Introduction to Yoneda lemma and its applications

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Definition of a locally small category

Definition A *locally small category* $\mathcal{C}$ is a mathematical structure consisting of

- a class of *objects*, denoted by $\mathrm{Ob}(\mathcal{C})$,
- for any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the set of all *morphisms* from X to Y, also denoted as $\mathcal{C}(X, Y)$
- and for any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$
$$(g, f) \mapsto g \circ f$$

called *composition*.

Definition of a category cont.

satisfying the following axioms:

(i) Associativity of composition: For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $h \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, one has

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(ii) For every $X \in \mathrm{Ob}(\mathcal{C})$ there exist a morphism id_X (called the *identity morphism* of X), with the property that for all $W, Y \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, one has

$$id_X \circ f = f \text{ and } g \circ id_X = g$$

Opposite category

Definition Let $\mathcal{C}$ be a category. Its *opposite* category $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$ and for objects X and Y of $\mathcal{C}^{op}$, one sets

$$Hom_{\mathcal{C}^{op}}(X, Y) := Hom_{\mathcal{C}}(Y, X).$$

A morphism $f : Y \to X$ is denoted by $f^{op} : X \to Y$, if considered in the category $\mathcal{C}^{op}$. The composition of morphism $g^{op} : W \to Y$ and $f^{op} : Y \to X$ in the category $\mathcal{C}^{op}$ is defined by

$$f^{op} \circ g^{op} := (g \circ f)^{op}$$

For objects W, X and Y of $\mathcal{C}^{op}$.

Examples of category

1. Category of sets (**Set**): Objects are the sets; morphism are the function between two sets; composition law is the usual composition of the functions.
2. Category of groups (**Gr**): Objects are the groups; morphism are the group homomorphism; composition law is the usual composition of the functions.
3. Category of rings (**Ri**): Objects are the rings; morphism are the ring homomorphism; composition law is the usual composition of the functions.

Definition of a functor

Definition A *functor* $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ consist of

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and
- for any two object C, C' of $\mathcal{C}$, a function

$$F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$$

such that

(i) for any two composable morphisms $f : C \to C'$ and $g : C' \to C''$ in $\mathcal{C}$, one has $F(g \circ f) = F(g) \circ F(f)$, and,

(ii) for any object C of $\mathcal{C}$, one has $F(id_C) = id_{F(C)}$.

Examples of functor

1. Let $\mathcal{C}$ be a category and let c be an object in $\mathcal{C}$. For an object x in $\mathcal{C}$ define $F_c(x) := Hom_{\mathcal{C}}(c, x)$, and for a morphism $f : x \to y$ in $\mathcal{C}$ define

$$F_c(f) := Hom_{\mathcal{C}}(c, x) \to Hom_{\mathcal{C}}(c, y), \quad g \mapsto f \circ g$$

The functor $F_c : \mathcal{C} \to \textbf{Set}$ is called the covariant functor *represented* by the object c.

$$\begin{tikzcd}
F_c : \mathcal{C} \arrow[r] & \textbf{Set} \\
x \arrow[r, mapsto] \arrow[d, "f"'] & Hom_{\mathcal{C}}(c, x) \arrow[d, "f_*"] \\
y \arrow[r, mapsto] & Hom_{\mathcal{C}}(c, y)
\end{tikzcd}$$

Remark: F_c also donate as $\mathcal{C}(c, -)$

Examples of functor cont.

2. Let $\mathcal{C}$ be a category and let c be an object in $\mathcal{C}$. For an object x in $\mathcal{C}$ define $F^c(x) := Hom_{\mathcal{C}}(x, c)$, and for a morphism $f \in Hom_{\mathcal{C}^{op}}(x, y)$, define
$$F^c(f) := Hom_{\mathcal{C}}(y, c) \to Hom_{\mathcal{C}}(x, c), \quad g \mapsto g \circ f$$
The functor $F^c : \mathcal{C}^{op} \to \mathbf{Set}$ is called the contravariant functor *represented* by the object c.

$$F^c : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & Hom_{\mathcal{C}}(x, c) \\ y \arrow[r, mapsto] & Hom_{\mathcal{C}}(y, c) \arrow[u, "f^*"'] \end{tikzcd}$$

