LECTURE NOTES

CATEGORY THEORY LEARNING SEMINAR
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Contents

1. Lecture 1 (1/07) by Deewang

Definition 1.1. A category $\mathscr{C}$ consists of

- collection of *objects*, denoted $\mathrm{obj}(\mathscr{C})$; and
- a collection of *morphisms* (also called *arrows* or *maps*)

such that

- Each morphism has specified *source* (or *domain*) and *target* (or *domain*) objects. For a morphism f we will sometimes denote the source as $s(f)$ and target as $t(f)$. The notation

  $$f : X \to Y$$

  tells us that f is a morphism between the source $s(f) = X$ and target $t(f) = Y$.

  The collection of morphisms from X to Y is denoted $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ or $\mathrm{Mor}_{\mathscr{C}}(X, Y)$ or $\mathscr{C}(X, Y)$. **We will assume that these are sets; that is, all our categories will be *locally small*.**

- For each object X, there exists an *identity morphism* $1_X : X \to X$.
- For any pair of morphisms f, g with $t(f) = s(g)$, there exists a morphism, the *composite morphism* gf with $s(gf) = s(f)$ and $t(gf) = t(g)$.

  That is, there's a binary operation

  $$\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z), \quad (f, g) \mapsto gf$$

This data is subject to the following axioms:

- For any morphism $f : X \to Y$, we have

  $$1_Y f = f \quad \text{and} \quad f 1_X = f$$

- For composable morphisms f, g and h, we have

  $$(fg)h = f(gh)$$

Example 1.2. Concrete examples.

Name	**Objects**	**Morphisms**
Set	Sets	Functions
Grp	Groups	Group homomorphisms
Top	Topological Spaces	Continuous Functions
$\mathsf{Top}_{\mathsf{Open}}$	Topological Spaces	Open Functions
Rng	Rings	Ring Homomorphisms
Ring	Unital Rings	Unital Ring Homomorphisms

Name	Objects	Morphisms
Mod_A	A-modules	Module homomorphisms
$\mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$	Abelian Groups	Group homomorphisms
$\mathsf{Vec}_k = \mathsf{Mod}_k$ (k a field)	Vector spaces	Linear transformations
G-Set	G-sets	G-equivariant maps
Set_*	Pointed Sets (X, x_0) where $x_0 \in X$ is called the basepoint	Basepoint preserving functions; that is, functions $f : X \to Y$ such that $f(x_0) = y_0$
Top_*	Pointed Topological Spaces	Basepoint preserving continuous functions
SmMan	Smooth Manifolds	Smooth Maps
Meas	Measurable Spaces	Measurable Functions

Example 1.3. Where morphisms are not maps but equivalence classes of maps.

Name	Objects	Morphisms
HTop	Topological Spaces	Homotopy classes of continuous functions
$\mathsf{Measure}$	Measurable Spaces	Equivalence classes of measurable functions where the set where they differ has measure zero

Example 1.4. Where morphisms are not maps, or objects are not sets.

Name	Objects	Morphisms
Mat_A (A a ring)	Positive Integers	$\mathrm{Hom}(n, m) := \mathrm{Mat}_{m\times n}(A)$. Composition is given by matrix multiplication, and the identity morphism is the identity matrix.
$(\mathsf{P}, \leqslant)$ a poset	Elements of P	$\mathrm{Hom}_{\mathsf{P}}(x, y) = \begin{cases} \{x \to y\} & \text{if } x \leqslant y \\ \varnothing & \text{otherwise} \end{cases}$ Composition is given by transitivity of the relation, and the identity morphism is given by reflexivity.
$\mathsf{B}G$ G a group	$\bullet$ unique (dummy) object	$\mathrm{Hom}_{\mathsf{B}G}(\bullet, \bullet) := G$ Composition is given by the group multiplication, and the identity morphism is given by the group identity element.

1.1. Problems

Problem 1.1. Come up with an example different from the ones given above.

Problem 1.2. Verify that the example in Example 1.4 are indeed categories.

Problem 1.3 (Slice Categories)**.** Let $\mathscr{C}$ be a category and fix an object X, we define the *slice category of $\mathscr{C}$ under X* denoted $X/\mathscr{C}$ as follows

- Objects of $X/\mathscr{C}$ are morphisms $a_Y : X \to Y$ with source X, we usually depict them as

$$\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd}$$

- Morphism between $a_Y : X \to Y$ and $a_Z : X \to Z$ is defined to be a morphism $f : Y \to Z$ (in $\mathscr{C}$) such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "a_Y"'] \arrow[dr, "a_Z"] & \\ Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

  commutes; that is, $a_Z = f a_Y$.

Verify $X/\mathscr{C}$ is indeed a category. Can you describe what would be the *slice category of $\mathscr{C}$ over X*, denoted as $\mathscr{C}/X$.

Remark 1.5. Set_* can be realised as $*/\mathrm{Set}$, where $*$ denotes a singleton.

2. Lecture 2 (1/14) by Vaibhav

Discussion 2.1 (A short historical note)**.** The need for the language of category theory was first realised by Samuel Eilenberg and Saunders MacLane when they discovered a curious connection between purely algebraic and topological objects.

MacLane	Eilenberg
was studying *group extensions*.	was studying *solenoids*.
Given groups G and H, a group E is a *group extension of G by H* if $H \cong E/G$	A solenoid, loosely, is a collection $(S_i, f_i)_{i \in \mathbb{Z}_{\geqslant 0}}$ where S_i are circles and $f_i : S_{i+1} \to S_i$ is the map that wraps S_{i+1} around S_i, n_i times ($n_i \in \mathbb{Z}_{\geqslant 2}$).
There's a group $\mathrm{Ext}(G, H)$ that classifies extensions, up to isomorphism.	Given a solenoid $\Sigma \subset S^3$, one studies continuous functions such that $f(S^3 - \Sigma) \subset S^3$. These are classified, up to homotopy, by the homology group $H^1(S^3 - \Sigma, \mathbb{Z})$.

Eilenberg and MacLane discovered that the group $H^1(S^3 - \Sigma, \mathbb{Z})$, that arises topologically, is isomorphic to the group $\mathrm{Ext}(\mathbb{Z}, \Sigma^*)$, which is a purely algebraic object (here Σ^* is an appropriately chosen group called the *character group of the solenoid* Σ). This discovery was detailed in their 1942 paper *Group Extensions and Homology*.

In language we haven't seen yet, but will soon, Ext and H^1 are *functors* (Definition 2.14) and the above isomorphism is not just an isomorphism of groups but a *natural isomorphism of functors* (Definition 3.3), as was noted by Eilenberg and MacLane. To make sense of this rigorously they had to create some new language, which they did in their 1945 paper *General Theory of Natural Equivalences*. With that, category theory entered the mathematical landscape.

Definition 2.2. Given a category $\mathscr{C}$, we define the notion of a subcategory $\mathscr{D}$.

- The objects of $\mathscr{D}$ is a sub-collection of the objects of $\mathscr{C}$.
- The morphisms in $\mathscr{D}$ are a sub-collection of the morphisms in $\mathscr{C}$, and includes the identity morphisms for each object of $\mathscr{D}$.
- The morphisms are closed under composition, that is, if f and g are two composable morphisms in $\mathscr{D}$, then gf is also a morphism in $\mathscr{D}$.

Example 2.3. Subcategories of some familiar categories

Category $\mathscr{C}$	**Subcategory $\mathscr{D}$**	**Objects of $\mathscr{D}$**	**Morphisms of $\mathscr{D}$**
Set	FinSets	Finite Sets	Functions

Set	InjSet	Sets	Injective Functions
Grp	Ab	Abelian Groups	(Group) Homomorphisms
Grp	FinGrp	Finite Groups	Homomorphisms
Grp	$\mathsf{SurjGrp}$	Groups	Surjective Homomorphisms
Mod_A	mod_A	Finitely Generated Modules	Homomorphisms
$\mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$	$\mathsf{FinAb} = \mathsf{mod}_{\mathbb{Z}}$	Finitely Generated Abelian Groups	Homomorphisms
$\mathsf{Vec}_k = \mathsf{Mod}_k$ k a field	$\mathsf{fdVec}_k = \mathsf{mod}_k$	Finite Dimensional Vector Spaces	Linear Transformations
Top	$\mathsf{Top}_{\mathsf{Open}}$	Topological Spaces	Open Functions
Top	CW	CW Complexes	Cellular Maps
Rng	CRing	Commutative Unital Rings	Unital Ring Homomorphisms
SmMan	Subm	Smooth Manifolds	Submersions
$\mathsf{B}G$	$\mathsf{B}H$	same object $\bullet$ as $\mathsf{B}G$	$H \leqslant G$, a subgroup

Definition 2.4 (Opposite Category)**.** Let $\mathscr{C}$ be any category, then we define $\mathscr{C}^{\text{op}}$, the *opposite category of* $\mathscr{C}$.

- $\text{obj}(\mathscr{C}^{\text{op}}) = \text{obj}(\mathscr{C})$
- For any two objects X and Y,

$$\text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y) := \text{Hom}_{\mathscr{C}}(Y, X)$$

Therefore a morphism $f^{\text{op}} \in \text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y)$ written formally as

$$f^{\text{op}} : X \to Y$$

is a morphism $f : Y \to X$ in $\mathscr{C}$.

Equivalently, a morphism $f : Y \to X$ in $\mathscr{C}$ *corresponds to a morphism* $f^{op} : X \to Y$ in $\mathscr{C}^{\text{op}}$.

We relegate proving this indeed gives a category to Problem 2.1.

Remark 2.5. Any time we prove or define something, we're really proving or defining two things simultaneously: in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$. This is the principle of duality, and we will encounter this again and again. An example of this phenomenon arises in the next definition (see Problem 2.3).

Definition 2.6 (Some special types of morphism)**.** A morphism $f : X \to Y$ in a category $\mathscr{C}$ is called

- an *isomorphism* if there exists a morphism $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, bend right, "\mathrm{id}_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

  that is, the above diagrams commute. An isomorphisms is also called an *invertible morphism*, with g above being denoted as f^{-1} and called as its *inverse*.

  If $f : X \to Y$ is an isomorphism, we write $X \cong Y$.

  Example 2.7.

Categories $\mathscr{C}$	**Isomorphisms in $\mathscr{C}$**
Set	Bijections
Grp & Ab	Group Isomorphisms
Mod_A	Module Isomorphisms
Vec_k	Linear Isomorphisms
Rng	Ring Isomorphisms

Categories $\mathscr{C}$	**Isomorphisms in $\mathscr{C}$**
Top	Homeomorphisms
HTop	Homotopy Equivalences
SmMan	Diffeomorphisms
Mat_A	Invertible Matrices
$(\mathsf{P}, \leqslant)$	Equality

- a *monomorphism* if there exist morphisms $g_1, g_2 : W \rightrightarrows X$

$$\begin{tikzcd} W \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

  such that if $fg_1 = fg_2$, then $g_1 = g_2$.

- an *epimorphism* if there exist morphisms $h_1, h_2 : Y \rightrightarrows Z$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z \end{tikzcd}$$

  such that if $h_1 f = h_2 f$, then $h_1 = h_2$.

Remark 2.8. Monomorphisms, epimorphisms and isomorphisms are categorical generalisations of an injective, a surjective and a bijective map (see Problems 2.6 and 2.7). But it's important to note that in general, morphisms that are both a monomorphism and an epimorphism are *not* isomorphisms. See Problem 2.6 (c) for an example, and Problem 2.7 for a more satisfactory conclusion.

Definition 2.9. A category $\mathscr{C}$ is called a *groupoid* if every morphism is an isomorphism.

Example 2.10. Last time we saw how a group G gives rise to a category with a single object that we called $\mathsf{B}G$. Since every group element has an inverse, this category has the property that all its morphisms are invertible. The notion of a groupoid captures this notion more generally.

In fact, in this manner *a group is a groupoid with one object*.

To a group G, we can associate the groupoid $\mathsf{B}G$. Conversely, given a groupoid $\mathscr{G}$ with one object $\bullet$, we recover the group as $\mathrm{Hom}_{\mathscr{G}}(\bullet, \bullet)$.

Example 2.11 (Maximal Groupoid)**.** Any category $\mathscr{C}$ has a subcategory called the *maximal groupoid*, $\mathscr{C}^{\cong}$, where any subcategory of $\mathscr{C}$ that's a groupoid is then a subcategory of $\mathscr{C}^{\cong}$.

- $\mathrm{obj}(\mathscr{C}^{\cong}) = \mathrm{obj}(\mathscr{C})$;
- $\mathrm{Hom}_{\mathscr{C}^{\cong}}(X, Y) = \{f \in \mathrm{Hom}_{\mathscr{C}}(X, Y) \ : \ f \text{ is an isomorphism}\}$.

This is a subcategory since the composition of two isomorphisms is again an isomorphism.

Remark 2.12. Given an object X in a category $\mathscr{C}$, the *automorphism group of* X is defined to be

$$\mathrm{Aut}_{\mathscr{C}}(X) := \mathrm{Hom}_{\mathscr{C}^{\cong}}(X, X)$$

It's indeed a group with respect to composition.

Example 2.13 (Fundamental Groupoid)**.** Given a topological space X, we have an associated groupoid $\Pi(X)$ called the *fundamental groupoid*.

- The objects of $\Pi(X)$ are points of X;
- For points x and y, the morphism from x to y are homotopy classes of paths from x to y, that is

$$\mathrm{Hom}_{\Pi(X)}(x, y) := \{\gamma : [0,1] \to X \ : \ \gamma \text{ is continuous, and } \gamma(0) = x, \ \gamma(1) = y\}/\text{homotopy}$$

This is a groupoid, since given a path $\gamma : x \to y$ we can always create an inverse path $\gamma^{-1} : y \to x$ given as $\gamma^{-1}(t) = \gamma(1-t)$.

For a point $x_0 \in X$, *the fundamental group of X at basepoint x_0* is[1]

$$\pi_1(X, x_0) := \mathrm{Aut}_{\Pi(X)}(x_0)$$

Our guiding principle is that morpshisms take precedence, and whenever we have objects we should give a notion of morphisms. So, if were to give a category of all categories CAT, what is the right notion of a morphisms between categories?

Definition 2.14 (Functors)**.** Given categories $\mathscr{C}$ and $\mathscr{D}$, a *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

consists of,

- for every object X of $\mathscr{C}$, an object $F(X)$ of $\mathscr{D}$.
- for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$

$$\begin{tikzcd} X \arrow[d, "f"] & \mapsto & F(X) \arrow[d, "F(f)"] \\ Y & & F(Y) \end{tikzcd}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object X of $\mathscr{C}$
- Given morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, we have $F(gf) = F(g)F(f)$;

$$\begin{tikzcd} & F(Y) \arrow[dr, "F(g)"] & \\ F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(gf)"'] & & F(Z) \end{tikzcd}$$

  that is, the above diagram commutes.

Sometimes the functors defined previously are called *covariant functors* to distinguish them from the functors we now define below

Definition 2.15 (Contravariant Functors)**.** Given categories $\mathscr{C}$ and $\mathscr{D}$, a *contravriant functor*

$$F : \mathscr{C} \to \mathscr{D}$$

consists of,

- for every object X of $\mathscr{C}$, an object $F(X)$ of $\mathscr{D}$.
- for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$

$$\begin{tikzcd} X \arrow[d, "f"] & \mapsto & F(X) \\ Y & & F(Y) \arrow[u, "F(f)"] \end{tikzcd}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object X of $\mathscr{C}$
- Given morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, we have $F(gf) = F(f)F(g)$;

$$\begin{tikzcd} & F(Y) \arrow[dl, "F(f)"'] & \\ F(X) & & F(Z) \arrow[ul, "F(g)"'] \arrow[ll, "F(gf)"] \end{tikzcd}$$

  that is, the above diagram commutes.

Remark 2.16. The notion of a contravariant functor is not a new concept. Problem 2.11 tells us that it's simply a (covariant) functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Example 2.17.

Functors	**What they do to Objects**	**What they do to Morphisms**
$\mathscr{P}_* : \mathsf{Set} \to \mathsf{Set}$ *direct image fuctor*	$A \mapsto \mathscr{P}(A)$ Power set of A	$A \xrightarrow{f} B$ $\downarrow$ $\mathscr{P}(A) \to \mathscr{P}(B) : S \mapsto f(S)$
$\mathscr{P}^* : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ *inverse image fuctor*	$A \mapsto \mathscr{P}(A)$ Power set of A	$A \xrightarrow{f} B$ $\downarrow$ $\mathscr{P}(B) \to \mathscr{P}(A) : T \mapsto f^{-1}(T)$
$U : \mathscr{C} \to \mathsf{Set}$ $\mathscr{C} = \mathsf{Mod}_A, \mathsf{Grp},$ $\mathsf{Rng}, \mathsf{Top}, \ldots$ *forgetful fuctor*	$A \mapsto U(A)$ the underlying set of A, "forget its additional structure"	$A \xrightarrow{f} B$ $\downarrow$ $U(f) : U(A) \to U(B)$ the underlying function f, "forget its additional structure"
$h^X : \mathscr{C} \to \mathsf{Set}$ X is any object in $\mathscr{C}$ *covariant* Hom	$T \mapsto \mathrm{Hom}_{\mathscr{C}}(X, T)$	$T \xrightarrow{f} S$ $\downarrow$ $\mathrm{Hom}_{\mathscr{C}}(X, T) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathscr{C}}(X, S)$ $\phi \longmapsto f\phi$

$h_X : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$ X is any object in $\mathscr{C}$ *contravariant* Hom	$T \mapsto \mathrm{Hom}_{\mathscr{C}}(T, X)$	$T \xrightarrow{f} S$ $\downarrow$ $\mathrm{Hom}_{\mathscr{C}}(S, X) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathscr{C}}(T, X)$ $\psi \longmapsto \psi f$
$\pi_1 : \mathsf{HTop}_* \to \mathsf{Grp}$ *fundamental group* (example of covariant Hom)	$(X, x_0) \mapsto \pi_1(X, x_0) := [S^1, X]_*$ basepoint preserving, homotopy classes of continuous functions	$(X, x_0) \xrightarrow{f} (Y, y_0)$ $\downarrow$ $\pi_1(X, x_0) \to \pi_1(Y, y_0) : [\gamma] \mapsto [f \circ \gamma]$
$(-)^{\mathrm{ab}} : \mathsf{Grp} \to \mathsf{Ab}$ *abelianisation*	$G \mapsto G^{\mathrm{ab}} := G/[G, G]$ $[G, G]$ is the commutator subgroup of G	$G \xrightarrow{f} H$ $\downarrow$ $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ induced by the universal property of quotients
$T_*(-) : \mathsf{SmMan}_* \to \mathsf{Vec}_{\mathbb{R}}$ *tangent space*	$(M, p) \mapsto T_p M$ the tangent space at p	$(M, p) \xrightarrow{F} (N, q)$ $\downarrow$ $dF_p : T_p M \to T_q N$ the differential at p
$T(-) : \mathsf{SmMan} \to \mathsf{SmMan}$ *tangent bundle*	$M \mapsto TM$ the tangent bundle	$M \xrightarrow{F} N$ $\downarrow$ $dF : TM \to TN$ the total differential
$C^0(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cring}$ *continuous functions pullback* (example of contravariant Hom: $\mathrm{Hom}_{\mathsf{Top}}(-, \mathbb{R})$)	$X \mapsto C^0(X, \mathbb{R})$ $\mathbb{R}$-valued continuous functions, "forget its additional structure"	$X \xrightarrow{f} Y$ $\downarrow$ $C^0(Y, \mathbb{R}) \to C^0(X, \mathbb{R}) : \phi \mapsto \phi \circ f$
$(-)^* : \mathsf{Vec}_k^{\mathrm{op}} \to \mathsf{Vec}_k$ *dual* (example of contravariant Hom: $\mathrm{Hom}_{\mathsf{Vec}_k}(-, k)$)	$V \mapsto V^* := \mathrm{Hom}_{\mathsf{Vec}_k}(V, k)$	$V \xrightarrow{f} W$ $\downarrow$ $W^* \to V^* : \phi \mapsto \phi \circ f$

$(-)^{-1} : \mathsf{B}G^{\mathrm{op}} \to \mathsf{B}G$	$\bullet \mapsto \bullet$	$\bullet \xrightarrow{g} \bullet$ $\downarrow$ $\bullet \xrightarrow{g^{-1}} \bullet$
$(-)^{\intercal} : \mathsf{Mat}_A^{\mathrm{op}} \to \mathsf{Mat}_A$ *transpose*	$n \mapsto n$	$n \xrightarrow{A} m$ $\downarrow$ $m \xrightarrow{A^{\intercal}} n$
$(-)^{\times} : \mathsf{Ring} \to \mathsf{Grp}$ *unit functor*	$A \mapsto A^{\times}$ group of units	$A \xrightarrow{f} B$ $\downarrow$ $f\vert_{A^{\times}} : A^{\times} \to B^{\times}$
$\mathbb{Z}[-] : \mathsf{Grp} \to \mathsf{Ring}$ *group ring functor*	$G \mapsto \mathbb{Z}[G]$ group ring	$G \xrightarrow{f} H$ $\downarrow$ $\mathbb{Z}[f] : \mathbb{Z}[G] \to \mathbb{Z}[H]$ linearly extend f
$\mathrm{Spec}(-) : \mathsf{CRing} \to \mathsf{Set}$ *spectrum*	$A \mapsto \mathrm{Spec}(A)$ $\mathrm{Spec}(A) :=$ $\{\mathfrak{p} \subseteq A \ : \ \mathfrak{p}$ is a prime ideal$\}$	$A \xrightarrow{f} B$ $\downarrow$ $\mathrm{Spec}(B) \to \mathrm{Spec}(A) : \mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$
$\mathrm{Aut}(-) : K/\mathsf{Field} \to \mathsf{Grp}$ *field automorphisms*	$E \mapsto \mathrm{Aut}(E/K)$ automorphisms of E fixing K point-wise	$E \hookrightarrow F$ $\downarrow$ $\mathrm{Aut}(F/K) \to \mathrm{Aut}(E/K) : \sigma \mapsto \sigma\vert_E$
$(-)^G : G\text{-}\mathsf{Set} \to \mathsf{Set}$ *fixed points functor*	$X \mapsto X^G$ G-fixed points of X	$X \xrightarrow{f} Y$ $\downarrow$ $f\vert_{X^G} : X^G \to Y^G$

2.1. Problems

Problem 2.1.

(a) Show that $\mathscr{C}^{\mathrm{op}}$ is indeed a category. What should the composition law be?

(b) Show that $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$.

Problem 2.2. Recall the notion of *slice categories* from Problem 1.3. Prove that

$$(\mathscr{C}^{\mathrm{op}}/X)^{\mathrm{op}} = X/\mathscr{C}$$

A similar argument also gives you $(X/\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}/X$.

Problem 2.3 (Example of Duality)**.** Prove that giving an epimorphism in $\mathscr{C}$ is the same as giving a monomorphism in $\mathscr{C}^{\mathrm{op}}$. Using Problem 2.1 (b), deduce the vice versa.

This is an example of *duality*.

Problem 2.4. A morphism $f : X \to Y$ in a category $\mathscr{C}$ is called a *split epimorphism* if there exists a morphism $\sigma : Y \to X$ (called a *section*) such that $f\sigma = \mathrm{id}_Y$

$$\begin{tikzcd} Y \arrow[r, "\sigma"] \arrow[dr, bend right, "\mathrm{id}_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

that is, the above diagram commutes. Prove that f is an epimorphism (and σ a monomorphism).

Problem 2.5. A split epimorphism in $\mathscr{C}^{\mathrm{op}}$ is called a *split monomorphism* in $\mathscr{C}$, where the section of the split epimorphism in $\mathscr{C}^{\mathrm{op}}$ is called a *retract* in $\mathscr{C}$.

Carefully write down this definition as it would be stated in $\mathscr{C}$.

Duality tells you that a split monomorphism is a monomorphism, and the retract is an epimorphism.

Problem 2.6. A monomorphism and epimorphism are categorical generalisation of injective and surjective maps respectively. The following problems should help you understand why.

(a) Prove that in the categories Set and Ab, a morphism is a monomorphism if and only if it's an injective map, and a morphism is a monomorphism if and only if it's a surjective map. [2]

(b) (Challenge) Prove similarly for Grp.

(c) Prove that every surjective map is an epimorphism in Ring but the following morphism is an epimorphism and not a surjective map

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$$

Deduce then that morphisms that are both a monomorphism and an epimorphism are *not* necessarily isomorphisms.

Problem 2.7. It's actually more reasonable[3] to note that *monomorphisms model injective maps* and *split epimorphisms model surjective maps* and epimorphisms and split monomorphisms tag along as just dual notions. The following problem should help you make sense of this.

Show that if a morphism f in a category $\mathscr{C}$ is a monomorphism and a split epimorphism, then f is an isomorphism. State what the dual statement will be.

Problem 2.8.

