# LECTURE NOTES

## CATEGORY THEORY LEARNING SEMINAR
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PROBLEMS TALLY: 3 + 14 + 8 + 7 + 8 = 40

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# Contents

**1 Lecture 1 (1/7) by Deewang** **2**
  1.1 Problems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4

**2 Lecture 2 (1/14) by Vaibhav** **5**
  2.1 Problems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12

**3 Lecture 3 (1/21) by Deewang** **16**
  3.1 Problems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

**4 Lecture 4 (1/28) by Vaibhav** **24**
  4.1 Problems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 28

**5 Lecture 5 (2/04) by David** **30**
  5.1 Problems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 32

# 1. Lecture 1 (1/7) by Deewang

**Definition 1.1.** A category $\mathscr{C}$ consists of

- collection of *objects*, denoted $\mathrm{obj}(\mathscr{C})$; and
- a collection of *morphisms* (also called *arrows* or *maps*)

such that

- Each morphism has specified *source* (or *domain*) and *target* (or *domain*) objects. For a morphism $f$ we will sometimes denote the source as $s(f)$ and target as $t(f)$. The notation

  $$f : X \to Y$$

  tells us that $f$ is a morphism between the source $s(f) = X$ and target $t(f) = Y$.

  The collection of morphisms from $X$ to $Y$ is denoted $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ or $\mathrm{Mor}_{\mathscr{C}}(X, Y)$ or $\mathscr{C}(X, Y)$. **We will assume that these are sets; that is, all our categories will be *locally small*.**

- For each object $X$, there exists an *identity morphism* $1_X : X \to X$.
- For any pair of morphisms $f$, $g$ with $t(f) = s(g)$, there exists a morphism, the *composite morphism* $gf$ with $s(gf) = s(f)$ and $t(gf) = t(g)$.

  That is, there's a binary operation

  $$\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z), \ \ (f, g) \mapsto gf$$

This data is subject to the following axioms:

- For any morphism $f : X \to Y$, we have

  $$1_Y f = f \quad \text{and} \quad f 1_X = f$$

- For composable morphisms $f$, $g$ and $h$, we have

  $$(fg)h = f(gh)$$

**Example 1.2.** Concrete examples.

| **Name** | **Objects** | **Morphisms** |
|---|---|---|
| Set | Sets | Functions |
| Grp | Groups | Group homomorphisms |
| Top | Topological Spaces | Continuous Functions |
| $\mathsf{Top}_{\mathsf{Open}}$ | Topological Spaces | Open Functions |
| Rng | Rings | Ring Homomorphisms |
| Ring | Unital Rings | Unital Ring Homomorphisms |

| **Name** | **Objects** | **Morphisms** |
|---|---|---|
| $\mathsf{Mod}_A$ | $A$-modules | Module homomorphisms |
| $\mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$ | Abelian Groups | Group homomorphisms |
| $\mathsf{Vec}_k = \mathsf{Mod}_k$ ($k$ a field) | Vector spaces | Linear transformations |
| $G$-$\mathsf{Set}$ | $G$-sets | $G$-equivariant maps |
| $\mathsf{Set}_*$ | Pointed Sets<br>$(X, x_0)$ where $x_0 \in X$<br>is called the basepoint | Basepoint preserving functions;<br>that is, functions $f : X \to Y$<br>such that $f(x_0) = y_0$ |
| $\mathsf{Top}_*$ | Pointed Topological Spaces | Basepoint preserving continuous functions |
| $\mathsf{SmMan}$ | Smooth Manifolds | Smooth Maps |
| $\mathsf{Meas}$ | Measurable Spaces | Measurable Functions |

**Example 1.3.** Where morphisms are not maps but equivalence classes of maps.

| **Name** | **Objects** | **Morphisms** |
|---|---|---|
| $\mathsf{HTop}$ | Topological Spaces | Homotopy classes of continuous functions |
| $\mathsf{Measure}$ | Measurable Spaces | Equivalence classes of measurable functions where the set where they differ has measure zero |

**Example 1.4.** Where morphisms are not maps, or objects are not sets.

| **Name** | **Objects** | **Morphisms** |
|---|---|---|
| $\mathsf{Mat}_A$<br>($A$ a ring) | Positive Integers | $\mathrm{Hom}(n, m) := \mathrm{Mat}_{m \times n}(A)$.<br>Composition is given by matrix multiplication, and the identity morphism is the identity matrix. |
| $(\mathsf{P}, \leqslant)$<br>a poset | Elements of $\mathsf{P}$ | $\mathrm{Hom}_{\mathsf{P}}(x, y) = \begin{cases} \{x \to y\} & \text{if } x \leqslant y \\ \varnothing & \text{otherwise} \end{cases}$<br>Composition is given by transitivity of the relation, and the identity morphism is given by reflexivity. |
| $\mathsf{B}G$<br>$G$ a group | $\bullet$<br>unique (dummy) object | $\mathrm{Hom}_{\mathsf{B}G}(\bullet, \bullet) := G$<br>Composition is given by the group multiplication, and the identity morphism is given by the group identity element. |

