

# 1. Pseudoinverse matrices. Skeletonization. Singular value decomposition (SVD)

## Problem 1. Skeletonization. Pseudoinverse matrix

Find the pseudoinverse matrix to matrix  $A$  using skeletonization:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution: Let's start with skeletonization and then find the pseudoinverse matrix by the formula:

$$A^+ = G^*(G, G^*)^{-1}(F^*, F)^{-1}F^*$$

$$A = \left[ \begin{array}{c|c} \overbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}}^F & \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \end{array} \right] \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \left[ \begin{array}{c|c} \overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}^G & \end{array} \right]$$

Let's check:

$$F \cdot G = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Correct. Now we need to find the pseudoinverse matrix:

$$G^*(G, G^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6}$$

### Note

Matrix  $(G, G^*)$  is called **Gramm Matrix** and contains results of scalar products.

$$\begin{aligned} (F^*, F)^{-1}F^* &= \left( \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \\ &= \frac{1}{6} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \end{aligned}$$

All things considered, we can obtain pseudoinverse matrix  $A^+$ :

$$\frac{1}{36} \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 13 & -4 & 0 \\ -4 & 4 & 0 \\ 5 & 4 & 0 \end{bmatrix}$$

**Singular Value Decomposition of matrix  $A \in M_{m \times n}(\mathbb{C})$  is a decomposition of a kind:**

$$A = U\Sigma V^*,$$

**where  $U \in M_{m \times m}(\mathbb{C})$  and  $V \in M_{n \times n}(\mathbb{C})$  are unitary matrices:**

$$U^*U = UU^* = UU^{-1} = I$$

$$V^*V = VV^* = VV^{-1} = I$$

**and  $\Sigma \in M_{m \times n}(\mathbb{C})$  is a diagonal matrix of a kind:**

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ & & & & 0 \end{bmatrix},$$

**where  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$  – singular values. ( $r = \text{rank } A$ ).**

## Problem 2. Singular Value Decomposition

**Find singular value decomposition of a matrix:**

$$A = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

**Solution: Let's start with finding  $V \in M_{2 \times 2}(\mathbb{R})$  and  $\Sigma \in M_{2 \times 2}(\mathbb{R})$ :**

$$A^T A = \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 25 & 25 \end{bmatrix}.$$

**Next step is to find eigenvalues and eigenvectors for the obtained matrix.**

$$\det \left( \begin{bmatrix} 25 - \lambda & 25 \\ 25 & 25 - \lambda \end{bmatrix} \right) = 0 \iff \lambda^2 - 50\lambda = 0.$$

**We can obtain eigenvalues:  $\lambda_1 = 50$ ;  $\lambda_2 = 0$ . Then singular values is equal to:  $\sigma_1 = \sqrt{50}$ ,  $\sigma_2 = 0$ . Then:**

$$\Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**After that we can find eigenvectors:**

$$\begin{bmatrix} -25 & 25 \\ 25 & -25 \end{bmatrix} \vec{x} = 0$$

**$\vec{x}_1$ , for example, can be equal  $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . We need to normalize it, so  $\vec{x}_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Similarly,  $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{\sqrt{2}}{2}$ .**

$$V = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Let's find matrix  $U$  though:

$$AA^T = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & 32 \end{bmatrix}$$

Repeating the same scenario:

$$\det \left( \begin{bmatrix} 18 - \lambda & 0 & 24 \\ 0 & -\lambda & 0 \\ 24 & 0 & 32 - \lambda \end{bmatrix} \right) \iff \lambda^3 - 50\lambda^2 = 0.$$

Then we can obtain singular values  $\sigma_1 = \sqrt{50}, \sigma_2 = 0, \sigma_3 = 0$ .

$$\begin{bmatrix} -32 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & -18 \end{bmatrix} \vec{x} = 0$$

Solving by Gaussian elimination:

$$\left[ \begin{array}{ccc|c} -4 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & 0 & -3 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} -1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} \frac{3}{4} \\ 0 \\ 1 \end{bmatrix} \cdot \frac{4}{5}. \text{ Similarly,}$$

$$\left[ \begin{array}{ccc|c} 18 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 32 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \cdot \frac{3}{5}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

All things considered, the result of singular value decomposition:

$$U = \frac{1}{5} \begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

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