. By the induction principle, we conclude that $P(n)$ holds for all n.

Example A.2. In this example we deduce some standard properties of natural numbers from Peano axioms.

First, let us consider the following statement $P(n)$: $0+n=n$. By the above definition of addition, we have $m+0=m$ for all $m\in\mathcal{N}$, hence $0+0=0$. This gives the basis of the induction: $P(0)$ is true. To prove the induction step, let us assume that the statement $P(n)$ is true for some n, that is, $0+n=n$. Using this fact and the definition of addition, we obtain: $0+(n+1)=s(0+n)=s(n)=n+1$, i.e., we have deduced the statement $P(n+1)$. Since both the basis of the induction and the inductive step are true, we conclude that $P(n)$ is true for all n.

Now, let us prove the associativity property

$$(l+m)+n=l+(m+n)$$

for all natural l,m,n (we denote this statement by $S_{l,m}(n)$). Again, let us apply the induction (on the number n). For $n=0$, we get the trivial statement $S_{l,m}(0): l+m=l+m$, which is obviously true, so, the basis of the induction $S_m(0)$ is proved. To show the induction step, it remains to show that $S_{l,m}(n+1)$ is true for all l,m provided that $S_{l,m}(n)$ is. Using the equalities $S_{l,m}(n)$, we have $(l+m)+(n+1)=s((l+m)+n)=s(l+(m+n))=l+s(m+n)=l+(m+(n+1))$. This gives $S_{l,m}(n+1)$. Hence, the proof is complete.

Now, let us prove the following statement $Q_m(n)$

$$m+(n+1)=(m+1)+n.$$

We proceed by the induction on the variable n. For $n=0$, we have $m+1=m+1$: this is the basis $Q_m(0)$ of the induction. To show the induction step, let us assume that for some n the statement $Q_m(n)$ is true for all m. We have to show $Q_m(n+1)$. Using the assumption, we have

$$m+(n+2)=s(m+(n+1))=s((m+1)+n)=(m+1)+(n+1).$$

So, we have deduced $Q_m(n+1)$. By the induction principle, the equality $Q_m(n)$ holds for all m and n.

Finally, let us show the commutativity property

$$m + n = n + m$$

for all $m, n \in \mathcal{N}$. We proceed by the induction on n. For $n = 0$ we get the above statement $P(m)$, hence the initial step holds. To prove the induction step, we use the induction assumption (the equality $m + n = n + m$) and the statement $Q_n(m)$:

$$m + (n + 1) = s(m + n) = s(n + m) = n + (m + 1) = (n + 1) + m.$$

Thus, the induction step is proved, and the induction is complete. □

Exercise A.3. Using the definition of multiplication of natural numbers given in Exercise A.2, prove the following standard properties:
1. $a(bc) = (ab)c$.
2. $ab = ba$.

The following version of Theorem A.1 is called a *weak induction principle*.

Corollary A.2. *Let $P(n)$ be a statement depending on element n of a set of natural numbers $\mathcal{N}$. Suppose that the following two assumption hold:*
1. $P(n_0)$ is true for some $n_0 \in \mathcal{N}$.
2. $P(n + 1)$ is true provided that $P(n)$ is true, where $n \geq n_0$.
Then $P(n)$ is true for every $n \geq n_0$.

Proof. Let $P'(n)$ be the following statement: '$P(n_0 + n)$ is true'. Then the above conditions on $P(n)$ are equivalent to the conditions of Theorem A.1 for the statement $P'(n)$. Hence, we apply the induction principle and deduce that $P'(n)$ is true for all n. This means that $P(n)$ is true for all $n \geq n_0$. □

Example A.3. Let us solve the inequality

$$2^n > 3n, \tag{A.2}$$

where $n > 0$ is an integer.

It is easy to check that the inequality fails for $0 < n \leq 3$, while for $n = 4$ it holds: $2^4 > 3 \cdot 4$. It is natural to assume that the inequality holds for all $n \geq 4$. How to prove the assumption?

Let us apply the weak induction principle with $n_0 = 4$. The statement $P(n)$ is then the inequality (A.2). The initial step $P(4)$ is done. To prove the induction step, we try to deduce $P(n + 1)$ from $P(n)$, where $n \geq 4$. Consider the left hand side 2^{n+1} of $P(n + 1)$. According to $P(n)$, we have

$$2^{n+1} = 2 \cdot 2^n > 2 \cdot 3n = 6n.$$

Since $n \geq 4$, we have $6n \geq 3n + 3n > 3n + 3 = 3(n+1)$. Thus, we obtain an inequality $2^{n+1} > 3(n+1)$, which is equivalent to $P(n+1)$. By Corollary A.2, the inequality (A.2) holds for all $n \geq 4$.

A more general version of the induction principle is given by the following Strongest Induction Principle.

Corollary A.3. *Let $P(n)$ be a statement depending on element n of a set of natural numbers $\mathcal{N}$. Suppose that the following two assumption hold:*
1. *$P(n_0)$ is true for some $n_0 \in \mathcal{N}$.*
2. *$P(n+1)$ is true provided that $P(k)$ is true for all $n \geq k \geq n_0$.*
   Then $P(n)$ is true for every $n \geq n_0$.

Proof. Let $P'(n)$ be the following statement: '$P(k)$ is true for all $n \geq k \geq n_0$'. Then we can apply Corollary A.2 to the statement $P'(n)$. □

Example A.4. *Problem.* Evaluate the determinant of the following matrix of order $n \times n$

$$A_n = \begin{pmatrix} 3\ 2\ 0 \ldots 0\ 0 \\ 1\ 3\ 2 \ldots 0\ 0 \\ 0\ 1\ 3 \ldots 0\ 0 \\ \ldots \\ 0\ 0\ 0 \ldots 1\ 3 \end{pmatrix}$$

Solution. Let us denote $D_n = \det A_n$. By direct calculations, we have $D_1 = 3$, $D_2 = 7$. It is natural to formulate a *conjecture*, called $P(n)$:

$$D_n = 2^{n+1} - 1, \text{ where } n \geq 1.$$

To prove the above conjecture, we apply the strongest induction principle (Corollary A.3). We put $n_0 = 1$: then the initial step $P(1)$ is given by the equality $D_1 = 3$. To show the induction step, let us evaluate D_{n+1}. If $n = 1$, then $D_{n+1} = D_2 = 7$, and the conjecture holds. For $n \geq 2$ we have

$$D_{n+1} = \det A_{n+1} = \det \left(\begin{array}{cc|ccc} 3 & 2 & 0 & \ldots & 0 \\ 1 & 3 & 2 & \ldots & 0 \\ \hline 0 & 1 & & & \\ \ldots & \ldots & & A_{n-1} & \\ 0 & 0 & & & \end{array}\right)$$

$$= 3 \cdot \det A_n - 2 \cdot 1 \cdot \det A_{n-1} - 2 \cdot 2 \cdot 0 = 3D_n - 2D_{n-1}.$$

Using $P(n)$ and $P(n-1)$, we get

$$D_{n+1} = 3(2^{n+1} - 1) - 2(2^n - 1) = 3 \cdot 2^{n+1} - 2^{n+1} - 1 = 2 \cdot 2^{n+2} - 1.$$

So, we have deduce $P(n+1)$. This complete the induction step. Thus, the conjecture is true for all $n \geq 1$. $\square$

Other methods of evaluating such determinants will be discussed in Appendix B.

A.3 Problems

1. Show that
$$1 + 3 + \cdots + (2n-1) = n^2.$$
2. Find the sum
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n}.$$
3. Prove the following *Bernoulli's inequality*
$$(1+x)^n \geq 1 + nx \text{ for every natural } n \text{ and real } x \geq 0.$$
4. Show that the number $x^n + \frac{1}{x^n}$ is integer provided that $x + \frac{1}{x}$ is integer.
5. Describe all natural n such that $2^n > n^2$.
6. Suppose that an automatic machine sells two type of phone cards, for \$3 and \$5 (for 30 min and 70 min of phone calls, respectively). Show that any integer amount greater than \$7 can be exchanged for the cards without change.
7. Show that
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$
8. Compute
(a)
$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n$$
(b)
$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^n$$
9. Show that the matrix
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^n + \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^n$$
has the form
$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$
for some number c.

10. Show that the determinant of order n

$$\begin{vmatrix} 0 & 1 & 0 \dots & 0 & 0 \\ -1 & 0 & 1 \dots & 0 & 0 \\ 0 & -1 & 0 \dots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \dots & -1 & 0 \end{vmatrix}$$

is equal to $(1 + (-1)^n)/2$.

Methods of Evaluating Determinants B

In addition to examples in Sect. 3.4, we discuss here some advanced methods of evaluating the determinants of various special matrices. This Appendix is mainly based on [25, Sect. 1.5]. For further methods of determinant evaluation, we refer the reader to [15].

B.1 Transformation of Determinants

Sometimes we can prove some equalities of determinants without directly evaluating them.

Let us consider the following problems.

Example B.1. Problem. Prove that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Solution. We can add one column of a determinant with any other column multiplied by some constant without changing the value of the determinant (see the property (a) in Sect. 3.4). Let us use this method here to get

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & bc + a\,(a+b+c) - 1\,(ab+ac+bc) \\ 1 & b & ac + b\,(a+b+c) - 1\,(ab+ac+bc) \\ 1 & c & ab + c\,(a+b+c) - 1\,(ab+ac+bc) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & bc + a^2 + ab + ac - ab - ac - bc \\ 1 & b & ac + b^2 + ab + ac - ab - ac - bc \\ 1 & c & ab + c^2 + ab + ac - ab - ac - bc \end{vmatrix}$$

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$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Example B.2. Problem. Prove that

$$\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}.$$

Solution. The answer follows from

$$\frac{1}{x^2y^2z^2} \begin{vmatrix} 0 & xyz & xyz & xyz \\ x & 0 & xz^2 & xy^2 \\ y & yz^2 & 0 & x^2y \\ z & y^2z & x^2z & 0 \end{vmatrix} = \frac{xyz \cdot xyz}{x^2y^2z^2} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}.$$

B.2 Methods of Evaluating Determinants of High Order

B.2.1 Reducing to Triangular Form

One of useful methods of calculation determinant is to reduce a matrix to a triangular form via elementary transformations and then calculate its determinant as a product of diagonal elements by Example 3.5.

Example B.3. Reducing the below matrix

$$\begin{vmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \dots & a_n \end{vmatrix}$$

to the triangular form, subtract the first row from all other rows to get

$$\begin{vmatrix} a_1 & x & x & \dots & x \\ x-a_1 & a_2-x & 0 & \dots & 0 \\ x-a_1 & 0 & a_3-x & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x-a_1 & 0 & \dots & \dots & a_n-x \end{vmatrix}.$$

Take out $a_1 - x$ from the first column, $a_2 - x$ from the second one, and so on, to obtain

$$(a_1 - x)(a_2 - x)\dots(a_n - x)\cdot\begin{vmatrix} \frac{a_1}{a_1-x} & \frac{x}{a_2-x} & \dots & \dots & \frac{x}{a_n-x} \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{vmatrix}.$$

Put $a_1/(a_1 - x) = 1 + x/(a_1 - x)$ and add all columns to the first one to get

$$(a_1 - x)(a_2 - x)\dots(a_n - x)\,.$$

$$\begin{vmatrix} 1 + \frac{a_1}{a_1-x} + \dots + \frac{x}{a_n-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \dots & \frac{x}{a_n-x} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

The last matrix has an upper triangular form. It follows from Example 3.5 that its determinant is the product of the diagonal entries $\left(1 + \frac{x}{a_1-x} + \dots + \frac{x}{a_n-x}\right)\cdot 1^{n-1}$. Thus,

$$\det A = (a_1 - x)(a_2 - x)\dots(a_n - x)\left(1 + \frac{x}{a_1 - x} + \dots + \frac{x}{a_n - x}\right).$$

B.2.2 Method of Multipliers

Let D be the determinant of a matrix $A = \|a_{ij}\|_{n\times n}$ of order n.

Proposition B.1. *(a) D is a polynomial on n^2 variables a_{ij}, and the degree of this polynomial is equal to n.*
(b) The polynomial D is linear as a polynomial on the elements $a_{i1},\dots,a_{in}$ of the i-th row of the matrix A.

Proof. Using decomposition by the i-th row, we have $D = \sum\limits_{j=1}^{n} a_{ij}A_{ij}$, where the cofactors A_{ij} do not depend on the elements of the i-th row. So, D is linear as a polynomial of $a_{i1},\dots,a_{in}$. This proves (b). To prove (a), one can assume, by the induction arguments, that the minor determinants M_{ij} of order $n-1$ are polynomials of degree $n-1$. Since $A_{ij} = (-1)^{i+j}M_{ij}$, we conclude that D is a sum of polynomials of degree n. □

To calculate a determinant D, one can now consider it as a polynomial of some variables. Then as a polynomial it can be divided on linear multipliers. Thus, comparing elements of D with elements of multiplication of linear multipliers one can evaluate we can find a formula for D.