## 1.1. Problems

**Problem 1.1.** Come up with an example different from the ones given above.

**Problem 1.2.** Verify that the example in Example 1.4 are indeed categories.

**Problem 1.3** (Slice Categories)**.** Let $\mathscr{C}$ be a category and fix an object $X$, we define the *slice category of $\mathscr{C}$ under $X$* denoted $X/\mathscr{C}$ as follows

- Objects of $X/\mathscr{C}$ are morphisms $a_Y : X \to Y$ with source $X$, we usually depict them as

$$\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd}$$

- Morphism between $a_Y : X \to Y$ and $a_Z : X \to Z$ is defined to be a morphism $f : Y \to Z$ (in $\mathscr{C}$) such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "a_Y"'] \arrow[dr, "a_Z"] & \\ Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

  commutes; that is, $a_Z = f a_Y$.

Verify $X/\mathscr{C}$ is indeed a category. Can you describe what would be the *slice category of $\mathscr{C}$ over $X$*, denoted as $\mathscr{C}/X$.

**Remark 1.5.** $\text{Set}_*$ can be realised as $*/\text{Set}$, where $*$ denotes a singleton.

## 2. Lecture 2 (1/14) by Vaibhav

**A short historical note.** The need for the language of category theory was first realised by Samuel Eilenberg and Saunders MacLane when they discovered a curious connection between purely algebraic and topological objects.

| MacLane | Eilenberg |
|---|---|
| was studying *group extensions*. | was studying *solenoids*. |
| Given groups $G$ and $H$, a group $E$ is a *group extension of G by H* if $H \cong E/G$ | A solenoid, loosely, is a collection $(S_i, f_i)_{i \in \mathbb{Z}_{\geqslant 0}}$ where $S_i$ are circles and $f_i : S_{i+1} \to S_i$ is the map that wraps $S_{i+1}$ around $S_i$, $n_i$ times ($n_i \in \mathbb{Z}_{\geqslant 2}$). |
| There's a group $\mathrm{Ext}(G, H)$ that classifies extensions, up to isomorphism. | Given a solenoid $\Sigma \subset S^3$, one studies continuous functions such that $f(S^3 - \Sigma) \subset S^3$. These are classified, up to homotopy, by the homology group $H^1(S^3 - \Sigma, \mathbb{Z})$. |

Eilenberg and MacLane discovered that the group $H^1(S^3 - \Sigma, \mathbb{Z})$, that arises topologically, is isomorphic to the group $\mathrm{Ext}(\mathbb{Z}, \Sigma^*)$, which is a purely algebraic object (here $\Sigma^*$ is an appropriately chosen group called the *character group of the solenoid* $\Sigma$). This discovery was detailed in their 1942 paper *Group Extensions and Homology*.

In language we haven't seen yet, but will soon, Ext and $H^1$ are *functors* (Definition 2.13) and the above isomorphism is not just an isomorphism of groups but a *natural isomorphism of functors* (Definition 3.3), as was noted by Eilenberg and MacLane. To make sense of this rigorously they had to create some new language, which they did in their 1945 paper *General Theory of Natural Equivalences*. With that, category theory entered the mathematical landscape.

**Definition 2.1.** Given a category $\mathscr{C}$, we define the notion of a subcategory $\mathscr{D}$.

- The objects of $\mathscr{D}$ is a sub-collection of the objects of $\mathscr{C}$.
- The morphisms in $\mathscr{D}$ are a sub-collection of the morphisms in $\mathscr{C}$, and includes the identity morphisms for each object of $\mathscr{D}$.
- The morphisms are closed under composition, that is, if $f$ and $g$ are two composable morphisms in $\mathscr{D}$, then $gf$ is also a morphism in $\mathscr{D}$.

**Example 2.2.** Subcategories of some familiar categories

| **Category $\mathscr{C}$** | **Subcategory $\mathscr{D}$** | **Objects of $\mathscr{D}$** | **Morphisms of $\mathscr{D}$** |
|---|---|---|---|
| Set | FinSets | Finite Sets | Functions |

| | | | |
|---|---|---|---|
| $\mathsf{Set}$ | $\mathsf{InjSet}$ | Sets | Injective Functions |
| $\mathsf{Grp}$ | $\mathsf{Ab}$ | Abelian Groups | (Group) Homomorphisms |
| $\mathsf{Grp}$ | $\mathsf{FinGrp}$ | Finite Groups | Homomorphisms |
| $\mathsf{Grp}$ | $\mathsf{SurjGrp}$ | Groups | Surjective Homomorphisms |
| $\mathsf{Mod}_A$ | $\mathsf{mod}_A$ | Finitely Generated Modules | Homomorphisms |
| $\mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$ | $\mathsf{FinAb} = \mathsf{mod}_{\mathbb{Z}}$ | Finitely Generated Abelian Groups | Homomorphisms |
| $\mathsf{Vec}_k = \mathsf{Mod}_k$<br>$k$ a field | $\mathsf{fdVec}_k = \mathsf{mod}_k$ | Finite Dimensional Vector Spaces | Linear Transformations |
| $\mathsf{Top}$ | $\mathsf{Top}_{\mathsf{Open}}$ | Topological Spaces | Open Functions |
| $\mathsf{Top}$ | $\mathsf{CW}$ | CW Complexes | Cellular Maps |
| $\mathsf{Rng}$ | $\mathsf{CRing}$ | Commutative Unital Rings | Unital Ring Homomorphisms |
| $\mathsf{SmMan}$ | $\mathsf{Subm}$ | Smooth Manifolds | Submersions |
| $\mathsf{B}G$ | $\mathsf{B}H$ | same object $\bullet$ as $\mathsf{B}G$ | $H \leqslant G$,<br>a subgroup |

**Definition 2.3** (Opposite Category)**.** Let $\mathscr{C}$ be any category, then we define $\mathscr{C}^{\text{op}}$, the *opposite category of $\mathscr{C}$*.

