1. Pseudoinverse matrices. Skeletonization. Singular value decomposition (SVD)

Problem 1. Skeletonization. Pseudoinverse matrix

Find the pseudoinverse matrix to matrix A using skeletonization:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution: Let's start with skeletonization and then find the pseudoinverse matrix by the formula:

$$A^+ = G^*(G, G^*)^{-1}(F^*, F)^{-1}F^*$$

$$A = \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{matrix}}}^{F} & \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{matrix}}}^{G} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{bmatrix}$$

Let's check:

$$F \cdot G = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Correct. Now we need to find the pseudoinverse matrix:

$$G^*(G, G^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6}$$

Note

Matrix (G, G^*) is called Gramm Matrix and contains results of scalar products.

$$(F^*, F)^{-1}F^* = \left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{6} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

All things considered, we can obtain pseudoinverse matrix A^+:

$$\frac{1}{36} \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 13 & -4 & 0 \\ -4 & 4 & 0 \\ 5 & 4 & 0 \end{bmatrix}$$

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