Example B.4. Let

$$D = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$$

Consider the columns A_1, A_2, A_3, A_4 of D. Add all columns to the first one. Then we obtain linear multiplier $(x + y + z)$.

Consider

$$\begin{array}{lll} A_1 + A_2 - A_3 - A_4 & & (y + z - x) \\ A_1 + A_3 - A_2 - A_4 & - \text{multiplier} & (x - y + z) \\ A_1 + A_4 - A_2 - A_3 & & (x + y - z) \end{array}$$

These multipliers are mutual. Hence, D is divisible by their product

$$\widetilde{D} = (x + y + z)\,(y + z - x)\,(x - y + z)\,(x + y - z)$$

According to Proposition B.1 a), the degree of the polynomial D is 4, so, it is equal to the degree of $\widetilde{D}$. It follows that $D = c\,\widetilde{D}$, where c is a scalar multiplier.

In the decomposition of $\widetilde{D}$, we obtain z^4 with coefficient -1, and in D we have z^4 with coefficient $+1$. Hence $c = -1$, that is,

$$D = -\,(x + y + z)\,(y + z - x)\,(x - y + z)\,(x + y - z)\,.$$

B.2.3 Recursive Definition of Determinant

The method is to decompose determinant by row or column and reduce it to the determinant of the same form but lower order. An example of application of this idea has been given in Example A.4 in Appendix A. Here we give some general formulae for determinants of that kind.

One of the possible forms is

$$D_n = pD_{n-1} + qD_{n-2}\ ,\ n > 2$$

If $q = 0$ then

$$D_n = p^{n-1} D_1.$$

If $q = 0$ then consider quadratic equation

$$x^2 - px - q = 0.$$

If its roots are α and β, then

$$p = \alpha + \beta,$$

$$q = -\alpha\beta$$

and

$$D_n = (\alpha + \beta)\, D_{n-1} - \alpha\beta D_n.$$

Suppose that $\alpha \neq \beta$. Then one can prove (by induction, see Appendix A) a formula for D_n

$$D_n = c_1\alpha^n + c_2\beta^n,$$

where

$$c_1 = \frac{D_2 - \beta D_1}{\alpha\,(\alpha - \beta)},$$

$$c_2 = -\frac{D_2 - \alpha D_1}{\beta\,(\alpha - \beta)}.$$

If $\alpha = \beta$ then we can obtain the following formula

$$D_n = (c_1 n + c_2)\,\alpha^{n-2},$$

where

$$c_1 = D_2 - \alpha D_1,$$

$$c_2 = 2\alpha D_1 - D_2.$$

This formula is again can be proved by induction.

Example B.5. Evaluate

$$\begin{vmatrix} 5 & 3 & 0 & 0 & \dots & 0 & 0 \\ 2 & 5 & 3 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 5 \end{vmatrix}_{n\times n}$$

Decompose by first row

$$D_n = 5D_{n-1} - 6D_{n-2}.$$

Quadratic equation gives the following solution:

$$x^2 - 5x + 6 = 0$$
$$\alpha = 2 \quad \beta = 3$$

Hence

$$D_n = c_1\alpha^n + c_2\beta^n = 3^{n+1} - 2^{n+1}.$$

B.2.4 Representation of a Determinant as a Sum of Two Determinants

By the linearity property of determinants (see Property *(a)* in p. 57), a complicated determinant can sometimes be presented as a sum of simpler ones.

Example B.6. Let

$$D_n = \begin{vmatrix} a_1 + b_1 & a_1 + b_2 & \dots & a_1 + b_n \\ a_2 + b_1 & a_2 + b_2 & \dots & a_2 + b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n + b_1 & a_n + b_2 & \dots & a_n + b_n \end{vmatrix} =$$

$$= \begin{vmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 + b_1 & a_2 + b_2 & \dots & a_2 + b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n + b_1 & a_n + b_2 & \dots & a_n + b_n \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & \dots & b_n \\ a_2 + b_1 & a_2 + b_2 & \dots & a_2 + b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n + b_1 & a_n + b_2 & \dots & a_n + b_n \end{vmatrix}$$

So after n steps we obtain 2^n determinants as summands.

If in each decomposition we take as first components the numbers a_i, and for second component numbers b_j then the rows will be either of the form

$$a_i, \dots, a_i$$

or of the form

$$b_1, b_2, \dots, b_n.$$

In the first case two are proportional, and in the second case even equal. If $n > 2$ in each determinant we have at least two rows of one type, i.e., for $n > 2$ we have $D_n = 0$. For $n = 1$ and 2, we have

$$D_1 = a_1 + b_1$$

$$D_2 = \begin{vmatrix} a_1 & a_1 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ a_2 & a_2 \end{vmatrix} = (a_1 - a_2)(b_2 - b_1).$$

B.2.5 Changing the Elements of Determinant

Consider

$$D = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

and

$$D' = \begin{bmatrix} a_{11}+x & \dots & a_{1n}+x \\ \vdots & \ddots & \vdots \\ a_{n1}+x & \dots & a_{nn}+x \end{bmatrix}.$$

Using the method of Sect. B.2.4, one can deduce that

$$D' = D + \begin{vmatrix} x & \dots & x \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ x & \dots & x \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ x & \dots & x \end{vmatrix}.$$

It follows that

$$D' = D + x \sum_{i,\ j=1}^{n} A_{ij},$$

where A_{ij} are cofactors of a_{ij}.

Example B.7. Let us evaluate

$$D' = \begin{vmatrix} a_1 & x & \dots & x \\ x & a_2 & \dots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \dots & a_n \end{vmatrix}.$$

Subtract x from all elements. Then

$$D = \begin{vmatrix} a_1 - x & 0 & \dots & 0 \\ 0 & a_2 - x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n - x \end{vmatrix} = (a_1 - x)\dots(a_n - x).$$

For any $i \neq j$ we have $A_{ij} = 0$, and for $i = j$

$$A_{ii} = (a_1 - x)\dots(a_{i-1} - x)\,(a_{i+1} - x)\dots(a_n - x)$$

Hence

$$D' = (a_1 - x)\dots(a_n - x) + x \sum_{i=1}^{n} A_{ii} =$$

by simple algebraic transformations

$$= x\,(a_1 - x)\,(a_2 - x)\dots(a_n - x)\left(\frac{1}{x} + \frac{1}{a_1 - x} + \dots + \frac{1}{a_n - x}\right).$$

B.2.6 Two Classical Determinants

The determinant

$$V(x_1,\dots,x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} \tag{B.1}$$

is called Vandermonde[1] determinant.

Theorem B.2.

$$V(x_1,\dots,x_n) = \prod_{1\le i<j\le n}(x_j - x_i).$$

Proof. By Proposition B.1 (b), the total degree of the polynomial $V(x_1,\dots,x_n)$ is equal to the sum

$$0 + 1 + \dots + (n-1) = n(n-1)/2.$$

If we subtract the i-th row from the j-th one, we get a new matrix with j-th row of the form $(0, x_j - x_i, x_j^2 - x_i^2, \dots, x_j^{n-1} - x_i^{n-1})^T$. Here each term $x_j^k - x_i^k = (x_j - x_i)(x_j^{k-1} + x_j^{k-2}x_i + \dots + x_i^{k-1})$ is divisible by $(x_j - x_i)$, hence $V(x_1,\dots,x_n)$ is divisible by $(x_j - x_i)$ for all $1 \le i < j \le n$. It follows that D is divisible by a polynomial

$$\widetilde{V}(x_1,\dots,x_n) \equiv \prod_{1\le i<j\le n}(x_j - x_i).$$

Since $\deg \widetilde{V}(x_1,\dots,x_n) = n(n-1)/2 = \deg V(x_1,\dots,x_n)$, it follows that $V(x_1,\dots,x_n) = c\widetilde{V}(x_1,\dots,x_n)$, where c is a number.

Using the decomposition by the last row, we see that the coefficient of x_n^{n-1} in $V(x_1,\dots,x_n)$ is $V(x_1,\dots,x_{n-1})$. At the same time, it is easy to see that the coefficient of x_n^{n-1} in $\widetilde{V}(x_1,\dots,x_n)$ is $\widetilde{V}(x_1,\dots,x_{n-1})$. By the induction arguments we conclude that $c = 1$. $\square$

[1] Alexandre Theophile Vandermonde (1735–1796), French mathematician and musician, one of the founders of the theory of determinants.

The matrix

$$C = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix}$$

is called *circulant matrix*.

Theorem B.3. *The determinant* $C(a_0, \dots, a_{n-1})$ *of the above matrix (circulant determinant) is equal to*

$$f(\varepsilon_0) \dots f(\varepsilon_{n-1}),$$

where $f(x) = a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ *and* $\varepsilon_0, \dots, \varepsilon_{n-1}$ *are different complex* n*-th roots of unity (for the definition of the complex roots, see Sect. C.2).*

Proof. Consider the product $Q = CV^T$, where V is the Vandermonde matrix (B.1) on $\varepsilon_0, \dots, \varepsilon_{n-1}$, that is,

$$V = \begin{bmatrix} 1 & \varepsilon_0 & \dots & \varepsilon_0^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon_{n-1} & \dots & \varepsilon_{n-1}^{n-1} \end{bmatrix}.$$

Then $Q = \|q_{ij}\|_{n\times n}$, where $q_{1j} = f(\varepsilon_j)$ and $q_{ij} = (C_i, V_j) = \sum_{k=0}^{i-2} a_{n+1+k-i}\varepsilon_{j-1}^k + \sum_{k=i-1}^{n-1} a_{1+k-i}\varepsilon_{j-1}^k = \varepsilon_{j-1}^{i-1} f(\varepsilon_{j-1})$ for $i \geq 2$. Therefore,

$$\det Q = f(\varepsilon_0)\dots f(\varepsilon_{n-1})|\varepsilon_{j-1}^{i-1}|_{n\times n} = f(\varepsilon_0)\dots f(\varepsilon_{n-1})V(\varepsilon_0, \dots, \varepsilon_{n-1}).$$

On the other hand, we have

$$\det Q = \det(CV^T) = \det C \det V = (\det C)V(\varepsilon_0, \dots, \varepsilon_{n-1}).$$

Since $V(\varepsilon_0, \dots, \varepsilon_{n-1}) \neq 0$, we have $\det C = f(\varepsilon_0)\dots f(\varepsilon_{n-1})$. $\square$

B.3 Problems

1. Prove that if all elements of a 3×3 matrix are equal ± 1, then the determinant of this matrix is an even number.
2. Without evaluating the determinants, show that:

   (a)

$$\begin{vmatrix} a_1 & b_1 & a_1x + b_1y + c_1 \\ a_2 & b_2 & a_2x + b_2y + c_2 \\ a_3 & b_3 & a_3x + b_3y + c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(b)

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b)$$

(c)

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

3. Evaluate the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix}.$$

4. Solve the equation

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1-x & 1 & \dots & 1 \\ 1 & 1 & 2-x & \dots & 1 \\ \dots. & \dots & \dots & \ddots & \dots \\ 1 & 1 & 1 & \dots & n-x \end{vmatrix} = 0.$$

5. Using the third row, evaluate the determinant

$$\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}.$$

6. Evaluate the determinant

$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}.$$

Evaluate the determinants in exercises 7–10 by reducing each of them to the triangular form. (Corresponding matrices in 7–9 are of order $n \times n$.)

7.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{vmatrix}.$$

8.

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}.$$

9.

$$\begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ 2 & 3 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \dots & \dots & \dots & \ddots & \dots \\ 2 & 2 & 2 & \dots & 3 \end{vmatrix}.$$

10.

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -x & x & 0 & \dots & 0 \\ 0 & -x & x & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix}.$$

Evaluate the determinants in questions 11 and 12 (by using linear multipliers).

11.

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_0 & x & a_2 & \dots & a_n \\ a_0 & a_1 & x & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & a_2 & \dots & x \end{vmatrix}$$

12.

$$\begin{vmatrix} -x & a & b & c \\ a & -x & c & b \\ b & c & -x & a \\ c & b & a & -x \end{vmatrix}$$

Evaluate the determinants in questions 13–15 by using the recursive definition. (Corresponding matrices in Problems 14 and 15 are of order $n \times n$.)

13.

$$\begin{vmatrix} a_1 & 1 & 1 & \dots & 1 \\ 1 & a_2 & 0 & \dots & 0 \\ 1 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & a_n \end{vmatrix}$$

14.

$$\begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{vmatrix}$$

15.

$$\begin{vmatrix} 3 & 2 & 0 & \dots & 0 \\ 1 & 3 & 2 & \dots & 0 \\ 0 & 1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 3 \end{vmatrix}$$

16. Evaluate the following determinant representing it as a sum of determinants:

$$\begin{vmatrix} x_1 & a_2 & \dots & a_n \\ a_1 & x_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & x_n \end{vmatrix}$$

Hint: insert $x_i = (x_i - a_i) + a_i$.