- $\text{obj}(\mathscr{C}^{\text{op}}) = \text{obj}(\mathscr{C})$
- For any two objects $X$ and $Y$,

$$\text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y) := \text{Hom}_{\mathscr{C}}(Y, X)$$

Therefore a morphism $f^{\text{op}} \in \text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y)$ written formally as

$$f^{\text{op}} : X \to Y$$

*is a morphism* $f : Y \to X$ in $\mathscr{C}$.

Equivalently, a morphism $f : Y \to X$ in $\mathscr{C}$ *corresponds to a morphism* $f^{op} : X \to Y$ in $\mathscr{C}^{\text{op}}$.

We relegate proving this indeed gives a category to Problem 2.1.

**Remark 2.4.** Any time we prove or define something, we're really proving or defining two things simultaneously: in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$. This is the principle of duality, and we will encounter this again and again. An example of this phenomenon arises in the next definition (see Problem 2.3).

**Definition 2.5** (Some special types of morphism)**.** A morphism $f : X \to Y$ in a category $\mathscr{C}$ is called

- an *isomorphism* if there exists a morphism $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, bend right, "\mathrm{id}_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

  that is, the above diagrams commute. An isomorphisms is also called an *invertible morphism*, with $g$ above being denoted as $f^{-1}$ and called as its *inverse*.

  If $f : X \to Y$ is an isomorphism, we write $X \cong Y$.

  **Example 2.6.**

| **Categories $\mathscr{C}$** | **Isomorphisms in $\mathscr{C}$** |
|---|---|
| Set | Bijections |
| Grp & Ab | Group Isomorphisms |
| $\mathsf{Mod}_A$ | Module Isomorphisms |
| $\mathsf{Vec}_k$ | Linear Isomorphisms |
| Rng | Ring Isomorphisms |

| **Categories $\mathscr{C}$** | **Isomorphisms in $\mathscr{C}$** |
|---|---|
| Top | Homeomorphisms |
| HTop | Homotopy Equivalences |
| SmMan | Diffeomorphisms |
| $\mathsf{Mat}_A$ | Invertible Matrices |
| $(\mathsf{P}, \leqslant)$ | Equality |

- a *monomorphism* if there exist morphisms $g_1, g_2 : W \rightrightarrows X$

$$\begin{tikzcd} W \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

  such that if $fg_1 = fg_2$, then $g_1 = g_2$.

- an *epimorphism* if there exist morphisms $h_1, h_2 : Y \rightrightarrows Z$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z \end{tikzcd}$$

  such that if $h_1 f = h_2 f$, then $h_1 = h_2$.

**Remark 2.7.** Monomorphisms, epimorphisms and isomorphisms are categorical generalisations of an injective, a surjective and a bijective map (see Problems 2.6 and 2.7). But it's important to note that in general, morphisms that are both a monomorphism and an epimorphism are *not* isomorphisms. See Problem 2.6 (c) for an example, and Problem 2.7 for a more satisfactory conclusion.

**Definition 2.8.** A category $\mathscr{C}$ is called a *groupoid* if every morphism is an isomorphism.

**Example 2.9.** Last time we saw how a group $G$ gives rise to a category with a single object that we called $\mathrm{B}G$. Since every group element has an inverse, this category has the property that all its morphisms are invertible. The notion of a groupoid captures this notion more generally.

In fact, in this manner *a group is a groupoid with one object*.

To a group $G$, we can associate the groupoid $\mathrm{B}G$. Conversely, given a groupoid $\mathscr{G}$ with one object $\bullet$, we recover the group as $\mathrm{Hom}_{\mathscr{G}}(\bullet, \bullet)$.

**Example 2.10** (Maximal Groupoid)**.** Any category $\mathscr{C}$ has a subcategory called the *maximal groupoid*, $\mathscr{C}^{\cong}$, where any subcategory of $\mathscr{C}$ that's a groupoid is then a subcategory of $\mathscr{C}^{\cong}$.

- $\mathrm{obj}(\mathscr{C}^{\cong}) = \mathrm{obj}(\mathscr{C})$;
- $\mathrm{Hom}_{\mathscr{C}^{\cong}}(X, Y) = \{f \in \mathrm{Hom}_{\mathscr{C}}(X, Y) \ : \ f \text{ is an isomorphism}\}$.

This is a subcategory since the composition of two isomorphisms is again an isomorphism.