Remark: F^c also donate as $\mathcal{C}(-, c)$

Definiton of the natural transformation

Definition Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between categories $\mathcal{C}$ and $\mathcal{D}$. A *natural transformation* between F and G is a family $\phi = (\phi_C)_{C \in Ob(\mathcal{C})}$ of morphisms $\phi_C \in Hom_{\mathcal{D}}(F(C), G(C))$ such that, for any morphism $f \in Hom_{\mathcal{C}}(C, C')$, the diagram

$$\begin{tikzcd} F(C) \arrow[r, "\phi_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\phi_{C'}"] & G(C') \end{tikzcd}$$

commutes, i.e., $G(f) \circ \phi_C = \phi_{C'} \circ F(f)$.

Example of natural transformation

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Show that $(id_{F(C)})_{C \in Ob(C)}$ is a natural transformation from F to F. It is called the identity natural transformation from F to F and it is denoted by $id_F : F \to F$.

Proof: For any $f \in Hom_{\mathcal{C}}(C, C')$, $C, C' \in Ob(\mathcal{C})$. The following diagram commute

$$\begin{tikzcd} F(C) \arrow[r, "id_{F(C)}"] \arrow[d, "F(f)"'] & F(C) \arrow[d, "F(f)"] \\ F(C') \arrow[r, "id_{F(C')}"] & F(C') \end{tikzcd}$$

since $F(f) \circ id_{F(C)} = F(f) = id_{F(C')} \circ F(f)$. ■

Yoneda lemma

Theorem For any functor $F : \mathcal{C} \longrightarrow \mathbf{Set}$, whose domain $\mathcal{C}$ is locally small and any object $c \in \mathcal{C}$, there is a bijection

$$Hom(\mathcal{C}(c, -), F) \cong Fc$$

that associates a natural transformation $\alpha : \mathcal{C}(c, -) \Longrightarrow F$ to the element $\alpha_c(1_c) \in Fc$. Moreover, this correspondence is natural in both c and F.

Category of **Mat**

The category of **Mat** has

- Non-negative integers $0, 1, \ldots, k, m, n, \ldots$ as objects.
- Matrices as morphisms: An arrow(morphism) $m \leftarrow n$ is an $m \times n$ matrix A
- The composition is defined by matrix multiplication. (e.g.The composite of a $k \times m$ and a $m \times n$ matrix defined by $k \times n$ matrix.)

$$\begin{tikzcd} & m \arrow[dl, "B"'] & \\ k & & n \arrow[ul, "A"'] \arrow[ll, dashed, "B*A"] \end{tikzcd}$$

Category of **Mat** Cont.

- The identity arrow $n \xleftarrow{I_n} n$ is given by the identity matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

  For any arrow $m \leftarrow n$

$$I_m(m \leftarrow n) = (m \leftarrow n)I_n = (m \leftarrow n)$$

- Matrix multiplication is an associative operation

K-column functor in Mat

The K-column functor h_k : **Mat** $\longrightarrow$ **Set** is defined by

- For all $n \in \mathbb{N}_0$, there exist $h_k(n)$ such that is a set of matrices have k columes and n rows
$$h_k(n) = \{n \times k \text{ matrices}\} = \{n \xleftarrow{C} k\}$$
- For a matrix $m \xleftarrow{A} n$ the function $h_k(m) \xleftarrow{A\times -} h_k(n)$ is given by left multiplication

$$\left(m \xleftarrow{AC} k\right) = \left(m \xleftarrow{A} n\right) \times \left(n \xleftarrow{C} k\right)$$

Remark: One can easily verify this is a functor from **Mat** to **Set** by using the definition of functor we introduced earlier.

Natural transformation between column functors

A natural transformation $h_k \xrightarrow{\alpha} h_j$ is given by a family of function $h_k(n) \xrightarrow{\alpha_n} h_j(n)$ for each $n \in \mathbb{N}_0$ so that for each matrix $m \xleftarrow{A} n$ the following diagram commutes:

$$\begin{tikzcd} h_k(n) \arrow[r, "\alpha_n"] \arrow[d, "A\times -"'] & h_j(n) \arrow[d, "A\times -"] \\ h_k(m) \arrow[r, "\alpha_m"'] & h_j(m) \end{tikzcd}$$

In other words, α is a naturally-defined operation on column functors.