17. Let $x_0, x_1, \dots, x_n$ are variables and $p_0, p_1, \dots, p_n$ are polynomials of the form $p_j = a_j x^j +$ (lower terms). Show that

$$\begin{vmatrix} p_0(x_0) & p_1(x_0) & \dots & p_n(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ p_0(x_n) & p_1(x_n) & \dots & p_n(x_n) \end{vmatrix} = a_0 \dots a_n V(x_0, \dots, x_n).$$

Complex Numbers

C

Besides the dot product, in the plane $\mathbb{R}^2$ there is another product of vectors, called *complex multiplication*. In contrast to the dot product, the complex product of two vectors is again a vector in $\mathbb{R}^2$. It is given by the formula

$$(x, y)(x', y') = (xx' - yy', xy' + yx').$$

The properties of complex multiplication (given below) are close to the usual multiplication of real numbers, and the elements of $\mathbb{R}^2$ are referred as *complex numbers*. The set $\mathbb{R}^2$ of complex numbers is also denoted by $\mathbb{C}$. In particular, the vectors in the horizontal axis are identified with real numbers, that is, the real number x corresponds to the vector $(x, 0)$ (therefore, we have $\mathbb{R} \subset \mathbb{C}$). For example, the number 1 is identified with the vector $(1, 0)$, the first vector of the canonical basis. The horizontal axis is called a *real axis*. The sum (product) of two real numbers a and b is identified with the sum (product) of corresponding complex numbers $(a, 0)$ and $(b, 0)$, so that one can consider the complex numbers as an extension of the real number system.

The second vector of the canonical basis $(0, 1)$, being considered as a complex number, is denoted by i and called a *imaginary unit*, or *the square root of* -1 (because $i^2 = (-1, 0) = -1$). The vertical axis is called an *imaginary axis*, and every vector of the form $(0, x) = x \cdot i$ is called a *pure imaginary number*. It follows that any complex number $z = (x, y)$ is a sum of a real number and a pure imaginary number, that is,

$$z = (x, y) = x(1, 0) + y(0, 1) = x + yi.$$

The first entry x is called a *real part* of z and denoted as $\Re z$. The second entry y is called an *imaginary part* of z and denoted as $\Im z$.

F. Aleskerov et al., *Linear Algebra for Economists*, Springer Texts in Business and Economics, DOI 10.1007/978-3-642-20570-5,

Main Properties of Complex Multiplication

The next properties can easily be checked directly.

Let $z = (x, y), z' = (x', y')$ and $z'' = (x'', y'')$ be three complex numbers.

1. $zz' = z'z$, (commutativity)
2. $z(z'z'') = (zz')z''$, (associativity)
3. $z(z' + z'') = zz' + zz''$, (distributivity)
4. $1 \cdot z = z, 0 \cdot z = 0$, where $0 = \mathbf{0} = (0, 0)$.

Example C.1. If $z = (1, 2) = 1 + 2i$ and $z' = (4, 3) = 4 + 3i$, then $zz' = (1 + 2i)(4 + 3i) = 4 + 3i + 8i + 6i^2 = 4 + 3i + 8i + 6(-1) = -2 + 11i$.

C.1 Operations with Complex Numbers

Let us introduce some other operations with complex numbers.

C.1.1 Conjugation

A conjugation is a reflection of a vector in the real axis, that is, the conjugated complex number $z = (a, b) = a + bi$ is

$$\bar{z} = (a, -b) = a - bi$$

(see Fig. C.1).

In particular, $\bar{\bar{z}} = z$.

The conjugation has the following nice connections with the standard operations.

1. $\overline{z + z'} = \bar{z} + \bar{z}'$.
2. $\overline{zz'} = \bar{z}\bar{z}'$.
3. If $z = (a, 0) \in \mathbb{R}$, then $\bar{z} = z$.
4. $z\bar{z} = a^2 + b^2 \in \mathbb{R}_+$.

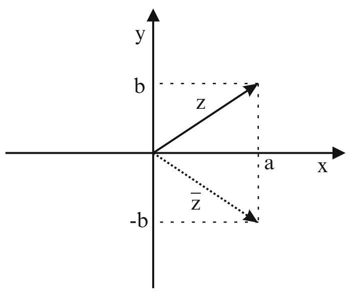


Fig. C.1 A conjugation

One can express the real and imaginary parts of a complex number via conjugation as

$$\Re z = (1/2)(z + \bar{z})$$

and

$$\Im z = (i/2)(\bar{z} - z).$$

C.1.2 Modulus

The *modulus* $|z|$ of a complex number $z = (a, b)$ is its length as a vector in $\mathbb{R}^2$, that is,

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}.$$

The modulus of the product of two complex numbers is the product of their moduli, that is,

$$|zz'| = |z||z'|$$

(because $|z||z'| = \sqrt{z\bar{z}}\sqrt{z'\bar{z'}} = \sqrt{zz'\overline{zz'}} = |zz'|$).

Example C.2. $|3 + 4i| = \sqrt{(3+4i)\overline{(3+4i)}} = \sqrt{(3+4i)(3-4i)} = \sqrt{3^2 + 4^2} = 5.$

C.1.3 Inverse and Division

The inverse of a nonzero complex number $z = a + bi$ is defined as $z^{-1} = |z|^{-2}\bar{z}$. Then $z^{-1}z = |z|^{-2}\bar{z}z = |z|^{-2}|z|^2 = 1$, so that one can define a *division* of complex numbers as $z'/z = z'z^{-1} = |z|^{-2}z'\bar{z}$, that is,

$$\frac{a' + b'i}{a + bi} = \frac{(a' + b'i)(a - bi)}{(a + bi)(a - bi)} = (a^2 + b^2)^{-1}(a' + b'i)(a - bi).$$

Note that the denominator $z = a + bi$ must be nonzero.

Example C.3.

$$\frac{18 + i}{3 - 4i} = \frac{(18 + i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{50 + 75i}{3^2 + 4^2} = 2 + 3i.$$

C.1.4 Argument

An *argument* of a nonzero complex number $z = a+bi$ is an angle φ between the real axis and the vector z (see Fig. C.2). Obviously, $\sin\varphi = b/|z|$ and $\cos\varphi = a/|z|$, so, φ is defined uniquely up to the period 2π. The set of all such possible ϕ is denoted

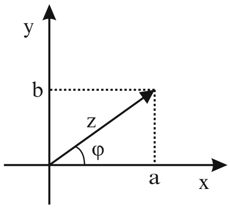


Fig. C.2 The angle φ is the argument of z

by $\operatorname{Arg} z$, while the unique argument φ such that $0 \leq \phi < 2\pi$ is denoted by $\arg z$. This means that

$$\operatorname{Arg} z = \{\arg z + 2\pi k | k \in \mathbb{Z}\}.$$

If $\varphi \in \operatorname{Arg} z$ is one the values of the argument of z, then it follows from the above sine and cosine values that

$$z = |z|(\cos\varphi + i \sin\varphi).$$

Let z and z' be two complex numbers with arguments α and β. One can check that

$$zz' = |z|(\cos\alpha + i \sin\alpha)|z'|(\cos\beta + i \sin\beta) = |z||z'|(\cos(\alpha+\beta) + i \sin(\alpha+\beta)), \tag{C.1}$$

so that the argument of the product of complex number is equal to the sum of their arguments. Analogously, we have

$$z/z' = |z|/|z'|(\cos(\alpha-\beta) + i \sin(\alpha-\beta)).$$

It follows that for each integer n we have

$$z^n = |z|^n(\cos(n\alpha) + i \sin(n\alpha)). \tag{C.2}$$

Example C.4. Let us calculate z^{100}, where $z = i + 1$. We have $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\cos(\arg z) = \sin(\arg z) = 1/\sqrt{2}$, so that $\arg z = \pi/4$. Therefore,

$$z^{100} = (\sqrt{2})^{100}\left(\cos\frac{100\pi}{4} + i \sin\frac{100\pi}{4}\right),$$

where $\frac{100\pi}{4} = 25\pi = 12 \cdot 2\pi + \pi$. Thus,

$$z^{100} = 2^{50}(\cos\pi + i \sin\pi) = -2^{50}.$$

C.1.5 Exponent

The exponent of a complex number $z = a + bi$ is defined as

$$e^{a+bi} = e^a(cosb + i \sin b)$$

(Euler[1] formula). It follows that for each nonzero complex z we have

$$z = |z|e^{i \arg z}.$$

Example C.5. $e^{i\pi} = e^0(cos\pi + i \sin \pi) = -1.$

Other properties of exponents of complex numbers follows from the equation (C.1):

$$e^{z+z'} = e^z e^{z'}, e^{-z} = 1/e^z, e^{z-z'} = e^z/e^{z'}.$$

C.2 Algebraic Equations

Many equations which have no real solutions have complex ones. The simplest example is the equation $z^2 + 1 = 0$, which has two complex solutions, $z = i$ and $z = -i$, and no real ones. More generally, each quadratic equation

$$ax^2 + bx + c = 0$$

with real coefficients a, b and c always have complex solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where in the case $D = b^2 - 4ac < 0$ we take $\sqrt{D} = i\sqrt{|D|}$.

Moreover, some simple algebraic equations have quite more complex solutions than real ones. Consider an equation

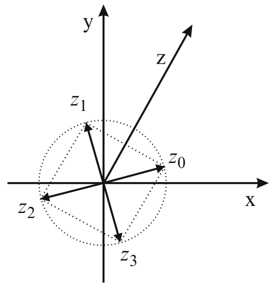
$$z^n = c,$$

where c is a nonzero complex number. If $\arg z = \alpha$ and $\arg c = \phi$, the formula (C.2) gives the equation

$$|z|^n(\cos(n\alpha) + i \sin(n\alpha)) = |c|(\cos\phi + i \sin\phi),$$

[1]Leonhard Euler (1707–1783) was a great Swiss mathematician who made enormous contributions to a wide range of mathematics and physics including analytic geometry, trigonometry, geometry, calculus and number theory. Most of his life he had been working in Russia (St. Petersburg) and Prussia (Berlin).

Fig. C.3 The fourth roots of a complex number z



therefore,

$$|z| = \sqrt[n]{|c|} \text{ and } \alpha = \frac{\phi + 2\pi k}{n}$$

for some integer k. Since $0 \leq \alpha < 2\pi$, we get n different values for α, that is, $\alpha_k = \frac{\phi+2\pi k}{n}$, where $k = 0, \ldots, n-1$. This gives exactly n pairwise different solutions of the above equation, that is,

$$z_k = \sqrt[n]{|c|}(\cos\alpha_k + i\sin\alpha_k) \text{ for } k = 0, \ldots, n-1.$$

All these n complex numbers could be referred as roots of degree n of c. All of them belong to a circle centered in the zero of radius $\sqrt[n]{|c|}$ and placed in the vertices of a regular polygon of n vertices inscribed to the circle (see Fig. C.3 for the case $n = 4$).

Example C.6. Solve the equation $z^3 = i$.

In the notation above, we have $n = 3$, $|c| = 1$ and $\phi = \pi/2$, so that $|z| = \sqrt[n]{1} = 1$, $\alpha_0 = \frac{\pi/2+2\pi 0}{3} = \pi/6$, $\alpha_1 = \frac{\pi/2+2\pi}{3} = 5\pi/6$ and $\alpha_2 = \frac{\pi/2+4\pi}{3} = 3\pi/2$. We obtain three solutions of the from $z_k = 1(\cos\alpha_k + i\sin\alpha_k)$ for $k = 0, 1, 2$, that is, $z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z_2 = -i$.

Theorem C.1 (Fundamental theorem of algebra). *Every non-constant polynomial with complex coefficients of a variable x has at least one complex root.*

Proof. See [33, Theorem 3.3.1]. □

We have discussed the finding of the roots for the polynomials of degree two and for polynomials of the form $x^n - z$. For general polynomials of any degree d greater than 4, there is no an universal formula for finding the root (Abel's theorem[2]).