**Remark 2.11.** Given an object $X$ in a category $\mathscr{C}$, the *automorphism group of $X$* is defined to be

$$\mathrm{Aut}_{\mathscr{C}}(X) := \mathrm{Hom}_{\mathscr{C}^{\cong}}(X, X)$$

It's indeed a group with respect to composition.

**Example 2.12** (Fundamental Groupoid)**.** Given a topological space $X$, we have an associated groupoid $\Pi(X)$ called the *fundamental groupoid*.

- The objects of $\Pi(X)$ are points of $X$;
- For points $x$ and $y$, the morphism from $x$ to $y$ are homotopy classes of paths from $x$ to $y$, that is

$$\mathrm{Hom}_{\Pi(X)}(x, y) := \{\gamma : [0,1] \to X \ : \ \gamma \text{ is continuous, and } \gamma(0) = x, \ \gamma(1) = y\}/\text{homotopy}$$

This is a groupoid, since given a path $\gamma : x \to y$ we can always create an inverse path $\gamma^{-1} : y \to x$ given as $\gamma^{-1}(t) = \gamma(1-t)$.

For a point $x_0 \in X$, *the fundamental group of $X$ at basepoint $x_0$* is[1]

$$\pi_1(X, x_0) := \mathrm{Aut}_{\Pi(X)}(x_0)$$

Our guiding principle is that morpshisms take precedence, and whenever we have objects we should give a notion of morphisms. So, if were to give a category of all categories CAT, what is the right notion of a morphisms between categories?

**Definition 2.13** (Functors)**.** Given categories $\mathscr{C}$ and $\mathscr{D}$, a *functor*

$$F : \mathscr{C} \to \mathscr{D}$$

consists of,

- for every object $X$ of $\mathscr{C}$, an object $F(X)$ of $\mathscr{D}$.
- for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$

$$\begin{tikzcd} X \arrow[d, "f"] & \mapsto & F(X) \arrow[d, "F(f)"] \\ Y & & F(Y) \end{tikzcd}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object $X$ of $\mathscr{C}$
- Given morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, we have $F(gf) = F(g)F(f)$;

$$\begin{tikzcd} & F(Y) \arrow[dr, "F(g)"] & \\ F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(gf)"'] & & F(Z) \end{tikzcd}$$

  that is, the above diagram commutes.

Sometimes the functors defined previously are called *covariant functors* to distinguish them from the functors we now define below

**Definition 2.14** (Contravariant Functors)**.** Given categories $\mathscr{C}$ and $\mathscr{D}$, a *contravriant functor*

$$F : \mathscr{C} \to \mathscr{D}$$

consists of,

- for every object $X$ of $\mathscr{C}$, an object $F(X)$ of $\mathscr{D}$.
- for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$

$$\begin{tikzcd} X \arrow[d, "f"] & \mapsto & F(X) \\ Y & & F(Y) \arrow[u, "F(f)"] \end{tikzcd}$$

such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object $X$ of $\mathscr{C}$
- Given morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, we have $F(gf) = F(f)F(g)$;

$$\begin{tikzcd}
 & F(Y) \arrow[dl, "F(f)"'] & \\
F(X) & & F(Z) \arrow[ul, "F(g)"'] \arrow[ll, "F(gf)"]
\end{tikzcd}$$

that is, the above diagram commutes.

**Remark 2.15.** The notion of a contravariant functor is not a new concept. Problem 2.11 tells us that it's simply a (covariant) functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

**Example 2.16.**

| **Functors** | **What they do to Objects** | **What they do to Morphisms** |
|---|---|---|
| $\mathscr{P}_* : \mathsf{Set} \to \mathsf{Set}$<br>*direct image fuctor* | $A \mapsto \mathscr{P}(A)$<br>Power set of $A$ | $A \xrightarrow{f} B$<br>$\downarrow$<br>$\mathscr{P}(A) \to \mathscr{P}(B) : S \mapsto f(S)$ |
| $\mathscr{P}^* : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$<br>*inverse image fuctor* | $A \mapsto \mathscr{P}(A)$<br>Power set of $A$ | $A \xrightarrow{f} B$<br>$\downarrow$<br>$\mathscr{P}(B) \to \mathscr{P}(A) : T \mapsto f^{-1}(T)$ |
| $U : \mathscr{C} \to \mathsf{Set}$<br>$\mathscr{C} = \mathsf{Mod}_A, \mathsf{Grp},$<br>$\mathsf{Rng}, \mathsf{Top}, \ldots$<br>*forgetful fuctor* | $A \mapsto U(A)$<br>the underlying set of $A$,<br>"forget its additional structure" | $A \xrightarrow{f} B$<br>$\downarrow$<br>$U(f) : U(A) \to U(B)$<br>the underlying function $f$,<br>"forget its additional structure" |
| $h^X : \mathscr{C} \to \mathsf{Set}$<br>$X$ is any object in $\mathscr{C}$<br>*covariant* Hom | $T \mapsto \mathrm{Hom}_{\mathscr{C}}(X, T)$ | $T \xrightarrow{f} S$<br>$\downarrow$<br>$\mathrm{Hom}_{\mathscr{C}}(X, T) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathscr{C}}(X, S)$<br>$\phi \longmapsto f\phi$ |