(a) Prove that an isomorphism in $\mathscr{C}$ is an isomorphism in $\mathscr{C}^{\mathrm{op}}$. That is, if a morphism f in $\mathscr{C}$ is an isomorphism, then the associated morphism f^{op} is an isomorphism in $\mathscr{C}^{\mathrm{op}}$.

(b) Prove that a functor sends isomorphisms to isomorphisms. That is, given a functor $F : \mathscr{C} \to \mathscr{D}$ and an isomorphism f in $\mathscr{C}$. Prove $F(f)$ is an isomorphism.

Problem 2.9. Prove that the following association gives you a functor

$$\mathrm{Aut}_{\mathscr{C}}(-) : \mathscr{C}^{\cong} \to \mathsf{Grp},\ X \mapsto \mathrm{Aut}_{\mathscr{C}}(X);$$

what does it do to morphisms?

Using Problem 2.8, this tells us that if $X \cong Y$ then $\mathrm{Aut}_{\mathscr{C}}(X) \cong \mathrm{Aut}_{\mathscr{C}}(Y)$. Applying this result to the fundamental groupoid of a topological space (Example 2.13) tells us that if the topological space is path connected, the fundamental groups at two different basepoints are isomorphic.

Problem 2.10.

(a) What's a functor $F : \mathsf{B}G \to \mathsf{B}H$, where G and H are groups?

(b) Consider posets $(\mathsf{P}, \leqslant_{\mathsf{P}})$ and $(\mathsf{Q}, \leqslant_{\mathsf{Q}})$ as categories.

 (b1) What's a functor $F : \mathsf{P} \to \mathsf{Q}$?

 (b2) What's a functor $F : \mathsf{P}^{\mathrm{op}} \to \mathsf{Q}$?

Problem 2.11. Prove that the notion of a contravariant functor $F : \mathscr{C} \to \mathscr{D}$ is equivalent to the notion of a (covariant) functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Problem 2.12. Prove that the examples in Example 2.17 are in fact functors.

Problem 2.13. Prove that the association

$$Z(-) : \mathsf{Grp} \to \mathsf{Ab},\ Z \mapsto Z(G),$$

sending a group to its center, is not functorial.

Prove that if we restrict this association to the subcategory $\mathsf{SurjGrp}$ then it is actually a functor.

Problem 2.14. For a commutative unital ring A, we define

$$\mathrm{MaxSpec}(A) = \{\mathfrak{m} \subseteq A \ :\ \mathfrak{m} \text{ is a maximal ideal}\} \subseteq \mathrm{Spec}(A)$$

Prove that the association

$$\mathrm{MaxSpec}(-) : \mathsf{CRing} \to \mathsf{Set},\ A \mapsto \mathrm{MaxSpec}(A)$$

is not functorial in the same way $\mathrm{Spec}(-)$ is.

Notes

1. Once can also define this as $\pi_1(X, x_0) = \mathrm{Hom}_{\mathsf{Top}_*}((S^1, *), (X, x_0))/\text{homotopy}$. Showing that these two definitions are the same is a good exercise in elementary topology.

2. One way to prove the "mono" part for Set is to reduce the definition of monomorphism to a singleton $*$. It is thus called a *generator* (or *separator*) of Set. A similar argument works for Ab; namely, it has a separator (for example, $\mathbb{Z}$). One way to prove the "epi" part for Set is to reduce the definition of epimorphism to the set of truth values $\Omega = \{\mathrm{T}, \mathrm{F}\}$. It is thus called a *cogenerator* (or *classifer*) of Set. However, a similar argument fails for Ab since it has no classifier.

3. Here *reasonable* means it behaves as one intended expect. In contrast, for example, in the category of rings, an epimorphism may not be a surjective homomorphism, see (c) of Problem 2.6; in the category of fields, an injective homomorphism is a split monomorphism if and only if it is an isomorphism, which is too restrictive.

3. Lecture 3 (1/21) by Deewang

Example 3.1. For a group G, consider the category $\mathsf{B}G$ and any other category $\mathscr{C}$. A functor

$$X : \mathsf{B}G \to \mathscr{C}$$

specifies an object X in $\mathscr{C}$ (the image of the unique object in $\mathsf{B}G$) and an automorphism $g_X : X \to X$ for each $g \in G$ (the image of the isomorphisms $g \in G$ in $\mathsf{B}G$). That is, the functor affords a morphism

$$G \to \mathrm{Aut}_{\mathscr{C}}(X).$$

This is subject to the following two properties

- $(gh)_X = g_X h_X$
- $e_X = 1_X$

That is, the functor $X : \mathsf{B}G \to \mathscr{C}$ defines an *action* of the group G on the object X in $\mathscr{C}$.

- When $\mathscr{C} = \mathsf{Set},\ X$ is called a G-set.
- When $\mathscr{C} = \mathsf{Vec}_k,\ X$ is called a G-representation.
- When $\mathscr{C} = \mathsf{Top},\ X$ is called a G-space.

This notion is also given the name *left action*, in which case a *right action* is a functor $X : \mathsf{B}G^{\mathrm{op}} \to \mathscr{C}$.

How would one express the notion of a G-equivariant map in this language? It will have to be some notion of *morphism of functors* as we've realised objects with a G-action as functors.

Example 3.2. For a field k, recall that any finite dimensional k-vector space V is isomorphic to its dual $V^* = \mathrm{Hom}_{k\text{-line}}(V, k)$. This is proven by constructing an explicit *dual basis*: choose a basis $\{e_1, \dots, e_n\}$ of V, then a basis of V^* is given by $\{e_1^*, \dots, e_n^*\}$ where

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases};$$

this isomorphism is then given by $e_i \mapsto e_i^*$.

A related construction is the double dual $V^{**} := (V^*)^*$. Since the association of a dual describes a functor $(-)^* : \mathsf{fdVec}_k^{\mathrm{op}} \to \mathsf{fdVec}_k$, for a finite dimensional vector space V, we get $V^* * = (V^*)^* \cong V^* \cong V$ where the isomorphism sends e_i to the dual dual basis e_i^{**}.

Turns out, there's a cleaner way to give an isomorphism between V and V^{**} without making a choice of basis. For any $v \in V$, the *evaluation map*

$$\mathrm{ev}_v : V^* \to k,\ \phi \mapsto \phi(v)$$

is a linear functional on V^*, that is, an element of V^{**}. The assignment $v \mapsto \mathrm{ev}_v$ defines a *basis-free natural* isomorphism $V \cong V^{**}$.

What distinguishes the isomorphism between a finite-dimensional vector space and its double dual from the isomorphism between a finite-dimensional vector space and its single dual is that the former assembles into the components of a *natural transformation*, a notion that we describe below.

Definition 3.3. Given categories $\mathscr{C}$ and $\mathscr{D}$ and functors $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of

- a morphism $\alpha_X : F(X) \to G(X)$ in $\mathscr{D}$ for each object X in $\mathscr{C}$, the collection of which we call the *components* of the natural transformation,

such that

- for any morphism $f : X \to Y$ in $\mathscr{C}$, the following square of morphisms in $\mathscr{D}$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

  commutes.

We usually say that the morphisms $\alpha_X : F(X) \to G(X)$ are natural in X to implicitly imply the existence of this commutative square.

A *natural isomorphism* is a natural transformation $\alpha : F \Rightarrow G$ in which every component α_X is an isomorphism in $\mathscr{D}$.

Example 3.4. Some natural transformations between already introduced functors (Example 2.17).

Source	**Target**	**Natural Transformation**	**Components**
Vec_k	Vec_k	$1_{\mathsf{Vec}} \Rightarrow (-)^{**}$ (isomorphism)	$V \to V^{**} : v \mapsto \mathrm{ev}_v$
Set	Set	$1_{\mathsf{Set}} \Rightarrow \mathscr{P}_*$	$A \to \mathscr{P}(A) : a \mapsto \{a\}$
$\mathscr{C}$	Set	$h^Y \Rightarrow h^X$ given a morphism $f : X \to Y$	$\mathrm{Hom}(Y,T) \to \mathrm{Hom}(X,T) : \phi \mapsto \phi f$
$\mathscr{C}^{\mathrm{op}}$	Set	$h_X \Rightarrow h_Y$ given a morphism $f : X \to Y$	$\mathrm{Hom}(T,X) \to \mathrm{Hom}(T,Y) : \psi \mapsto f\psi$
$\mathsf{B}G$	$\mathscr{C}$	$X \Rightarrow Y$ for objects X, Y in $\mathscr{C}$	single component $f : X \to Y$ such that $g_Y f = f g_X$ for all $g \in G$ (a G-equivariant map)

Top	Set	$h^* \Rightarrow U$ (isomorphism)	$\mathrm{Hom}_{\mathsf{Top}}(*, X) \to U(X) : f \mapsto f(*)$
CRing	Set	$h^{\mathbb{Z}[t^{\pm}]} \Rightarrow (-)^{\times}$ (isomorphism)	$\mathrm{Hom}_{\mathsf{CRing}}(\mathbb{Z}[t^{\pm}], A) \to A^{\times} : f \mapsto f(t)$
G-Set	Set	$h^* \Rightarrow (-)^G$ (isomorphism)	$\mathrm{Hom}_{G\text{-}\mathsf{Set}}(*, X) \to X^G : f \mapsto f(*)$
Grp	Grp	$1_{\mathsf{Grp}} \Rightarrow (-)^{\mathrm{ab}}$	$G \twoheadrightarrow G/[G,G]$ canonical projection

Discussion 3.5. In contrast with the first example, the identity functor and the single dual functor on finite-dimensional vector spaces are not naturally isomorphic. Looking beyond the one technical obstruction, that the identity functor is covariant while the dual functor is contravariant, which is beside the point, the more significant is the essential failure of naturality.

Given a linear map $T : V \to W$ between finite dimensional vector spaces, we obtain the diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_{\mathbf{e}_V}"'] & W \arrow[d, "\phi_{\mathbf{e}_W}"] \\ V^* & W^* \arrow[l, "T^*"] \end{tikzcd}$$

where $\phi_{\mathbf{e}_V}$ and $\phi_{\mathbf{e}_W}$ are isomorphism described in Example 3.2 respect to the choice of basis $\mathbf{e}_V$ and $\mathbf{e}_W$ of V and W respectively. The only "naturality" condition that can be read from this diagram is

$$\phi_{\mathbf{e}_V} = T^* \circ \phi_{\mathbf{e}_W} \circ T,$$

but taking $T = 0$ gives us that $\phi_{\mathbf{e}_V} = 0$ contradicting the fact that $\phi_{\mathbf{e}_V}$ was an isomorphism. A line of enquiry would be to consider what happens if assume T to be an isomorphism, you will then note that one still cannot escape the failure of this notion of naturality.

Consider the subcategory Euc of Euclidean vector spaces of $\mathsf{fdVec}_{\mathbb{R}}$, that is, the subcategory of inner product spaces. Then it's important to note that there is a natural isomorphism $1_{\mathsf{Euc}} \Rightarrow (-)^*$ given by (components are) $V \to V^* : v \mapsto \langle v, - \rangle$.

Example 3.6. We describe two functors $\mathcal{O}, \mathcal{C} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$

$\mathcal{O} : X \mapsto \mathcal{O}(X)$, set of open sets

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, mapsto, "\mathcal{O}"] & Y \\ \mathcal{O}(Y) \arrow[r, "f^{-1}"] & \mathcal{O}(X) \end{tikzcd}$$

$$\mathcal{C} : X \mapsto \mathcal{C}(X), \text{set of closed sets} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, mapsto, "\mathcal{C}"] \\ \mathcal{C}(Y) \arrow[r, "f^{-1}"] & \mathcal{C}(X) \end{tikzcd}$$

Then there's a natural isomorphism $\mathcal{O} \Rightarrow \mathcal{C}$ with components as $\mathcal{O}(X) \to \mathcal{C}(X) : U \mapsto X \setminus U$.

Example 3.7. Recall that a monoid is a set equipped with a binary product for which there exists a netural element (that is, a set that satisfies all the group axioms but the one about existence of inverses). A morphism of monoids is a function that commutes with the binary products, similar to a group homomorphism. We can then consider the category Mon of monoids, where the objects are monoids and morphisms are (monoid) morphisms.

Given any ring with unity A, A is a monoid with respect to multiplication. So is the set of $n \times n$ matrices $M_n(A)$ with respect to multiplication. These assemble to give functors

$$M_n(-), U : \mathsf{Ring} \rightrightarrows \mathsf{Mon}$$

Consider now the determinant map $\det_A : M_n(A) \to U(A)$, this is a monoid morphism since $\det_A(XY) = \det_A(X) \det_A(Y)$. The determinant assembles to give us (that is, $\det_A$ are the components) a natural transformation

$$\det : M_n(-) \Rightarrow U$$

Example 3.8. Consider the category HTop_* and denote

$$[X, Y]_* := \mathrm{Hom}_{\mathsf{HTop}_*}((X, x_0), (Y, y_0)) = \{f : X \to Y \ : \ f \text{ is continuous}, f(x_0) = y_0\}/\text{homotopy}$$

leaving the base points implicit.

We've already seen the functor $\pi_1 : \mathsf{HTop}_* \to \mathsf{Grp}$ where $(X, x_0) \mapsto [S^1, X]_*$, the fundamental group of X at basepoint x_0. We can similarly define, for any $n \geqslant 1$ functors

$$\pi_n : \mathsf{Htop}_* \to \mathsf{Grp}, \ (X, x_0) \mapsto \pi_n(X, x_0) := [S^n, X]_*$$

$\pi_n(X, x_0)$ are called the n^{th} *homotopy groups of X at basepoint x_0*. It's a fact that for $n > 1$, the functor π_n takes values in Ab.

There's another functor

$$H_n : \mathsf{HTop}_* \to \mathsf{Ab}, \ (X, x_0) \mapsto H_n(X, \mathbb{Z})$$

$H_n(X, \mathbb{Z})$ are called the n^{th} *singular homology group of X with coefficients in $\mathbb{Z}$*.

For any pointed space X (and n), there's a group homomorphism

$$h_n(X) : \pi_n(X, x_0) \to H_n(X, \mathbb{Z})$$

called the Hurewicz homomorphism, which assemble to give a natural tranformation

$$h_n : \pi_n \Rightarrow H_n$$

We have talked about categories, functors and natural transformations but we have yet to discuss or introduce the right notion of "sameness" for categories. Naively, one would hope that the following would be that notion.

Definition 3.9. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, then we say $\mathscr{C}$ and $\mathscr{D}$ *isomorphic* to each other if there exist functors

$$F : \mathscr{C} \leftrightarrows \mathscr{D} : G$$

such that $GF = 1_{\mathscr{C}}$ and $FG = 1_{\mathscr{D}}$, where $1_{\mathscr{C}}$ and $1_{\mathscr{D}}$ are the obvious identity functors. We then write $\mathscr{C} \cong \mathscr{D}$.

Unfortunately, turns out this notion is too strong and rarely satisfied in practice and even rarely needed in practice. Following is a weaker notion than an isomorphism of categories but is the right notion of "sameness"

Definition 3.10. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, then we say $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* to each other if there exist functors

$$F : \mathscr{C} \leftrightarrows \mathscr{D} : G$$

such that there exist natural isomorphisms $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$. We then write $\mathscr{C} \simeq \mathscr{D}$.

Example 3.11. For a field k, we consider the categories Mat_k, fdVec_k, $\mathsf{fdVec}_k^{\mathsf{basis}}$. The only category we haven't seen before if $\mathsf{fdVec}_k^{\mathsf{basis}}$, the objects of bVec_k are finite dimensional vector spaces with a chosen basis are morphisms are arbitrary (not necessarily basis-preserving) linear maps. These three categories are relates by a few functors

$$\begin{tikzcd} \mathsf{Mat}_k \arrow[r, shift left, "k^{(-)}"] & \mathsf{fdVec}_k^{\mathsf{basis}} \arrow[l, shift left, "D"] \arrow[r, shift left, "U"] & \mathsf{fdVec}_k \arrow[l, shift left, "C"] \end{tikzcd}$$

where

- the functor $k^{(-)}$ sends a non-negative integer n to the vector space k^n equipped with the standard basis, and an $m \times n$ matrix to itself as it defines a linear map $k^n \to k^m$.
- The functor U is the forgetful functor.
- The functor C is defined by choosing a basis for each vector space.
- The functor D sends a vector space to its dimension, and a linear map between two vector spaces to its matrix representation with respect to the chosen bases.

The functors define an equivalence of categories

$$\mathsf{Mat}_k \simeq \mathsf{fdVec}_k \simeq \mathsf{fdVec}_k^{\mathsf{basis}}$$

One can prove this directly or we can prove it using a very useful characterisation of an equivalence of categories that we now give.

Definition 3.12. A functor $F : \mathscr{C} \to \mathscr{D}$ is

- **full** if for objects X, Y, the map

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$$

is surjective.

- **faithful** if for objects X, Y, the map

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$$

is injective.

- **essentially surjective (on objects)** if for every object W in $\mathscr{D}$, there is some object X in $\mathscr{C}$ such that $F(X) \cong W$.

Remark 3.13. Fullness and faithfulness are *local conditions* on morphisms, not *global* as a global condition would apply "everywhere". A full functor need not be surjective on morphisms (one reason is because such a functor may not be essentially surjective), and a faithful functor need not be injective on morphisms.

- A faithful functor that is injective on objects is called an *embedding*, and identifies the domain category as a subcategory of the codomain. In this case, faithfulness implies that the functor is (globally) injective on arrows.
- A full and faithful functor, called *fully faithful* for short, that is injective on objects defines a *full embedding* of the domain category into the codomain category. The domain then defines a *full subcategory* of the codomain.

Theorem 3.14. *A functor defines an equivalence of categories if and only if it is full, faithful and essentially surjective on objects.*

3.1. Problems

Problem 3.1.

- Given categories $\mathscr{C}$ and $\mathscr{D}$, describe a category (that is, verify the axioms) $[\mathscr{C},\mathscr{D}]$ where the objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and morphisms natural tranformations. This category is also sometimes denoted as $\mathscr{D}^{\mathscr{C}}$.
- Show that a natural isomorphism is precisely an isomorphism in the category $[\mathscr{C},\mathscr{D}] = \mathscr{D}^{\mathscr{C}}$.

Problem 3.2. One feature of "higher structures", like categories, is that they has several "levels". In particular, a category has two levels: objects and morphisms.

In contrast, a set has only one level: elements. We categorify a set with objects being its elements and the only morphisms being identity maps.

A "correct" notion of maps between such structures will have to respect the levels. So a *functor* (or a *map*/0-*morphism*) needs to preserve the levels; a *natural transformation* (or a 1-*morphism*) between them will have to respect a level shifted up by 1.

(a) What should be a *functor* from a set to a category? (It has to map elements to objects as they are both at lowest level.)

(b) What should be a *natural transformation* between functors from a set to a category? (It has to map elements to morphisms as the latter are in one higher level than the former)

(c) What should be a *functor* from a category to a set? (It has to map objects to elements as they are both at lowest level.)

(d) What should be a *natural transformation* between functors from a category to a set?

Problem 3.3. Prove that the notion of an equivalence of categories defines an equivalence relation.

Problem 3.4. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

(a) Prove that if F is faithful then it need not be injective on morphisms.

(b) Prove that if F is faithful and injective on objects (that is, if $F(X) = F(Y)$ then $X = Y$), then it is injective on morphisms.

(c) Prove that if F is fully faithful then it need not be injective on objects. But show that it is "injective up to isomorphism", that is if $F(X) \cong F(Y)$ then $X \cong Y$.

Problem 3.5. Prove Theorem 3.14. More precisely, let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

(a) Suppose F defines an equivalence, prove that F is full, faithful and essentially surjective on objects. Prove faithfulness before fullness.

(b) Conversely, suppose that F is full, faithful and essentially surjective on objects. For each object W in $\mathscr{D}$, choose (axiom of choice is being invoked here) an object $G(W)$ of $\mathscr{C}$ and an isomorphism $\varepsilon_W : F(G(W)) \to W$. Prove that G extends to a functor in such a way that ε, with components ε_W, is a natural isomorphism $FG \Rightarrow 1_{\mathscr{D}}$. Then construct a natural isomorphism $\eta : 1_{\mathscr{C}} \Rightarrow GF$, thus proving that F is an equivalence.

Problem 3.6. Prove that the categories in Example 3.11 are equivalent.

Problem 3.7. It's not often that the opposite categories can be identified with familiar categories, it's a rare phenomenon. It's also rare for the categories to be equivalent to their oppposite versions. Here are some simple examples of this rare phenomena.

(a) Prove $(-)^* : \mathsf{fdVec}_k^{\mathrm{op}} \to \mathsf{fdVec}_k$ defines an equivalence of categories, see Example 2.17 for how it's defined. Through the equivalence established in Example 3.11, the functor $(-)^*$ translates to the functor $(-)^{\intercal} : \mathsf{Mat}_k^{\mathrm{op}} \to \mathsf{Mat}_k$ which also then defines an equivalence (isomorphism, in fact) of categories.

(b) Prove $(-)^{-1} : \mathsf{B}G^{\mathrm{op}} \to \mathsf{B}G$ defines an equivalence (isomorphism, in fact) of categories, see Example 2.17 for how it's defined.

(c) Let X be any set, we can consider the poset $(\mathscr{P}(X), \subseteq)$, where $\mathscr{P}(X)$ is the power set of X, with respect to set containment. This then gives a category, as described previously in Example 1.4 on how posets give rise to categories. Prove that the functor

$$(-)^c : (\mathscr{P}(X), \subseteq)^{\mathrm{op}} \to (\mathscr{P}(X), \subseteq),\ A \mapsto A^c := X \setminus A$$

defines an equivalence (isomorphism, in fact) of categories.

(d) You can either take the following two statements for granted, or explore the details yourself.

- the opposite category of unital commutative ring is equivalent to the category of affine schemes.
- the opposite category of sets is equivalent to the category of complete atomic boolean algebras. When restricted to finite sets, the opposite category of finite sets is equivalent to the category of finite boolean algebras.

(e) Consider the category Γ, called *Segal's category*, described as following.

- Objects of Γ are finite sets.
- For finite sets S and T

$$\mathrm{Hom}_\Gamma(S, T) := \{\theta : S \to \mathscr{P}(T)\ :\ \theta(\alpha) \cap \theta(\beta) = \varnothing, \text{ whenever } \alpha \neq \beta\}$$

  where $\mathscr{P}(-)$ denotes the power set.
- The composite of $\theta : S \to \mathscr{P}(T)$ and $\phi : T \to \mathscr{P}(U)$ is the function

$$\psi : S \to \mathscr{P}(U),\ \alpha \mapsto \bigcup_{\gamma \in \theta(\alpha)} \phi(\gamma)$$

  What do the identity morphisms look like?

Prove that Γ is equivalent to the opposite of the category $\mathsf{FinSets}_*$ of finite pointed sets (describe this to yourself).

Problem 3.8. A category $\mathscr{C}$ is *skeletal* if it contains just one object in each isomorphism class. A *skeleton* $\mathrm{sk}(\mathscr{C})$ of a category $\mathscr{C}$ is a full subcategory such that every object of $\mathscr{C}$ is isomorphic to precisely one object in $\mathrm{sk}(\mathscr{C})$.

(a) Prove that a skeleton of a category is skeletal.

(b) Show that any two skeletons of a category are isomorphic.

(c) Show that $\mathrm{sk}(\mathscr{C})$ of a category $\mathscr{C}$ is equivalent to $\mathscr{C}$. Therefore (a) and Problem 3.3 give us that *every* skeleton of $\mathscr{C}$ is equivalent to $\mathscr{C}$. This also shows that an equivalence need not be injective on objects.

(d) Show that two categories are equivalent if and only if they have isomorphic skeletons.

Example 3.11 and Problem 3.6 exhibit that the skeleton of fdVec_k is the category Mat_k.

(e) Let $\mathsf{FinSets}_{\mathsf{bij}}$ be the maximal groupoid of the category $\mathsf{FinSets}$ of finite sets. Prove that the skeleton of $\mathsf{FinSets}_{\mathsf{bij}}$ is the category whose objects are positive integers and with $\mathrm{Hom}(n, n) = \Sigma_n$, the symmetric group on n letters. The sets of morphisms between distinct positive integers are all empty.

4. Lecture 4 (1/28) by Vaibhav

Recall that we assumed our categories to be *locally small*. That is, we have assumed in our categories $\mathscr{C}$, for any two objects X, Y, that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a set. We introduce some more size-related notions for categories.

Definition 4.1. A category $\mathscr{C}$ is said to be

- *small* if $\mathscr{C}$ is locally small, and $\mathrm{obj}(\mathscr{C})$ is a set. For example, a set itself can be treated as a category with its elements as objects and the only morphisms being identity morphisms; in this way, a set is an example of a small category.
- *essentially small* if $\mathscr{C}$ is equivalent to a small category. Equivalently, if $\mathrm{sk}(\mathscr{C})$, the skeleton of $\mathscr{C}$, is small (see Problem 3.8). For example, the category of finite sets (resp. finite dimensional vector spaces) is essentially small; this follows from Problem 3.8(e) (resp. Example 3.11 and Problem 3.6).