If $x = x_1$ is a root of a polynomial $f(x)$ of degree d, then one can decompose

[2]Niels Henrik Abel (1802–1829) was a famous Norwegian mathematician. In spite of his short life, he made an extremely important contribution both to algebra and calculus. One of the most significant international prize for mathematician is called the Abel Prize. Abel had proved his theorem at age of 19.

$$f(x) = (x - x_1)g(x),$$

where $g(x)$ is another polynomial of degree $d - 1$ (with complex coefficients). If the degree $d - 1$ is positive, the polynomial $g(x)$ must have another complex root, say, x_2. Proceeding such decomposition up to degree 0, we get the decomposition of the from

$$f(x) = c(x - x_1)(x - x_2)\dots(x - x_d),$$

where c is a complex number and $x_1, \dots, x_d$ are roots of the polynomial $f(x)$. Combining the identical terms, we get (after a possible re-numerating of x_i's) the decomposition

$$f(x) = c(x - x_1)^{k_1}(x - x_2)^{k_2}\dots(x - x_s)^{k_s}$$

for some $s \le d$, where the sum of the powers $k_1 + \dots + k_s$ is d. Each number k_j is called a *multiplicity* of the corresponding root x_j. The roots of multiplicity one are called *simple*. One can check that each root x_j of the polynomial should appear in this decomposition.

We have

Corollary C.2. *The number of roots of any polynomial $p(x)$ of degree $d > 0$ is not greater than d. Moreover, the sum of all multiplicities of the roots is equal to d.*

Example C.7. Problem. Solve the equation $z^2 - 4iz - 7 - 4i = 0$.

Solution. By the well-known formula (which is easy to be checked), the number $z = \frac{-(-4i)+d}{2}$ satisfies the equation, where $d^2 = (-4i)^2 - 4(-7-4i) = 12 + 16i$. We need to find d. Let $d = x + iy$, where x and y are real numbers. Then $d^2 = (x^2 - y^2) + 2ixy$, so that the equality of two 2-dimensional vectors $d^2 = 12 + 16i$ gives a system of two equations

$$\begin{cases} x^2 - y^2 = 12, \\ 2xy = 16. \end{cases}$$

This system has two solutions for $d = x + iy$, that is, $d_1 = 4 + 2i$ and $d_2 = -4 - 2i$. This leads us to two roots of the equation above: $z_1 = (4i + d_1)/2 = 2 + 3i$ and $z_2 = (4i + d_2)/2 = -2 + i$. Since the equation has degree two, there are no other roots but these two.

To find a decomposition of a polynomial with real coefficients, the following statement is often useful.

Theorem C.3. *If a complex number z is a root of a polynomial $f(x)$ with real coefficients, then the conjugated number $\bar{z}$ is also a root of $f(x)$. Moreover, the multiplicities of these roots are the same.*

Proof. If $f(x) = a_d x^d + \dots + a_1 x + a_0$ and $f(z) = 0$, then $f(\bar{z}) = a_d \bar{z}^d + \dots + a_1 \bar{z} + a_0 = \bar{a}_d \bar{z}^d + \dots + \bar{a}_1 \bar{z} + \bar{a}_0 = \overline{a_d z^d} + \dots + \overline{a_1 z} + \overline{a_0} = \overline{f(z)} = \bar{0} = 0$.

The claim about the multiplicities follows from the induction argument applied to the polynomial $g(x)$ of degree $d-2$ such that $f(x) = (x-z)(x-\bar{z})g(x)$. $\square$

Example C.8. Let us find the decomposition of a polynomial $f(z) = z^4 - 5z^3 + 7z^2 - 5z + 6$. One can check that $f(i) = 0$, so that $z = i$ is a root of the polynomial. If follows that the number $\bar{z} = -i$ is another root. From the decomposition

$$f(z) = (z-i)(z+i)g(z)$$

one can find $g(z) = z^2 - 5z + 6$. It follows that $g(z) = (z-2)(z-3)$, so that

$$f(z) = (z-i)(z+i)(z-2)(z-3).$$

C.3 Linear Spaces Over Complex Numbers

Recall from the definition in the beginning of Chap. 6 that a linear space is a set admitting two operations, that is, addition and multiplication by a real number (dot product) such that these operations satisfies the linearity properties. One can extend this definition by allowing the multiplication of by complex numbers, not only by real ones. Such a vector space is called a *vector space over complex numbers*, or simply a *complex vector space*. In contrast, the vector space in the sense of Chap. 6 is called a *vector space over real numbers*, or a *real vector space*. The definition of complex vector space repeats the definition of real one literally but the world 'real' is replaced by 'complex'.

The notions of linear dependence and independence of vectors, dimension and basis of a vector space, subspace, isomorphism etc. for complex vector spaces repeat the correspondent definitions given in Chap. 6 verbatim. The definition and the properties of linear transformation and its matrix from Chap. 8 are transferred to the case of complex vector spaces verbatim as well.

Example C.9. The set $\mathbb{C}$ itself is a one-dimensional complex vector space. Any its nonzero element $z = a + bi \neq 0$ form a basis of it, since for each $w = x + yi \in C$ one has $w = \alpha z$, where $\alpha = w/z \in \mathbb{C}$.

Example C.10. The set $\mathbb{C}^n$ of n-tuples of complex numbers $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n \in \mathbb{C}$ is an n-dimensional vector space with standard operations

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and

$$\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n).$$

Its dimension is equal to n by the same reason as for the real space $\mathbb{R}^n$, see Example 6.5.

Given a real vector space V, one can embed it in a complex vectors space by the following way. Let $V^{\mathbb{C}}$ be the set of pairs of vectors from V. Each such pair we denote by $(\mathbf{u}, \mathbf{v}) = \mathbf{u} + i\mathbf{v}$. The addition of pairs and their multiplication by complex numbers are defined as

$$(\mathbf{u} + i\mathbf{v}) + (\mathbf{u}' + i\mathbf{v}') = (\mathbf{u} + \mathbf{u}') + i(\mathbf{v} + \mathbf{v}')$$

and

$$(a + bi)(\mathbf{u} + i\mathbf{v}) = (a\mathbf{u} - b\mathbf{v}) + i(a\mathbf{v} + b\mathbf{u}).$$

Exercise C.1. Show that $V^{\mathbb{C}}$ is a linear space over complex numbers.

Such a complex vector space $V^{\mathbb{C}}$ is called a *complexification* of V.

Example C.11. $\mathbb{C} = \mathbb{R}^{\mathbb{C}}$ and $\mathbb{C}^n = (\mathbb{R}^n)^{\mathbb{C}}$.

Given a vector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ in a complexification $V^{\mathbb{C}}$, one can define its real and imaginary parts as $\Re\mathbf{w} = \mathbf{u}$ and $\Im\mathbf{w} = \mathbf{v}$, the both are vectors in the real vector space V. Any basis of V is also a basis of the complexification $V^{\mathbb{C}}$, but one can also construct other bases in $V^{\mathbb{C}}$ which do not belong to V. For any linear transformation $f : U \to V$ of two real vector spaces U and V, one can also define its complexification $f^{\mathbb{C}} : U^{\mathbb{C}} \to V^{\mathbb{C}}$ by the obvious formula $f^{\mathbb{C}}(\mathbf{u} + i\mathbf{v}) = f(\mathbf{u}) + if(\mathbf{v})$. Note that if we fix two bases in the real vector spaces U and V, then the matrices of the linear transformations f and $f^{\mathbb{C}}$ in these bases are the same.

In particular, we prove in Corollary 9.5 that any linear operator in a finite-dimensional complex vector space has an eigenvector. This means that if f is a linear operator in a real vector space V, then its complexification $f^{\mathbb{C}}$ has an eigenvector in $V^{\mathbb{C}}$. In particular, it follows that the matrix of f has an eigenvalue.

C.4 Problems

1. Calculate
$$\frac{(1+i)(2+i)}{3-i}.$$
2. Calculate
$$\frac{i-3}{2-3i} + \frac{i+3}{2+3i}.$$
3. Calculate
$$\frac{(1+i)^4}{(i-1)^5}.$$
4. Solve the system of linear equations with complex coefficients
$$\begin{cases} (1+2i)x - 2iy = 5 + 9i \\ (-1+3i)x + (1+i)y = -6 + 4i. \end{cases}$$

5. Find all real solutions of the equation

$$(2i-2)x-(i+1)y=2i-10.$$

6. Solve the equation

$$z^2+2z+37=0.$$

7. Solve the equation

$$iz^2+(3i-2)z+12+4i=0.$$

8. Solve the equation

$$z^2+6z-4iz+5-12i=0.$$

9. Solve the equation

$$z^2-5z+4iz+9-i=0.$$

10. Calculate i^{100}.
11. Calculate $(1-i)^n$, where n is a positive integer.
12. Solve the equation $z^6=i$.
13. Solve the equation $z^4=-128+128\sqrt{3}i$.
14. Plot all the solutions of the equation $z^6=117+44i$ in the complex plane.
15. Find all solutions of the equation $z^5=5e^{5i}$.
16. Find the multiplicity of the root $z=2$ of the polynomial $z^5-6z^4+13z^3-14z^2+12z-8$.
17. Find the multiplicity of the root $z=i+1$ of the polynomial $z^5-6z^4+16z^3-24z^2+20z-8$.
18. Prove that $|z_1+z_2|\leq|z_1|+|z_2|$.
19. Prove the equation (C.1).
20. Let z and z' be two nonzero vectors in $\mathbb{R}^2$ (that is, complex numbers) such that $|z|=|z'|$ and $z\perp z'$. Find all possible values of z'/z.
21. Let $\varepsilon_0,\ldots,\varepsilon_{n-1}$ be different complex n-th roots of unity, where $n\geq 2$. Show that $\varepsilon_0+\cdots+\varepsilon_{n-1}=0$ and $\varepsilon_0\ldots\varepsilon_{n-1}=(-1)^{n-1}$.
22. Prove that $e^{z_1+z_2}=e^{z_1}e^{z_2}$.
23. Solve the equation $e^z=e$ for a complex number z.
24. Let A be a linear operator in $\mathbb{C}^2$ given by the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find its matrix in the basis

$$\mathbf{f}_1 = (i, 1), \mathbf{f}_2 = (1, i).$$

25. Let V be an n-dimensional complex vector space. Show that V is also a real vector space, where the multiplication by the real numbers is defined by the same way as the multiplication by complex ones (as the real numbers form a subspace of complex numbers). Find the dimension of V as a real vector space.

Pseudoinverse

D

Consider a general linear system

$$A\mathbf{x} = \mathbf{b}. \tag{D.1}$$

If the matrix A is square and non-singular, we have the solution

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

But if A is singular or even non-square, problems of two types arise. First, if the system (D.1) is consistent and the rank of A is less than the number of variables, we have a multiple solution which can be found algorithmically by, say, Gaussian elimination (see Sect. 5.2). But in many practical problems, we need an explicit formula for a (particular) solution depending on the vector $\mathbf{b}$ in the right side. How to get such an explicit formula?

Second, a problem arises if the system (D.1) is inconsistent. In practical problems, we need in this case an approximate solution (see Sect. (7.3)). Is there a formula for expressing it?

An answer to the both questions is given by a construction due to Moore[1] and Penrose[2] called a *pseudoinverse*[3]. For many generalizations and applications of this construction, we refer the reader to [2, 5].

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[1]Eliakim Hastings Moore (1862–1932), an American mathematician who made a significant contribution to algebra and logics. In 1935, he gave a definition pseudoinverse under the name *general reciprocal*.

[2]Roger Penrose (1931) is a famous English physicists and mathematician. He was awarded the Wolf foundation prise (1988, shared with Stephen Hawking) for the work which has "greatly enlarged our understanding of the origin and possible fate of the Universe". In 1955, he re-discovered the Moore definition of the pseudoinverse and demonstrated its connection with the least square approximation.

[3]The terms *generalized inverse* and *Moore–Penrose inverse* are also used.

F. Aleskerov et al., *Linear Algebra for Economists*, Springer Texts in Business and Economics, DOI 10.1007/978-3-642-20570-5,

D.1 Definition and Basic Properties

Definition D.1. Let A be an $m \times n$ matrix. The matrix A^+ is called a *pseudoinverse*[4] of A if the following four conditions hold

$$\begin{aligned}
&1.\ AA^+A = A\\
&2.\ A^+AA^+ = A^+\\
&3.\ \left(AA^+\right)^T = AA^+\\
&4.\ \left(A^+A\right)^T = A^+A.
\end{aligned}$$

Later (see Theorem D.5) we will show that every matrix A admits a pseudoinverse.

Note that since both A^+A and AA^+ exist, A^+ must be an $m \times n$ matrix.

The first property give us the right to call it *the* pseudoinverse.