| | | |
|---|---|---|
| $h_X : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$<br>$X$ is any object in $\mathscr{C}$<br>*contravariant* Hom | $T \mapsto \mathrm{Hom}_{\mathscr{C}}(T, X)$ | $T \xrightarrow{f} S$<br>$\downarrow$<br>$\mathrm{Hom}_{\mathscr{C}}(S, X) \xrightarrow{- \circ f} \mathrm{Hom}_{\mathscr{C}}(T, X)$<br>$\psi \longmapsto \psi f$ |
| $\pi_1 : \mathsf{HTop}_* \to \mathsf{Grp}$<br>*fundamental group*<br>(example of covariant Hom) | $(X, x_0) \mapsto \pi_1(X, x_0) := [S^1, X]_*$<br>basepoint preserving, homotopy classes of continuous functions | $(X, x_0) \xrightarrow{f} (Y, y_0)$<br>$\downarrow$<br>$\pi_1(X, x_0) \to \pi_1(Y, y_0) : [\gamma] \mapsto [f \circ \gamma]$ |
| $(-)^{\mathrm{ab}} : \mathsf{Grp} \to \mathsf{Ab}$<br>*abelianisation* | $G \mapsto G^{\mathrm{ab}} := G/[G, G]$<br>$[G, G]$ is the commutator subgroup of $G$ | $G \xrightarrow{f} H$<br>$\downarrow$<br>$f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$<br>induced by the universal property of quotients |
| $T_*(-) : \mathsf{SmMan}_* \to \mathsf{Vec}_{\mathbb{R}}$<br>*tangent space* | $(M, p) \mapsto T_p M$<br>the tangent space at $p$ | $(M, p) \xrightarrow{F} (N, q)$<br>$\downarrow$<br>$dF_p : T_p M \to T_q N$<br>the differential at $p$ |
| $T(-) : \mathsf{SmMan} \to \mathsf{SmMan}$<br>*tangent bundle* | $M \mapsto TM$<br>the tangent bundle | $M \xrightarrow{F} N$<br>$\downarrow$<br>$dF : TM \to TN$<br>the total differential |
| $C^0(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cring}$<br>*continuous functions pullback*<br>(example of contravariant Hom: $\mathrm{Hom}_{\mathsf{Top}}(-, \mathbb{R})$) | $X \mapsto C^0(X, \mathbb{R})$<br>$\mathbb{R}$-valued continuous functions,<br>"forget its additional structure" | $X \xrightarrow{f} Y$<br>$\downarrow$<br>$C^0(Y, \mathbb{R}) \to C^0(X, \mathbb{R}) : \phi \mapsto \phi \circ f$ |
| $(-)^* : \mathsf{Vec}_k^{\mathrm{op}} \to \mathsf{Vec}_k$<br>*dual*<br>(example of contravariant Hom: $\mathrm{Hom}_{\mathsf{Vec}_k}(-, k)$) | $V \mapsto V^* := \mathrm{Hom}_{\mathsf{Vec}_k}(V, k)$ | $V \xrightarrow{f} W$<br>$\downarrow$<br>$W^* \to V^* : \phi \mapsto \phi \circ f$ |

| | | |
|---|---|---|
| $(-)^{-1} : \mathsf{B}G^{\mathrm{op}} \to \mathsf{B}G$ | $\bullet \mapsto \bullet$ | $\bullet \xrightarrow{g} \bullet$<br>$\big\downarrow$<br>$\bullet \xrightarrow{g^{-1}} \bullet$ |
| $(-)^{\intercal} : \mathsf{Mat}_A^{\mathrm{op}} \to \mathsf{Mat}_A$<br>*transpose* | $n \mapsto n$ | $n \xrightarrow{A} m$<br>$\big\downarrow$<br>$m \xrightarrow{A^{\intercal}} n$ |
| $(-)^{\times} : \mathsf{Ring} \to \mathsf{Grp}$<br>*unit functor* | $A \mapsto A^{\times}$<br>group of units | $A \xrightarrow{f} B$<br>$\big\downarrow$<br>$f\|_{A^{\times}} : A^{\times} \to B^{\times}$ |
| $\mathbb{Z}[-] : \mathsf{Grp} \to \mathsf{Ring}$<br>*group ring functor* | $G \mapsto \mathbb{Z}[G]$<br>group ring | $G \xrightarrow{f} H$<br>$\big\downarrow$<br>$\mathbb{Z}[f] : \mathbb{Z}[G] \to \mathbb{Z}[H]$<br>linearly extend $f$ |
| $\mathsf{Spec}(-) : \mathsf{CRing} \to \mathsf{Set}$<br>*spectrum* | $A \mapsto \mathsf{Spec}(A)$<br>$\mathsf{Spec}(A) :=$<br>$\{\mathfrak{p} \subseteq A \ : \ \mathfrak{p}$ is a prime ideal$\}$ | $A \xrightarrow{f} B$<br>$\big\downarrow$<br>$\mathsf{Spec}(B) \to \mathsf{Spec}(A) : \mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ |
| $\mathrm{Aut}(-) : K/\mathsf{Field} \to \mathsf{Grp}$<br>*field automorphisms* | $E \mapsto \mathrm{Aut}(E/K)$<br>automorphisms of $E$<br>fixing $K$ point-wise | $E \hookrightarrow F$<br>$\big\downarrow$<br>$\mathrm{Aut}(F/K) \to \mathrm{Aut}(E/K) : \sigma \mapsto \sigma\|_E$ |
| $(-)^G : G\text{-}\mathsf{Set} \to \mathsf{Set}$<br>*fixed points functor* | $X \mapsto X^G$<br>$G$-fixed points of $X$ | $X \xrightarrow{f} Y$<br>$\big\downarrow$<br>$f\|_{X^G} : X^G \to Y^G$ |