Remark 4.2. We have introduced (or now know) mathematical objects with different levels of structures (assume our categories are small).

- ○ sets (treated as a small category) have
  - ▷ objects (elements) $\bullet$
- ○ categories have
  - ▷ objects $\bullet$
  - ▷ morphisms $\bullet \longrightarrow \bullet$
- ○ the category of categories has
  - ▷ objects (categories) $\bullet$ "0-dimensional".
  - ▷ morphisms (functors) $\bullet \longrightarrow \bullet$ "1-dimensional".
  - ▷ 2-morphisms (natural transformations) $\begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$ "2-dimensional".
    - ◇ (identities) Given a functor $F : \mathscr{C} \to \mathscr{D}$, we can define an identity natural tranformation $1_F : F \Rightarrow F$, where the components are $1_{F(X)} : F(X) \to F(X)$.
    - ◇ (vertical composition)

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \bullet \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & \bullet \end{tikzcd}$$

- ◇ (horizontal composition)

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, bend left=50, "A"] \arrow[r, bend right=50, "B"'] \arrow[r, phantom, "\Downarrow\beta"] & \bullet \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "AF"] \arrow[r, bend right=50, "BG"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \bullet \end{tikzcd}$$

We can collect these to into a construction called a 2-category.

Definition 4.3. For a category $\mathscr{C}$, an object I is called an *initial object* if there exists a unique morphism $I \to X$ for any object X of $\mathscr{C}$.

Dually, an object T is called a *final* or *terminal object* if there exists a unique morphism $X \to T$ for any object X of $\mathscr{C}$ (i.e., it's an initial object in $\mathscr{C}^{\mathrm{op}}$).

In other words, I is an initial object if and only if $\mathrm{Hom}_{\mathscr{C}}(I, X) \cong *$ (the singleton set).

Example 4.4. An initial object in Set is $\varnothing$, the unique map $\varnothing \hookrightarrow X$ can be understood to be the inclusion of the empty set as a subset of any set X. A terminal object in Set is the singleton set $\{1\}$, the unique map $X \to \{1\}$ is the constant function for any set X. See Problem 4.1 for more examples and ideas.

Discussion 4.5. We can interpret the notion of an initial object in terms of functors.

For an object Z in a category $\mathscr{C}$, recall the functor $h^Z = \mathrm{Hom}_{\mathscr{C}}(Z, -) : \mathscr{C} \to \mathrm{Set}$ where $X \mapsto \mathrm{Hom}_{\mathscr{C}}(Z, X)$ and

$$\begin{array}{ccc} X & & \mathrm{Hom}_{\mathscr{C}}(Z,X) \\ \Big\downarrow{\scriptstyle f} & \longmapsto & \Big\downarrow{\scriptstyle f\circ -} \\ Y & & \mathrm{Hom}_{\mathscr{C}}(Z,Y) \end{array}$$

For any category $\mathscr{C}$ and any singleton set $*$, we can define the constant functor $* : \mathscr{C} \to \mathrm{Set}$ where any object $X \mapsto *$ and any morphism $f \mapsto 1_*$.

Then, an object I in $\mathscr{C}$ is an initial object if and only if there exists a natural isomorphism

$$\rho : \mathrm{Hom}_{\mathscr{C}}(I, -) \overset{\sim}{\Longrightarrow} *$$

That is, for any object X we have a bijection $\rho_X : \mathrm{Hom}_{\mathscr{C}}(I, X) \xrightarrow{\sim} *$ and for any morphism $X \xrightarrow{f} Y$ the following square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(I,X) \arrow[r, "\rho_X"] \arrow[d, "f\circ -"'] & * \arrow[d, "1_*"] \\ \mathrm{Hom}_{\mathscr{C}}(I,Y) \arrow[r, "\rho_Y"'] & * \end{tikzcd}$$

commutes.

We can think of this as saying I witnesses the structure of the constant functor $*$; we say that I *represents* the functor $*$.

Definition 4.6. A functor $F : \mathscr{C} \to \mathrm{Set}$ is *representable* if there exists an object X of $\mathscr{C}$ and a natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X, -) \overset{\sim}{\Longrightarrow} F$$

Note: if F were a contravariant functor, that is a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$, then for it to be a representable functor we would want a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(-, X) = \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, -) \overset{\sim}{\Longrightarrow} F$.

We then say F *is represented by* X.

Therefore, a category $\mathscr{C}$ has an initial object if and only if the constant functor $*$ is representable.

Definition 4.7. Given a functor $F : \mathscr{C} \to \mathsf{Set}$, a *representation of* F or a *universal object for* F is a pair (X, α), where X is an object in $\mathscr{C}$ and α is a choice of a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(X, -) \overset{\sim}{\Longrightarrow} F$.

Example 4.8. We give an example and a non-example of representable functors.

- Recall the functor $\mathscr{P}^* : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ which sends a set $A \mapsto \mathscr{P}(A)$ to its power set, and a function $f : A \to B$ is sent to

$$f^* : \mathscr{P}(B) \to \mathscr{P}(A),\ S \mapsto f^{-1}(S)$$

  The functor is representable, and is represented by the set $\Omega = \{\mathrm{T}, \mathrm{F}\}$.

  We claim that we obtain a natural isomorphism $C : \mathrm{Hom}_{\mathsf{Set}}(-, \Omega) \Longrightarrow \mathscr{P}^*$ with the components as the bijections

$$\begin{aligned} C_A : \mathrm{Hom}_{\mathsf{Set}}(A, \Omega) &\longrightarrow \mathscr{P}^*(A) \\ \chi &\longmapsto \chi^{-1}(\mathrm{T}) \\ \left(\chi_S : a \mapsto \begin{cases} \mathrm{T} & \text{if } a \in S \\ \mathrm{F} & \text{if } a \notin S \end{cases} \right) &\longleftarrow\!\shortmid S \end{aligned}$$

  for any set A. We now verify that the components $(C_A)_A$ do assemble to give a natural isomorphism; that is, given any function $f : A \to B$, we verify the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Set}}(B, \Omega) \arrow[r, "C_B"] \arrow[d, "-\circ f"'] & \mathscr{P}(B) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathsf{Set}}(A, \Omega) \arrow[r, "C_A"'] & \mathscr{P}(A) \end{tikzcd}$$

  Consider any function $\chi \in \mathrm{Hom}_{\mathsf{Set}}(B, \Omega)$, then

$$f^* \circ C_B(\chi) = f^*(\chi^{-1}(\mathrm{T})) = f^{-1}(\chi^{-1}(\mathrm{T})) = f^{-1} \circ \chi^{-1}(\mathrm{T})$$
$$C_A \circ (- \circ f)(\chi) = C_A(\chi \circ f) = (\chi \circ f)^{-1}(\mathrm{T})$$

  We know from properties of inverse image that $(\chi \circ f)^{-1}(\mathrm{T}) = f^{-1} \circ \chi^{-1}(\mathrm{T})$, hence the diagram indeed commutes. Thus, we have proven that C is a natural isomorphism and therefore (Ω, C) is a representation of $\mathscr{P}^*$.

- Consider the abelianisation functor $(-)^{\mathrm{ab}} : \mathsf{Grp} \to \mathsf{Ab}$ which sends a group $G \mapsto G^{\mathrm{ab}}$ to its abelianisation where $G^{\mathrm{ab}} := G/[G, G]$. A group homomorphism $f : G \to H$ is sent to the induced group homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. We prove that this functor is not representable, and for that we use Problem 4.5. Recall that monomorphisms in these categories are simply injective functions (see Problem 2.8).

Consider the non-abelian group S_3, and its subgroup A_3, which is abelian. Therefore $A_3^{\mathrm{ab}} \cong A_3$ and one can prove that $S_3^{\mathrm{ab}} \cong C_2$, the cyclic group of order 2. Consider the inclusion group homomorphism $\iota : A_3 \hookrightarrow S_3$, which is indeed injective. But the induced map $\iota^{\mathrm{ab}} : A_3 \to C_2$ is necessarily the trivial homomorphism and hence not injective. Thus $(-)^{\mathrm{ab}}$ is not a representable functor.

Remark 4.9. Representable functors are rare and form a very special class of functors. Some of the biggest questions in algebraic geometry, for example, in the past century were if certain nice geometric functors were representable, and what to do if they were found to be not.

Definition 4.10 (a first attempt)**.** A *universal property* of an object X is expressed by a representable functor F together with a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(X, -) \cong F$ (or $\mathrm{Hom}_{\mathscr{C}}(-, X) \cong F$ if F is contravariant).

We will give a slightly better description once we have discussed the Yoneda lemma, and we can then also interpret this as an initial object in some category.

Example 4.11. Let R be a commutative ring with unity and M and N are R-modules, we consider the functor

$$\mathrm{Bil}_R(M \times N, -) : \mathsf{Mod}_R \to \mathsf{Set},$$

where any R-module T is sent to the set of R-bilinear maps $\mathrm{Bil}_R(M \times N, R)$. These are maps

$$f : M \times N \to T$$

such for any $m \in M$ and $n \in N$, the maps $f(m, -) : N \to T$ and $f(-, n) : M \to T$ are R-linear.

This functor has as a representation $(M \otimes_R N, \iota)$, where $M \otimes_R N$ is an R-module called the *tensor product* and $\iota : M \times N \to M \otimes_R N$, $(m, n) \mapsto m \otimes n$ is an R-bilinear map. The map ι affords the natural isomorphism

$$\mathrm{Hom}_R(M \otimes_R N, T) \cong \mathrm{Bil}_R(M \times N, T)$$

which we illustrate by exhibiting the universal property of $M \otimes_R N$.

The universal property is as follows: for any R-module T and R-bilinear map $f : M \times N \to T$ there exists a unique R-linear map $\tilde{f} : M \otimes_R N \to T$ such that the diagram

$$\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr, "\iota"'] & & T \\
& M \otimes_R N \arrow[ur, dashed, "\exists! \tilde{f}"'] &
\end{tikzcd}$$

commutes. That is, we have the bijection

$$\mathrm{Hom}_R(M \otimes_R N, T) \xrightarrow{\sim} \mathrm{Bil}_R(M \times N, T) : g \mapsto g \circ \iota$$

Example 4.12. Let G be a group and N a normal subgroup of G, we consider the functor

$$\mathrm{Kil}_N(G, -) : \mathsf{Grp} \to \mathsf{Set},$$

where any group H is sent to the set

$$\mathrm{Kil}_N(G, H) = \{\phi : G \to H \ : \ N \subseteq \ker \phi\} \subseteq \mathrm{Hom}_{\mathsf{Grp}}(G, H)$$

This functor has as a representation $(G/N, \pi)$, where G/N is the usual quotient group (of left cosets) and $\pi : G \to G/N,\ g \mapsto gN$ is the natural projection map. The map π affords the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Grp}}(G/N, H) \cong \mathrm{Kil}_N(G, H)$$

which we illustrate by exhibiting the universal property of G/N.

The universal property is as follows: for any module H and group homomorphism $\phi : G \to H$ such that $N \subseteq \ker \phi$ there exists a unique group homomorphism $\tilde{\phi} : G/N \to H$ such that the diagram

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & N \\ & G/N \arrow[ur, dashed, "\exists! \tilde{\phi}"'] & \end{tikzcd}$$

commutes. That is, we have the bijection

$$\mathrm{Hom}_{\mathsf{Grp}}(G/N, H) \xrightarrow{\sim} \mathrm{Kil}_N(G, H) : \psi \mapsto \psi \circ \pi$$

4.1. Problems

Problem 4.1.

(a) Look at the categories in Examples 1.2, 1.3 and 1.4, and try determining or describing the initial and terminal objects in those categories.

(b) Prove that an initial object of a category is *unique up to unique isomorphism*. Precisely put, suppose I and I' are two initial objects of a category $\mathscr{C}$, prove that there exists a unique isomorphism $f : I \to I'$.

Applying this to $\mathscr{C}^{\mathrm{op}}$ gives us that terminal objects are also unique upto unique isomorphism, this again is an example of a duality argument.

(c) Let I and T be initial and terminal objects in a category $\mathscr{C}$, prove that if there exists a morphism $f : T \to I$, then f is necessarily an isomorphism.

(d) An object that's both an initial and terminal object is called a *zero object*. Did you encounter any zero objects while solving (a)?

(e) Prove that a zero object exists in a category $\mathscr{C}$ if and only if $\mathscr{C}$ has an initial object, a terminal object and a morphism from the terminal to the initial object.

Problem 4.2. Formulate a similar statement for terminal objects to the one that's been formulated for initial objects in Discussion 4.5.

Problem 4.3. For objects X in a category $\mathscr{C}$, consider the slice category $\mathscr{C}/X$. Prove that 1_X is a terminal object in $\mathscr{C}/X$. We think of this as "X is the terminal object of $\mathscr{C}/X$".

Using Problem 2.2 (or not), what can you say about the initial object of the slice category $X/\mathscr{C}$.

Problem 4.4. Prove that if I is an initial object in a category $\mathscr{C}$, then the slice category $I/\mathscr{C}$ is isomorphic to $\mathscr{C}$.

Using Problem 2.2 (or not), prove that if T is a terminal object in a category $\mathscr{C}$, then $\mathscr{C}/T \cong \mathscr{C}$.

Problem 4.5. Prove that if $F : \mathscr{C} \to \mathsf{Set}$ is representable, then F preserves monomorphisms, i.e., sends every monomorphism in $\mathscr{C}$ to an injective function. What would be the statement if F was contravariant?

Hint: it's enough to prove this for the functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ for some object X of $\mathscr{C}$ (why?).

Using this, produce an example of a non-representable functor (covariant or contravariant), different from the one seen in Example 4.8.

Problem 4.6. Prove that

(a) the forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ is represented by the group $\mathbb{Z}$.

(b) the forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$ is represented by the ring R treated as an R-module.

(c) the functor $U(-)^n : \mathsf{Ring} \to \mathsf{Set}$, where any ring R is sent to the set R^n, is represented by the ring $\mathbb{Z}[t_1, \dots, t_n]$.

(d) the functor $\mathcal{O} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ defined in Example 3.6, where we send a space to its set of open sets, is represented by the *Sierpinski space* $\mathcal{S} = \{0, 1\}$ where the open sets are $\varnothing$, $\{0\}$ and $\mathcal{S}$.

(e) the functor $\mathrm{Hom}_{\mathsf{Set}}(-, X) \times \mathrm{Hom}_{\mathsf{Set}}(-, Y) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$, $T \mapsto \mathrm{Hom}_{\mathsf{Set}}(T, X) \times \mathrm{Hom}_{\mathsf{Set}}(T, Y)$ is represented by the cartesian product $X \times Y$.

Problem 4.7. Given a group G, prove that a functor $E : \mathsf{B}G \to \mathsf{Set}$, i.e., a left G-set E (see Example 3.1), is representable if and only if there is an isomorphism $G \cong E$ of left G-sets.

This implies that the action of G on E is free (every stabilizer group is trivial) and transitive (the orbit of any point is the entire set), and that E is non-empty. One thinks of this E being a group that has forgotten its identity element. Such a G-set is called a *G-torsor*.

5. Lecture 5 (2/04) by David

Recall the notion of a representable functor: given a functor $F : \mathscr{C} \to \mathsf{Set}$, we say F is representable if there exists an object X in $\mathscr{C}$ such that we have a natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X, -) \cong F.$$

We may then ask the following questions, that were either directly or implicitly brought up last time.

- Is this X unique?
- How does this relate to initial (or terminal) objects?
- How does this relate to universal properties?

First, recall the notation $\mathscr{D}^{\mathscr{C}}$ from Problem 3.1.

Theorem 5.1 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be locally small and $F : \mathscr{C} \to \mathsf{Set}$ a functor. For any object X in $\mathscr{C}$ there's a bijection*

$$\mathrm{Nat}(h^X, F) \cong F(X),$$

where the former is the set of natural transformations from $h^X = \mathrm{Hom}_{\mathscr{C}}(X, -)$ to F, that is, the set of morphisms from h^X to F in $\mathsf{Set}^{\mathscr{C}}$.

Similarly, if $G : \mathscr{C}^{op} \to \mathsf{Set}$ is a contravariant functor, then for any object X in $\mathscr{C}$ there's a bijection

$$\mathrm{Nat}(h_X, G) \cong G(X),$$

where the former is the set of morphisms from $h_X = \mathrm{Hom}_{\mathscr{C}}(-, X)$ to G in $\mathsf{Set}^{\mathscr{C}^{op}}$.

We will see a proof soon.

Corollary 5.2 (Yoneda Embedding)**.** *The functors*

$$\mathscr{Y}^* : \mathscr{C}^{op} \to \mathsf{Set}^{\mathscr{C}},\ X \mapsto h^X$$

and

$$\mathscr{Y}_* : \mathscr{C} \to \mathsf{Set}^{\mathscr{C}^{op}},\ X \mapsto h_X$$

are fully faithful.

Proof. Recall that $\mathscr{Y}^*$ is fully faithful if

$$\mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Nat}(h^X, h^Y) : f \mapsto \mathscr{Y}^*(f) = - \circ f$$

is a bijection for any objects X and Y. Yoneda Lemma (Theorem 5.1) tells us that when applied to $F = h^Y = \mathrm{Hom}_{\mathscr{C}}(Y, -)$ we get

$$\mathrm{Hom}_{\mathscr{C}}(Y, X) = \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) \cong \mathrm{Hom}_{\mathsf{Set}^{\mathscr{C}}}(\mathscr{Y}^*(X), \mathscr{Y}^*(Y)) = \mathrm{Nat}(h^X, h^Y).$$

The proof of Yoneda Lemma (see end) tells us that this bijection in this case is indeed the map needed. Therefore $\mathscr{Y}^*$ is fully faithful.

The second statement for $\mathscr{Y}_*$ similarly follows from the contravariant version of the Yoneda Lemma.

In particular,

$$\mathrm{Nat}(h^X, h^Y) \cong \mathrm{Hom}_{\mathscr{C}}(Y, X) \quad \text{and} \quad \mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

That is, natural transformations between representable functors correspond to maps between the representing objects. $\square$

Example 5.3 (Revisiting Example 4.11)**.** For a field k, consider two vector spaces V and W. Recall the functor

$$\mathrm{Bil}_k(V \times W, -) : \mathsf{Vec}_k \to \mathsf{Set},$$

where any k-vector space U is sent to the set of k-bilinear maps $\mathrm{Bil}_R(V \times W, U)$. A representation of this functor is $V \otimes_k W$, that is

$$\mathrm{Hom}_k(V \otimes_k W, U) \cong \mathrm{Bil}_k(V \times W, U)$$

Yoneda Lemma (rather, the proof) tells us that the object $V \otimes_k W$ representing $\mathrm{Bil}_k(V \times W, -)$ is uniquely determined by a "universal element" of $\mathrm{Bil}_k(V \times W, V \otimes_k W)$, which is the bilinear map

$$\iota : V \times W \to V \otimes_k W,\ (v, w) \mapsto v \otimes w$$

We can use Problem 5.4, a consequence of the Yoneda embedding, to prove the following facts about tensor products.

(1) $k \otimes_k V \cong V$ for any vector space V;

(2) $V \otimes_k W \cong W \otimes_k V$ for any pair of vector spaces V, W;

(3) $(U \otimes_k V) \otimes_k W \cong U \otimes_k (V \otimes_k W)$ for any triple of vector spaces U, V and W.

We sketch the proof of (2) and leave the remaining two as exercises. We show that we have a natural isomorphism $\mathrm{Bil}_k(V \times W, -) \cong \mathrm{Bil}_k(W \times V, -)$, then Problem 5.4 immediately gives us $V \otimes_k W \cong W \otimes_k V$.

The components for this natural isomorphism are given as, for any vector space U

$$\mathrm{Bil}_k(V \times W, U) \to \mathrm{Bil}_k(W \times V, U)$$
$$f \mapsto \tilde{f}, \text{ where } \tilde{f}(w, v) := f(v, w)$$

One checks this is a bijection and natural in U, and thus we have $V \otimes_k W \cong W \otimes_k V$.

Sketch of Proof of Theorem 5.1 (Yoneda Lemma). We want to establish a bijection

$$\mathrm{Nat}(h^X, F) \cong F(X)$$

Consider a natural transformation $\eta : h^X \to F$, then consider its component at X, it is a function $\eta_X : \mathrm{Hom}_{\mathscr{C}}(X, X) \to F(X)$. Then $x_\eta := \eta_X(1_X) \in F(X)$. So we have a function,

$$\Phi : \mathrm{Nat}(h^X, F) \to F(X),\ \eta \mapsto x_\eta$$

Furthermore, given a morphism $u : X \to A$ for some object A in $\mathscr{C}$, by naturality of η we have a commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[r, "\eta_X"] \arrow[d, "u\circ -"'] & F(X) \arrow[d, "F(u)"] \\ \mathrm{Hom}_{\mathscr{C}}(X,A) \arrow[r, "\eta_A"'] & F(A) \end{tikzcd}$$

Taking $1_X \in \mathrm{Hom}_{\mathscr{C}}(X,X)$ along the two sides of the square we get

$$\eta_A(u) = \eta_A(u \circ 1_X) = F(u) \circ \eta_X(1_X) = F(u)(x_\eta)$$

This motivates us to give the following function

$$\Psi : F(X) \to \mathrm{Nat}(h^X, F),\ x \mapsto \eta_x$$

where η_x is the natural transformation with components, for any object A in $\mathscr{C}$, given by

$$\eta_{x,A} : \mathrm{Hom}_{\mathscr{C}}(X,A) \to F(A),\ u \mapsto F(u)(x)$$

One readily checks that this is indeed natural in A, and that Φ and Ψ are inverses of each other. $\square$

Revisiting the Proof of Corollary 5.2 (Yoneda Embedding). Let the notation be as in the statement of Corollary 5.2, we now apply the proof of the Yoneda Lemma to the functor $F = h^Y$ and see what the bijection Ψ is explicitly in this case. The claim is that Ψ is equal to the function

$$\mathrm{Hom}_{\mathscr{C}}(Y,X) \to \mathrm{Nat}(h^X, h^Y) : f \mapsto \mathscr{Y}^*(f) = - \circ f.$$

In the language of the theorem, we have

$$\Psi : \mathrm{Hom}_{\mathscr{C}}(Y,X) \to \mathrm{Nat}(h^X, h^Y),\ f \mapsto \eta_f,$$

where η_f is the natural transformation with components, for any object A in $\mathscr{C}$, given by

$$\eta_{f,A} : \mathrm{Hom}_{\mathscr{C}}(X,A) \to \mathrm{Hom}_{\mathscr{C}}(Y,A),\ u \mapsto h^Y(u)(f).$$

Recall that $h^Y(u) = u \circ -$, and therefore $h^Y(u)(f) = u \circ f$. Hence,

$$\eta_{f,A} : u \mapsto u \circ f, \qquad \text{i.e.,} \quad \eta_{f,A} = - \circ f,$$

for any A, which is precisely the claim. $\square$

Discussion 5.4. Suppose $F : \mathscr{C} \to \mathrm{Set}$ is representable with representation (X, α), that is $\alpha : h^X \Rightarrow F$ is a natural isomorphism. Now, the Yoneda Lemma tells us that α necessarily corresponds to an element of $F(X)$, which is precisely $\xi := \alpha_X(1_X)$ (the map Φ). We can, in fact, construct α from just ξ (the map Ψ). ξ is then called a *universal element* and a representation of F is just given to be (X, ξ). More precisely, $\xi \in F(X)$ is an element such that

$$\alpha_{\xi,A} : \mathrm{Hom}_{\mathscr{C}}(X,A) \to F(A),\ u \mapsto F(u)(x)$$

is a bijection, i.e., for every $a \in F(A)$ there exists a unique $u \in \mathrm{Hom}_{\mathscr{C}}(X, A)$ such that $a = F(u)(\xi)$. For a functor F with a representation (X, ξ), a *universal property* is the description of the natural isomorphism $h^X \Rightarrow F$ given by ξ.

One notes in Examples 4.11 and 4.12, ι and π precisely the image of the identity maps under the given bijections towards the end; they are indeed the universal elements.

5.1. Problems

Problem 5.1. Consider the category $\mathsf{Set}^{\mathscr{C}^{\mathrm{op}}}$, prove that the subcategory of representable functors is equivalent to $\mathscr{C}$.

Problem 5.2. Using the Yoneda embedding (Corollary 5.2) with respect to $\mathscr{C} = \mathsf{Mat}_A$ (see Example 1.4) prove that *every row operation on matrices with n rows is defined by left multiplication by some $n \times n$ matrix*.

Problem 5.3. Using the Yoneda embedding (Corollary 5.2) with respect to $\mathscr{C} = \mathsf{B}G$ (see Example 1.4) prove *Cayley's Theorem: any group is isomorphic to a subgroup of a permutation group*.