Theorem D.1. *Let A be a matrix. If its pseudoinverse exists, then it is unique.*

Proof. Suppose B and C are two pseudoinverses of A, that is, both matrices B and C satisfy the conditions (1–4) of Definition D.1. Then we have:

$$AB = (ACA)B = (AC)(AB) = (AC)^T(AB)^T$$

$$= C^TA^TB^TA^T = C^T(ABA)^T = C^TA^T = (AC)^T = AC.$$

By the same way, we have

$$BA = B(ACA) = (BA)(CA) = (BA)^T(CA)^T$$

$$= A^TB^TA^TC^T = (ABA)^TC^T = A^TC^T = (CA)^T = CA.$$

Thus

$$B = BAB = (BA)B = (CA)B = C(AB) = CAC = C.$$

□

Example D.1. Suppose that A is a non-singular square matrix. Then the pseudoinverse A^+ exists and

$$A^+ = A^{-1}.$$

Proof. We have $AA^{-1}A = AI = A$ and $A^{-1}AA^{-1} = A^{-1}I = A^{-1}$. If A is of order n, then A^{-1} is of order n as well, so that

[4]There are other versions of definition of generalized inverse, or pseudoinverse, matrix. The most useful of the definitions require only *some* of the Penrose conditions (1–4) from Definition D.1, e.g., (1) and (2). On the definitions and properties of these more general versions of inverse matrix, see [5, Chap. 6] and [2]; see also [8, pp. 94–98], [26, pp. 203–205].

$$AA^{-1} = I_n = A^{-1}A.$$

This means that the matrix A^{-1} satisfies the conditions (1–4) of Definition D.1. □

Example D.2. Let

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

be a nonzero column vector. Then the row vector

$$C = \lambda A^T$$

is a pseudoinverse of A, where $\lambda = 1/(a_1^2 + \cdots + a_n^2)$.

Proof. We have $AC = \lambda(a_i a_j)_{n\times n}$ and $CA = \lambda(AA^T) = 1$, so that the conditions (3) and (4) hold for the matrix C. In addition, we have $ACA = A1 = A$ and $CAC = 1C = C$, so that the conditions (1) and (2) hold as well. □

Example D.3. Suppose that an $m \times n$ matrix A has the form

$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right],$$

where B is an $r \times r$ non-singular submatrix. Then the pseudoinverse A^+ is an $n \times m$ matrix of the form

$$C = \left[\begin{array}{c|c} B^{-1} & 0 \\ \hline 0 & 0 \end{array}\right].$$

Proof. It is easy to see that AC and CA are square matrices of the form

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

of orders m and n, respectively. It follows that $(AC)^T = AC$ and $(CA)^T = CA$. According to Lemma 9.16, we have $ACA = (AC)A = A$ and $CAC = (CA)C = C$. By Definition D.1, this means that $A^+ = C$ is the pseudoinverse. □

D.1.1 The Basic Properties of Pseudoinverse

5. $(A^+)^+ = A$
6. $(A^T)^+ = (A^+)^T$

Proof. Exercise. □

7. rank $A^+ = \operatorname{rank} A$

Proof. Recall that $\operatorname{rank}(AB) \leq \operatorname{rank} A$ and $\operatorname{rank}(AB) \leq \operatorname{rank} B$ for any two matrices A and B conformable for multiplication (see Problem 24 in Chap. 2). Then

$$\operatorname{rank} A = \operatorname{rank}(AA^+A) \leq \operatorname{rank}(AA^+) \leq \operatorname{rank} A^+$$

and

$$\operatorname{rank} A^+ = \operatorname{rank}(A^+AA^+) \leq \operatorname{rank}(AA^+) \leq \operatorname{rank} A.$$

□

Exercise D.1. Show that if $\operatorname{rank} A = 1$, then $A^+ = \frac{1}{\operatorname{Tr}(AA^T)} A^T$.

Hint. If $\operatorname{rank} A = 1$, then all columns of A are linear combinations of a single column, say, A^k, that is,

$$A = (\lambda_1 A^k | \dots | \lambda_n A^k) = (\lambda_1, \dots, \lambda_n) A^k.$$

D.2 Full Rank Factorization and a Formula for Pseudoinverse

Let us give the formula for a pseudoinverse of a matrix with linearly independent columns.

Theorem D.2. *Suppose that an $m \times n$ matrix A has full column rank, that is,* $\operatorname{rank} A = n$. *Then A has a pseudoinverse*

$$A^+ = (A^T A)^{-1} A^T.$$

We begin with

Lemma D.3. *Let A be an $m \times n$ with full column rank. Then the $n \times n$ matrix $A^T A$ is non-singular.*

Proof. Suppose that $\operatorname{rank}(A^T A) < n$. By Theorem 5.6, it follows that the system

$$A^T A\mathbf{x} = \mathbf{0}$$

has a nonzero solution $\mathbf{x}$. Then

$$\mathbf{x}^T A^T A\mathbf{x} = \mathbf{0},$$

$$(A\mathbf{x})^T A\mathbf{x} = \mathbf{0},$$

that is, $(A\mathbf{x}, A\mathbf{x}) = \mathbf{0}$. Hence $A\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} \neq \mathbf{0}$, we have $\operatorname{rank} A < n$, a contradiction. □

Proof of Theorem D.2. By Lemma D.3, the matrix $(A^TA)^{-1}$ exists. Let $C = (A^TA)^{-1}A^T$. We are going to show that C satisfies Definition D.1. Then $CA = I_n$, hence $(CA)^T = I_n^T = CA$. Since $C^T = A(A^TA)^{-1}$, we have $(AC)^T = C^TA^T = A(A^TA)^{-1}A^T = AC$. Moreover, $ACA = AI_n = A$ and $CAC = I_nC = C$. By Definition D.1, $C = A^+$. □

Example D.4. Let us calculate the pseudoinverse of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

We have rank $A = 2$ is equal to the number of columns, so that we can use the above formula. We have

$$A^TA = \begin{bmatrix} 3 & -1 \\ -1 & 13 \end{bmatrix},$$

so that

$$A^+ = (A^TA)^{-1}A^T = \begin{bmatrix} 3 & -1 \\ -1 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 13/38 & 1/38 \\ 1/38 & 3/38 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{38} \begin{bmatrix} 13 & 15 & -10 \\ 1 & 7 & 8 \end{bmatrix}.$$

In order to obtain similar formula in the general case of arbitrary matrix A, we need a presentation of A as a product of two matrices of full ranks, one of full column rank and one of full row rank.

Theorem D.4 (Full rank factorization). *Let A be an $m \times n$ matrix of rank r. Then there exist an $m \times r$ matrix F and an $r \times n$ matrix G (both of rank r) such that*

$$A = FG.$$

Proof. Consider any r linearly independent columns of A. Let F be the submatrix of A formed by these columns. Then F has size $m \times r$ and rank r. Each column A^k of A is a linear combination of columns of F, that is, $A^k = F\mathbf{G}^k$, where $\mathbf{G}^k$ is a column vector of dimension r. Then all n vectors $G^1, \ldots, G^n$ form a matrix G of size $r \times n$ such that $A = FG$. According to Problem 24 in Chap. 2, we have rank $G \geq r$. Since G has r rows, we have rank $G \leq r$, thus rank $G = r$. □

Note that for practical purposes, we may choose as F any matrix columns of which form a basis of the linear span of the columns of A.

Example D.5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Since $\det A = 0$ and $\det A_{(2)} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$, we conclude that $\operatorname{rank} A = 2$ and the first two columns (which form the submatrix $A_{(2)}$) are linearly independent. Then one can choose

$$F = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}.$$

Let us construct presentations of the columns of A as linear combinations of columns F^1 and F^2 of F. For the first two columns of A, we have $A^1 = F^1$ and $A^2 = F^2$, hence $A^1 = FG^1$ and $A^2 = FG^2$ with $G^1 = (1,0)^T$ and $G^2 = (0,1)^T$. To obtain a presentation of the third column A^3 via the columns of F, we have the system $FG^3 = A^3$ with unknown G^3, or

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} G^3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

We have the unique solution $G^3 = (-1,2)^T$. Finally, we have $A = FG$, where F is as above and

$$G = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

For another useful method to construct a full rank decomposition, see Problem 6 below.

Now, we a ready to deduce a general formula for pseudoinverse.

Theorem D.5. *For an arbitrary $m \times n$ matrix A, its pseudoinverse A^+ exists. If $A = FG$ is a full rank decomposition of A, then*

$$A^+ = G^T(GG^T)^{-1}(F^TF)^{-1}F^T.$$

Proof. By Lemma D.3, both matrices GG^T and F^TF are non-singular, so the matrix $C = G^T(GG^T)^{-1}(F^TF)^{-1}F^T$ exists. We will show that it satisfies the conditions of Definition D.1. Let $X = GG^T$ and $Y = F^TF$. Note that $X = X^T$ and $Y = Y^T$ are symmetric matrices. Then $AC = FXX^{-1}Y^{-1}F^T = FY^{-1}F^T$, so, $(AC)^T = AC$. Moreover, $CA = G^TX^{-1}Y^{-1}YG = G^TX^{-1}G$, hence $(CA)^T = CA$. Finally, $CAC = G^TX^{-1}Y^{-1}F^TFGG^TX^{-1}Y^{-1}F^T = G^TX^{-1}Y^{-1}YXX^{-1}Y^{-1}F^T = G^TX^{-1}Y^{-1}F^T = C$ and $ACA = FGG^TX^{-1}Y^{-1}F^TFG = FXX^{-1}Y^{-1}YG = FG = A$. By Definition D.1, $C = A^+$. $\square$

Example D.6. Let us calculate the pseudoinverse of the matrix A from Example D.5. We have

$$A^+ = G^T(GG^T)^{-1}(F^TF)^{-1}F^T,$$

where

$$GG^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

and

$$F^TF = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 66 & 78 \\ 78 & 93 \end{bmatrix},$$

so that

$$\begin{aligned} A^+ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 66 & 78 \\ 78 & 93 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5/6 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 31/18 & -13/9 \\ -13/9 & 11/9 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \\ &= \frac{1}{36} \begin{bmatrix} -23 & -6 & 11 \\ -2 & 0 & 2 \\ 19 & 6 & -7 \end{bmatrix}. \end{aligned}$$

For another useful method of pseudoinverse calculation, see [5, Theorem 1.3.1].

D.3 Pseudoinverse and Approximations

Suppose that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, that is, it has no exact solution. If the values of the coefficients are considered as approximate ones, then it is reasonable to find an approximate solution of the system, say, by the least-square method.

Recall from Sect. 7.3 that a vector $\mathbf{u} \in \mathbb{R}^n$ is called a *least square solution* of the system (D.1) if for every $\mathbf{x} \in \mathbb{R}^n$ we have $|A\mathbf{u} - \mathbf{b}| \le |A\mathbf{x} - \mathbf{b}|$. In Sect. 7.3, we have considered the case of full rank matrix A and shown that in this case the least square solution is unique. In the general case of arbitrary A, the solution is not necessarily unique. A least square solution $\mathbf{u}$ of the system (D.1) is called *minimal* if it has the smallest length among all others, that is, for any other least square solution $\mathbf{v}$ of the same system we have $|\mathbf{u}| \leq |\mathbf{v}|$.

Theorem D.6. *The minimal least-squares solution of smallest length of the linear system $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$-matrix, is unique and given by the formula*

$$\mathbf{u} = A^+\mathbf{b},$$

where A^+ is the pseudoinverse of A.

Lemma D.7. *For each $m \times n$-matrix A, $\operatorname{Im}(AA^+ - I) \perp \operatorname{Im} A$, where I is the corresponding identity matrix.*

Proof of Lemma D.7. For each matrix B, its kernel is the orthogonal complement to the span of its rows, that is, the span of columns of its transpose, $(\ker B)^\perp = \operatorname{Im} B^T$. Therefore, we have $\operatorname{Im}(AA^+ - I) = \left(\operatorname{Ker}(AA^+ - I)^T\right)^\perp = \left(\operatorname{Ker}\left((AA^+)^T - I^T\right)\right)^\perp = \left(\operatorname{Ker}(AA^+ - I)\right)^\perp$. So, it is sufficient to prove that $\operatorname{Im} A \subset \operatorname{Ker}(AA^+ - I)$. Indeed, let $\mathbf{y} = A\mathbf{x} \in \operatorname{Im} A$. Then

$$(AA^+ - I)\mathbf{y} = (AA^+ - I)A\mathbf{x} = (AA^+A - A)\mathbf{x} = \mathbf{0x} = \mathbf{0},$$

hence $\mathbf{y} \in \operatorname{Ker}(AA^+ - I)$. □

Proof of Theorem D.6. First, let us proof that the vector $\mathbf{u} = A^+\mathbf{b}$ is a least square solution. For each $\mathbf{x} \in \mathbb{R}^n$, we have $A\mathbf{x}-\mathbf{b} = (A\mathbf{x}-AA^+\mathbf{b})+(AA^+\mathbf{b}-\mathbf{b}) = \mathbf{c_x}+\mathbf{d}$, where $c_x = A\mathbf{x} - AA^+\mathbf{b} = A(\mathbf{x} - A^+\mathbf{b}) \in \operatorname{Im} A$ and $\mathbf{d} = (AA^+ - I)\mathbf{b} \in \operatorname{Im}(AA^+ - I)$. By Lemma D.7, we have $\mathbf{c_x} \perp \mathbf{d}$. By Pythagorean theorem,

$$|A\mathbf{x} - \mathbf{b}|^2 = |\mathbf{c_x} + \mathbf{d}|^2 = |\mathbf{c_x}|^2 + |\mathbf{d}|^2 \geq |\mathbf{d}|^2.$$

Put $\mathbf{x} = \mathbf{u}$. Then $\mathbf{c_x} = A\mathbf{u}-AA^+\mathbf{b} = \mathbf{0}$, so that the value of $|A\mathbf{x}-\mathbf{b}|$ has its minimal value $|\mathbf{d}| = |(AA^+ - I)\mathbf{b}|$. This means that $\mathbf{u}$ is a least square solution.