## 2.1. Problems

**Problem 2.1.**

(a) Show that $\mathscr{C}^{\mathrm{op}}$ is indeed a category. What should the composition law be?

(b) Show that $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$.

**Problem 2.2.** Recall the notion of *slice categories* from Problem 1.3. Prove that

$$(\mathscr{C}^{\mathrm{op}}/X)^{\mathrm{op}} = X/\mathscr{C}$$

A similar argument also gives you $(X/\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}/X$.

**Problem 2.3** (Example of Duality)**.** Prove that giving an epimorphism in $\mathscr{C}$ is the same as giving a monomorphism in $\mathscr{C}^{\mathrm{op}}$. Using Problem 2.1 (b), deduce the vice versa.

This is an example of *duality*.

**Problem 2.4.** A morphism $f : X \to Y$ in a category $\mathscr{C}$ is called a *split epimorphism* if there exists a morphism $\sigma : Y \to X$ (called a *section*) such that $f\sigma = \mathrm{id}_Y$

$$\begin{tikzcd}
Y \arrow[r, "\sigma"] \arrow[dr, bend right, "\mathrm{id}_Y"'] & X \arrow[d, "f"] \\
 & Y
\end{tikzcd}$$

that is, the above diagram commutes. Prove that $f$ is an epimorphism (and $\sigma$ a monomorphism).

**Problem 2.5.** A split epimorphism in $\mathscr{C}^{\mathrm{op}}$ is called a *split monomorphism* in $\mathscr{C}$, where the section of the split epimorphism in $\mathscr{C}^{\mathrm{op}}$ is called a *retract* in $\mathscr{C}$.

Carefully write down this definition as it would be stated in $\mathscr{C}$.

Duality tells you that a split monomorphism is a monomorphism, and the retract is an epimorphism.

**Problem 2.6.** A monomorphism and epimorphism are categorical generalisation of injective and surjective maps respectively. The following problems should help you understand why.

(a) Prove that in the categories Set and Ab, a morphism is a monomorphism if and only if it's an injective map, and a morphism is a monomorphism if and only if it's a surjective map. [2]

(b) (Challenge) Prove similarly for Grp.

(c) Prove that every surjective map is an epimorphism in Ring but the following morphism is an epimorphism and not a surjective map

$$\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$$

Deduce then that morphisms that are both a monomorphism and an epimorphism are *not* necessarily isomorphisms.

**Problem 2.7.** It's actually more reasonable[3] to note that *monomorphisms model injective maps* and *split epimorphisms model surjective maps* and epimorphisms and split monomorphisms tag along as just dual notions. The following problem should help you make sense of this.

Show that if a morphism $f$ in a category $\mathscr{C}$ is a monomorphism and a split epimorphism, then $f$ is an isomorphism. State what the dual statement will be.

**Problem 2.8.**

(a) Prove that an isomorphism in $\mathscr{C}$ is an isomorphism in $\mathscr{C}^{\mathrm{op}}$. That is, if a morphism $f$ in $\mathscr{C}$ is an isomorphism, then the associated morphism $f^{\mathrm{op}}$ is an isomorphism in $\mathscr{C}^{\mathrm{op}}$.

(b) Prove that a functor sends isomorphisms to isomorphisms. That is, given a functor $F : \mathscr{C} \to \mathscr{D}$ and an isomorphism $f$ in $\mathscr{C}$. Prove $F(f)$ is an isomorphism.

**Problem 2.9.** Prove that the following association gives you a functor

$$\mathrm{Aut}_{\mathscr{C}}(-) : \mathscr{C}^{\cong} \to \mathsf{Grp},\ X \mapsto \mathrm{Aut}_{\mathscr{C}}(X);$$

what does it do to morphisms?

Using Problem 2.8, this tells us that if $X \cong Y$ then $\mathrm{Aut}_{\mathscr{C}}(X) \cong \mathrm{Aut}_{\mathscr{C}}(Y)$. Applying this result to the fundamental groupoid of a topological space (Example 2.12) tells us that if the topological space is path connected, the fundamental groups at two different basepoints are isomorphic.

**Problem 2.10.**

(a) What's a functor $F : \mathsf{B}G \to \mathsf{B}H$, where $G$ and $H$ are groups?

(b) Consider posets $(\mathsf{P}, \leqslant_{\mathsf{P}})$ and $(\mathsf{Q}, \leqslant_{\mathsf{Q}})$ as categories.