Problem 5.4. Suppose $F : \mathscr{C} \to \mathsf{Set}$ is representable by objects X and Y in $\mathscr{C}$, that is

$$\mathrm{Hom}_{\mathscr{C}}(X, -) \cong F \cong \mathrm{Hom}_{\mathscr{C}}(Y, -),$$

prove that $X \cong Y$ using Yoneda embedding (Corollary 5.2) and Problem 3.4.

Use this to prove that any two initial objects are isomorphic.

Problem 5.5. Prove statements (1) and (3) in Example 5.3.

Problem 5.6. Given an object X in a category $\mathscr{C}$, what's the universal element (see Discussion 5.4) of the functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ (or $\mathrm{Hom}_{\mathscr{C}}(-, X)$, for that matter)?

Problem 5.7. Consider Example 5.3. Let V and W be k-vector spaces.

(a) Construct a *"category of bilinear maps"* out of $V \times W$.

(b) Prove that $\iota : M \times N \to M \otimes_k N$ is an initial object in this category.

More generally, this refers to the *category of elements*, see Problem 5.8 (f), (g), (h).

Problem 5.8. Given functors $F : \mathscr{D} \to \mathscr{C}$ and $G : \mathscr{E} \to \mathscr{C}$, we describe the *comma category* $F \downarrow G$. It has

- as objects triples (D, E, f), where D is an object in $\mathscr{D}$ and E in $\mathscr{E}$, and $f : F(D) \to G(E)$ is a morphism in $\mathscr{C}$.
- as morphisms $(D, E, f) \to (D', E', f')$ pairs of morphisms (h, k) where $h : D \to D'$ is a morphism in $\mathscr{D}$ and $k : E \to E'$ in $\mathscr{E}$ such that the diagram

$$\begin{tikzcd} F(D) \arrow[r, "f"] \arrow[d, "F(h)"'] & G(E) \arrow[d, "G(k)"] \\ F(D') \arrow[r, "f'"'] & G(E') \end{tikzcd}$$

  commutes.

Convince yourself this is indeed a category. What are the identity morphisms? How's composition defined?

Consider the following questions.

(a) Describe two canonical projection functors $H_{\mathscr{D}} : F \downarrow G \to \mathscr{D}$ and $H_{\mathscr{E}} : F \downarrow G \to \mathscr{E}$.

(b) Let $\mathscr{D} = \mathscr{C}$ and $F = 1_{\mathscr{C}}$, and let $\mathscr{E} = *$, the singleton set treated as a category. Let $X = G(*)$, then prove that $F \downarrow G$ is the slice category $\mathscr{C}/X$, which is therefore sometimes denoted as $\mathscr{C} \downarrow X$.

(c) Similarly, provide a description of $X/\mathscr{C}$ as a comma category. That is, make sense of the alternate notation $X \downarrow \mathscr{C}$.

(d) Describe the projection functors given in (a) in (b) and (c).

(e) If we let $\mathscr{D} = \mathscr{E} = \mathscr{C}$ and $F = G = 1_{\mathscr{C}}$, the resulting category is called the *arrow category of* $\mathscr{C}$ and denoted as either $\mathscr{C}^{\to}$ or $\mathrm{Arr}(\mathscr{C})$ or $\mathscr{C}^2$ or, of course, $1_{\mathscr{C}} \downarrow 1_{\mathscr{C}}$. Describe this category, i.e., its objects and morphisms.

(f) Let $F : \mathscr{C} \to \mathrm{Set}$ be a functor and consider $* \to \mathrm{Set}$ which we also denote by $*$. The category $* \downarrow F$ is called the *category of elements of* F and sometimes denoted as $\int F$ or $\mathrm{el}(F)$. Prove that $\int F$ has the following description: it has
   - as objects pairs (C, x) where C is an object in $\mathscr{C}$ and $x \in F(C)$.
   - as morphisms $(C, x) \to (C', x')$ a morphism $f : C \to C'$ such that $F(f)(x) = x'$.

(g) Prove that if F is representable, that is $F \cong \mathrm{Hom}_{\mathscr{C}}(X, -)$, then $\int F$ is equivalent to the category $X/\mathscr{C}$. In particular, it has an initial element, namely 1_X.

(h) Conversely, prove that if $\int F$ has an initial object, then F is representable.

(i) Describe the functor $\int F \to \mathscr{C}$ as given in (a). Given an object C in $\mathscr{C}$ describe the objects in $\int F$ that get sent to C under this functor, this is called the fiber over C.

6. Lecture 6 (2/11) by David

Example 6.1 (Product of Sets). Given two (assume non-empty for ease) sets X and Y, let's investigate the cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$. There are two obvious maps that we can always consider in this case, the projections

$$\begin{tikzcd} & X \\ X \times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & Y \end{tikzcd}$$

where

$$p_X : (x, y) \mapsto x \quad \text{and} \quad p_Y : (x, y) \mapsto y$$

$(X \times Y, p_X, p_Y)$ possesses a defining property. To see this, suppose we consider any other set Z with functions

$$f_X : Z \to X \quad \text{and} \quad f_Y : Z \to Y.$$

Then we can describe an induced function

$$f : Z \to X \times Y,\ z \mapsto (f_X(z), f_Y(z)),$$

which is such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$. Diagramatically, we describe this as follows

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "f_X"] \arrow[drr, bend right, "f_Y"'] \arrow[r, dashed, "\exists f"] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\ & & Y \end{tikzcd}$$

where the triangles commute.

Claim. *The function f is the unique function with the property $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$. That is, f is the unique function that fits into the diagram above.*

Suppose $g : Z \to X \times Y$ is any other functions such that $f_X = p_X \circ g$ and $f_Y = p_Y \circ g$. Then necessarily, we have $g : z \mapsto (f_X(z), f_Y(z))$ and therefore $f = g$. Thus,

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "f_X"] \arrow[drr, bend right, "f_Y"'] \arrow[r, dashed, "\exists! f"] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\ & & Y \end{tikzcd}$$

This is describing the *universal property* of $(X \times Y, p_X, p_Y)$ (what functor are they representation of?). One thinks of this, morally, as saying that the product $(X \times Y, p_X, p_Y)$ is the best approximation of the diagram

$$\begin{array}{c} X \\ \\ Y \end{array}$$

from the left. It's important to note that the maps p_X and p_Y play a fundamental role.

Attempting Definition 6.2 (Take 1: Intuition as Approximation)**.** Fix a category $\mathscr{C}$.

- *Limits* are the best "best approximation" of a diagram "on the left". For example,

$$\begin{tikzcd}
X_1 \arrow[dr, "a"] & \\
& X_2 \\
X_3 \arrow[ur, "b"] \arrow[dr, "c"] & \\
& X_4 \\
X_5 \arrow[ur, "d"'] &
\end{tikzcd}$$

consider this diagram

$$\begin{tikzcd}
& X_1 \arrow[dr, "a"] & \\
& & X_2 \\
L \arrow[uur, bend left, "1"] \arrow[urr, bend left, "2"] \arrow[r, "3"] \arrow[drr, bend right, "4"] \arrow[ddr, bend right, "5"] & X_3 \arrow[ur, "b"'] \arrow[dr, "c"] & \\
& & X_4 \\
& X_5 \arrow[ur, "d"'] &
\end{tikzcd}$$

a limit (in red) is mapping in to the diagram

So, a limit is first an object along with morphisms that map into the diagram such that all triangles commute.

In our example, commutativity makes some morphisms superfluous, as $2 = a \circ 1 = b \circ 3$ and $4 = c \circ 3 = d \circ 5$. Therefore, it suffices to consider

$$\begin{tikzcd}
& X_1 \arrow[dr, "a"] & \\
& & X_2 \\
L \arrow[uur, bend left, "1"] \arrow[r, "3"] \arrow[ddr, bend right, "5"] & X_3 \arrow[ur, "b"'] \arrow[dr, "c"] & \\
& & X_4 \\
& X_5 \arrow[ur, "d"'] &
\end{tikzcd}$$

The "best approximation" property of the limit $(L, 1, 3, 5)$ translates to the following diagram for any object N mapping in to the diagram

$$\begin{tikzcd}
 & & X_1 \arrow[dr, "a"] & \\
 & & & X_2 \\
N \arrow[uurr, bend left] \arrow[r, dashed, "\exists!"] \arrow[rr, bend right] \arrow[ddrr, bend right] & L \arrow[uur, bend left, "1"] \arrow[r, "3"] \arrow[ddr, bend right, "5"] & X_3 \arrow[ur, "b"] \arrow[dr, "c"] & \\
 & & & X_4 \\
 & & X_5 \arrow[ur, "d"] &
\end{tikzcd}$$

where the unique map $N \to L$ makes all possible diagrams commute.

- Dually, *Colimits* are the best "best approximation" of a diagram "on the right". For our example diagram above, this translates to a colimit (in blue), an object along with morphisms, mapping out of the diagram such that all triangles commute.

$$\begin{tikzcd}
X_1 \arrow[dr, "a"] \arrow[ddrr, bend left, "1'"] & & & & X_1 \arrow[dr, "a"] & & \\
 & X_2 \arrow[dr, bend left, "2'"] & & & & X_2 \arrow[dr, bend left, "2'"] & \\
X_3 \arrow[ur, "b"] \arrow[dr, "c"] \arrow[rr, "3'"] & & C & \rightsquigarrow & X_3 \arrow[ur, "b"] \arrow[dr, "c"] & & C \\
 & X_4 \arrow[ur, bend right, "4'"] & & & & X_4 \arrow[ur, bend right, "4'"] & \\
X_5 \arrow[ur, "d"] \arrow[uurr, bend right, "5'"] & & & & X_5 \arrow[ur, "d"] & &
\end{tikzcd}$$

since $1' = 2' \circ a$, $3' = 2' \circ b = 4' \circ c$ and $5' = 4' \circ d$. The "best approximation" property of the colimit $(C, 2', 4')$ translates to the following diagram for any object M mapping out of the diagram

$$\begin{tikzcd}
X_1 \arrow[dr, "a"] & & & \\
 & X_2 \arrow[dr, bend left, "2'"] \arrow[drr, bend left] & & \\
X_3 \arrow[ur, "b"] \arrow[dr, "c"] & & C \arrow[r, dashed, "\exists!"] & M \\
 & X_4 \arrow[ur, bend right, "4'"] \arrow[urr, bend right] \arrow[dl, "d"] & & \\
X_5 & & &
\end{tikzcd}$$

where the unique map $C \to M$ makes all possible diagrams commute.

Compare the limit to a notion of minimum, and colimit to the notion of a maximum. We make this rigorous in Problem 6.2.

Attempting Definition 6.3 (Take 2: Cones)**.** Consider a diagram in a category $\mathscr{C}$, for example, such as

$$\cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots$$

Definition. A *cone over a diagram with summit* C is an object C in $\mathscr{C}$ with maps

$$\begin{tikzcd}
 & & C \arrow[dll, dashed] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, dashed] & & \\
\cdots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that all the triangles commute, illustrated here with our example.

Example 6.4.

- The cone of the diagram below (in black) is

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, "\gamma"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

  where, since the triangles commute, we have $\gamma = f\alpha = g\beta$; so we can exclude γ. Therefore we get a diagram, given on the right, such that $f\alpha = g\beta$.

- The cone of the diagram below (in black) is

$$\begin{tikzcd}
 & C \arrow[dl, bend right, "\alpha"'] \arrow[dr, bend left, "\beta"] & \\
X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Y
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, "\alpha"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

  where, since the triangles commute, we have $\beta = f\alpha = g\alpha$; so we can exclude β. Therefore we get a diagram, given on the right, such that $f\alpha = g\alpha$.

Definition 6.5. The *limit* of a diagram is the *universal cone over the diagram*.

Discussion 6.6. We still haven't really given a precise definition of a limit, the Yoneda lemma helps us out. You're better served reading this remark after you've attempted a few problems and have gained familiarity with specific examples of limits.

- A *$\mathscr{J}$-shaped diagram* in a category $\mathscr{C}$ is a functor $F : \mathscr{J} \to \mathscr{C}$, where $\mathscr{J}$, called the indexing category, is a small category: a set's worth of objects and morphisms.

  For example, the (naive) diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

  is interpreted as a diagram, in the sense introduced above, as follows: let $\mathscr{J}$ be the category visualised as follows

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

  That is, it's a category with three objects where the only morphisms, apart from identity morphisms, are where two of the objects map to the third one. Then the naive diagram above is the image of the $\mathscr{J}$-shaped diagram (functor) $F : \mathscr{J} \to \mathscr{C}$ which sends the three objects and two morphisms appropriately to X, Y, Z, f and g (write down this functor precisely for yourself).

- For a fixed object C in $\mathscr{C}$, one can the consider the *constant $\mathscr{J}$-shaped diagram* $\Delta(C) : \mathscr{J} \to \mathscr{C}$. This is the functor that sends every object and morphism in $\mathscr{J}$ to C and 1_C respectively.

- Thinking of diagrams in this way, a cone over a diagram F with summit C is simply a natural transformation

$$\eta : \Delta(C) \Rightarrow F$$

- Fixing a diagram F, there is a functor

$$\mathrm{Cone}(-, F) : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$$

  that sends any object C to $\mathrm{Cone}(C, F)$ which is the set of cones over F with summit C.

- Then a *limit of the diagram F*, if it exists, is a representation (in the sense of Discussion 5.4) of the functor $\mathrm{Cone}(-, F)$. That is, a limit is $(\lim F, \lambda)$ where $\lim F$ is an object in $\mathscr{C}$ and $\lambda : \Delta(\lim F) \Rightarrow F$ is a cone over F with summit $\lim F$ that induces a natural isomorphism

$$\mathrm{Nat}(\Delta(-), F) = \mathrm{Cone}(-, F) \cong \mathrm{Hom}_{\mathscr{C}}(-, \lim F)$$

  This, in particular, tells us that the limit, whenever it exists, is unique upto a unique isomorphism.

- Equivalently (see Problem 5.8 (h)) $\lim F$ is a terminal object in $\int \mathrm{Cone}(-, F)$.

Example 6.7 (Some Special Limits)**.**

(i) The limit (in colour) of the diagram (in black); the indexing category is on the right

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

is called the *pullback* or *fiber(ed) product*; this square is also sometimes called a *Cartesian square*. Here $fp_X = gp_Y$.

(ii) The limit (in colour) of the diagram (in black); the indexing category is on the right

$$\begin{tikzcd} \mathrm{eq}(f,g) \arrow[r, "\alpha"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

is called the *equaliser*. Here $f\alpha = g\alpha$.

(iii) The limit (in colour) of the diagram (in black); the indexing category is below

$$\begin{tikzcd} & & & \prod_i X_i \arrow[dlll, dashed] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr, dashed] & & & \\ \cdots & X_{n-2} & X_{n-1} & X_n & X_{n+1} & X_{n+2} & \cdots \end{tikzcd}$$

$$\cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$

is called the *product*.

(iv) The limit (in colour) of the diagram (in black); the indexing category is below

$$\begin{tikzcd} & & & & \varprojlim P_i \arrow[dllll, dashed, "f_4"'] \arrow[dlll, "f_3"'] \arrow[dll, "f_2"'] \arrow[dl, "f_1"] \arrow[d, "f_0"] \\ \cdots \arrow[r, dashed, "\pi_4"'] & P_3 \arrow[r, "\pi_3"'] & P_2 \arrow[r, "\pi_2"'] & P_1 \arrow[r, "\pi_1"'] & P_0 \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r, dashed] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

is an example of a limit called the *sequential limit* (or *projective limit* or *inverse limit*). Here $\pi_{i+1} f_{i+1} = f_i$ for all $i \geqslant 0$.

Attempting Definition 6.8. We can repeat the discussion in Attempting Definition 6.3 in a dual manner: considering cones *under* a diagram.

Definition 6.9. A *colimit* for a diagram is the *universal cone* under it.

Dually to the discussion in Discussion 6.6, we produce a functor for a diagram F

$$\mathrm{Cone}(F, -) : \mathscr{C} \to \mathsf{Set}$$

The colimit, if it exists, will be a representation of this functor, if and only if $\int \mathrm{Cone}(F, -)$ has an initial object. We will have exhibited a natural isomorphism

$$\mathrm{Nat}(F, \Delta(-)) = \mathrm{Cone}(F, -) \cong \mathrm{Hom}_{\mathscr{C}}(\operatorname{colim} F, -)$$

More precisely, a colimit will be $(\operatorname{colim} F, \kappa)$ where $\operatorname{colim} F$ is an object in $\mathscr{C}$ and $\kappa : F \Rightarrow \Delta(\operatorname{colim} F)$ is a cone under F with summit $\operatorname{colim} F$ which induces the above natural isomorphism.

Therefore, colimits, just like limits, are unique up to unique isomorphism whenever they exist.

Example 6.10 (Some Special Colimits)**.**

(i) The colimit (in colour) of the diagram (in black); the indexing category is on the right

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_X"] \\ Y \arrow[r, "i_Y"'] & X \amalg_Z Y \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

is called the *pushout* or *fibered coproduct*. Here $i_X f = i_Y g$.

(ii) The colimit (in colour) of the diagram (in black); the indexing category is on the right

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] & \operatorname{coeq}(f, g) \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

is called the *coequaliser*. Here $\alpha f = \alpha g$.

(iii) The colimit (in colour) of the diagram (in black); the indexing category is below

$$\begin{tikzcd} & & & \coprod_i X_i & & & \\ \cdots \arrow[urrr, dashed] & X_{n-2} \arrow[urr] & X_{n-1} \arrow[ur] & X_n \arrow[u] & X_{n+1} \arrow[ul] & X_{n+2} \arrow[ull] & \cdots \arrow[ulll, dashed] \end{tikzcd}$$

$$\cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$

is called the *coproduct*.

(iv) The colimit (in colour) of the diagram (in black); the indexing category is below

$$\begin{tikzcd} & \varinjlim I_j & & & \\ I_0 \arrow[r, "\iota_0"'] \arrow[ur, "g_0"] & I_1 \arrow[r, "\iota_1"'] \arrow[u, "g_1"] & I_2 \arrow[r, "\iota_2"'] \arrow[ul, "g_2"] & I_3 \arrow[r, dashed, "\iota_3"'] \arrow[ull, "g_3"] & \cdots \arrow[ulll, dashed, "g_4"] \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r, dashed] & \cdots \end{tikzcd}$$

is an example of a colimit called the *sequential colimit* (or *direct limit*; more generally, *inductive limit*). Here $g_{j+1}\iota_j = g_j$ for all $j \geqslant 0$.

Example 6.11. Here are some important objects defined as limit and colimits in various field of mathematics.

- There's a projective system, where p is a prime,

$$\cdots \xrightarrow{\pi_4} \mathbb{Z}/p^4\mathbb{Z} \xrightarrow{\pi_3} \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi_1} \mathbb{Z}/p\mathbb{Z}$$

  where each π_i is the usual reduction map. The *p-adic integers* is defined to be $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

- There are directed systems

$$\mathbb{R}^0 \hookrightarrow \mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots$$

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots$$

  where each inclusion is the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$.

  The *infinite dimensional Euclidean space* and the *infinite dimensional sphere* are defined to be $\mathbb{R}^\infty := \varinjlim \mathbb{R}^n$ and $S^\infty := \varinjlim S^n$ respectively.

- If you have seen CW complexes or relative CW complexes, they can be defined very cleanly as a colimit (of their cells) in Top.

- Recall the notion of a *solenoid* described at the beginning of Lecture 2: it was described as a collection $(S_i, f_i)_{i\in\mathbb{Z}_{\geqslant 0}}$ where S_i are circles and $f_i : S_{i+1} \to S_i$ is the map that wraps S_{i+1} around S_i, n_i times ($n_i \in \mathbb{Z}_{\geqslant 2}$).

  This is actually describing a projective system

$$\cdots \xrightarrow{f_4} S_4 \xrightarrow{f_3} S_3 \xrightarrow{f_2} S_2 \xrightarrow{f_1} S_1$$

  and a *solenoid* $S := \varprojlim S_i$.

  If $P = (n_i)_{i\geqslant 0}$ is in fact a sequence of prime numbers, that is, n_i is a prime number for each i, then S is called a *P-adic solenoid*.

Remark 6.12. In concrete cases, one needs to actually determine if limits (and colimits) exist and what they are, usually by explicitly constructing them.

We shall see that for categories that are "sets with extra steps", i.e. categories $\mathscr{C}$ such that there's a faithful functor

$$U : \mathscr{C} \to \mathsf{Set}$$

(these are called *concrete categories* and U the *forgetful functor*), the limits are easy to describe. In contrast, the colimits can be much harder to describe (see Problem 6.8 (d), for example). We will see why this is so when we discuss *adjunction*.

You will see some examples of limits in the Problems. In the next lecture, we will construct all (small) limits in Set.

6.1. Problems

Problem 6.1. In this problem we will prove by hand that the product of two objects, assuming it exists, in a category $\mathscr{C}$ is *unique up to a unique isomorphism*.

(a) Let X and Y be objects, and suppose (P, p_X, p_Y) and (Q, q_X, q_Y) are products of X and Y. Following is the diagram exhibiting the universal property of P with respect to Q

$$\begin{tikzcd} & & X \\ Q \arrow[r, dashed, "\exists!\phi"] \arrow[urr, bend left, "q_X"] \arrow[drr, bend right, "q_Y"'] & P \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\ & & Y \end{tikzcd}$$

Draw the diagram exhibiting the universal property of Q with respect to P.

(b) Combine the two diagrams you obtained in (a) to obtain (commutative) diagrams as follows

$$\begin{tikzcd} & & X \\ Q \arrow[r, dashed, "\exists!"] \arrow[urr, bend left, "q_X"] \arrow[drr, bend right, "q_Y"'] & Q \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\ & & Y \end{tikzcd} \qquad \begin{tikzcd} & & X \\ P \arrow[r, dashed, "\exists!"] \arrow[urr, bend left, "p_X"] \arrow[drr, bend right, "p_Y"'] & P \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\ & & Y \end{tikzcd}$$

(c) What's the diagram exhibiting the universal property of P with respect to P? How about Q?

(d) Using (b) and (c), conclude that Q and P are isomorphic to each other via ϕ.

Problem 6.2. We'll look at some examples of products, to remind yourself of the categories mentioned, look at Examples 1.2, 1.3 and 1.4.

(a) Recall that any poset can be considered a category. So, given a poset $(\mathsf{P}, \leqslant)$, a product of two objects (i.e., elements of P) x, y, if it exists, is given by $\min\{x, y\}$.

Using this, find the product of two objects in the following posets.

(i) The poset $(\mathbb{Z}_{>0}, \leqslant)$ where $m \leqslant n$ if and only if $m \mid n$.

(ii) The poset $(\mathscr{P}(X), \leqslant)$, where $\mathscr{P}(X)$ is the power set of a set X, where $A \leqslant B$ if and only if $A \subseteq B$.

In the general theory, the (categorical) product of two elements x and y is called their *meet* and denoted $x \wedge y$. The *joint* $x \vee y$ is exactly the coproduct.

(b) Consider the category Mat_k, where k is a field. Prove that the product of n and m is $n + m$.

(c) Consider the category of fields, does it have products?

Problem 6.3.

(a) Prove that the cartesian product of two sets is indeed the categorical product in Set.

(b) Prove that the disjoint union of two sets is the categorical coproduct in Set.

Problem 6.4. Consider a category $\mathscr{C}$ where products exist.

(a) Prove that $\mathscr{C}$ has a terminal object T by showing that it's the empty product. Dually, what's the condition for $\mathscr{C}$ having an initial object with respect to coproducts?

(b) Describe the twist isomorphism $\tau : X \times Y \to Y \times X$ for any two objects X and Y.

(c) Prove that $X \times T \cong X$ for any object X, where T is a terminal object of $\mathscr{C}$. What's the dual statement?

(d) Prove that for objects X and Y, we have $X \times_T Y \cong X \times Y$. What's the dual statement?

(e) What's the product of two objects $X \to Z$ and $Y \to Z$ in the slice category $\mathscr{C}/Z$.

Problem 6.5. Discussion 6.6 tells us that the limit represents the cone functor. Let's make this explicit in the case of product. Fix a category $\mathscr{C}$, and let $\mathscr{J}$ be the category

$$\bullet \qquad \bullet$$

that is, it's a category with two objects and no morphisms other than identity morphisms.

(a) Prove a $\mathscr{J}$-shaped diagram, i.e., a functor $F : \mathscr{J} \to \mathscr{C}$ just picks two objects in $\mathscr{C}$, say X_1 and X_2.

(b) Make sense of this: product of two objects is the limit of a $\mathscr{J}$-shaped diagram.

(c) For any object Y in $\mathscr{C}$, prove that $\mathrm{Cone}(\Delta(Y), F) \cong \mathrm{Hom}_{\mathscr{C}}(Y, X_1) \times \mathrm{Hom}_{\mathscr{C}}(Y, X_2)$.