Now, let $\mathbf{x}$ be another least square solution of the same system $A\mathbf{x} = \mathbf{b}$. By the above, $\mathbf{c_x} = \mathbf{0}$, that is, $\mathbf{x}$ satisfies the linear system $A\mathbf{x}-AA^+\mathbf{b} = \mathbf{0}$, or $A\mathbf{x} = AA^+\mathbf{b}$. Since $\mathbf{u}$ form a solution of the last system, any other solution has the form

$$\mathbf{x} = \mathbf{u} + \mathbf{w},$$

where $\mathbf{w}$ is a solution of the corresponding homogeneous system $A\mathbf{w} = \mathbf{0}$. Then we have

$$(\mathbf{u}, \mathbf{w}) = \mathbf{u}^T\mathbf{w} = \mathbf{b}^T \left(A^+\right)^T \mathbf{w},$$

where $\left(A^+\right)^T = \left(A^+AA^+\right)^T = \left(A^+\right)^T \left(A^+A\right)^T = \left(A^+\right)^T A^+A$. Hence

$$(\mathbf{u}, \mathbf{w}) = \mathbf{b}^T \left(A^+\right)^T A^+A\mathbf{w} = 0,$$

that is, $\mathbf{u} \perp \mathbf{w}$. By Pythagorean theorem, it follows that

$$|\mathbf{x}|^2 = |\mathbf{u} + \mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{w}|^2 \geq |\mathbf{u}|^2.$$

Thus, $\mathbf{u}$ is a least square solution of the smallest possible length, and any other least square solution $\mathbf{x} = \mathbf{u} + \mathbf{w}$, where $\mathbf{w}$ is nonzero, has strictly larger length. □

Example D.7. Consider the following macroeconomic policy model

$$\begin{aligned} Y = C + I + G + X - M, C = 0.9Y_d, Y_d = Y - T, \\ T = 0.15Y, I = 0.25(Y - Y_{-1}) + 0.75G_I, G = G_C + G_I, \\ M = 0.02C + 0.08I + 0.06G_I + 0.03X, \\ N = 0.8Y, B = X - M, D = G - T, \end{aligned}$$

where Y – GDP, Y_{-1} – GDP in the previous year, Y_d – Disposable Income, C – Private Consumption, T – Tax Revenues, I – Private Investment, M – Imports, N – Employment, B – Current Account of the Balance of Payments, D – Budget Deficit, X – Exports, G_C – Public Consumption Expenditures (instrument), G_I – Public Investment Expenditures (instrument).

All variables, except N, are measured in \$ million. Employment (N) unit of measurement is 1,000 persons.

Suppose the following data is given: $Y_{-1} = \$1{,}200$ million, $X = \$320$ million.

The problem is the following.

i. Suppose that the government is interested in three targets: Employment (N), Balance of Payments (B) and Public Sector Deficit (D). Reduce the model into three equations by eliminating the 'irrelevant endogenous variables'.
ii. Does this model satisfy the Tinbergen's Theorem on the equality of the number of instruments and the number of targets? Why?
iii. Suppose that the government wants employment level (N) is 1,000. On the other hand the government is aware that the external borrowing limit of the country is 100 million, i.e., the country can not increase its current account deficit beyond this figure. Finally, government is eager to reduce public sector debt, and therefore aims at creating a budget surplus of \$120 million. In other words, the government's targets are as follows:

$$\begin{aligned} N &= 1{,}000, \\ B &= -100, \\ D &= -120. \end{aligned}$$

How should government determine the values of its instruments, i.e., government investment and government consumption, under these circumstances?

Partial Solution

Tinbergen's approach requires 'targets' to be given. Starting from this data and using the model, the solution gives the necessary magnitudes of the instruments to achieve the desired target levels.

i. The *reduced form* of the model looks like

$$\begin{cases} N = \beta_{10} + \beta_{11}G_I + \beta_{12}G_C, \\ B = \beta_{20} + \beta_{21}G_I + \beta_{22}G_C, \\ D = \beta_{30} + \beta_{31}G_I + \beta_{32}G_C, \end{cases}$$

or in matrix terms

$$\begin{bmatrix} N \\ B \\ D \end{bmatrix} = \begin{bmatrix} \beta_{10} \\ \beta_{20} \\ \beta_{30} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} \begin{bmatrix} G_I \\ G_C \end{bmatrix}.$$

Using the above equations and through substitution the system can be reduced to the following form

$$\begin{cases} N = 1355.665 + 64.236G_I + 39.409G_C, \\ B = 274.581 - 2.954G_I - 1.739G_C, \\ D = -254.187 - 11.044G_I - 6.389G_C, \end{cases}$$

or in matrix terms

$$\begin{bmatrix} N \\ B \\ D \end{bmatrix} = \begin{bmatrix} 1355.665 \\ 274.581 \\ -254.187 \end{bmatrix} + \begin{bmatrix} -3.161 & -1.861 \\ 64.236 & 39.409 \\ -11.044 & -6.389 \end{bmatrix} \begin{bmatrix} G_I \\ G_C \end{bmatrix}.$$

This may be re-written as

$$\mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{x}$$

with

$$\mathbf{y} = \begin{bmatrix} N \\ B \\ D \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1355.665 \\ 274.581 \\ -254.187 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} G_I \\ G_C \end{bmatrix},$$

or

$$A\mathbf{x} = \mathbf{c},$$

where

$$\mathbf{c} = \mathbf{b} - \mathbf{y} = \begin{bmatrix} 1355.665 - N \\ 274.581 - B \\ -254.187 - D \end{bmatrix}.$$

The problem is then to find $\mathbf{y}$, given $\mathbf{c}$.

ii. No, it does not. The number of instruments (=2) is less than the number of targets (=3). The matrix of coefficients

$$A = \begin{bmatrix} -3.161 & -1.861 \\ 64.236 & 39.409 \\ -11.044 & -6.389 \end{bmatrix}$$

is an 3×2 matrix and rank $A = 2$. Therefore it does not have an inverse.

iii. In [31, pp. 37–42] the problem of inequality of targets and instruments is discussed. Tinbergen [31, pp. 39–40] correctly points out that when the number of targets exceed the number of instruments, an inconsistency problem arises.

In this case, one can calculate the pseudoinverse of the matrix A, which always exists. Notice that the rank of A is 2, i.e., A has full column rank. Then by Theorem D.2 the matrix A has the pseudoinverse of the following form

$$A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} -0.261 & -0.288 & -1.532 \\ 0.451 & 0.470 & 2.498 \end{bmatrix}.$$

A least square approximate solution can be obtained by calculating

$$\mathbf{x}^* = A^+ \mathbf{c},$$

that is,

$$\mathbf{x}^* = \begin{bmatrix} -0.261 & -0.288 & -1.532 \\ 0.451 & 0.470 & 2.498 \end{bmatrix} \begin{bmatrix} 1355.665 - 1000 \\ 274.581 + 100 \\ -254.187 + 120 \end{bmatrix} = \begin{bmatrix} 4.701 \\ 1.231 \end{bmatrix}.$$

Thus $G_I = \$4.701$ million and $G_C = \$1.231$ million. These figures indicate that the government has to target extremely low levels for government expenditures.

D.4 Problems

1. Calculate $[1, 0]^+$.
2. Calculate

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}^+ .$$

3. Calculate $[3, 2, 1, 0]^+$.

4. Calculate
$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}^+ .$$
5. Calculate
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}^+ .$$
6.* Let A be an $m \times n$ matrix of rank r and let
$$K = \left[\frac{G}{0}\right]$$
be its canonical form (see Definition 2.4), where G is an upper $r \times n$ submatrix without zero rows and 0 denotes a zero submatrix. Let $i_1, \dots, i_r$ be the numbers of columns where the leading coefficients of the echelons appears, and let F be a submatrix of A formed by its columns $i_1, \dots, i_r$. Prove that
$$A = FG$$
and this is a full rank factorization of A.
7. Find a full rank factorization of the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$
8. Calculate
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^+ .$$
9. Let E_{ij} be an $n \times n$ matrix such that its element in i-th row and j-th column is unit and all other elements are zeroes. Find its full rank decomposition and pseudoinverse.
10. Prove the formulae:
   (a) $\mathrm{Im}(AA^+) = \mathrm{Im}(AA^T) = \mathrm{Im}\, A$.
   (b) $\mathrm{Ker}(AA^+) = \mathrm{Ker}(AA^T) = \mathrm{Ker}\, A^T$.
   (c) $\mathrm{Im}\, A^+ = \mathrm{Im}\, A^T$.
   (d) $\mathrm{Ker}\, A^+ = \mathrm{Ker}\, A^T$.
   (e) $\mathrm{Im}\, A^+ = (\mathrm{Ker}\, A)^\perp$.
   (f) $\mathrm{Ker}\, A^+ = (\mathrm{Im}\, A)^\perp$.

11. Find the solution of smallest length of the linear system

$$\begin{cases} 2x + 3y + 2z = 7, \\ 3x + 4y - z \ \ = 6. \end{cases}$$

12. Find the least-square solution of smallest length of the linear system

$$\begin{cases} x - 3y + t & = -1, \\ 2y - 3z & = -1, \\ x - 2y + z + t & = 0, \\ x - 2z + t & = 8. \end{cases}$$

Answers and Solutions

E

Chapter 1

1. (a) $\sqrt{137}$; (b) $\sqrt{113}$; (c) $2\sqrt{5}$; **4.** (a) $y = -2.25x + 1$; (b) $y = (8/7)x$; (c) $x = \sqrt{2}$. **8.** $x = 10/7$. **10.** (a) $x = 40/13, y = -15/13$; (b)$y = 0.6x - 3$; (c) no solutions. **11.** $z_1 = (a_{11}b_{11} + a_{12}b_{21})x_1 + (a_{11}b_{12} + a_{12}b_{22})x_2, z_2 = (a_{21}b_{11} + a_{22}b_{21})x_1 + (a_{21}b_{12} + a_{22}b_{22})x_2$.

Chapter 2

1. (a) $\mathbf{x} = (3, -1, -3, -3)$; (b) $\mathbf{x} = (-0.2, -0.4, -1)$. **4.** Yes. **7.** (a) 2; (b) 3. **11.** 2^n. **12.** For example, $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \mathbf{0}$. **13.** For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. **17.** *Hint.* Use the property (1-b) of matrix multiplication. **18.** For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

19. (a) $x = \begin{bmatrix} a \\ a \end{bmatrix}$ for $a \in \mathbb{R}$; (b) $y = \begin{bmatrix} 3b & b \end{bmatrix}$ for $b \in R$. **22.** $\begin{bmatrix} 1 & 16 & 0 \\ 0 & 10 & 0 \\ 0 & -180 & 5 \end{bmatrix}$.

23. (a) In AB, the i-th and j-th rows are interchanged as well; (b) c times j-th row of AB will be added to i-th row of AB; (c) i-th and j-th columns of AB are interchanged as well; (d) c times j-th column of AB will be added to i-th column of AB. **24.** *Hint.* Use Problem 23. **27.** $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where $bc = -a^2$. **28.** Either $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$, where $a^2 + bc = 1$, or $A = \pm I_2$.

F. Aleskerov et al., *Linear Algebra for Economists*, Springer Texts in Business and Economics, DOI 10.1007/978-3-642-20570-5,

29. $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, rank = 2. **30.** $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Chapter 3

1. (a) -2; (b) 0; (c) -1; (d) $-2b^3$; (e) $\sin(\alpha-\beta)$; (f) 0. **2.** (a) 2; (b) 30; (c) $abc + 2x^3 - (a+b+c)x^2$; (d) $\alpha^2+\beta^2+\zeta^2+1$; (e) $\sin(\beta-\zeta)+\sin(\zeta-\alpha)+\sin(\alpha-\beta)$. **3.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \sqrt{2} - \frac{3}{\sqrt{2}} \\ \sqrt{2} - \frac{3}{\sqrt{2}} \end{bmatrix}$ **4.** $f(A) = \mathbf{0}$. **7.** 1875. **8.** $(-1)^n d$. **9.** (a) 1 for $n = 4k$ and $n = 4k+1$, -1 for $n = 4k+2$ and $n = 4k+3$, where k is an integer; (b) $n+1$; c) $1 + (-1)^{n-1}2^n$. **10**. $\det X = 1$. *Solution.* We have

$$X^3 - I_n = (X - I_n)(X^2 + X + I_n) = (X - I_n)\mathbf{0} = \mathbf{0},$$

hence $X^3 = I_n$. Then $\det(X^3) = \det I_n = 1$, that is, $(\det X)^3 = 1$ and $\det X = 1$.