  (b1) What's a functor $F : \mathsf{P} \to \mathsf{Q}$?

  (b2) What's a functor $F : \mathsf{P}^{\mathrm{op}} \to \mathsf{Q}$?

**Problem 2.11.** Prove that the notion of a contravariant functor $F : \mathscr{C} \to \mathscr{D}$ is equivalent to the notion of a (covariant) functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

**Problem 2.12.** Prove that the examples in Example 2.16 are in fact functors.

**Problem 2.13.** Prove that the association

$$Z(-) : \mathsf{Grp} \to \mathsf{Ab},\ Z \mapsto Z(G),$$

sending a group to its center, is not functorial.

Prove that if we restrict this association to the subcategory $\mathsf{SurjGrp}$ then it is actually a functor.

**Problem 2.14.** For a commutative unital ring $A$, we define

$$\mathrm{MaxSpec}(A) = \{\mathfrak{m} \subseteq A \ : \ \mathfrak{m} \text{ is a maximal ideal}\} \subseteq \mathrm{Spec}(A)$$

Prove that the association

$$\mathrm{MaxSpec}(-) : \mathsf{CRing} \to \mathsf{Set},\ A \mapsto \mathrm{MaxSpec}(A)$$

is not functorial in the same way $\mathrm{Spec}(-)$ is.

# Notes

1. Once can also define this as $\pi_1(X, x_0) = \mathrm{Hom}_{\mathsf{Top}_*}((S^1, *), (X, x_0))/\text{homotopy}$. Showing that these two definitions are the same is a good exercise in elementary topology.

2. One way to prove the "mono" part for Set is to reduce the definition of monomorphism to a singleton $*$. It is thus called a *generator* (or *separator*) of Set. A similar argument works for Ab; namely, it has a separator (for example, $\mathbb{Z}$). One way to prove the "epi" part for Set is to reduce the definition of epimorphism to the set of truth values $\Omega = \{\mathrm{T}, \mathrm{F}\}$. It is thus called a *cogenerator* (or *classifer*) of Set. However, a similar argument fails for Ab since it has no classifier.

3. Here *reasonable* means it behaves as one intended expect. In contrast, for example, in the category of rings, an epimorphism may not be a surjective homomorphism, see (c) of Problem 2.6; in the category of fields, an injective homomorphism is a split monomorphism if and only if it is an isomorphism, which is too restrictive.

# 3. Lecture 3 (1/21) by Deewang

**Example 3.1.** For a group $G$, consider the category $\mathsf{B}G$ and any other category $\mathscr{C}$. A functor

$$X : \mathsf{B}G \to \mathscr{C}$$

specifies an object $X$ in $\mathscr{C}$ (the image of the unique object in $\mathsf{B}G$) and an automorphism $g_X : X \to X$ for each $g \in G$ (the image of the isomorphisms $g \in G$ in $\mathsf{B}G$). That is, the functor affords a morphism

$$G \to \mathrm{Aut}_{\mathscr{C}}(X).$$

This is subject to the following two properties

- $(gh)_X = g_X h_X$
- $e_X = 1_X$

That is, the functor $X : \mathsf{B}G \to \mathscr{C}$ defines an *action* of the group $G$ on the object $X$ in $\mathscr{C}$.

- When $\mathscr{C} = \mathsf{Set}$, $X$ is called a $G$-set.
- When $\mathscr{C} = \mathsf{Vec}_k$, $X$ is called a $G$-representation.
- When $\mathscr{C} = \mathsf{Top}$, $X$ is called a $G$-space.

This notion is also given the name *left action*, in which case a *right action* is a functor $X : \mathsf{B}G^{\mathrm{op}} \to \mathscr{C}$.

How would one express the notion of a $G$-equivariant map in this language? It will have to be some notion of *morphism of functors* as we've realised objects with a $G$-action as functors.

**Example 3.2.** For a field $k$, recall that any finite dimensional $k$-vector space $V$ is isomorphic to its dual $V^* = \mathrm{Hom}_{k\text{-line}}(V, k)$. This is proven by constructing an explicit *dual basis*: choose a basis $\{e_1, \ldots, e_n\}$ of $V$, then a basis of $V^*$ is given by $\{e_1^*, \ldots, e_n^*\}$ where

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases};$$

this isomorphism is then given by $e_i \mapsto e_i^*$.

A related construction is the double dual $V^{**} := (V^*)^*$. Since the association of a dual describes a functor $(-)^* : \mathsf{fdVec}_k^{\mathrm{op}} \to \mathsf{fdVec}_k$, for a finite dimensional vector space $V$, we get $V^** = (V^*)^* \cong V^* \cong V$ where the isomorphism sends $e_i$ to the dual dual basis $e_i^{**}$.

Turns out, there's a cleaner way to give an isomorphism between $V$ and $V^{**}$ without making a choice of basis. For any $v \in V$, the *evaluation map*

$$\mathrm{ev}_v : V^* \to k,\ \phi \mapsto \phi(v)$$

is a linear functional on $V^*$, that is, an element of $V^{**}$. The assignment $v \mapsto \mathrm{ev}_v$ defines a *basis-free natural* isomorphism $V \cong V^{**}$.