(d) Conclude that a product $(X \times Y, p_X, p_Y)$ represents the functor

$$Y \mapsto \mathrm{Hom}_{\mathscr{C}}(Y, X_1) \times \mathrm{Hom}_{\mathscr{C}}(Y, X_2)$$

(e) Explicitly describe the bijection

$$\mathrm{Hom}_{\mathscr{C}}(Y, X_1 \times X_2) \cong \mathrm{Hom}_{\mathscr{C}}(Y, X_1) \times \mathrm{Hom}_{\mathscr{C}}(Y, X_2)$$

Problem 6.6. Let $\mathscr{C}$ be a category and let $(X \times Y, p_X, p_Y)$ be a product of the objects X and Y in $\mathscr{C}$. Show that if there exists a map $f : X \to Y$, then p_X is an epimorphism in $\mathscr{C}$. Formulate a similar statement for p_Y.

Formulate a dual statement for coproducts.

Problem 6.7. Recall from Example 6.11 that $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n/\mathbb{Z}$.

(a) Prove that the limit can be explicitly described as follows

$$\left\{ (a_n)_{n\geqslant 1} \in \prod_{n\geqslant 1} \mathbb{Z}/p^n\mathbb{Z} \;:\; \pi_i(a_{i+1}) = a_i, \text{ for all } i \geqslant 1 \right\}.$$

(b) Prove that one can then describe

$$\mathbb{Z}_p = \left\{ \sum_{n\geqslant 0} \alpha_n p^n \;:\; 0 \leqslant \alpha_i < p, \text{ for all } i \geqslant 0 \right\}$$

Problem 6.8. Consider the category of groups Grp and the subcategory of abelian subgroups Ab.

(a) Prove that the product in Grp and Ab is the direct product of groups.

(b) What is the equalise of the diagram in Grp

$$\begin{tikzcd} G \arrow[r, shift left, "\phi"] \arrow[r, shift right, "e"'] & H \end{tikzcd}$$

where e denotes the trivial map. How about the coequaliser of this diagram in Ab?

(c) Describe the pullback of

$$\begin{tikzcd} & G \arrow[d, "\phi"] \\ H \arrow[r, "\psi"'] & K \end{tikzcd}$$

in Grp. What if $\psi = e$, the trivial map? What if $H = *$, the trivial group?

(d) Describe the pushout of, where $N \trianglelefteq G$

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & G \\ * & \end{tikzcd}$$

in Grp.

(d) Colimits in subcategories can be vastly different from those in the ambient category. We illustrate this with following problem.

(i) Let G and H be groups, assume they have presentations

$$G = \langle S_G \mid R_G \rangle \quad \text{and} \quad H = \langle S_H \mid R_H \rangle,$$

then the *free product of G and H* is the group with presentation

$$G * H := \langle S_G \sqcup S_H \mid R_G \sqcup R_H \rangle$$

Prove that $G * H$ is the coproduct of G and H in Grp.

(ii) If G and H are abelian, as long as one of them is non-trivial, $G * H$ is not abelian. Therefore, the coproduct of abelian groups in Grp is not an abelian group. One may then be lead to deduce that coproducts of abelian groups don't exist in Ab. This will be a false deduction, coproducts in Ab exist.

Prove that given abelian group A and B, the coproduct of A and B in Ab is $A \times B$.

(iii) Let A and B be abelian groups, prove that

$$A \times B \cong (A * B)^{\mathrm{ab}}.$$

Problem 6.9. If you've seen topology, prove the following are pushout diagrams in Top. That is, the pushout of the diagram (in black) are objects and morphisms (which you need to describe) in colour.

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, hook] & D^2 \arrow[d] \\ S^1 \arrow[r] & S^2 \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d] & \mathbb{R} \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

Problem 6.10. Consider the category of unital commutative rings CRing

(a) Make sense of

$$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$$

If you've figured this out, try proving an analogous result for localisation of rings in general. More precisely, make sense of

$$S^{-1}A = \varinjlim_{s \in S} A_s$$

where $A_s = A[1/s]$ is the localisation at $s \in S$.

(b) Prove that the following is a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_A C \end{tikzcd}$$

If you've seen algebraic geometry and (pre)sheaves, consider the following questions

(c) Make an educated guess as what should be the pullback of this diagram

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[d, "\phi"] \\ \operatorname{Spec} C \arrow[r, "\psi"'] & \operatorname{Spec} A \end{tikzcd}$$

is in the category of affine schemes.

7. Lecture 7 (2/18) by Vaibhav

We quickly recall the notion of a limit in a category $\mathscr{C}$. We start with a diagram indexed by $\mathscr{J}$, i.e., a functor

$$F : \mathscr{J} \to \mathscr{C},$$

where $\mathscr{J}$ is a small category. Let $A_i := F(i)$ for $i \in \mathscr{J}$, then the limit of F is an object $\lim_{\mathscr{J}} F$ or $\lim_{\mathscr{J}} A_i$ of $\mathscr{C}$ along with morphisms $f_j : \lim_{\mathscr{J}} A_i \to A_j$ for each $j \in \mathscr{J}$ (the cone over F with summit $\lim_{\mathscr{J}} A_i$), such that if $j \to k$ is a morphism in $\mathscr{J}$, then

$$\begin{tikzcd}
& \lim_{\mathscr{J}} A_i \arrow[dl, "f_j"'] \arrow[dr, "f_k"] & \\
A_j \arrow[rr, "F(j\to k)"'] & & A_k
\end{tikzcd}$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property.

More precisely, given any other object W along with maps $g_i : W \to A_i$ commuting with the $F(j \to k)$, i.e., $g_k = F(j \to k)g_j$ (a cone over F with summit W), then there is a unique map

$$g : W \to \lim_{\mathscr{J}} A_i$$

so that $g_i = p_i g$ for all i.

$$\begin{tikzcd}
& W \arrow[d, dashed, "\exists! g"] \arrow[ddl, bend right, "g_j"'] \arrow[ddr, bend left, "g_k"] & \\
& \lim_{\mathscr{J}} A_i \arrow[dl, "f_j"'] \arrow[dr, "f_k"] & \\
A_j \arrow[rr, "F(j\to k)"'] & & A_k
\end{tikzcd}$$

That is, all triangles in the diagram above commute.

Example 7.1 (Profinite Integers)**.** Consider the poset of positive integers with partial order given by divisibility, more precisely, say $m \mid n$ if and only if $m \leqslant n$. We categorify this poset, which we call $\mathscr{N}$, by saying there's a morphism $n \to m$ if and only if $m \mid n$. We give a diagram

$$F : \mathscr{N} \to \mathsf{CRing}$$

where $F(n) := \mathbb{Z}/n\mathbb{Z}$, and $\pi_{nm} =: F(n \to m) : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the usual reduction map. We call the limit $\lim_{\mathscr{N}} \mathbb{Z}/n\mathbb{Z}$ the *profinite integers* and denote it as $\widehat{\mathbb{Z}}$. A vast generalisation of the Chinese Remainder Theorem gives us

$$\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$$

where $\mathbb{Z}_p$ is the ring of p-adic integers described in Example 6.11. Can you describe the indexing category of the diagram $\mathbb{Z}_p$ is a limit of?

Discussion 7.2. You will see all kinds of limits in your journey, but it'll be useful to study in the "prototypical" category of sets. Set is nice for a few reasons.

- It's straightforward.
- It turns out to have all *(small)* limits. This is not true for all categories.
- You can breakdown limits in certain (locally small) categories as limits in Set.

A limit is called small if the indexing category we're taking the limit over is small, which has been our assumption all along. You should be able to deduce what a small colimit is.

Definition 7.3. A category $\mathscr{C}$ is called

- *complete* if it has all small limits.
- *cocomplete* if it has all small colimits.

Set is not only complete but also comcomplete, although the colimits are a bit harder to describe. We will completely describe small limits in Set, and Problem 7.7 describes a particular class of colimits.

Example 7.4 (Limits in Set)**.** Let's work with the category Set and consider a diagram

$$F : \mathscr{J} \to \mathsf{Set},$$

where $\mathscr{J}$ is small. Now, the limit is a pair $(L = \lim_{\mathscr{J}} F, (p_i)_{i\in\mathscr{J}})$ where $(p_i)_{i\in\mathscr{J}}$ assemble to give a cone over F with summit L, i.e. an element of $\mathrm{Cone}(L, F)$ (see Attempting Definition 6.3 and Discussion 6.6). The universal property of $(L, (p_i)_i)$ is that the map, for any set S,

$$\mathrm{Hom}_{\mathsf{Set}}(S, L) \to \mathrm{Cone}(S, F) : g \mapsto (p_i \circ g)_i$$

is a bijection. Our aim is to get a handle on L and the p_i's, how do we do that?

Recall that, for any set X

$$\mathrm{Hom}_{\mathsf{Set}}(*, X) \to X : f \mapsto f(*),$$

where $*$ is a singleton set, is a bijection, with an inverse given by $x \mapsto (* \mapsto x)$. So, we will name an element of $\mathrm{Hom}_{\mathsf{Set}}(*, X)$ with its image in X, i.e., $x : * \to L$ is a function where $x(*) = x \in X$. Hence, we do not distinguish between the map $x : * \to X$ and element $x \in X$.

Let's go back to our bijection of sets given by the universal property of L for $S = *$, it now reads

$$\mathrm{Hom}_{\mathsf{Set}}(*, L) \to \mathrm{Cone}(*, F) : \ell \mapsto (p_i \circ \ell)_i$$

Let's track this more

$$\begin{aligned} L &\cong \mathrm{Hom}_{\mathsf{Set}}(*, L) \\ &\cong \mathrm{Cone}(*, F) \\ &= \{(a_i : * \to F(i))_i \ : \ \text{any possible triangle commutes}\} \\ &\cong \left\{ (a_i)_i \in \prod_i F(i) \ : \ \text{translate the commutativity of triangles to conditions on } a_i\text{'s} \right\} \end{aligned}$$

Why don't we take this as the definition of L? We will have to check that $\mathrm{Cone}(*, F)$ satisfies the universal property of a limit. In this way, $p_i : L \to F(i)$ are simply the projection maps.

Example 7.5 (Products in Set)**.** Let $\mathcal{J}$ be discrete, i.e., it's a small category where the only morphisms are identity morphisms (a set categorified). It can be visualised as

$$\cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$

Consider a diagram $F : \mathcal{J} \to \mathsf{Set}$; we will compute the limit $\lim_{\mathcal{J}} F$ using our discussion in Example 7.4. That is, we can take

$$\lim_{\mathcal{J}} F = \mathrm{Cone}(*, F)$$

Let $A_i := F(i)$, we now investigate $\mathrm{Cone}(*, F)$. It's the set of a collection of maps as follows

$$\begin{tikzcd}
 & & & * \arrow[llld, dashed] \arrow[lld, "a_{n-2}"'] \arrow[ld, "a_{n-1}"'] \arrow[d, "a_n"] \arrow[rd, "a_{n+1}"] \arrow[rrd, "a_{n+2}"] \arrow[rrrd, dashed] & & & \\
\cdots & A_{n-2} & A_{n-1} & A_n & A_{n+1} & A_{n+2} & \cdots
\end{tikzcd}$$

which necessary corresponds to $(a_i)_i \in \prod_i A_i$. Therefore $\lim_{\mathcal{J}} F = \prod_i A_i$.

Example 7.6 (Fibered Product in Set)**.** Let $\mathcal{J}$ be the category with three objects where the only morphisms, apart from identity morphisms, are where two of the objects map to the third one. It can be visualised as

$$\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

Consider a diagram $F : \mathcal{J} \to \mathsf{Set}$, where the image is given as

$$\begin{tikzcd}
 & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

We will compute the limit $\lim_{\mathcal{J}} F$ using our discussion in Example 7.4. That is, we can take

$$\lim_{\mathcal{J}} F = \mathrm{Cone}(*, F)$$

and investigate $\mathrm{Cone}(*, F)$. It's the set of a collection of maps as follows

$$\begin{tikzcd}
* \arrow[rd, "z" description] \arrow[r, bend left, "x"] \arrow[d, bend right, "y"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

such that the triangles commute, that is it's a triple (x, y, z) such that $z = g \circ y = f \circ x$. Therefore it necessarily corresponds to a pair $(x, y) \in X \times Y$ such that $f(x) = g(y)$. Hence

$$X \times_Z Y = \lim_{\mathscr{J}} F = \{(x, y) \times X \times Y \ : \ f(x) = g(y)\} .$$

7.1. Problems

Problem 7.1. Fix a category $\mathscr{C}$

(a) Consider a diagram

$$F : \mathscr{J} \to \mathscr{C}$$

where $\mathscr{J}$ is small. Assume $\mathscr{J}$ has an initial object e, prove that $F(e) = \lim_{\mathscr{J}} F$.

(b) State and prove the dual statement for colimits.

(c) Recall that the limit of the diagram indexed by

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

is the fibered product. What about the colimit?

Problem 7.2. Let $\mathscr{J}$ be a poset considered as a category; in particular, for any two objects in $\mathscr{J}$ there's at most one map between them. Explicitly describe $\lim_{\mathscr{J}} F$ where $F : \mathscr{J} \to \mathsf{Set}$ is a diagram in Set.

Problem 7.3. Let X be a set, and consider the power set $\mathscr{P}(X)$ considered as a poset with respect to set containment. Treating this poset as a category, find all (small) limits and colimits in this category. They will be very familiar objects.

Problem 7.4. In Example 7.6, we have seen what the fibered product looks like in general in the category of sets. Consider the following special case

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ U \arrow[r, hook, "i"'] & X \end{tikzcd}$$

where U is a subset of X and $i : U \hookrightarrow X$ is the canonical inclusion map, and $f : Y \to X$ is any function. What is the limit of this diagram, that is, $U \times_X Y$.

We're asking you to give a more concrete expression for this limit beyond realising it as a subset of $U \times Y$. Similarly for the questions below.

(a) What's the fibered product if $U = \{x\}$, for some $x \in X$?

(b) What's the fibered product if $Y = V$ is also a subset of X and $f = j : V \hookrightarrow X$ is the canonical inclusion map?

Problem 7.5. While we haven't talked about how colimits, in general, look like in Set, the following special case is quite tractable. Find the colimit (pushout) of the diagram in the category of sets

$$\begin{tikzcd} A \cap B \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & A \\ B & \end{tikzcd}$$

where i and j are the canonical inclusion maps. Problem 7.3 can serve as an inspiration.

Problem 7.6. Let G be a non-trivial group, prove that the category $\mathsf{B}G$ has fibered products but no products.

Problem 7.7. A nonempty partially ordered set $(S, \geqslant)$ is directed (or filtered) if for each $x, y \in S$, there is a z such that $x \geqslant z$ and $y \geqslant z$.

Suppose $\mathcal{J}$ is a directed set treated as a category. Let $F : \mathcal{J} \to \mathsf{Set}$ be a diagram; show that, with the obvious maps to it

$$\operatorname{colim}_{\mathcal{J}} F = \left\{ (a_i, i) \in \coprod_{i \in \mathcal{J}} A_i \right\} \Big/ \left(\begin{array}{c} (a_i, i) \sim (a_j, j) \text{ if and only if there are } f : A_i \to A_k \text{ and} \\ g : A_j \to A_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } A_k \end{array} \right)$$

where $A_i = F(i)$ and "in the diagram" means there's a morphism $i \to k$ and $j \to k$ in $\mathcal{J}$ such that $f = F(i \to k)$ and $g = F(j \to k)$.

First prove that $\sim$ is indeed an equivalence relation.

Problem 7.8. Let's consider the following, similar but distinct, diagrams in the category of unital commutative rings CRing, where $\mathbb{F}_p[t]$ is the ring of polynomials in a single variable t over the finite field $\mathbb{F}_p$.

$$\underline{P}^{\leftarrow} : \quad \cdots \xrightarrow{\text{Fr}} \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \mathbb{F}_p[t]$$

$$\underline{P}^{\rightarrow} : \quad \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \mathbb{F}_p[t] \xrightarrow{\text{Fr}} \cdots$$

where $\text{Fr} : \mathbb{F}_p[t] \to \mathbb{F}_p[t]$, $f \mapsto f^p$ is called the *Frobenius map*. Prove that the Frobenius map is equal to the map $t \mapsto t^p$.

A strategy to compute $\lim \underline{P}^{\leftarrow}$ and $\operatorname{colim} \underline{P}^{\rightarrow}$, vaguely put, is to "replace the terms by their images or preimages so that the maps become inclusions, and then use Problem 7.3 as an inspiration".

Realising this strategy prove that $\lim \underline{P}^{\leftarrow} = \mathbb{F}_p$ (we use equality loosely), and that $\operatorname{colim} \underline{P}^{\rightarrow}$ can be described as $\mathbb{F}_p[t, t^{1/p}, t^{1/p^2}, \ldots] =: \mathbb{F}_p[t^{1/p^\infty}]$, obtained by adjoining all "p^{th} roots of t".

Prove that Frobenius maps on these rings is an isomorphism.

Remark 7.7. This limit and a colimit is a general phenomenon for any ring of characteristic p. More precisely, a ring A (object of CRing) is said to be of characteristic p if the canonical map $\mathbb{Z} \to A$ factors through $\mathbb{F}_p$, equivalently if $p\mathbb{Z}$ is in the kernel. In this case, the the Frobenius map $\mathrm{Fr} : A \to A$, $a \mapsto a^p$ is a ring homomorphism. We can then consider the two diagrams above, then

$$A^{\text{perf}} := \varprojlim_{\mathrm{Fr}} A \quad \text{and} \quad A_{\text{perf}} := \varinjlim_{\mathrm{Fr}} A$$

are called the *perfection of A* and the *perfect closure of A* respectively; sometimes these notations (scripts) are switched. Both these rings are *perfect*, in the sense that the (induced) Frobenius maps on them is an isomorphism.

Problem 7.9. Recall the notion of a presheaf $\mathscr{F}$ on a topological space X.

(a) For any point $x \in X$, one defines the *stalk*

$$\mathscr{F}_x := \operatorname*{colim}_{U \ni x} \mathscr{F}(U)$$

Use Problem 7.7 to obtain an explicit description of $\mathscr{F}_x$: identify the directed set the colimit is indexed over, then identify the equivalence relation, and finally write down the resulting set given by Problem 7.7.

(b) Suppose $\mathscr{F}$ was a sheaf (of sets) and U an open set such that $U = U_1 \cup U_2$. Let $U_{12} := U_1 \cap U_2$. Prove, using Example 7.6, that

$$\mathscr{F}(U) = \mathscr{F}(U_1) \times_{\mathscr{F}(U_{12})} \mathscr{F}(U_2)$$

What does this tell you when U_1 and U_2 are disjoint, that is, when $U_{12} = \varnothing$.

Can you generalise this to an arbitrary cover? That is, given an open set U and a cover $U = \cup_{i \in I} U_i$, find a diagram D such that $\mathscr{F}(U) = \lim D$.

8. Lecture 8 (2/25) by Deewang

Discussion 8.1 (Optimisation)**.** Let's pose the following questions

(a) Recall the category Rng, the objects are sets that satisfy the ring axioms except the requirement that they possess a multiplicative identity, and the morphisms are ring homomorphisms that are not required to preserve the multiplicative identities, if and when they exist. We call an object of this category a *rng* and morphisms *rng (homo)morphisms*.

But we like Ring more, having a multiplicative identity makes several things easier. So, we ask ourselves if given a rng R can we produce a ring $\widetilde{R}$ out of it. So that this ring $\widetilde{R}$ is the "right" or the "optimal" choice, we would like it to satisfy a universal property. There are two possible choices for what this universal property could be.

(i) We have a rng homomorphism $u : R \to \widetilde{R}$ such that for any rng homomorphism $\phi : R \to S$, where S is actually a ring, there exists a unique **ring** homomorphism $\widetilde{\phi} : \widetilde{R} \to S$ such that $\widetilde{\phi} \circ u = \phi$ as rng homomorphisms. That is, the following diagram exists and commutes.

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "u"'] & S \\ \widetilde{R} \arrow[ur, dashed, "\exists!\widetilde{\phi}"'] & \end{tikzcd}$$

(ii) We have a rng homomorphism $v : \widetilde{R} \to R$ such that for any rng homomorphism $\psi : S \to R$, where S is actually a ring, there exists a unique **ring** homomorphism $\widetilde{\psi} : S \to \widetilde{R}$ such that $v \circ \widetilde{\psi} = \psi$ as rng homomorphisms. That is, the following diagram exists and commutes.

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[dr, dashed, "\exists!\psi"'] & R \\ & \widetilde{R} \arrow[u, "v"'] \end{tikzcd}$$

The answer is that such a ring exists but it only satisfies the universal property in (i). There exists no ring that can satisfy the universal property in (ii). This is because, and it may sound a bit random, Rng has the zero ring as its initial object, while the initial object in Ring is $\mathbb{Z}$. This reasoning will be clarified soon.

(b) Consider a function $f : X \to Y$, with no conditions on X and Y. One wonders the following two questions.

(i) Suppose Y was a topological space, with no conditions on its topology. Can we give X a topology such that f itself becomes a continuous map of topological spaces?

(ii) Suppose X was a topological space, instead, with no conditions on its topology. Similarly as in (i), Can we give Y a topology such that f itself becomes a continuous map of topological spaces?

The answer to both (i) and (ii) is yes, we can (see Example 8.4).

We can translate (a) into our language of category theory in a slightly different way. In (i) we are first treating a ring S as a rng; more concretely what we are doing is first considering the forgetful functor

$$U : \mathsf{Ring} \to \mathsf{Rng}$$

which "forgets" that a ring possesses a multiplicative identity, and then considering $U(S)$. Therefore, $\phi \in \mathrm{Hom}_{\mathsf{Rng}}(R, U(S))$.

Next, we are really asking for a functor $F : \mathsf{Rng} \to \mathsf{Ring}$ with the universal property in (i) which translates into the following bijection

$$\tau_{R,S} : \mathrm{Hom}_{\mathsf{Ring}}(F(R), S) \to \mathrm{Hom}_{\mathsf{Rng}}(R, U(S)) \to, \ \widetilde{\phi} \mapsto \widetilde{\phi} \circ u$$

We would also like these bijections to be compatible, that is, natural in R and S.

In (ii) we are really asking for a functor $G : \mathsf{Rng} \to \mathsf{Ring}$ such that there's a bijection

$$\eta_{RS} : \mathrm{Hom}_{\mathsf{Rng}}(U(R), S) \to \mathrm{Hom}_{\mathsf{Ring}}(R, G(S))$$

natural in its entries. Something similar is being asked in (b).

Codifying examples like these brings us to *adjunction*.

Definition 8.2. Fix categories $\mathscr{C}$ and $\mathscr{D}$, assumed to be locally small. An *adjunction* consists of a pair of functors

$$F : \mathscr{C} \to \mathscr{D} \quad \text{and} \quad G : \mathscr{D} \to \mathscr{C}$$

together with a bijection

$$\tau_{X,Y} : \mathrm{Hom}_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X, G(Y))$$

for each object X in $\mathscr{C}$ and Y in $\mathscr{D}$, that is natural in both entries. More precisely, if $f : C \to X$ is a morphism in $\mathscr{C}$ and $g : Y \to D$ is a morphism in $\mathscr{D}$ then we want the following two squares induced by f and g respectively to commute.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(X), Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "-\circ F(f)"'] & \mathrm{Hom}_{\mathscr{C}}(X, G(Y)) \arrow[d, "-\circ f"] \\
\mathrm{Hom}_{\mathscr{D}}(F(C), Y) \arrow[r, "\tau_{C,Y}"'] & \mathrm{Hom}_{\mathscr{C}}(C, G(Y))
\end{tikzcd} \qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(X), Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "g\circ -"'] & \mathrm{Hom}_{\mathscr{C}}(X, G(Y)) \arrow[d, "G(g)\circ -"] \\
\mathrm{Hom}_{\mathscr{D}}(F(X), D) \arrow[r, "\tau_{X,D}"'] & \mathrm{Hom}_{\mathscr{C}}(X, G(D))
\end{tikzcd}$$

Here we say F is *left adjoint* to G and G is *right adjoint* to F, sometimes indicated as $F \dashv G$.

Therefore, in the examples in Discussion 8.1 we were asking if the forgetful functors

$$\mathsf{Ring} \to \mathsf{Rng} \quad \text{and} \quad \mathsf{Top} \to \mathsf{Set}$$

admit a right or left adjoint. Let's see some examples.

Remark 8.3. The use of the word adjoint for such pairs of functors is inspired by the notion of an adjoint operator in linear algebra (or more generally, functional analysis). Recall that given an operator $T : V \to W$ on inner product spaces (or more generally, Hilbert spaces), the *adjoint operator* $T^* : W \to V$ is the (most cases unique) operator such that $\langle Tv, w\rangle_W = \langle v, T^*w\rangle_V$.

