R be a ring (commutative & has $1 \in R$)

* R is called a local ring if it has only one maximal ideal. $\mathfrak{m}$

$k := R/\mathfrak{m}$; the residue field of R.

eg $\mathbb{C}[[x]]$ or $\mathbb{C}[\![x]\!]$ Ring of formal power series

$$\mathbb{C}[\![x]\!] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \mathbb{C} \right\}$$

add^n: $$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

identity 0 power series ; $0 = \sum_{i=0}^{\infty} 0 x^i$

mult^n: $$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right)$$

$$= \sum_{k=0}^{\infty} c_k x^k$$

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} = \left(a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0\right)$$

aside: $(a_3x^3 + \underline{a_2x^2} + \underline{a_1x} + \underline{a_0})(b_3x^3 + \underline{b_2x^2} + \underline{b_1x} + \underline{b_0})$

$$= (a_2b_0 + a_1b_1 + a_0b_2)\,x^2 + (a_3b_0 + a_0b_3)\,x^3 + \cdots$$

$$1 = 1 + 0\cdot x + 0\cdot x^2 + \cdots$$

$\mathbb{C}[x]$ subring of $\mathbb{C}[[x]]$

$$\underbrace{a_nx^n + \cdots + a_1x + a_0} = a_0 + a_1x + \cdots + a_nx^n + 0\cdot x^{n-1} + \cdots$$

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① Claim: $(x) = \{x \cdot p(x) \mid p(x) \in \mathbb{C}[[x]]\}$ is the ONLY maximal ideal.

$$\mathrm{ev}_0 : \mathbb{C}[[x]] \longrightarrow \mathbb{C}$$
$$\sum_{i=0}^{\infty} a_i x^i \longmapsto a_0$$

Surjective: $z \in \mathbb{C}$, then $p(x) = z + 0\cdot x + 0\cdot x^2 + \cdots$

Then $\mathrm{ev}_0(p(x)) = z$

kernel: $\ker \mathrm{ev}_0 = (x)$

(i) $(x) \subseteq \ker \mathrm{ev}_0$

$x \cdot p(x) \in (x)$

$$ev_0(x \cdot p(x)) = ev_0(x \cdot (a_0 + a_1 x + a_2 x^2 + \cdots))$$

$$= ev_0(a_0 x + a_1 x^2 + a_2 x^3 + \cdots)$$

$$= 0 \qquad \Rightarrow \qquad x \cdot p(x) \in \ker ev_0$$

Hence $(x) \subseteq \ker ev_0$

(ii) $\ker ev_0 \subseteq (x)$

$q(x) \in \ker ev_0$

$$0 = ev_0(q(x)) = ev_0(\underbrace{a_0 + a_1 x + a_2 x^2 + \cdots}) = a_0$$

$$q(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= x(a_1 + a_2 x + a_3 x^2 + \cdots) \in (x)$$

Hence $\ker ev_0 \subseteq (x)$

Therefore $\ker ev_0 = (x)$

Hence by FIT

$$\mathbb{C}[\![x]\!]/(x) \cong \mathbb{C}, \text{ a field}$$

$m = (x)$ is maximal!

② If $p(x) \notin \mathfrak{m}$, then $p(x) \in \mathbb{C}[[x]]^\times$

$\underbrace{\exists\, q(x) \in \mathbb{C}[[x]] \text{ st } p(x) \cdot q(x) = 1}$ ☆

More precisely, $\underbrace{\mathbb{C}[[x]] \setminus \mathfrak{m} = \mathbb{C}[[x]]^\times}$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

$1 = p(x) \cdot q(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2$
$+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) x^3 + (a_0 b_4 + a_1 b_3 +$
$a_2 b_2 + a_3 b_1 + a_4 b_0) x^4 + \cdots$

$\to 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$

$1 = a_0 b_0 \Rightarrow b_0 = 1/a_0$

$0 = a_0 b_1 + a_1 b_0 \Rightarrow b_1 = (-a_1 b_0) b_0$

$0 = a_0 b_2 + a_1 b_1 + a_2 b_0 \Rightarrow b_2 = -(a_1 b_1 + a_2 b_0) b_0$

$0 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \Rightarrow b_3 = -(a_3 b_0 + a_2 b_1 + a_1 b_2) b_0$

$0 = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 = \vdots$

$p(x) \notin (x)$ i.e. $p(x) \neq x \cdot (f(x)) \Rightarrow \boxed{a_0 \neq 0}$

If $a_0 = 0$; $p(x) = a_1 x + a_2 x^2 + \cdots = x(a_1 + a_2 x + \cdots) \in (x)$ $\times$

$$a_0 b_0 = 1 \Rightarrow b_0 = 1/a_0$$

$k>0$ $$\underbrace{\sum_{i=0}^{k} a_i b_{k-i} = 0}\Rightarrow b_k = -b_0 \sum_{i=1}^{k} a_i b_{k-i}$$

$$0 = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0$$

$$\Rightarrow a_0 b_k = -(a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0)$$

$$\Rightarrow b_k = -\frac{1}{a_0}(a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0)$$

$$= -b_0 (a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0)$$

$$= -b_0 \sum_{i=1}^{k} a_i b_{k-i}$$

$p(x) \notin (x) = m$, then $p(x) \in \mathbb{C}[[x]]^\times$

$$\mathbb{C}[[x]] \setminus m = \mathbb{C}[[x]]^\times$$

Let n be another maximal ideal st $n \neq m$

$\exists\ r(x) \in n$ st $\underline{r(x) \notin m}$, then $\underset{s(x)}{r(x)} \in \mathbb{C}[[x]]^\times$

n a maximal ideal contains a unit ✗

$$r(x) \in n$$

$$1 = s(x) \cdot r(x) \in n$$

$$1 \in n$$

every element of the ring $\mathbb{C}[[x]]$ is in n

$$a(x) \in \mathbb{C}[[x]]$$

$$a(x) \cdot 1 \in n \Rightarrow \mathbb{C}[[x]] \subseteq n$$

$$n = \mathbb{C}[[x]]$$

I, $(I,+) \leq (R,+)$

& $x \in I$, $r \in R$ then $rx \in I$

So $n = m$. m is the unique maximal ideal.