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1. Sequence. Convergence of sequences.

Let $x : \mathbb{N} \to \mathbb{R}$. Then we can say that sequence was defined and there is a valid notation: $x(n) = x_n$.

Definition

Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ some sequence, we can say that it converge to $l \in \mathbb{R}$ (or $l = \lim\limits_{n\to\infty} x$), iff:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)\ |x_n - l| < \varepsilon$$

Example 1:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Longleftrightarrow$$
$$\Leftrightarrow (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N) : \left|\frac{1}{n}\right| < \varepsilon \Longleftrightarrow$$
$$\Leftrightarrow n > \frac{1}{\varepsilon},\ N = \left[\frac{1}{\varepsilon}\right] + 1.$$
$$(\forall \varepsilon > 0)\left(N = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}\right)(\forall n > N) n > N \longrightarrow$$
$$\to n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon.$$

Theorem

Numeric sequence can't have more than one limit.

Theorem: Properties of limit of consequence

Let $\{x_n\}_{n=1}^{\infty}$ some sequence. We can define some properties of it:

- if $\{x_n\}_{n=1}^{\infty}$ converges then $\{x_n\}_{n=1}^{\infty}$ is bounded;
- if $\lim\limits_{n\to\infty} x_n = l \neq 0$, then
$$(\exists N \in \mathbb{N})(\forall n > N)$$
$$(\operatorname{sgn}(x_n) = \operatorname{sgn}(l)) \wedge |x_n| > \frac{|l|}{2};$$
- if $\lim\limits_{n\to\infty} x_n = l_1,\ \lim\limits_{n\to\infty} y_n = l_2$:
$$(\forall n \in \mathbb{N})\ x_n \leq y_n \Rightarrow l_1 \leq l_2$$
- if $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = l$:
$$(\forall n \in \mathbb{N}) x_n \leq y_n \leq z_n$$

then $\lim\limits_{n\to\infty} y_n = 1$.

Theorem: Arithmetic operations with limits

If $\lim\limits_{n\to\infty} x_n = l_1,\ \lim\limits_{n\to\infty} y_n = l_2$, then:

- $x_n \pm y_n$ converges to $l_1 \pm l_2$;
- $x_n \cdot y_n$ converges to $l_1 \cdot l_2$;
- if in addition $y_n \neq 0$, then $(\forall n \in \mathbb{N})$, $l_2 \neq 0$, then $\frac{x_n}{y_n}$ converges to $\frac{l_1}{l_2}$.

Definition: Infinitesimal

Infinitesimal sequence is called sequence converged to zero.

Theorem

Product of an infinitesimal sequence and bounded one is infinitesimal.

Definition: I

finitely large sequence is called a sequence with infinite limit.

Theorem

$\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}\backslash\{0\}$ infinitesimal iff $\left\{\frac{1}{x_n}\right\}_{n=1}^{\infty}$ infinitely large.