Apart from the visual similarity, one other similarity is that just as the adjoint operator allows us to switch between inner product computations in V and W, adjoint pairs of functors allow us to switch between morphisms in $\mathscr{C}$ and $\mathscr{D}$. A fact that has interesting consequences: how these functors interact with limits and colimits, for example; as we shall see soon.

Example 8.4 (Free-Forgetful Adjunction)**.** The examples treated in Example 8.1 belong to a large class of examples of adjunctions called the *free-forgetful adjunctions*. These are the examples where the obvious (is it?) forgetful functors have left adjoints that are dubbed *free*. If the forgetful functors has a right adjoint, which is rare, it is dubbed *cofree*.

In these examples, we don't treat the inclusion of a full subcategory as a forgetful functor, they are treated in the next example, Example 8.5.

Forgetful Functors	**Left Adjoint F on Objects (the Free Functors)**
$U : \mathsf{Set}_* \to \mathsf{Set}$	$F(X) = X_+ := * \amalg X$ we add a basepoint
$U : \mathsf{Top}_* \to \mathsf{Top}$	$F(X) = X_+ := * \amalg X$ we add a basepoint and give it the disjoint topology
$U : X/\mathscr{C} \to \mathsf{Cat}$ $(X \xrightarrow{f} Y) \mapsto Y$ assume $\mathscr{C}$ has coproducts	$F(Y) = (X \xrightarrow{\iota_X} X \amalg Y)$ last two examples are a special case of this; so are some below U has a right adjoint if and only if X is initial
$U : \mathsf{Top} \to \mathsf{Set}$	$F(X) = (X, \text{discrete})$ we equip X with the discrete topology this is (b)(i) in Example 8.1
$U : \mathsf{Grp} \to \mathsf{Set}$	$F(X) = F_X$ the free group of reduced words on X
$U : \mathsf{Mod}_R \to \mathsf{Set}$ specialise to $R = \mathbb{Z}$ for Ab; and $R = k$, a field, for Vec_k	$F(X) = R[X] := \bigoplus_{x \in X} R$ free R-module on X
$U : \mathsf{Ring} \to \mathsf{Rng}$	$F(R) = \mathbb{Z} \times R$ see Problem 8.2 for details, the multiplicative identity is $(1, 0)$

$U : \mathsf{Mon} \to \mathsf{Set}$	$F(X) = \coprod_{n \geqslant 0} X^{\times n}$ the set of finite lists of elements of X, including the empty list $= X^{\times 0}$
$U : \mathsf{Alg}_R \to \mathsf{Mod}_R$ Alg_R is the category of R-algebras, where R is a ring specialise to $R = \mathbb{Z}$ for $\mathsf{Ring} \to \mathsf{Ab}$; and $R = k$, a field, for $\mathsf{Alg}_k \to \mathsf{Vec}_k$	$F(M) = T^{\bullet}_R(M) := \bigoplus_{n \geqslant 0} M^{\otimes n}$ tensor algebra of M, where $M^{\otimes 0} = R$
$U : \mathsf{CAlg}_R \to \mathsf{Set}$ CAlg_R is the category of commutative R-algebras, R a ring specialise to $R = \mathbb{Z}$ for CRing; and $R = k$, a field, for kicks	$F(X) = R[\{x\}_{x \in X}]$ the polynomial algebra over R with the set of indeterminants being X
$U : \mathsf{CAlg}_R \to \mathsf{Mod}_R$ specialise to $R = \mathbb{Z}$ for CRing; and $R = k$, a field, for kicks	$F(M) = \mathsf{Sym}^{\bullet}_R(M)$ the symmetric algebra over M which is (non-canonically) isomorphic to the polynomial algebra over R in generators of M this is a choice-free version of the example above
$\mathrm{Res}^G_H : G\text{-}\mathsf{Set} \to H\text{-}\mathsf{Set}$ G a group, $H \leqslant G$ a subgroup	$F(X) = \mathrm{Ind}^G_H(X) = G \times_H X := G \times X / \sim$ where $\sim$ is generated by $(gh, x) \sim (g, hx)$, and $g \cdot [(h, x)] := [(gh, x)]$
$U : G\text{-}\mathsf{Set} \to \mathsf{Set}\ (= *\text{-}\mathsf{Set})$ $* \leqslant G$ trivial subgroup of G	$F(X) = G \times X$ where $g \cdot (h, x) := (gh, x)$
$\phi^* : \mathsf{Mod}_S \to \mathsf{Mod}_R$ *restriction of scalars* via a ring homorphism $\phi : R \to S$	$F(M) = \phi_!(M) = S \otimes_R M$ *extension of scalars*
$U : \mathsf{Mod}_R \to \mathsf{Ab}\ (= \mathsf{Mod}_{\mathbb{Z}})$ via the canonical map $\mathbb{Z} \to R$	$F(A) = R \otimes_{\mathbb{Z}} A$
$\mathrm{Res}^G_H : \mathsf{Mod}_{kG} \to \mathsf{Mod}_{kH}$ *restriction of scalars* induced by $H \hookrightarrow G$	$F(M) = \mathrm{Ind}^S_R(M) = kG \otimes_{kH} M$ *induced representation*
$U : \mathsf{Mod}_{kG} \to \mathsf{Vec}_k (= \mathsf{Mod}_{k*})$ $* \hookrightarrow G$, trivial subgroup	$F(V) = \mathrm{Ind}^G_*(M) = kG \otimes_k V$
$U : \mathsf{Vec}_{\mathbb{C}}\ (= \mathsf{Mod}_{\mathbb{C}}) \to \mathsf{Vec}_{\mathbb{R}}\ (= \mathsf{Mod}_{\mathbb{R}})$ *restriction of scalars* via $\mathbb{R} \hookrightarrow \mathbb{C}$	$F(V) = V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ *complexification*

Here are some examples of cofree functors.

Forgetful Functors	Right Adjoint F' on Objects (the Cofree Functors)
$U : \mathscr{C}/X \to \mathsf{Cat}$ $(Y \xrightarrow{f} X) \mapsto Y$ assume $\mathscr{C}$ has products	$F'(Y) = (X \times Y \xrightarrow{\pi_X} X)$ U has a left adjoint if and only if X is terminal
$U : \mathsf{Top} \to \mathsf{Set}$	$F'(X) = (X, \text{indiscrete})$ we equip X with the indiscrete topology this is (b)(ii) in Example 8.1
$\mathrm{Res}^G_H : G\text{-}\mathsf{Set} \to H\text{-}\mathsf{Set}$ G a group, $H \leqslant G$ a subgroup	$F'(X) = \mathrm{CoInd}^G_H(X) = \mathrm{Hom}_{H\text{-}\mathsf{Set}}(G, X)$ where $(g' \cdot f)(g) := f(gg')$
$U : G\text{-}\mathsf{Set} \to \mathsf{Set}\ (= *\text{-}\mathsf{Set})$ $* \leqslant G$ trivial subgroup of G	$F'(X) = \mathrm{Hom}_{\mathsf{Set}}(G, X)$ where $(g' \cdot f)(g) := f(gg')$
$\phi^* : \mathsf{Mod}_S \to \mathsf{Mod}_R$ *restriction of scalars* via a ring homorphism $\phi : R \to S$	$F(M) = \phi_*(M) = \mathrm{Hom}_S(R, M)$ *co-extension of scalars* where $(r' \cdot f)(r) = f(rr')$
$\mathrm{Res}^G_H : \mathsf{Mod}_{kG} \to \mathsf{Mod}_{kH}$ *restriction of scalars* induced by $H \hookrightarrow G$	$F'(M) = \mathrm{CoInd}^G_H(M) = \mathrm{Hom}_{kH}(kG, M)$ *co-induced representation* where $(g' \cdot f)(g) := f(gg')$
$U : \mathsf{Vec}_{\mathbb{C}}\ (= \mathsf{Mod}_{\mathbb{C}}) \to \mathsf{Vec}_{\mathbb{R}}\ (= \mathsf{Mod}_{\mathbb{R}})$ *restriction of scalars* via $\mathbb{R} \hookrightarrow \mathbb{C}$	$F'(V) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, V)$ *co-complexification* where $(z \cdot f)(w) := f(wz)$

None of the functors that forget algebraic structure on fields, into commutative unital rings, abelian groups or sets, admit left or right adjoints (see Problem 8.3).

Example 8.5 (Reflector-Inclusion Adjunction)**.** Another class of examples, which are really a subclass of free-forgetful adjunctions but with a more specific flavour, are the reflector-inclusion adjunctions.

We call a full subcategory $\mathscr{D}$ of $\mathscr{C}$ *reflective*, if the inclusion functor $i : \mathscr{D} \hookrightarrow \mathscr{C}$ has a left adjoint which we call the *reflector*. Dually, $\mathscr{D}$ is said to be *coreflective* when the inclusion functor has a right adjoint, which is then called the *coreflector*.

Inclusions	Left Adjoint F on Objects (the Reflector)
$\mathsf{Ab} \hookrightarrow \mathsf{Grp}$	$F(G) = G^{\mathrm{ab}} := G/[G, G]$ the abelianisation

$\mathsf{Ab} \hookrightarrow \mathsf{AbSemiGrp}$ (resp. AbMon) $\mathsf{AbSemiGrp}$ (resp. AbMon) is the category of abelian semigroups (resp. monoids)	$F(S) = \mathrm{Gr}(S) = K(S)$ the *group completion* or *Grothendieck group* of S (see Problems 8.4 and 8.5)
$\mathsf{Grp} \hookrightarrow \mathsf{Mon}$	$F(M)$ the *group completion* of M
$\mathsf{Mon} \hookrightarrow \mathsf{SemiGrp}$	$F(S) := S \cup \{e\}$ add an extra element and declare it neutral
$\mathsf{CRing} \hookrightarrow \mathsf{Ring}$	$F(R) = R/[R, R]$ where $[R, R]$ is the commutator ideal
$\mathsf{Field} \hookrightarrow \mathsf{IntDom}$ IntDom is the category of integral domains with morphisms being injective ring homomorphisms	$F(A) = \mathrm{Frac}(A)$ field of fractions of A
$\mathsf{Haus} \hookrightarrow \mathsf{Top}$ Haus is the category of Hausdorff spaces	$F(X) = H(X) = X/\sim$ where $x \sim y$ if and only if $f(x) = f(y)$ for every continuous map from X to a Hausdorff space.
$\mathsf{CHaus} \hookrightarrow \mathsf{Top}$ CHaus is the category of compact Hausdorff spaces	$F(X) = \beta X$ *Stone-Čech compactifiction*
$\mathsf{Cauchy} \hookrightarrow \mathsf{Metric}$ Metric is the category of metric spaces with morphisms being uniformly continuous maps. Cauchy is the full subcategory of complete metric spaces	$F(X) = \widehat{X}$ *metric completion* the ring of Cauchy sequences modulo the (maximal ideal of) null sequences
$\mathsf{Sh}_X \hookrightarrow \mathsf{PSh}_X$ category of sheaves an presheaves on a topological space X	$F(\mathscr{F}) = a(\mathscr{F})$ *sheafification*

Here are two examples of coreflectors.

Inclusions	**Right Adjoint F' on Objects (the Coreflector)**
$\mathsf{TorAb} \hookrightarrow \mathsf{Set}$ TorAb is the subcategory of torsion abelian groups, where all elements have finite order	$F'(A) = TA$ the torsion subgroup

$\mathsf{Grp} \hookrightarrow \mathsf{Mon}$	$F'(M) = M^*$ the group of invertible elements

Discussion 8.6. The adjunctions in Examples 8.4 and 8.5 are all given or exhibited by a universal property. That is, in all of these there's a universal map such that the relevant bijection is given by composing with it. As hinted in our discussion in Example 8.1 (a)(i).

Example 8.7 (Adjunctions of Posets)**.** Recall how we categorify a poset (or more generally, a pre-order), also recall from Problem 2.12 that functors $F : \mathsf{P} \to \mathsf{Q}$ and $G : \mathsf{Q} \to \mathsf{P}$ between posets P and Q are simply order-preserving functions.

An adjunction $F \dashv G$, called a *monotone Galois connection*, asserts that

$$Fa \leqslant b \text{ if and only if } a \leqslant Gb$$

for all $a \in \mathsf{P}$ and $b \in \mathsf{Q}$. In this context, F is the lower adjoint and G is the upper adjoint.

Here are some examples.

- The inclusion of posets $i : \mathbb{Z} \hookrightarrow \mathbb{R}$, with the usual ordering $\leqslant$, has both left and right adjoints, namely, the *ceiling* and *floor functions* respectively.

  For any integer n and real number r,

  $$n \leqslant r \text{ if and only if } n \leqslant \lfloor r \rfloor,$$

  The floor function is order-preserving, defining a functor $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ that is right adjoint to the inclusion i.

  Dually,

  $$r \leqslant n \text{ if and only if } \lceil r \rceil \leqslant n,$$

  and this order-preserving function defines a functor $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ that is left adjoint to the inclusion.

  This is also tells us that $\mathbb{Z}$ is a reflective and coreflective subcategory $\mathbb{R}$, in the sense of Example 8.5.

- Let X be a topological space, then its power set $\mathscr{P}(X)$ and its set of closed sets $\mathcal{C}(X)$ are posets with respect to set containment. Then the inclusion $\mathcal{C}(X) \hookrightarrow \mathscr{P}(X)$ has a left adjoint, given by the *closure*. Since, for any set S and closet set A

  $$S \subseteq A \text{ if and only if } \overline{S} \subseteq A$$

  Since the closure is order preserving, it defines a functor $\overline{(\cdot)} : \mathscr{P}(X) \to \mathcal{C}(X)$.

  Similarly, the inclusion $\mathcal{O}(X) \hookrightarrow \mathscr{P}(X)$, where $\mathcal{O}(X)$ is the set of open sets of X, has a right adjoint, given by the *interior*.

- Let $f : X \to Y$ be any function, then it induces the *inverse image functor* on the posets of power sets of X and Y
$$f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X),\ T \mapsto f^{-1}(T)$$
This functor has both a left and right adjoint.

  The left adjoint is the *direct image functor*
$$f_* : \mathscr{P}(X) \to \mathscr{P}(Y),\ S \mapsto f(S)$$
  since
$$f(S) \subseteq T \text{ if and only if } S \subseteq f^{-1}(T)$$

  The right adjoint is the *extension by zero functor*
$$f_! : \mathscr{P}(X) \to \mathscr{P}(Y),\ S \mapsto \left\{ y \in Y \ : \ f^{-1}(y) \subseteq S \right\}$$
  where $f_!$ is read as *f-lower shriek*. It immediately follows from definitions that
$$f^{-1}(T) \subseteq S \text{ if and only if } T \subseteq f_!(S)$$
  These names are inspired by sheaf theory.

Example 8.8 (Beyond Free-Forgetful)**.** As mentioned in Discussion 8.6, the free-forgetful functors essentially are exhibited or witnessed by universal properties. But the notion of an adjoint functor is more general. Let's look at some more examples that sit more as instances of duality and symmetry.

We ease into this with an example that invokes the vibe of a free-forgetful adjunction

(1) The unit functor
$$(-)^{\times} : \mathsf{Ring} \to \mathsf{Grp}$$
has a left adjoint given by the group ring functor
$$\mathbb{Z}[-] : \mathsf{Grp} \to \mathsf{Ring}$$
More generally, we have the adjoint pair $R[-] : \mathsf{Grp} \rightleftarrows \mathsf{Alg}_R : (-)^{\times}$, where R is a ring.

We now consider the example that started it all, the motivating example in Daniel M. Kan's 1958 paper *Adjoint Functors* where he first introduces the notion of an adjoint.

- **Tensor-Hom Adjunction**. For simplicity let's fix a commutative ring R, and consider an R-module M. Then we have two functors
$$- \otimes_R M : \mathsf{Mod}_R \to \mathsf{Mod}_R \quad \text{and} \quad \mathrm{Hom}_R(M, -) : \mathsf{Mod}_R \to \mathsf{Mod}_R$$
These two functors define an adjoint pair, that is we have bijections
$$\mathrm{Hom}_R(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$$
natural in L and N. Let's prove this.

For any R-modules L and N, we have have the following bijection

$$\begin{aligned} \tau_{L,N} : \mathrm{Hom}_R(L \otimes_R M, N) &\xrightarrow{\sim} \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)) \\ \phi &\longmapsto (x \mapsto (m \mapsto \phi(x \otimes m))) \\ (x \otimes m \mapsto \psi(x)(m)) &\longleftarrow\!\shortmid\ \psi \end{aligned}$$

For ease, we note that the morphism $\tau_{L,N}(\phi)$ can be written down as

$$\tau_{L,N}(\phi)(x)(-) = \phi(x \otimes -).$$

Now for maps $f : L \to L'$ and $g : N \to N'$, letting $F = - \otimes_R M$ and $G = \mathrm{Hom}_R(M, -)$, we have $F(f) = f \otimes 1$ and $G(g) = g \circ -$ and the following diagram

$$\begin{tikzcd} \mathrm{Hom}_R(L' \otimes_R M, N) \arrow[r, "- \circ Ff"] \arrow[d, "\tau_{L',N}"'] & \mathrm{Hom}_R(L \otimes_R M, N) \arrow[r, "g \circ -"] \arrow[d, "\tau_{L,N}"] & \mathrm{Hom}_R(L \otimes M, N') \arrow[d, "\tau_{L,N'}"] \\ \mathrm{Hom}_R(L', \mathrm{Hom}_R(M, N)) \arrow[r, "- \circ f"'] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)) \arrow[r, "Gg \circ -"'] & \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N')) \end{tikzcd}$$

For a $\phi' \in \mathrm{Hom}_R(L' \otimes_R M, N)$, we look at its image in $\mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$ along the boundary of the first square

$$\tau_{L,N}(\phi' \circ Ff)(x)(-) = \phi' \circ (f \otimes 1)(x \otimes -) = \phi'(f(x) \otimes -)$$

$$(\tau_{L',N}(\phi') \circ f)(x)(-) = \tau_{L',N}(\phi')(f(x))(-) = \phi'(f(x) \otimes -)$$

Therefore the first square commutes.

Now, for a $\phi \in \mathrm{Hom}_R(L \otimes_R M, N)$, we look at its image in $\mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N'))$ along the boundary of the second square

$$\tau_{L,N'}(g \circ \phi)(x)(-) = g \circ \phi(x \otimes -)$$

$$Gg \circ \tau_{L,N}(\phi)(x)(-) = Gg(\phi(x \otimes 1)) = g \circ \phi(x \otimes -)$$

Hence the second square commutes.

Thus proving that (F, G) form an adjoint pair, that is, $F \dashv G$.

- **"Set-Theoretic Tensor-Hom Adjunction" a.k.a. Currying**. Consider a set Y. Then we have two functors

$$- \times Y : \mathsf{Set} \to \mathsf{Set} \quad \text{and} \quad \mathrm{Hom}_{\mathsf{Set}}(Y, -) : \mathsf{Set} \to \mathsf{Set}$$

These two functors define an adjoint pair, that is we have bijections

$$\mathrm{Hom}_{\mathsf{Set}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Set}}(X, \mathrm{Hom}_{\mathsf{Set}}(Y, Z))$$

natural in X and Y.

This is often to convert a function that takes multiple arguments (the left hand side) into a sequence of functions that each take a single argument (the right hand side). This adjunction is proved similar to the above example.

Recall that we sometimes denote the set of functions from sets $X \to Y$ by Y^X. Therefore currying tells us that

$$Z^{X\times Y} \cong (Z^Y)^X.$$

- **"Topological (Homotopy-Theoretic) Tensor-Hom Adjunction"**. We will first set things up; the category under consideration is Top_* and the base point of a pointed space is implicit so we'll call them X instead of (X, x_0).

  Given two pointed spaces X and Y, $\mathrm{Hom}_{\mathsf{Top}_*}(X, Y)$, the pointed set of basepoint-preserving continuous maps, can be made into a pointed topological space by giving it, what we call, the *compact-open topology*; the resulting topological space is denoted as $\mathrm{Map}_*(X, Y)$.

  Given two pointed spaces X and Y, we can consider their *smash product* $X \wedge Y$.

  Given a pointed locally compact topological space K, we can consider the functors

  $$- \wedge K : \mathsf{Top}_* \to \mathsf{Top}_* \quad \text{and} \quad \mathrm{Map}_*(K, -) : \mathsf{Top}_* \to \mathsf{Top}_*$$

  These two functors define an adjoint pair, that is we have bijections

  $$\mathrm{Hom}_{\mathsf{Top}_*}(X \wedge K, Y) \cong \mathrm{Hom}_{\mathsf{Top}_*}(X, \mathrm{Map}_*(K, Y))$$

  natural in X and Y. This adjunction is proved similar to the above example.

  This bijection is compatible homotopy, and therefore descends to a bijection in HTop_* where we indicate the set of morphisms as $[-.-]_*$. So we have a bijection, natural in X and Y

  $$[X \wedge K, Y]_* \cong [X, \mathrm{Map}_*(K, Y)]_*$$

  In the special case when $K = S^1$, then functor $- \wedge S^1$ is called the *(reduced) suspension functor* and denoted Σ, and the functor $\mathrm{Map}_*(S^1, -)$ is called the *loop space functor* and denoted Ω, which gives us the *suspension-loop adjunction*

  $$[\Sigma X, Y]_* \cong [X, \Omega Y]_*$$

8.1. Problems

Problem 8.1. Pick some (or all) examples given in Examples 8.4, 8.5 and 8.8 (1), and work out the details. That is, prove that the the given pairs are indeed adjoints.

Problem 8.2 (Dorroh Extension)**.** Let R be a rng, and we define $\widetilde{R} = \mathbb{Z} \times R$ as an abelian group (pointwise addition) and we define the multiplication on it as follows

$$(n, r) \cdot (n', r') := (nn', nr' + n'r + rr')$$

(we sometimes write (n, r) as $n \oplus r$ which tells you why multiplication is defined this way, but note that rings have no reliable notion of direct sum).

(a) Prove that $\widetilde{R}$ is a ring with multiplicative identity given by $(1,0)$.

(b) Prove that there is a rng homomorphism $u : R \to \widetilde{R}$ such that $(\widetilde{R}, u)$ has the following universal property: For any rng homomorphism $\phi : R \to S$, where S is a **ring**, there exists a unique **ring** homomorphism $\widetilde{\phi} : \widetilde{R} \to S$ such that $\widetilde{\phi} \circ u = \phi$ as rng homomorphisms. That is, the following diagram exists and commutes.

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "u"'] & S \\ \widetilde{R} \arrow[ur, dashed, "\exists!\widetilde{\phi}"'] & \end{tikzcd}$$

(c) Prove that the association $R \mapsto \widetilde{R}$ is functorial and the above universal property describes the adjunction $\widetilde{(-)} \dashv U$ where $U : \mathsf{Ring} \to \mathsf{Rng}$ is the forgetful functor.

(d) Given a rng R, an R-module M is defined in an obvious way: it satisfies all the axioms that a module over a ring does except the one involving the multiplicative identity of the ring, for obvious reasons.

Let M be an R-module, where R is a rng. Consider the following action of $\widetilde{R}$ on M

$$(n,r) \cdot m := n \cdot m + r \cdot m$$

where $n \cdot m = \underbrace{m + \cdots + m}_{n \text{ times}}$, and $r \cdot m$ is the usual action of R on M.

Prove that this action defines an $\widetilde{R}$-module structure on M.

(d) Continuing from (c), let $\widetilde{M}$ be the abelian group M equipped with the $\widetilde{R}$-module structure. Prove that the association $M \mapsto \widetilde{M}$ is functorial, and describes an equivalence between the categories of R-modules and $\widetilde{R}$-modules.

Problem 8.3. Prove that none of the following functors

$$\mathsf{Field} \xrightarrow{U} \mathsf{Ring}, \quad \mathsf{Field} \xrightarrow{U} \mathsf{Ab}, \quad \mathsf{Field} \xrightarrow{(-)^\times} \mathsf{Ab}, \quad \mathsf{Field} \xrightarrow{U} \mathsf{Set}$$

that forget algebraic structure on fields admit left or right adjoints.

Problem 8.4 (Group Completion, or Grothendieck Group)**.** Recall that an abelian semigroup $(S,+)$ is a set S equipped with an associative binary product $+$ which is commutative. A homomorphism of semigroups is one that commutes with the binary products, just like group homomorphisms.

Given a semigroup S, define

$$K(S) := S \times S/\sim$$

where $(a,b) \sim (c,d)$ if and only if there exists an $x \in S$ such that $a + d + x = b + c + x$. One thinks of the equivalence class $[(a,b)] \in K(S)$ as "$a - b$".

(a) Prove that $K(S)$ is an abelian group with the group operation $+_K$ described as follows

$$[(a,b)] +_K [(c,d)] := [(a+c, b+d)];$$

that is, "$(a-b) +_K (c-d) = (a+c) - (b+d)$".

(b) Prove that there is a abelian semigroup homomorphism $\iota : S \to K(S)$ such that $(K(S), \iota)$ has the following universal property.