Chapter 4

1. (a) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$; (b) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$; (c) $\begin{bmatrix} -7/3 & 2 & -1/3 \\ 5/3 & -1 & -1/3 \\ -2 & 1 & 1 \end{bmatrix}$;

(d) $\frac{1}{9} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$; (e) $\begin{bmatrix} -1 & 1 & 16 & -9 \\ -8 & 7 & 125 & -70 \\ -10 & 9 & 160 & -90 \\ -1 & 1 & 18 & -10 \end{bmatrix}$. **2.** (a) $\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$;

(b) $\begin{bmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 \end{bmatrix}$; (c) $\frac{1}{n-1} \begin{bmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & 2-n & 1 & \dots & 1 \\ 1 & 1 & 2-n & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2-n \end{bmatrix}$.

3. (a) $\begin{bmatrix} -1 & a-10 \\ 2 & 7.5-0.5a \end{bmatrix}$; (b) $\begin{bmatrix} 10.5 & 6.5 \\ 11 & -7 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; (d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

7. (a) 3; (b) 3 for $\lambda = \pm 1$, 4 for $\lambda \neq \pm 1$. **9.** rank $\begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \leq 1$.

10. $\operatorname{rank}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 2.$

Chapter 5

1. (a) $(x_1, x_2, x_3, x_4) = (1, 1, -1, -1)$. (b) $x_1 = (x_3 - 9x_4 - 2)/11$, $x_2 = (-5x_3 + x_4 + 10)/11$, $x_3, x_4 \in \mathbb{R}$. **2.** $x_1 = 1 - x_2 - x_3$ for $\lambda = 1$, $x_1 = x_2 = x_3 = 1/(\lambda+2)$ for $\lambda \neq 1$. **3.** $x_1 = -53/208$, $x_2 = 59/16$, $x_3 = -29/16$, $x_4 = 27/13$. **4.** No solution. **5.** $(x_1, x_2, x_3, x_4) = (3, 0, -5, 11)$. **6.** $(x_1, x_2, x_3, x_4, x_5) = (3, -5, 4, -2, 1)$. **7.** $(x_1, x_2, x_3, x_4, x_5) = (1/2, -2, 3, 2/3, -1/5)$. **8.** $x_1 = (-15x_2 + x_4 - 6)/10$, $x_3 = (4x_4 + 1)/5$, $x_2, x_4 \in \mathbb{R}$. **9.** $f(x) = x^2 - 5x + 3$. **10.** $f(x) = 2x^3 - 5x^2 + 7$. **12.** No solution if $\alpha_1 + \beta_1 = 0$ and $\alpha_0 + \beta_0 \neq 0$; unique solution $p_i = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 + \beta_1}$, $q_i^d = q_i^s = \frac{\alpha_0 + \beta_0}{\alpha_1 + \beta_1}$ (can be obtained by Kramer's rule); infinitely many solutions $q_i^d = q_i^s = \alpha_0 - \alpha_1 p_i$, p_i is arbitrary if $\alpha_1 + \beta_1 = \alpha_0 + \beta_0 = 0$.

13. i. For 'normal' goods, one expects demand to increase (decline) as its price falls (increases). This explains the negativity of the coefficients of own prices of all three goods. If the coefficient of the price of another good is positive (negative) in its demand function, this implies these two goods are *substitutes (complements)*[1].

ii. Equate supply and demand for each good, and rearrange the equations and express the linear equation system in matrix form as

$$\begin{bmatrix} -0,25 & -0,02 & 0,01 \\ 0,01 & -0,34 & 0,01 \\ -0,03 & 0,02 & -0,16 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -40 \\ -54 \\ -35 \end{bmatrix}$$

The solution is

$$p_1 \approx 154.87$$
$$p_2 \approx 169.58$$
$$p_3 \approx 210.91$$

(The values are rounded)

iii. $p_1 \approx 169.44$, $p_2 \approx 193.92$, $p_3 \approx 223.72$ (the values are rounded).

14. i. Y, C, I, G, M, N and B are endogenous, G_C, G_I, Y_{-1}, X are exogenous.

ii. Yes, both are endogenous variables.

iii. No, because it is an endogenous variable. It can not be controlled by the policy maker.

[1]For a further discussion of substitutes and complements, see, for example, [29, pp. 57–58] or [32, pp. 111–112].

iv. No, Tinbergen's theorem asserts that the number of instruments should be equal to the number of targets. See (vi).
v. Through substitution one gets

$$B = -0.375G_I - 0.268G_C + 317,$$
$$N = 6.789G_I + 5.829G_C - 1874.$$

vi. Notice that the equations given in (v) can be expressed as

$$\mathbf{y} = \mathbf{b} + A\mathbf{x}$$

where $\mathbf{y}$ is the vector of target variables, $\mathbf{x}$ is the vector of instruments, A is the coefficients matrix and $\mathbf{b}$ is the vector of intercept terms that are fixed. A unique solution to the above system can be obtained, if A^{-1} exists. The matrix A has an inverse if it is a square matrix, i.e. the number of rows (which is equal to the number of targets) should be equal to the number of columns (which is equal to number of instruments). This condition is Tinbergen's theorem. Secondly, A should have full rank, i.e. target variables, as well as instruments, should not be linearly dependent, i.e. they must be different.

For the given values of target variables the solution of the above system is obtained by calculating

$$\mathbf{x} = A^{-1}(\mathbf{y} - \mathbf{b})$$

which gives $G_I \approx \$325.64$ billion and $G_C \approx \$131.02$ billion.

Chapter 6

1. No. **3.** $(\xi_1, \xi_2, \ldots, \xi_n)$. **4.** $(\xi_1, \xi_2 - \xi_1, \xi_3 - \xi_2, \ldots, \xi_n - \xi_{n-1})$. **5.** $\dim M_n = n^2$. All matrices with all zero entries but one entry equal to 1 ("matrix units") form a basis of M_n. **7.** Yes. **8.** Yes. **9.** E. g., $f : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto ax^3 + bx^2 + cx + d$. **10.** $n(n+1)/2$. **15.** $(1/3, 11/6, 7/6, 11/6)$. **16.** $(-7, 11, 3)$. **17.** $(-2, -1, -2, -7)$. **18.** (a) No. (b) No. (c) Yes for $c = 0$, no for all other c. **19.** Point ($\dim = 0$), line ($\dim = 1$), plane ($\dim = 2$), the space $\mathbb{R}^3$ itself ($\dim = 3$). **20.** The $n-1$ vectors $(1, 0, \ldots 0, 1), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1, 0)$ form a basis of $\mathcal{L}'$, so that $\dim \mathcal{L}' = n-1$. **21.** The dimension is equal to 3; some possible bases are $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ and $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_5\}$.

Chapter 7

1. (a) Yes. (b) No. **2.** (a) No. (b) No. **3.** $\mathbf{e}_1 = \mathbf{f}_1$, $\mathbf{e}_2 = \left(\frac{1-n}{n}, \ldots, \frac{1}{n}, \frac{1}{n}\right)$, $\mathbf{e}_3 = \left(0, \frac{2-n}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n}\right), \ldots,$ $\mathbf{e}_n = \left(0, \ldots, 0, -\frac{1}{2}, \frac{1}{2}\right)$. **6.** (a) E.g., $\mathbf{f}_3 = (2, 2, 1, 0)$, $\mathbf{f}_4 = (-2, 5, 6, 1)$. (b) E.g., $\mathbf{f}_3 = (1, -2, 1, 0)$, $\mathbf{f}_4 = (-25, -4, 17, 6)$. **7.** All 4 angles are equal to $\pi/3$. **10.** $\mathbf{e}_1 = \left(1/\sqrt{15}\right)(2,1,3,1)$, $\mathbf{e}_2 = \left(1/\sqrt{23}\right)(3,2,-3,-1)$, $\mathbf{e}_3 = \left(1/\sqrt{127}\right)(1,5,1,10)$. **11.** $2\sqrt{7}$. **12.** $\mathbf{y} = (2,1,1,3), \mathbf{x}-\mathbf{y} = (5,-5,-2,-1)$. **13.** (a) 5; (b) 2. **14.** $[1/3, 11/3]$. **15.** $p(x) = 10.13 + 0.091x$, $p(27) = 12.6$.

Chapter 8

2. (a) 34. (b) 4. **3.** (a) Yes. (b) Yes. (c) No. (d) No. (e) No. **5.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. **6.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
7. (a) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix}$. (b) Non-linear. (c) Non-linear. (d) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. **8.** $\begin{bmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -1 & 0 \end{bmatrix}$.
9. In the canonical basis: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, in the given basis: $\begin{bmatrix} 20/3 & -5/3 & 5 \\ -16/3 & 4/3 & -4 \\ 8 & -2 & 6 \end{bmatrix}$.
10. (a) $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 5 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} -2 & 0 & 1 & 0 \\ 1 & -4 & -8 & -7 \\ 1 & 4 & 6 & 4 \\ 1 & 3 & 4 & 7 \end{bmatrix}$. **11.** $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$. **12.** $\begin{bmatrix} 16 & 47 & -88 \\ 18 & 44 & -92 \\ 12 & 27 & -59 \end{bmatrix}$.
13. (a) $\begin{bmatrix} -70 & 17 \\ -243 & 59 \end{bmatrix}$; (b) $\begin{bmatrix} 23 & -29 \\ 27 & -34 \end{bmatrix}$; (c) $\begin{bmatrix} -10 & -9 \\ -1 & -1 \end{bmatrix}$.

Chapter 9

1. Basis: $\mathbf{e}_1 = (1,0)$. **2.** Basis: $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,0,1)$. **3.** (a) $\lambda_1 = 1$, $\mathbf{e}_1 = (1,0,0)$. (b) $\lambda_1 = 1$, $\mathbf{e}_1 = (1,0,0)$. **4.** $\lambda_1 = -2i$, $\mathbf{e}_1 = c(1,-i)$ and $\lambda_1 = 2i$, $\mathbf{e}_1 = c(1,i)$, where $c \in \mathbb{C}$. **8.** (a) $\lambda_1 = 1$, $\lambda_2 = 3$; (b) $\lambda_1 = \frac{1-\sqrt{5}}{2}$, $\lambda_2 = \frac{1+\sqrt{5}}{2}$. **9.** *Hint.* Put $B = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_n} \end{bmatrix}$. **10.** (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. (b) $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. **11.** The only eigenvalue is $\lambda = 0$; the eigenvectors are constant polynomials. **14.** (a) $\begin{bmatrix} 2 & 0 \\ -4 & 2 \end{bmatrix}$; (b) $2x_1^2 - 4x_1x_2 + 2x_2^2$, $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. **15.** Positive definite if $a > 0, b > 1/a, c > 0$; negative definite if $a < 0, b < 1/a, c < 0$.

Chapter 10

1. i. The total sales of firms of a sector to another firms of the same sector; ii. $d_1 = 600, d_2 = 230, d_3 = 250$; iii. $A = \begin{bmatrix} 0.2 & 0.05 & 0.15 \\ 1/11 & 4/11 & 7/55 \\ 1/4 & 1/6 & 1/6 \end{bmatrix} \approx \begin{bmatrix} 0.2 & 0.05 & 0.15 \\ 0.091 & 0.364 & 0.127 \\ 0.25 & 0.167 & 0.167 \end{bmatrix}$, $(I - A)^{-1} = \begin{bmatrix} 1.355 & 0.177 & 0.271 \\ 0.286 & 1.674 & 0.307 \\ 0.464 & 0.388 & 1.343 \end{bmatrix}$; iv. Yes (by Theorem 10.9, because $(I - A)^{-1} > 0$).

2. i. $A = \begin{bmatrix} \frac{20}{150} & \frac{40}{480} & \frac{10}{300} \\ \frac{30}{150} & \frac{200}{480} & \frac{100}{300} \\ \frac{20}{150} & \frac{60}{480} & \frac{50}{300} \end{bmatrix} \approx \begin{bmatrix} 0.133 & 0.083 & 0.033 \\ 0.2 & 0.417 & 0.333 \\ 0.133 & 0.125 & 0.167 \end{bmatrix}$, $(I - A)^{-1} \approx \begin{bmatrix} 1.220 & 0.202 & 0.130 \\ 0.580 & 1.971 & 0.812 \\ 0.282 & 0.328 & 1.342 \end{bmatrix}$.

ii. By (10.26), $\mathbf{p}^T = \mathbf{v}^T(I - A)^{-1} = (80/150, 180/480, 140/300)\,(I - A)^{-1} = [1, 1, 1]$.

iii. From the information given in flow of funds table, the price equation can be written by reading the columns of the table as

Outlay = payments made inputs+ wage + profit.

