1. Pseudoinverse matrices. Skeletonization. Singular value decomposition (SVD)

Problem 1. Skeletonization. Pseudoinverse matrix

Find the pseudoinverse matrix to matrix A using skeletonization:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution: Let's start with skeletonization and then find the pseudoinverse matrix by the formula:

$$A^+ = G^*(G, G^*)^{-1}(F^*, F)^{-1}F^*$$

$$A = \left[\overbrace{\boxed{\begin{matrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{matrix}}}^{F} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}\right] \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{matrix} \overbrace{\boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{matrix}}}^{G} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{matrix}\right]$$

Let's check:

$$F \cdot G = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Correct. Now we need to find the pseudoinverse matrix:

$$G^*(G, G^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6}$$

Note

Matrix (G, G^*) is called Gramm Matrix and contains results of scalar products.

$$(F^*, F)^{-1}F^* = \left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{6} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

All things considered, we can obtain pseudoinverse matrix A^+:

$$\frac{1}{36} \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 13 & -4 & 0 \\ -4 & 4 & 0 \\ 5 & 4 & 0 \end{bmatrix}$$

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Singular Value Decomposition of matrix $A \in M_{m\times n}(\mathbb{C})$ is a decomposition of a kind:

$$A = U\Sigma V^*,$$

where $U \in M_{m\times m}(\mathbb{C})$ and $V \in M_{n\times n}(\mathbb{C})$ are unitary matrices:

$$U^*U = UU^* = UU^{-1} = I$$
$$V^*V = VV^* = VV^{-1} = I$$

and $\Sigma \in M_{m\times n}(\mathbb{C})$ is a diagonal matrix of a kind:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_r \\ & 0 & & \end{bmatrix},$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ – singular values. ($r = \operatorname{rank} A$).

Problem 2. Singular Value Decomposition

Find singular value decomposition of a matrix:

$$A = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

Solution: Let's start with finding $V \in M_{2\times 2}(\mathbb{R})$ and $\Sigma \in M_{2\times 2}(\mathbb{R})$:

$$A^\top A = \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 25 & 25 \end{bmatrix}.$$

Next step is to find eigenvalues and eigenvectors for the obtained matrix.

$$\det\left(\begin{bmatrix} 25-\lambda & 25 \\ 25 & 25-\lambda \end{bmatrix}\right) = 0 \iff \lambda^2 - 50\lambda = 0.$$

We can obtain eigenvalues: $\lambda_1 = 50$; $\lambda_2 = 0$. Then singular values is equal to: $\sigma_1 = \sqrt{50}$, $\sigma_2 = 0$. Then:

$$\Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

After that we can find eigenvalues:

$$\begin{bmatrix} -25 & 25 \\ 25 & -25 \end{bmatrix} \vec{x} = 0$$

$\vec{x}_1$, for example, can be equal $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We need to normalize it, so $\vec{x}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\frac{\sqrt{2}}{2}$.

$$V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Let's find matrix U though:

$$AA^\top = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & 32 \end{bmatrix}$$

Repeating the same scenario:

$$\det\left(\begin{bmatrix} 18-\lambda & 0 & 24 \\ 0 & -\lambda & 0 \\ 24 & 0 & 32-\lambda \end{bmatrix}\right) \iff \lambda^3 - 50\lambda^2 = 0.$$

Then we can obtain singular values $\sigma_1 = \sqrt{50}, \sigma_2 = 0, \sigma_3 = 0.$

$$\begin{bmatrix} -32 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & -18 \end{bmatrix} \vec{x} = 0$$

Solving by Gaussian elimination:

$$\left[\begin{array}{ccc|c} -4 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} \frac{3}{4} \\ 0 \\ 1 \end{bmatrix} \cdot \frac{4}{5}$. Similarly,

$$\left[\begin{array}{ccc|c} 18 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 32 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \cdot \frac{3}{5}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

All things considered, the result of singular value decomposition:

$$U = \frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

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