For any abelian semigroup homomorphism $\phi : S \to A$, where A is an abelian group, there exists a unique abelian group homomorphism $\widetilde{\phi} : K(S) \to A$ such that $\widetilde{\phi} \circ \iota = \phi$ as abelian semigroup homomorphisms. That is, the following diagram exists and commutes.

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\iota"'] & A \\
K(S) \arrow[ur, dashed, "\exists ! \widetilde{\phi}"'] &
\end{tikzcd}$$

$(K(S), \iota)$ is called the *group completion* (or *groupification*) of S.

(c) Let A be an abelian group, prove that (A, id_A) satisfies the universal property that $(K(A), \iota)$ does above. We interpret that as saying "A is its own group completion (or groupification)" (as it should be).

(d) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, then $(\mathbb{N}, +)$ is a semigroup. Prove that $K(\mathbb{N}) \cong \mathbb{Z}$. This is one way how we construct the integers, by defining it to be $K(\mathbb{N})$.

(e) Prove that the association $S \mapsto K(S)$ is functorial and the above universal property describes the adjunction $K(-) \dashv I$ where $I : \mathsf{Ab} \hookrightarrow \mathsf{AbSemiGrp}$ is the inclusion of Ab as a full subcategory.

Problem 8.5. Recall that an abelian monoid $(M, +, 0)$ is a set M equipped with an associative binary product which is commutative and a neutral element 0. A homomorphism of monoid is one that commutes with the binary products, just like group homomorphisms.

Since M is, in particular, a semigroup, we can consider, as in Problem 8.4, the abelian group $K(M)$. In this case, the neutral element can just be treated as the class $[(m, 0)]$.

$(K(M), \iota)$ satisfies a similar commutative diagram as the one given in Problem 8.4(b), and is called the*group completion* (or *groupification*) of M. Furthermore $K(-)$ is a reflector of $I : \mathsf{Ab} \hookrightarrow \mathsf{AbMon}$, similar to what we saw in Problem 8.4(e).

Let $(M, +, 0)$ be an abelian monoid. Call an element $h \in M$ *high* if for all $x \in M$, there exists a $y \in M$ such that $h = x + y$. Let $H(M)$ be the set of all high elements of M.

(a) Prove that if $M = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then $H(M) = \varnothing$.

(b) Prove that if $H(M) \neq \varnothing$, then $H(M)$ is an abelian group under the same binary operation $+$ as M *but it's netural element is not necessarily the same neutral element as M.*

In this case, also describe a map $j : M \to H(M)$ of abelian monoids.

(c) Prove that $(H(M), j)$ has the following universal property.

For any abelian monoid homomorphism $\phi : M \to A$, where A is an abelian group, there exists a unique abelian group homomorphism $\widehat{\phi} : H(M) \to A$ such that $\widehat{\phi} \circ j = \phi$ as abelian monoid homomorphisms. That is, the following diagram exists and commutes.

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "j"'] & A \\ H(M) \arrow[ur, dashed, "\exists! \widehat{\phi}"'] & \end{tikzcd}$$

(d) If $H(M) \neq \varnothing$, prove that there exists an isomorphism of abelian groups $\phi : H(M) \to K(M)$ such that $\phi \circ j = \iota$, where ι is the map you described in Problem 8.4(b) for any (abelian) semigroup.

Remark 8.9. This is saying that if $H(M) \neq \varnothing$, then $H(M)$ itself is the group completion of M, and $H(-)$ extends to a functor which is left adjoint to $I : \mathsf{Ab} \hookrightarrow \mathsf{AbMon}$. In fact, the final part is also telling you that $H(-)$ and $K(-)$ are naturally isomorphic.

But this is not a practical way to construct group completion of a monoid M, as $H(M)$ is in most cases empty because we're essentially asking for inverses from within M.

Here's an example of an abelian monoid M where $H(M)$ is trivial. Let $M = \mathscr{P}(X)$, the power set of set X, then it's a commutative monoid with the operation given by intersection and the neutral element being X itself (resp. with the operation given by union and the neutral element being $\varnothing$). Then $H(\mathscr{P}(X)) = \{\varnothing\}$ (resp. $H(\mathscr{P}(X)) = \{X\}$) which is the trivial group with respect to the binary operation we have.

Problem 8.6. Look at Definition 8.2, prove that asking for the commutativity of the two diagrams there is equivalent to asking for commutativity of the single square below.

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(F(X), Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "g \circ - \circ F(f)"'] & \mathrm{Hom}_{\mathscr{C}}(X, G(Y)) \arrow[d, "G(g) \circ - \circ f"] \\ \mathrm{Hom}_{\mathscr{D}}(F(C), D) \arrow[r, "\tau_{C,D}"'] & \mathrm{Hom}_{\mathscr{C}}(C, G(D)) \end{tikzcd}$$

Problem 8.7. Given functors as follows

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "H"] & \mathscr{E} \arrow[l, shift left, "K"] \end{tikzcd}$$

such that $F \dashv G$ and $H \dashv K$, prove that $HF \dashv GK$.

9. Lecture 9 (3/04) by Deewang

Discussion 9.1. Suppose we have an adjunction $F \dashv G$, where $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$. That is, we have bijections, natural in their entries

$$\tau_{X,Y} : \mathrm{Hom}_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X, G(Y))$$

Fixing an object C in $\mathscr{C}$, this asserts that the functor $\mathrm{Hom}_{\mathscr{C}}(C, G(-))$ is represented by $F(C)$, so a representation will be given by $(F(C), \eta_C)$ where η_C is an element of $\mathrm{Hom}_{\mathscr{C}}(C, G(F(C)))$ that affords the required natural isomorphism. Specifically,

$$\eta_C = \tau_{C,F(C)}(1_{F(C)}).$$

The naturality of the adjunction gives us that η_C assembles into a natural transformation

$$\eta : 1_{\mathscr{C}} \Rightarrow GF$$

and is called the *unit* of the adjunction.

Dually, fixing an object D in $\mathscr{D}$, the adjunction asserts that the functor $\mathrm{Hom}_{\mathscr{D}}(F(-), D)$ is represented by $G(D)$, so a representation will be given by $(G(D), \varepsilon_D)$ where ε_D is an element of $\mathrm{Hom}_{\mathscr{D}}(F(G(D)), D)$ that affords the required natural isomorphism. Specifically,

$$\varepsilon_D = \tau^{-1}_{G(D),D}(1_{G(D)}).$$

The naturality of the adjunction gives us that ε_D assembles into a natural transformation

$$\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$$

and is called the *counit* of the adjunction.

We ask ourselves the question: are there conditions that we can ask η and ε to satisfy such that they characterise the adjunction $F \dashv G$. The answer is yes.

Theorem 9.2 (Adjunction via Unit-Counit)**.** *Given a pair of functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, we have $F \dashv G$ if and only if there exists a pair of natural transformations $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ satisfying the triangle identities, which is that the following triangle of natural transformations commute*

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\
 & G
\end{tikzcd}$$

To reiterate, the left-hand triangle asserts that a certain diagram commutes in $\mathscr{D}^{\mathscr{C}}$, while the right-hand triangle asserts that the dual diagram commutes in $\mathscr{C}^{\mathscr{D}}$.

Example 9.3. Let's find the unit η and counit ε of the tensor-hom adjunction. So, let's fix an R-module M, where R is a commutative ring. Then we have the following bijection natural in its entries

$$\begin{aligned} \tau_{L,N} : \mathrm{Hom}_R(L \otimes_R M, N) &\xrightarrow{\sim} \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)) \\ \phi &\longmapsto (x \mapsto (m \mapsto \phi(x \otimes m))) \\ (x \otimes m \mapsto \psi(x)(m)) &\longleftarrow\!\shortmid\, \psi \end{aligned}$$

Let $N = L \otimes_R M$, this gives us

$$\tau_{L, L\otimes_R M} : \mathrm{Hom}_R(L \otimes_R M, L \otimes_R M) \xrightarrow{\sim} \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, L \otimes_R M))$$

Then the component of η at L is $\eta_L = \tau_{L, L\otimes_R M}(1_{L\otimes_R M})$, that is, it's the morphism

$$\begin{aligned} \eta_L : L &\to \mathrm{Hom}_R(M, L \otimes_R M) \\ x &\mapsto (m \mapsto x \otimes m) \end{aligned}$$

Let $L = \mathrm{Hom}_R(M, N)$, this gives us

$$\tau_{\mathrm{Hom}_R(M,N), N} : \mathrm{Hom}_R(\mathrm{Hom}_R(M, N) \otimes_R M, N) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), \mathrm{Hom}_R(M, N))$$

Then the component of ε at N is $\varepsilon_N = \tau^{-1}_{\mathrm{Hom}_R(M,N), N}(1_{\mathrm{Hom}_R(M,N)})$, that is, it's the morphism

$$\begin{aligned} \varepsilon_N : \mathrm{Hom}_R(M, N) \otimes_R M &\to N \\ f \otimes m &\mapsto f(m) \end{aligned}$$

Corollary 9.4 (Unit-Counit for Monotone Galois Connections)**.** *If* P *and* Q *are posets and* $F : \mathsf{P} \rightleftarrows \mathsf{Q}$ *form a Galois connection, with* $F \dashv G$*, then* F *and* G *satisfy the following fixed point formulae*

$$FGF = F \quad \textit{and} \quad GFG = G.$$

Remark 9.5. Suppose $F \dashv G$ and $F \dashv H$, where $F : \mathscr{C} \rightleftarrows \mathscr{D} : G, H$, then Discussion 9.1 tells us that, fixing an object D in $\mathscr{D}$, both $G(D)$ and $H(D)$ represent $\mathrm{Hom}_{\mathscr{D}}(F(-), D)$. Therefore $G(D) \cong H(D)$, and this isomorphism is natural; one can deduce this from the given adjunctions. Hence G is naturally isomorphic to H.

Similarly, if $F \dashv K$ and $G \dashv K$, then we similarly have that F is naturally isomorphic to G.

Hence, any adjoint functor determines its adjoints up to natural isomorphism.

How about contravariant functors?

Definition 9.6. A pair of contravariant functors $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ and $G : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}$ *are mutually left adjoint* if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathscr{C}}(G(Y), X)$$

or *mutually right adjoint* if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(Y, F(X)) \cong \mathrm{Hom}_{\mathscr{C}}(X, G(Y)).$$

Dualizing Discussion 9.1 and Theorem 9.2, a pair of functors $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ and $G : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}$ that

- are mutually left adjoints come equipped with a pair of "counit" natural transformations $GF \Rightarrow 1_{\mathscr{C}}$ and $FG \Rightarrow 1_{\mathscr{D}}$;
- are mutually right adjoints come equipped with a pair of "unit" natural transformations $1_{\mathscr{C}} \Rightarrow GF$ and $1_{\mathscr{D}} \Rightarrow FG$.

The formulation of the triangle identities in each case is left to Problem 9.4.

Mutual right adjoints between posets form what is sometimes called an *antitone Galois connection*. The prototypical and eponymous example is the *Fundamental Theorem of Galois Theory*.

Example 9.7. Let LRS be the category of locally ringed spaces $(X, \mathscr{O}_X)$. Then the functors

$$\mathsf{Spec} : \mathsf{CRing}^{\mathrm{op}} \to \mathsf{LRS},\ A \mapsto \mathrm{Spec}(A) \quad \text{and} \quad \Gamma : \mathsf{LRS}^{\mathrm{op}} \to \mathsf{CRing},\ (X, \mathscr{O}_X) \mapsto \Gamma(X, \mathscr{O}_X)$$

are mutually right adjoint

$$\mathrm{Hom}_{\mathsf{CRing}}(A, \Gamma(X, \mathscr{O}_X)) \cong \mathrm{Hom}_{\mathsf{LRS}}(X, \mathrm{Spec}(A)).$$

Remark 9.8. Recall, from Discussion 6.6, that given a *$\mathscr{J}$-shaped diagram* in a category $\mathscr{C}$, that is a functor $D : \mathscr{J} \to \mathscr{C}$, the *limit of D*, if it exists, is a representation (in the sense of Discussion 5.4) of the functor $\mathrm{Cone}(-, D)$. That is, we have a natural isomorphism (natural in X)

$$\mathrm{Hom}_{\mathscr{C}^{\mathscr{J}}}(\Delta(X), D) = \mathrm{Nat}(\Delta(X), D) = \mathrm{Cone}(X, D) \cong \mathrm{Hom}_{\mathscr{C}}(X, \lim D)$$

Dually, we also have

$$\mathrm{Hom}_{\mathscr{C}^{\mathscr{J}}}(D, \Delta(X)) \cong \mathrm{Hom}_{\mathscr{C}}(\operatorname{colim} D, X)$$

Here $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{J}}$ is the functor which sends any object X to the constant diagram $\Delta(X) : \mathscr{J} \to \mathscr{C}$ which sends every object in $\mathscr{J}$ to X.

These isomorphisms, in fact, upgrade to adjunctions $\operatorname{colim} \dashv \Delta \dashv \lim$. Hence, we have: *A category $\mathscr{C}$ admits all limits of diagrams indexed by a small category $\mathscr{J}$ if and only if the constant diagram functor $\Delta : \mathscr{C} \to \mathscr{C}^{\mathscr{J}}$ admits a right adjoint, and admits all colimits of $\mathscr{J}$-shaped diagrams if and only if Δ admits a left adjoint.*

Theorem 9.9 (RAPL)**.** *Right adjoints preserve limits. Dually, left adjoints preserve colimits.*

That is, if $F : \mathscr{C} \to \mathscr{D}$ is a right adjoint (there exists a functor $H : \mathscr{D} \to \mathscr{C}$ such that $H \dashv F$) then

$$F(\lim A) \cong \lim F(A)$$

whenever $\lim A$ *exists, where* $\lim A$ *is the limit of a diagram $A : \mathscr{J} \to \mathscr{C}$.*

Sketch of Proof for Products. Let the adjunction be built from the bijections

$$\mathrm{Hom}_{\mathscr{C}}(H(X), Y) \leftrightarrow \mathrm{Hom}_{\mathscr{D}}(X, F(Y))$$

$$f^{\sharp} \leftrightarrow f^{\flat}$$

We'll explicitly prove this result for a baby case. Let $X \times Y$ be a product of objects X and Y in $\mathscr{C}$, we want to prove that

$$F(X \times Y) \cong F(X) \times F(Y);$$

we prove this by showing that $F(X \times Y)$ satisfies the universal property of products.

Consider the product diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd} \quad \underset{\text{apply } F}{\rightsquigarrow} \quad \begin{tikzcd} & F(X \times Y) \arrow[dl, "F(p_X)"'] \arrow[dr, "F(p_Y)"] & \\ F(X) & & F(Y) \end{tikzcd}$$

We show that $(F(X \times Y), F(p_X), F(p_Y))$ is the product of $F(X)$ and $F(Y)$; that is, that it satisfies the universal property. Consider any other object W in $\mathscr{D}$ such that

$$\begin{tikzcd} & W \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\ & F(X \times Y) \arrow[dl, "F(p_X)"'] \arrow[dr, "F(p_Y)"] & \\ F(X) & & F(Y) \end{tikzcd} \quad \underset{\text{apply adjunction}}{\rightsquigarrow} \quad \begin{tikzcd} & H(W) \arrow[ddl, bend right, "f_X^\sharp"'] \arrow[ddr, bend left, "f_Y^\sharp"] & \\ & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

Then by the universal property of the product we have

$$\begin{tikzcd} & H(W) \arrow[ddl, bend right, "f_X^\sharp"'] \arrow[ddr, bend left, "f_Y^\sharp"] \arrow[d, dashed, "h"] & \\ & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd} \quad \underset{\text{apply adjunction}}{\rightsquigarrow} \quad \begin{tikzcd} & W \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] \arrow[d, dashed, "h^\flat"] & \\ & F(X \times Y) \arrow[dl, "F(p_X)"'] \arrow[dr, "F(p_Y)"] & \\ F(X) & & F(Y) \end{tikzcd}$$

Uniqueness and commutativity with respect to $h^\flat$ is a consequence of the adjunction, and therefore $F(X \times Y)$ satisfies the universal property of the product. Hence

$$F(X \times Y) \cong F(X) \times F(Y)$$

given by a unique ϕ such that

$$p_{F(X)} \circ \phi = F(p_X) \quad \text{and} \quad p_{F(Y)} \circ \phi = F(p_Y).$$

The general proof follows as a result of the following adjunction isomorphisms and that Δ commutes with functors

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^{\mathcal{I}}}(\Delta(X), F(A)) &\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(H(\Delta(X)), A) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\Delta(H(X)), A) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(H(X), \lim A) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(X, F(\lim A))
\end{aligned}$$

That is, $F(\lim A)$ exhibits the defining universal property of the limit, and therefore $F(\lim A) \cong \lim F(A)$ canonically. $\square$

Discussion 9.10. This is precisely why products in categories like Grp, Ring, Mod_R and Top are so familiar, that is because the forgetful functors are all right adjoints (the left adjoints are listed in Example 8.4). The underlying sets of limits in groups, rings, R-modules and topological spaces, like products, pullbacks etc., are the limits in the category of sets.

On the other hand, forgetful functors rarely are left adjoints, therefore colimits in these categories are much different, for example the free product in Grp and the tensor product in CRing. But the forgetful functor from the category of topological spaces to sets does possess a right adjoint, making it a left adjoint, and therefore the colimits in Top look exactly like the ones in Set with an appropriate topology.

Example 9.11. We use Theorem 9.9 to show how it can be used to prove the non-existence of adjoints. Consider the forgetful functor $U : \mathsf{Ring} \to \mathsf{Rng}$, we have seen in Problem 8.2 that it has a left adjoint, the Dorroh extension.

Let's prove that U does not have any right adjoints. Suppose it did, then U would be a left adjoint and then the dual statement of Theorem 9.9 that it would preserve all small colimits. In particular, initial objects (which are empty coproducts). But $U(\mathbb{Z})$ is not the initial object of Rng, which is the 0 ring, while $\mathbb{Z}$ is the initial object of Ring. Hence we have arrived at a contradiction, and thus U has no right adjoint.

9.1. Problems

Problem 9.1. Pick some (or all) examples given in Examples 8.4, 8.5 and 8.8 (1), and work out the unit and counit (you should have proved these are adjunctions in Problem 8.1).

Problem 9.2.

(a) Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be a pair of functors such that $F \dashv G$ with unit η and counit ε. Write $\mathrm{Fix}(GF)$ for the full subcategory of $\mathscr{C}$ whose objects are those C in $\mathscr{C}$ such that η_C is an isomorphism, and dually consider $\mathrm{Fix}(FG) \hookrightarrow \mathscr{D}$.

Prove that the adjunction $(F, G, \eta, \varepsilon)$ restricts to an equivalence $(F', G', \eta', \varepsilon')$ between $\mathrm{Fix}(GF)$ and $\mathrm{Fix}(FG)$.

(b) Part (a) shows that every adjunction restricts to an equivalence between full subcategories in a canonical way.

Pick some (or all) examples given in Examples 8.4 and 8.8, and work out the full subcategories that are shown to be equivalent in this way.

Problem 9.3.

(a) Prove that for the examples in Example 8.5, the counit is an isomorphism.

(b) More generally, show that for any adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism.

Such adjunctions, in general, are called *reflections*. The essential image of the right adjoint, in this case, is then a reflective subcategory. Dualising we get *coreflections*, and the essential image of the left adjoint is then a coreflective subcategory.

Problem 9.4.

(a) Dualize Theorem 9.2 to define mutual left adjoints and mutual right adjoints as a pair of contravariant functors equipped with appropriate natural transformations.

(b) Repeat Problem 9.2 (a) for mutually left and right adjoints.

In the case of Example 9.7, the adjunction restricts to give an equivalence between the category of commutative unital rings and affine schemes.

Problem 9.5. Show that the inverse image (contravariant power set) functor $\mathscr{P}^* : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is mutually right adjoint to itself. Determine the units of this adjunction, as worked out in Problem 9.4 (a).

Problem 9.6. Which of the examples given in Examples 8.4, 8.5 and 8.8 (1) don't have right adjoints.

A. Group Objects

Definition A.1. Given a category $\mathscr{C}$ with products, in particular a terminal object pt, a *group object* G in $\mathscr{C}$ is an object equipped with three morphisms

multiplication	$m_G : G \times G \to G$
identity element (not the identity map)	$e_G : \mathrm{pt} \to G$
inverse	$i_G : G \to G$

such that following diagrams commute.

- *the identity is a left and right identity*

$$\begin{tikzcd} \mathrm{pt} \times G \arrow[r, "e_G \times 1_G"] \arrow[d, "\wr"'] & G \times G \arrow[d, "m_G"] \\ G \arrow[r, "1_G"'] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \times \mathrm{pt} \arrow[r, "1_G \times e_G"] \arrow[d, "\wr"'] & G \times G \arrow[d, "m_G"] \\ G \arrow[r, "1_G"'] & G \end{tikzcd}$$

- *multiplication is associative*

$$\begin{tikzcd} G \times G \times G \arrow[r, "m_G \times 1_G"] \arrow[d, "1_G \times m_G"'] & G \times G \arrow[d, "m_G"] \\ G \times G \arrow[r, "m_G"'] & G \end{tikzcd}$$

- *the inverse is a left and right inverse*

$$\begin{tikzcd} G \arrow[r, "{(i_G, 1_G)}"] \arrow[d] & G \times G \arrow[d, "m_G"] \\ \mathrm{pt} \arrow[r, "e_G"'] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "{(1_G, i_G)}"] \arrow[d] & G \times G \arrow[d, "m_G"] \\ \mathrm{pt} \arrow[r, "e_G"'] & G \end{tikzcd}$$

Example A.2. It follows immediately from the definition that when $\mathscr{C} = \mathsf{Set}$, the group objects are precisely groups. In fact, the definition of a group object is clearly modelled on the group axioms.

The group objects in $\mathscr{C} = \mathsf{Grp}$ are abelian groups, this is because we require the inverse to be a group homomorphism. The group objects in $\mathscr{C} = \mathsf{Top}$ are topological groups and in $\mathscr{C} = \mathsf{SmMan}$ are Lie groups. For any space Y, the loop space ΩY, see Example 8.8, is a group object in HTop_*.

The group objects in the category of schemes are called group schemes.

Proposition A.3. *To give a group object structure on an object G of $\mathscr{C}$ is equivalent to saying that, using Theorem 5.1 (Yoneda Lemma), the functor*

$$h_G = \mathrm{Hom}_{\mathscr{C}}(-, G) : \mathscr{C}^{op} \to \mathsf{Set}$$

factors through Grp*, where U is the forgetful functor*

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[rr, "h_G"] \arrow[dr, "G(-)"'] & & \mathsf{Set} \\ & \mathsf{Grp} \arrow[ur, "U"'] & \end{tikzcd}$$

That is, $G(-)$ is just the functor which associates any object X of $\mathscr{C}$ to the group whose underlying set is $\mathrm{Hom}_{\mathscr{C}}(X, G)$.

Definition A.4. If G and H are group objects in $\mathscr{C}$, we define a homomorphism of group objects as a morphism $G \to H$ in $\mathscr{C}$, such that for each object U of $\mathscr{C}$ the induced function $G(U) \to H(U)$ is a group homomorphism.

Equivalently, using Theorem 5.1 (Yoneda Lemma), a homomorphism is a morphism $\phi : G \to H$ such that the following diagram commutes

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "\phi \times \phi"'] & G \arrow[d, "\phi"] \\ H \times H \arrow[r, "m_H"'] & H \end{tikzcd}$$

Discussion A.5. The identity is clearly a homomorphism from a group object to itself. Furthermore, the composite of homomorphisms of group objects is still a homomorphism; thus, group objects in a category $\mathscr{C}$ forms a subcategory, which we denote by $\mathsf{Grp}(\mathscr{C})$. So, we have, for example $\mathsf{Grp} \simeq \mathsf{Grp}(\mathsf{Set})$ and $\mathsf{Ab} = \mathsf{Grp}(\mathsf{Grp})$.

Furthermore, we have Theorem 5.1 (Yoneda Lemma) embedding

$$\mathsf{Grp}(\mathscr{C}) \hookrightarrow \mathsf{Grp}^{\mathscr{C}^{\mathrm{op}}} = [\mathscr{C}^{\mathrm{op}}, \mathsf{Grp}]$$

Definition A.6. For a category $\mathscr{C}$ with coproducts, a group object in $\mathscr{C}^{\mathrm{op}}$ is called a *cogroup object*. The subcategory of cogroup objects of $\mathscr{C}$ is identified with $\mathsf{Grp}(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$. Working out the diagrams involved in the definition of a cogroup object, as in Definition A.1, is left as Problem A.4. Equivalently, the functor

$$h^G = \mathrm{Hom}_{\mathscr{C}}(G, -) : \mathscr{C} \to \mathsf{Set}$$

factors through Grp

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "h^G"] \arrow[dr] & & \mathsf{Set} \\ & \mathsf{Grp} \arrow[ur, "U"'] & \end{tikzcd}$$

Example A.7. For any space X, the suspension ΣX, see Example 8.8, is a cogroup object in HTop_*. Specifically, the n-spheres S^n are cogroup objects in HTop_*. The cogroup objects in $\mathscr{C} = \mathsf{Grp}$ are free groups. The cogroup objects in $\mathscr{C} = \mathsf{CRing}$ are commutative Hopf algebras.