Projection operation on column functors

Here is a natural transformation $h_k \xrightarrow{\pi} h_{k-1}$ that delete the k^{th} column. We can verify the naturality for matrix $m \xleftarrow{A} n$. Observe the following square commute:

$$\begin{tikzcd}
m \xleftarrow{(AC_1|AC_2|\dots|AC_k)} k \arrow[d, mapsto] & & n \xleftarrow{(C_1|C_2|\dots|C_k)} k \arrow[ll, mapsto] \arrow[d, mapsto] \\
m \xleftarrow{(AC_1|AC_2|\dots|AC_{k-1})} k-1 = m \xleftarrow{A(C_1|C_2|\dots|C_{k-1})} k-1 & & n \xleftarrow{(C_1|C_2|\dots|C_{k-1})} k-1 \arrow[ll, mapsto]
\end{tikzcd}$$

Unnatural column operation

The operation $h_k \xrightarrow{f} h_{k+1}$ that appends a column of ones is not natural. For matrix $m \xleftarrow{A} n$. Observe the following square is not commute:

$$
\begin{array}{ccc}
 & m \xleftarrow{(AC_1|AC_2|\dots|AC_k)} k & \longleftarrow\!\shortmid & n \xleftarrow{(C_1|C_2|\dots|C_k)} k \\
 & \downarrow & & \downarrow \\
m \xleftarrow{(AC_1|AC_2|\dots|AC_k|1)} k+1 \neq & m \xleftarrow{A(C_1|C_2|\dots|C_k|1)} k+1 & \longleftarrow\!\shortmid & n \xleftarrow{(C_1|C_2|\dots|C_k|1)} k+1
\end{array}
$$

Challenge

Challenge: Can we classify all naturally-defined column operations that transform matrices with k columns into matrices with j columes?
Answer: Yes, by Yoneda lemma!

Yoneda lemma in Mat

1. Every naturally defined column operation $h_k \xrightarrow{\alpha} h_j$ is uniquely determined by a single $k \times j$ matrix.
2. This $k \times j$ matrix, $k \xleftarrow{\alpha_k(I_k)} j$ is obtained by applying the column operation $h_k(k) \xrightarrow{\alpha_k} h_j(k)$ to the identity matrix $k \xleftarrow{I_k} k$
3. The column operation $h_k(n) \xrightarrow{\alpha_n} h_j(n)$ is then defined by right multiplication by matrix $k \xleftarrow{\alpha_k(I_k)} j$

$$\alpha_n(C) := \left(n \xleftarrow{C} k\right) \times \left(k \xleftarrow{\alpha_k(I_k)} j\right)$$

Permutation operations on column functors

The operation $h_k \xrightarrow{\sigma} h_k$ that swaps the first two columns is a natural column operation. And it is defined by right multiplication by the matrix

$$\sigma_k(I_k) = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Scalar operations on column functors

The operation $h_k \xrightarrow{\alpha} h_k$ that multiplies the first column by a scalar is a natural column operation. And it is defined by right multiplication by the matrix

$$\alpha_k(I_k) = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Addition operations on column functors

The operation $h_k \xrightarrow{\mu} h_k$ that adds the second column to the first column is a natural column operation. And it is defined by right multiplication by the matrix

$$\mu_k(I_k) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Conclusion

Conclusion: Yoneda lemma tells us: Every naturally-defined column operation $h_k \xrightarrow{\alpha} h_j$ is given by right multiplication by the matrix $k \xleftarrow{\alpha_k(I_k)} j$ obtained by applying the column operation α_k to the identity matrix $k \xleftarrow{I_k} k$.

Reference:

i Reading material: Math 200 lecture note written by Prof. Robert Boltje

ii Book: *Category Theory In Context* by Emily Riehl

iii ACT 2020 Tutorial: The Yoneda lemma in the category of matrices (by Emily Riehl) Link

iv Beamer template: Link

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