Since outlay is price times quantity, by dividing each side by corresponding output levels we can get the price equation for the Leontief model as

$$\mathbf{p} = \mathbf{p}A + \mathbf{w} + \pi, \tag{E.1}$$

where $\mathbf{p}$ is the prices vector as in (ii), $\mathbf{w}$ is the row vector of wage payments per unit of output, π is the row vector of profits per unit of output, and A is the input coefficients matrix as in (i). From (E.1) one gets

$$\mathbf{p}^T = (\mathbf{w} + \pi)(I - A)^{-1}. \tag{E.2}$$

In (iii) the question is to find the effect of a change in wage payments, on relative prices. We know that only $\mathbf{w}$ changes. So the new wage cost vector is $\mathbf{w}'$, substituting it to (E.2) we get

$$\mathbf{p}'^T = (\mathbf{w}' + \pi)(I - A)^{-1}. \tag{E.3}$$

From (E.2) and (E.3) we get

$$\mathbf{p}'^T - \mathbf{p} = (\mathbf{w}' - \mathbf{w})(I - A)^{-1}.$$

We have here $\mathbf{w}' = 1.2\mathbf{w}$, so that $\mathbf{p}'^T = \mathbf{p}^T + 0.2\mathbf{w}(I - A)^{-1} \approx [1.078, 1.085, 1.077]$. We see that the relative prices are changed.

3. i. From the original data the relative price vector (in terms of the price of the first commodity, p_j/p_1) can be computed as

$$\mathbf{p} = (1, 0.7754, 0.7537).$$

Notice that in this example $p_2/p_3 = 1.0288$.

When the technological progress takes place, all the coefficients in the second column of the above matrix declines by 10%.

Using the new matrix, we can calculate the relative price vector as

$$\mathbf{p} = (1, 0.7297, 0.7466).$$

Notice that in this example $p_2/p_3 = 0,9774$.

Notice that the technological progress led to a decline in the relative price of the manufacturing good, with respect to other two goods. On the other hand, relative price of the agricultural good increased with respect to other two goods.

ii. The Perron–Frobenius root of the matrix given in the table is 0.703 which gives the maximum rate of profit as 0.4225. After the technological change, the Perron–Frobenius root of the new matrix is 0.687, which corresponds to a higher rate of maximum profit 0.4556.

The finding indicates that, assuming competition which equalizes the rate of profit among sectors, an input saving technological progress in one sector, leads to an increase of the maximum rate of profit of the system as a whole.

4. (a) $\hat{\lambda} = 10, \hat{\mathbf{x}} = [1, 2]$; (b) $\hat{\lambda} = 0.961, \hat{\mathbf{x}} = [0.917, 0.398]$; (c) $\hat{\lambda} = 0.485, \hat{\mathbf{x}} = [0, 0.851, 0.526]$; (d) $\hat{\lambda} = 0.828, \hat{\mathbf{x}} = [0.247, 0.912, 0.448, 0.703]$.

5. productive matrices: b,c,d; irreducible matrices: a,b,c. **6.** (i) 18.9%; (ii) $[0.625, 0.539, 0.565]$; (iii) Let w be the wage rate, r the rate of profit, $\mathbf{a}_0$ the labor coefficients vector and $\mathbf{d}' = (1, 1, 1)$ the summation vector. Then the general expression for the wage rate can be derived as

$$w = \mathbf{d}'[I - (1 + r)A]^{-1}\mathbf{a}_0.$$

Substituting 10% for r one gets 5.379. **7.** Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ (price vector) and $\mathbf{a}_0 = (\mathbf{a}_{01}, \mathbf{a}_{02})$ (labor coefficients vector) with dimensions defined in accordance with the partition of the matrix A. Then it can be shown that (show this!) that

$$\begin{cases} \mathbf{p}_1 = (1 + r)\mathbf{p}_1 A_{11} + w\mathbf{a}_{01}, \\ \mathbf{p}_2 = (1 + r)[\mathbf{p}_1 A_{12} + \mathbf{p}_2 A_{22}] + w\mathbf{a}_{02}. \end{cases}$$

Here the first equation is sufficient to determine the maximum rate of profit. Therefore it is independent from the production technology of the second group of sectors (in technical terms, non-basics).

Chapter 11

6. (a) Square with vertices $(0,2), (-2,0), (0,-2), (2,0)$; (b) square with vertices $(2,2), (-2,2), (-2,-2), (2,-2)$. **7.** (a) Yes; (b) Yes; (c) Yes. **8.** (a) $H^{[+]}$; (b) $H^{[-]}$; (c) H. **10.** Not necessary. For example, the union of two lines $l_1 : y = 0$ and $l_2 : x = 0$ in $\mathbb{R}^2$ is not convex. **11.** *Hint.* The problem can be posed in the following way:

$$\max_{x_1, x_2} 75x_1 + 55x_2$$

under the conditions

$$\begin{cases} x_1 + x_2 \leq 100, \\ 7x_1 + 5x_2 \leq 1000, \\ 2x_1 + 3x_2 \leq 150, \\ x_1, x_2 \geq 0, \end{cases}$$

where x_1 acres of land should be allocated to wheat and x_2 acres of land should be allocated to barley.
Answer. $x_1 = 50, x_2 = 0$. Net revenue will be \$450.

12. *Hint.* Let X_1 be the amount invested in government bonds, X_2 the amount invested in auto company A, X_3 the amount invested in auto company B, X_4 the amount invested in textile company C, and X_5 the amount invested in textile company D.

Then the objective function is

$$\max 0.035X_1 + 0.055X_2 + 0.065X_3 + 0.06X_4 + 0.09X_5$$

with the constrains

$$\begin{cases} X_1 + X_2 + X_3 + X_4 + X_5 = 1000000, \\ X_2 + X_3 \leq 500000, \\ X_4 + X_5 \leq 500000, \\ X_1 - 0.35(X_2 + X_3) \geq 0, \\ X_5 - 0.65(X_4 + X_5) \leq 0 \end{cases}$$

and

$$X_1, X_2, X_3, X_4, X_5 \geq 0.$$

Answer. Projected return (value of the objective function): \$68,361, $X_1 = 129,630$, $X_2 = 0$, $X_3 = 370{,}370$, $X_4 = 175,000$, $X_5 = 325,000$. Thus the LP solution suggests that the portfolio manager should not invest in auto company A.

13. *Hint.* The problem can be formulated as follows.

The objective is to minimize risk under the conditions given above.

Objective function: $\min 4x + 9y$.

Fund availability constraint: $50x + 150y \leq 10{,}000{,}000$.

Required revenue constraint: Since the return in money market is 4%, one money market certificate will earns $0.04 \cdot 50 = 2$. The same calculation leads to $0.1 \cdot 150 = 15$ for the stock market certificate. Therefore the constraint can be written as

$$2x + 15y \geq 400{,}000.$$

Liquidity Constraint: $x \geq 90{,}000$.

The problem therefore can be formulated as

$$\min 4x + 9y$$

under the constrains

$$\begin{cases} 50x + 150y \leq 10{,}000{,}000, \\ 2x + 15y \geq 400{,}000, \\ x \geq 90{,}000, \\ x, y \geq 0. \end{cases}$$

Answer. The solution is $x = 90{,}000, y = 14{,}667$. The value of the objective function is 492,000.

14. *Hint.* This is a linear programming problem. It can be formulated as

$$\max W = C_1 + 1.1C_2$$

subject to

$$\begin{cases} X_1 \geq 0.1X_1 - 0.2X_2 - C_1, \\ X_2 \geq -0.3X_1 + 0.15X_2 - C_2, \\ 0.05X_1 + 0.07X_1 \leq 150, \\ C_1 \geq 1000, \\ C_2 \geq 440, \end{cases}$$

where C_i is the consumption of the output produced by sector i $(i = 1, 2)$, X_i is the output of sector i $(i = 1, 2)$.

Answer. $W = 1643.53$, $C1 = 11159.53, C2 = 440, X1 = 1522.83, X2 = 1055.12$.

15. *Hint.* All functions are linear. Therefore the problem can be formulated as a linear programming problem (using the given constraints and the assumption that all variables are non-negative). After making substitutions the problem can be formulated as

$$\max Y$$

subject to

$$
\begin{cases}
-4Y \leq -1200, \\
0.05C + 0.6Y \leq 300, \\
0.95C + 2.4Y \leq 1020, \\
-0.95C - 2.4Y \leq -920, \\
3.8Y \leq 1220, \\
-3.8Y \leq -1200, \\
C + 3Y \leq 1220, \\
-C - 3Y \leq 1200, \\
Y, C \geq 0,
\end{cases}
$$

where, in particular, $I = 4Y - 1200$.
Answer. $Y = 321.053, C = 157.341, I = 4Y - 1200 = 84.211$.

Appendix A

2. $1 - 1/n$. **5.** $n = 1, n \geq 5$. **8.** (a) $\begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$; (b) $\begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}$.

Appendix B

3. 0. **4.** $x = 0, 1, \dots, n-1$. **5.** $8a + 15b + 12c - 19d$. **6.** $abcd$. **7.** $(-1)^n$. **8.** $2n + 1$ **9.** $2n + 1$. **10.** $(a_0 + a_1 + \cdots + a_n)x^n$. **11.** $a_0(x - a_1)\dots(x - a_n)$. **12.** $(a - b + c + x)(a + b + c - x)(a + b - c + x)(a - b - c - x)$. **13.** $a_1 \dots a_n - a_3 \dots a_n - \cdots - a_2 \dots a_{i-1}a_{i+1} \dots a_n - \cdots - a_2 \dots a_{n-1}$. **14.** $n + 1$. **15.** $2^{n+1} - 1$. **16.** $\sum\limits_{j=1}^{n} a_j \prod\limits_{i \neq j} (x_i - a_i) + \sum\limits_{j=1}^{n} (x_j - a_j)$

Appendix C

1. i. **2.** $-(14/13)i$. **3.** -0.5. **4.** $x = 1.72 - 0.04i, y = -2/6 + 1.6i$. **5.** $x = 3, y = 4$. **6.** $z = -1 \pm 6i$. **7.** $z_1 = -4 - 4i, z_2 = 1 + 2i$. **8.** $z = -3 + 2i$. **9.** $z_1 = 1 + i, z_2 = 4 - 5i$. **10.** 1. **11.** 2^{4k} for $n = 8k$, $2^{4k}(1 - i)$ for $n = 8k + 1$, $-2^{4k+1}i$ for $n = 8k + 2$, $-2^{4k+1}(1 + i)$ for $n = 8k + 3$, -2^{4k+2} for $n = 8k + 4$, $2^{4k+2}(i - 1)$ for $n = 8k + 5$, $2^{4k+3}i$ for $n = 8k + 6$, $2^{4k+3}(1 + i)$ for $n = 8k + 7$. **12.** $z = \cos\left(\pi \frac{4k+1}{12}\right) + i \sin\left(\pi \frac{4k+1}{12}\right)$ for $k = 0, 1, \dots, 5$. **13.** $z = 4\left(\cos\left(\pi \frac{3k+1}{6}\right) + i \sin\left(\pi \frac{3k+1}{6}\right)\right)$ for $k = 0, 1, 2, 3$, that is, either $z = \pm 2\left(\sqrt{3} + i\right)$ or $z = \pm 2\left(-1 + \sqrt{3}i\right)$. **15.** $z = \sqrt[5]{5}e^i = \sqrt[5]{5}\,(\cos(1) + i \sin(1))$. **16.** 3. **17.** 2. **20.** ± 1. **23.** $1 + 2\pi k i, k \in \mathbb{Z}$. **24.** $\begin{bmatrix} -5 + i & -6 \\ 4 & 5 + i \end{bmatrix}$. **25.** $2n$.

Appendix D

1. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **2.** $\begin{bmatrix} 0 & \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \end{bmatrix}$. **3.** $\begin{bmatrix} \frac{3}{14} \\ \frac{1}{7} \\ \frac{1}{14} \\ 0 \end{bmatrix}$. **4.** $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 1 \end{bmatrix}$. **5.** $\begin{bmatrix} \frac{5}{6} & \frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{2}{3} \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. **8.** $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$ **9.** $\begin{bmatrix} 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 1_j & \dots & 0 \end{bmatrix}$. **11.** $\left[\frac{71}{93}, \frac{109}{93}, \frac{91}{93}\right] \approx$ $[0.763, 1.172, 0.978]$. **12.** $[3.7, 2.8, 1.2, 3.7]$.

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F. Aleskerov et al., *Linear Algebra for Economists*, Springer Texts in Business and Economics, DOI 10.1007/978-3-642-20570-5,