Discussion A.8. We have discussed the notion of a group object, and by Theorem 5.1 (Yoneda Lemma), the group objects can be considered a full subcategory of Grp-valued functors. The next natural thing to consider will be objects with an action of a group object. Because of Theorem 5.1 (Yoneda Lemma), we can, in general, talk about an action of a Grp-valued functor on a Set-valued functor and this definition will restrict to objects. Here's how we do this.

A *left action* α of a functor $G : \mathscr{C}^{\text{op}} \to \mathsf{Grp}$ on a functor $F : \mathscr{C}^{\text{op}} \to \mathsf{Set}$ is a natural transformation

$$\alpha : G \times F \to F,$$

such that for any object U of $\mathscr{C}$, the induced function

$$\alpha_U : G(U) \times F(U) \to F(U)$$

is an action of the group $G(U)$ on the set $F(U)$.

Then, a left action of a group object G on an object X is defined to be a left action of the functor h_G on the functor h_X. This is a bit high-brow, we reduce this definition down in terms of diagrams.

Definition A.9. Given a category $\mathscr{C}$ with a group object (G, m_G, e_G, i_G) and terminal object pt, a *left action of G on an object X* is a morphism

$$\alpha_X : G \times X \to X$$

such that following diagrams commute.

- *the identity of G acts like the identity on X:*

$$\begin{tikzcd} \text{pt} \times X \arrow[r, "e_G \times 1_X"] \arrow[d, "\wr"'] & G \times X \arrow[d, "\alpha_X"] \\ X \arrow[r, "1_G"'] & X \end{tikzcd}$$

- *the action is associative with respect to the multiplication on G*

$$\begin{tikzcd} G \times G \times X \arrow[r, "m_G \times 1_X"] \arrow[d, "1_G \times \alpha_X"'] & G \times X \arrow[d, "\alpha_X"] \\ G \times X \arrow[r, "\alpha_X"'] & X \end{tikzcd}$$

Let X and Y be objects of $\mathscr{C}$ with an action of G, a morphism $f : X \to Y$ is called G-equivariant if for all objects U of $\mathscr{C}$ the induced function $h_X(U) \to h_Y(U)$ is $G(U)$-equivariant.

Equivalently, using Theorem 5.1 (Yoneda Lemma), f is G-equivariant if the following diagram commutes

$$\begin{tikzcd} G \times X \arrow[r, "\alpha_X"] \arrow[d, "1_G \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\alpha_Y"'] & Y \end{tikzcd}$$

Discussion A.10. Suppose you're given an action of a group G on a set X, note that this is equivalent to saying there's a group homomorphism

$$G \to \mathsf{Sym}(X) = \mathrm{Aut}_{\mathsf{Set}}(X),\ g \mapsto (x \mapsto g \cdot x)$$

So, one may ask if there's a similar way to express that an object X in a category $\mathscr{C}$ has an action of a group object G in $\mathscr{C}$. One can talk naively about $\mathrm{Aut}_{\mathscr{C}}(X)$ and it will be a group under composition but it's not necessarily an object in $\mathscr{C}$ much less a group object in $\mathscr{C}$, so it's not clear what a "group homomorphism" means between G and $\mathrm{Aut}_{\mathscr{C}}(X)$.

So instead of wondering how we can interpret $\mathrm{Aut}_{\mathscr{C}}(X)$ as a group object, we expand our vision a bit and give a generalisation of this as a Grp-valued functor

Discussion A.11. For a category $\mathscr{C}$ with products and a terminal object pt, fix an object X. For an object U in $\mathscr{C}$ we consider the canonical projection $p_U : U \times X \to U$ which is an object of the slice category $\mathscr{C}/U$. Write

$$\mathrm{Aut}_U(U \times X) := \mathrm{Aut}_{\mathscr{C}/U}(p_U) = \{f \in \mathrm{Aut}_{\mathscr{C}}(U \times X)\ :\ p_U \circ f = p_U\}$$

These are the automorphisms of $U \times X$ compatible with the projection p_U.

This gives a functor (Problem A.7)

$$\underline{\mathrm{Aut}}_{\mathscr{C}}(X) : \mathscr{C}^{\mathrm{op}} \to \mathsf{Grp},\ U \mapsto \mathrm{Aut}_U(U \times X)$$

The group $\underline{\mathrm{Aut}}_{\mathscr{C}}(X)(\mathrm{pt})$ is canonically isomorphic to $\mathrm{Aut}_{\mathscr{C}}(X)$.

Proposition A.12. *Let G be a group object in $\mathscr{C}$, To give an action of G on X is equivalent to giving a natural transformation*

$$G(-) \Rightarrow \underline{\mathrm{Aut}}_{\mathscr{C}}(X)$$

of functors $\mathscr{C}^{op} \to \mathsf{Grp}$.

In particular, we have group homomorphisms $G(U) \to \underline{\mathrm{Aut}}_{\mathscr{C}}(X)(U)$, for each object U.

Specifically, for $U = \mathrm{pt}$ we obtain a group homomorphism

$$G(\mathrm{pt}) \to \mathrm{Aut}_{\mathscr{C}}(X).$$

Remark A.13. When $\mathscr{C} = \mathsf{Set}$, the functor $\underline{\mathrm{Aut}}_{\mathsf{Set}}(X)$ is represented (Problem A.8) for any set X by $\mathrm{Sym}(X) = \{f : X \to X \ : \ f \text{ is a bijection}\}$.

Therefore, by Theorem 5.1 (Yoneda Lemma), asking for a natural transformation $G(-) \Rightarrow \underline{\mathrm{Aut}}_{\mathsf{Set}}(X)$ of functors $\mathsf{Set}^{\mathrm{op}} \to \mathsf{Grp}$ is the same as asking for a group homomorphism $G \to \mathrm{Sym}(X)$, which was our motivating example.

A.1. Problems

Problem A.1. Define the notion of commutative (abelian) group object, and ring object in a category $\mathscr{C}$. You can try generalising other algebraic notions into this categorical framework, e.g. inner automorphisms of a group, a module etc.

Problem A.2. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are categories with products, and suppose that $F : \mathscr{C} \to \mathscr{D}$ is a functor that preserves finite products. Prove that F restrict to a functor $\mathsf{Grp}(\mathscr{C}) \to \mathsf{Grp}(\mathscr{D})$, that is, prove that given a group object G in $\mathscr{C}$, $F(G)$ is a group object in $\mathscr{D}$.

Problem A.3. Given a homomorphism $\phi : G \to H$ of group objects G and H in $\mathscr{C}$ with terminal object pt, prove that we have the following commutative diagrams

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "e_G"] \arrow[d, "e_H"'] & G \arrow[d, "\phi"] \\ H \arrow[r, "1_H"'] & H \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\phi"'] & G \arrow[d, "\phi"] \\ H \arrow[r, "i_H"'] & H \end{tikzcd}$$

Problem A.4. Given a category $\mathscr{C}$ with coproducts, in particular an initial object O, a *cogroup object* K in $\mathscr{C}$ is an object equipped with three morphisms

$$\begin{array}{cc} \textit{comultiplication} & \mu_K : K \to K \amalg K \\ \textit{coidentity element} & \eta_K : K \to \mathrm{O} \\ \textit{coinverse} & \iota_K : K \to K \end{array}$$

Write down the commutative diagrams we should have.

Problem A.5. Pick your favourite categories and work out what are the group and cogroup objects in them.

Problem A.6.

(a) Construct a functor $\mathrm{GL}_n : \mathsf{CRing} \to \mathsf{Grp}$ that associates to each commutative unital ring A, the group, $\mathrm{GL}_n(A)$, of invertible matrices with entries in A, i.e. the group of matrices with a non-zero determinant.

(b) Prove that the functor GL_n is representable, i.e., there exists a (commutative and unital) ring $B_{(n)}$ such that we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{CRing}}(B, -) \xRightarrow{\sim} \mathrm{GL}_n$$

You may find it easier to first work this out for $n = 2$. We've already seen $n = 1$, we usually denote the functor $\mathrm{GL}_1 = \mathbb{G}_m$, and it's precisely the unit group functor and is represented by the ring $\mathbb{Z}[t^{\pm}]$.

This shows that B is a cogroup object in CRing. In fact, this also shows that $\operatorname{Spec} B$ is a group object in the category of (affine) schemes.

Problem A.7. Prove that the association $\underline{\mathrm{Aut}}_{\mathscr{C}}(X)$ defined in Definition A.11 is indeed a functor.

Problem A.8. Prove that the functor $\underline{\mathrm{Aut}}_{\mathsf{Set}}(X)$ is represented by $\mathrm{Sym}(X)$.

Problem A.9.

(a) Given a cogroup object K in a category $\mathscr{C}$ with coproducts, describe what it means for an object X to have a *right co-action of K* by dualising Definition A.9.

(b) Prove that the coproduct of two rings A and B in CRing is $A \otimes_{\mathbb{Z}} B$.

(c) Prove that saying that a ring A is equipped with a co-action of the cogroup $\mathbb{Z}[t^{\pm}]$ is equivalent to saying that A has an integer-valued grading.

B. Abelian Categories

Discussion B.1. We know how useful the notion of kernels, images, cokernels are since we first learned linear algebra. More generally, the category of modules possesses really nice properties and notions. We want to axiomatise this so that we may recognise other categories that behave the way a category of modules does so that we can commence a similar exercise as the one we carry out with modules. This is the notion of an *abelian category*, for which we need to first define *additive categories*.

Definition B.2. A category $\mathscr{C}$ is said to be *additive* if it satisfies the following properties.

- For each object A, B in $\mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition, or equivalently the composition is bilinear.
- $\mathscr{C}$ has a zero object, denoted 0. That is, an object that is simultaneously both an initial and terminal object. The 0-morphism from A to B is the unique morphism in $\mathrm{Hom}_{\mathscr{C}}(A, B)$ given as $A \to 0 \to B$
- It has products of a pair of objects. That is, $A \times B$ makes sense for any pair of objects A, B in $\mathscr{C}$, and therefore all finite products exist inductively.

Definition B.3. Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories, then a functor $F : \mathscr{C} \to \mathscr{D}$ is called *additive* if the map

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)) : f \mapsto F(f)$$

is a homomorphism of abelian groups for all pairs of objects A, B in $\mathscr{C}$.

Example B.4. The prototypical example is the category of R-modules Mod_R which is clearly an additive category. We of course have even more structure, we can talk about kernels, images, the First Isomorphism theorem, for example, which we now build towards.

Definition B.5. Let $\mathscr{C}$ be a category with a 0-object (and thus 0-morphisms). A *kernel of a morphism* $f : A \to B$, if it exists, is the equaliser (recall from Example 6.7 (ii)) of the diagram

$$A \underset{0}{\overset{f}{\rightrightarrows}} B$$

That is, the kernel is a pair (K, ι), denoted $\ker f$, where $\iota : K \to A$ is a morphism with $f \circ \iota = 0$, which has the following universal property: for any pair (L, j) where $j : L \to A$ is a morphism with $f \circ j = 0$, there exists a unique $g : L \to K$ such that $\iota \circ g = j$.

Dually, a *cokernel of f*, if it exists, is the coequaliser of the diagram above, denoted $\operatorname{coker} f$.

Definition B.6. An additive category $\mathscr{C}$ is said to be *abelian* if it satisfies the following properties.

- Every morphism has a kernel and cokernel.

- Every monomorphism is the kernel of its cokernel.
- Every epimorphism is the cokernel of its kernel.

Example B.7. The prototypical and eponymous example is the category of abelian groups Ab, and in general the category of R-modules Mod_R. A more general example is the category of sheaves over a space where sheaves take values in an abelian category.

Lemma B.8. *Let $f : A \to B$ be a morphism in an abelian category $\mathscr{C}$. Then f is an isomorphism if and only if f is a monomorphism and an epimorphism.*

Proof. Any isomorphism is by definition a split monomorphism and epimorphism, and is therefore a monomorphism an an epimorphism, in particular.

Now, suppose that $f : A \to B$ is both an epimorphism and a monomorphism. Since $f : A \to B$ is, in particular, an epimorphism, we have $f = \operatorname{coker}(\ker f)$. By Problem B.5 (a) we have $\ker f = 0$, since f is also a monomorphism. That is, $\operatorname{coker}(0 \to A) = (B, f)$.

But as noted in Problem B.5 (b), we also have $\operatorname{coker}(0 \to A) = (A, 1_A)$. Therefore, as a colimit is unique up to unique isomorphism as a consequence of its universal property, there exists an isomorphism $\phi : A \to B$ such that $1_A \circ \phi = f$. Hence f is an isomorphism. $\square$

Proposition B.9 (First Isomorphism theorem)**.** *Let $f : A \to B$ be a morphism in an additive category. Suppose that $\ker f = (K, i)$ and $\operatorname{coker} f = (C, p)$. Moreover, suppose that $\operatorname{coker}(\ker f) = \operatorname{coker} i = (I, \pi)$ and $\ker(\operatorname{coker} f) = \ker p = (I', \kappa)$.*

Then there exists a unique morphism $\phi : I \to I'$ in $\mathscr{C}$ such that $f = \kappa \circ \phi \circ \pi$.

$$\begin{tikzcd}
K \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\pi"'] & B \arrow[r, "p"] & C \\
& I \arrow[r, dashed, "\phi"'] & I' \arrow[u, "\kappa"'] &
\end{tikzcd}$$

Furthermore, ϕ is an isomorphism if and only if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Proof. We first remark that π and κ are an epimorphism and a monomorphism respectively by Problem B.4. We first show the existence of ϕ.

Since $\operatorname{coker} i = (I, \pi)$ and $f \circ i = 0$, then by the universal property of the cokernel, there exists a unique morphism $g : I \to B$ such that $g \circ \pi = f$.

$$\begin{tikzcd}
K \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\pi"'] & B \arrow[r, "p"] & C \\
& I \arrow[ur, dashed, "g"'] & I' \arrow[u, "\kappa"'] &
\end{tikzcd}$$

Moreover, since $p \circ g \circ \pi = p \circ f = 0$ and π is an epimorphism, we obtain $p \circ g = 0$. Thus, since $\ker p = (I', \kappa)$, then by the universal property of the kernel, there exists a unique morphism $\phi : I \to I'$ such that $\kappa \circ \phi = g$.

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\pi"'] & B \arrow[r, "p"] & C \\ & I \arrow[ur, dashed, "g"] \arrow[r, dashed, "\phi"'] & I' \arrow[u, "\kappa"'] & \end{tikzcd}$$

Altogether, we have

$$\kappa \circ \phi \circ \pi = g \circ \pi = f$$

Assume that also $\psi : I \to I'$ is any other morphism such that $\kappa \circ \psi \circ \pi = f$. Then

$$\kappa \circ \psi \circ \pi = f = \kappa \circ \phi \circ \pi.$$

Since π is an epimorphism and κ a monomorphism, we obtain $\psi = \phi$. Therefore ϕ is the unique such morphism.

($\Leftarrow$) You can find a proof for this in [5, Chapter VIII].

($\Rightarrow$) Suppose ϕ is an isomorphism, we have

$$\operatorname{coker}(\ker f) \xrightarrow{\sim} \ker(\operatorname{coker} f)$$

If f is a monomorphism, by Problem B.5 (b), the source is f, giving us f is the kernel of its cokernel. On the other hand, if f is an epimorphism, again by Problem B.5 (b), the target is f, giving us f is the cokernel of its kernel. □

Remark B.10. In an additive category, or a category with a 0-object, given a morphism $f : A \to B$, we define the *image of* f as $\operatorname{im} f := \ker(\operatorname{coker} f)$ and the *coimage of* f as $\operatorname{coim} f := \operatorname{coker}(\ker f)$.

Let's investigate these objects in the category of abelian groups Ab. Consider a group homomorphism $f : A \to B$, then the kernel and image is what we know them to be. Turns out the cokernel is $A/\operatorname{im} f$. Precisely put,

$$\ker f = (\ker f,\ \iota : \ker f \hookrightarrow A) \quad \text{and} \quad \operatorname{coker} f = (A/\operatorname{im} f,\ \pi : A \twoheadrightarrow A/\operatorname{im} f)$$

Then, by our definition, $\operatorname{im} f := \ker \pi = \operatorname{im} f$ and $\operatorname{coim} f := \operatorname{coker} \iota = A/\ker f$. If we were to ask for an isomorphism

$$A/\ker f \cong \operatorname{im} f$$

we would be asking for the First Isomorphism theorem to hold.

Thus, Proposition B.9 tells us that one can equivalently define an abelian category to be an additive category where each morphism has a kernel and cokernel and the "First Isomorphism theorem holds". The proposition also tells us that every morphism $f : A \to B$ in an abelian category factorises as

$$A \to \operatorname{im} f \to B$$

where the first arrow is an epimorphism, and the second arrow a monomorphism.

Definition B.11. We say a sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

- is a *complex* at B if $g \circ f = 0$; and
- is *exact* at B if $\ker g = \operatorname{im} f$. More specifically, g has a kernel that is an image of f.

A sequence is a *complex* (resp. *exact*) if it is a complex (resp. exact) at each term.

A *short exact sequence* is a five-term exact sequence of the form

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

Discussion B.12. Several results about exact sequences, say in the category Mod_R, are proved by chasing elements, as you will see in Problems B.7 and **??**. One would like to be able to do the same in any abelian category, since the objects in an abstract abelian category are not guaranteed to be sets. But this can be justified by the

Freyd-Mitchell Embedding Theorem. If $\mathscr{C}$ is a locally small abelian category, then there is a ring R, which could be non-commutative, and an exact (see Definition B.13), fully faithful functor $\mathscr{C} \to \mathsf{Mod}_A$, which embeds $\mathscr{C}$ as a full subcategory.

What this means is that to prove something about a diagram in some abelian category, we may assume that it is a diagram of modules over some ring, and we may then "diagram-chase". Moreover, any fact about kernels, cokernels, and so on that holds in Mod_R holds in any abelian category.

Definition B.13. Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between abelian categories, and consider a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

Then we say

- F is *right-exact* if
$$F(A) \xrightarrow{f} F(A') \xrightarrow{g} F(A'') \longrightarrow 0$$
is exact
- F is *left-exact* if
$$0 \longrightarrow F(A') \xrightarrow{f} F(A) \xrightarrow{g} F(A'')$$
is exact
- F is *exact* if F is both left- and right-exact.

Example B.14. Let R be a ring, and consider the category Mod_R and fix an R-module M.

- $- \otimes_R M$ is an additive right-exact functor $\mathsf{Mod}_R \to \mathsf{Mod}_R$.
- $\mathrm{Hom}_R(M, -)$ is an additive left-exact functor $\mathsf{Mod}_R \to \mathsf{Mod}_R$.
- $\mathrm{Hom}_R(-, M)$ is an additive (contravariant) left-exact functor $\mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Mod}_R$.

More generally, if $\mathscr{C}$ be any abelian category, and C is an object of $\mathscr{C}$

- $\mathrm{Hom}_{\mathscr{C}}(C, -)$ is an additive left-exact functor $\mathscr{C} \to \mathsf{Ab}$.
- $\mathrm{Hom}_{\mathscr{C}}(-, C)$ is an additive (contravariant) left-exact functor $\mathscr{C}^{\mathrm{op}} \to \mathsf{Ab}$.

B.1. Problems

Problem B.1. In an additive category, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group for any pair of objects A, B in $\mathscr{C}$. Prove that the neutral element of this group is the 0-morphism.

Problem B.2. Let $\{A_i\}_{i \in I}$ be a collection of finite objects in an additive category $\mathscr{C}$; we know their product exists. Let $(P = \prod_{i \in I} A_i, (\pi_i : P \to A_i)_{i \in I})$ be the product.

Prove that there exist unique morphisms $\iota_j : A_j \to P$ in $\mathscr{C}$, for each $j \in J$, such that

$$\pi_i \circ \iota_j = \begin{cases} 0 & \text{if } i \neq j \\ 1_{A_i} & \text{if } i = j \end{cases} \qquad \text{for any } i, j \in J$$

and

$$\sum_{j \in I} \iota_j \circ \pi_J = 1_P$$

Moreover, prove that $(P, (\iota_j)_{j \in I})$ is a coproduct of $\{A_i\}_{i \in I}$ in $\mathscr{C}$.

$(P, (\pi_j)_{j \in I}, (\iota_j)_{j \in I})$ such that above two relations above are satisfied is called the *biproduct* or *direct sum* and denoted $\oplus_{i \in I} A_i$ with the maps left implicit.

Problem B.3. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between additive categories. Prove that

(a) if F is additive, then $F(0_{\mathscr{C}}) \cong 0_{\mathscr{D}}$, by showing that an object Z is a zero object if and only if $1_Z = Z \to 0 \to Z$.

(b) if F is additive, F preserves products, that is $F(A \times B) \cong F(A) \times F(B)$.

(c) if F is an equivalence, then F is additive.

Problem B.4. If $i : A \to B$ is a monomorphism, then we say that A is a *subobject* of B, where the map i is implicit; furthermore, we write $B/A := \operatorname{coker} i$. Dually, if $p : A \to B$ is an epimorphism, then we say that B is a *quotient object* of A, where p is implicit.

Let $\mathscr{C}$ be a category with a 0-object, and consider a morphism $f : A \to B$. Assume (K, ι) and (C, π), a kernel and cokernel of f respectively, exist, prove that ι is a monomorphism and π is an epimorphism. That is, K is a subobject of A and C is a quotient object of B.

Problem B.5. Let $\mathscr{C}$ be an abelian category.

(a) Prove that $i : A \to B$ is a monomorphism if and only if $\ker i = (0, 0 \to A)$. Formulate and prove the dual statement.

(b) Consider the morphism $0 \to A$, prove that its cokernel is $(A, 1_A)$. Formulate and prove the dual statement.

This problem then tells us, in particular, that if i and p is a monomorphism and epimorphism respectively, then

$$\operatorname{coker}(\ker i) = i \quad \text{and} \quad \ker(\operatorname{coker} p) = p$$

Problem B.6. Let $\mathscr{C}$ be an abelian category, and suppose you have the short exact sequence

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

Prove that

(a) f is a monomorphism and g is an epimorphism.

(b) f (resp. g) is an isomorphism if and only $A'' = 0$ (resp. $A' = 0$).

(c) $A = 0$ if and only if $A' = A'' = 0$.

Problem B.7 (Snake Lemma)**.** Consider the following diagram in Mod_R, where R is a ring,

$$\begin{tikzcd}
 & M' \arrow[r] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' &
\end{tikzcd}$$

where the rows are exact and the squares commute. Show that the following exact sequence in colour exists.

$$\begin{tikzcd}
 & \ker\alpha \arrow[r] \arrow[d, hook] & \ker\beta \arrow[r] \arrow[d, hook] & \ker\gamma \arrow[d, hook] \arrow[dddll, rounded corners] & \\
 & M' \arrow[r] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d, two heads] & N \arrow[r] \arrow[d, two heads] & N'' \arrow[d, two heads] & \\
 & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma &
\end{tikzcd}$$

here $\hookrightarrow$ denotes the canonical monomorphism, and $\twoheadrightarrow$ denotes the canonical epimorphism.

The connecting morphism $\ker\gamma \to \operatorname{coker}\alpha$ in the diagram is titular figurative "snake".

Problem B.8 (Four Lemma)**.** Consider the following diagram in Mod_R, where R is a ring,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

where the rows are exact and the squares commute. Show that

(a) if α is surjective, and β and δ are injective, then γ is injective.

(b) if δ is injective, and α and γ are surjective, then β is surjective.

Problem B.9 (Five Lemma)**.** Consider the following diagram in Mod_R, where R is a ring,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

where the rows are exact and the squares commute. Show that if α, β, δ and ε are isomorphisms, then γ is also an isomorphism.

Problem B.10. Formulate the notions of (left-/right-) exactness for a contravariant additive functor between abelian categories.

Problem B.11. Prove that the examples in Example B.14 are indeed what we claim them to be.

Problem B.12.

(a) Consider a pair of additive functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ between abelian categories such that $F \dashv G$. Prove that F is right-exact and G is left-exact.

(b) Consider the category Ab, then we have seen the adjunction $- \otimes_{\mathbb{Z}} A \dashv \mathrm{Hom}_{\mathbb{Z}}(A, -)$. Therefore, this immediately tells us that our assertions in Example B.14 are correct. Let $A = \mathbb{Q}$, prove that $- \otimes_{\mathbb{Z}} \mathbb{Q}$ is exact (that is, it is also left exact), while $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, -)$ is not (that is, it is not right exact).

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