What distinguishes the isomorphism between a finite-dimensional vector space and its double dual from the isomorphism between a finite-dimensional vector space and its single dual is that the former assembles into the components of a *natural transformation*, a notion that we describe below.

**Definition 3.3.** Given categories $\mathscr{C}$ and $\mathscr{D}$ and functors $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of

- a morphism $\alpha_X : F(X) \to G(X)$ in $\mathscr{D}$ for each object $X$ in $\mathscr{C}$, the collection of which we call the *components* of the natural transformation,

such that

- for any morphism $f : X \to Y$ in $\mathscr{C}$, the following square of morphisms in $\mathscr{D}$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

  commutes.

We usually say that the morphisms $\alpha_X : F(X) \to G(X)$ are natural in $X$ to implicitly imply the existence of this commutative square.

A *natural isomorphism* is a natural transformation $\alpha : F \Rightarrow G$ in which every component $\alpha_X$ is an isomorphism in $\mathscr{D}$.

**Example 3.4.** Some natural transformations between already introduced functors (Example 2.16).

| **Source** | **Target** | **Natural Transformation** | **Components** |
|---|---|---|---|
| $\mathsf{Vec}_k$ | $\mathsf{Vec}_k$ | $1_{\mathsf{Vec}} \Rightarrow (-)^{**}$<br>(isomorphism) | $V \to V^{**} : v \mapsto \mathrm{ev}_v$ |
| $\mathsf{Set}$ | $\mathsf{Set}$ | $1_{\mathsf{Set}} \Rightarrow \mathscr{P}_*$ | $A \to \mathscr{P}(A) : a \mapsto \{a\}$ |
| $\mathscr{C}$ | $\mathsf{Set}$ | $h^Y \Rightarrow h^X$<br>given a morphism $f : X \to Y$ | $\mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T) : \phi \mapsto \phi f$ |
| $\mathscr{C}^{\mathrm{op}}$ | $\mathsf{Set}$ | $h_X \Rightarrow h_Y$<br>given a morphism $f : X \to Y$ | $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y) : \psi \mapsto f\psi$ |
| $\mathsf{B}G$ | $\mathscr{C}$ | $X \Rightarrow Y$<br>for objects $X$, $Y$ in $\mathscr{C}$ | single component $f : X \to Y$<br>such that $g_Y f = f g_X$ for all $g \in G$<br>(a $G$-equivariant map) |

| | | | |
|---|---|---|---|
| Top | Set | $h^* \Rightarrow U$<br>(isomorphism) | $\mathrm{Hom}_{\mathsf{Top}}(*, X) \to U(X) : f \mapsto f(*)$ |
| CRing | Set | $h^{\mathbb{Z}[t^{\pm}]} \Rightarrow (-)^{\times}$<br>(isomorphism) | $\mathrm{Hom}_{\mathsf{CRing}}(\mathbb{Z}[t^{\pm}], A) \to A^{\times} : f \mapsto f(t)$ |
| $G$-Set | Set | $h^* \Rightarrow (-)^G$<br>(isomorphism) | $\mathrm{Hom}_{G\text{-}\mathsf{Set}}(*, X) \to X^G : f \mapsto f(*)$ |
| Grp | Grp | $1_{\mathsf{Grp}} \Rightarrow (-)^{\mathrm{ab}}$ | $G \twoheadrightarrow G/[G,G]$<br>canonical projection |

**Remark 3.5.** In contrast with the first example, the identity functor and the single dual functor on finite-dimensional vector spaces are not naturally isomorphic. Looking beyond the one technical obstruction, that the identity functor is covariant while the dual functor is contravariant, which is beside the point, the more significant is the essential failure of naturality.

Given a linear map $T : V \to W$ between finite dimensional vector spaces, we obtain the diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_{\mathbf{e}_V}"'] & W \arrow[d, "\phi_{\mathbf{e}_W}"] \\ V^* & W^* \arrow[l, "T^*"] \end{tikzcd}$$

where $\phi_{\mathbf{e}_V}$ and $\phi_{\mathbf{e}_W}$ are isomorphism described in Example 3.2 respect to the choice of basis $\mathbf{e}_V$ and $\mathbf{e}_W$ of $V$ and $W$ respectively. The only "naturality" condition that can be read from this diagram is

$$\phi_{\mathbf{e}_V} = T^* \circ \phi_{\mathbf{e}_W} \circ T,$$

but taking $T = 0$ gives us that $\phi_{\mathbf{e}_V} = 0$ contradicting the fact that $\phi_{\mathbf{e}_V}$ was an isomorphism. A line of enquiry would be to consider what happens if assume $T$ to be an isomorphism, you will then note that one still cannot escape the failure of this notion of naturality.

Consider the subcategory $\mathsf{Euc}$ of Euclidean vector spaces of $\mathsf{fdVec}_{\mathbb{R}}$, that is, the subcategory of inner product spaces. Then it's important to note that there is a natural isomoprhism $1_{\mathsf{Euc}} \Rightarrow (-)^*$ given by (components are) $V \to V^* : v \mapsto \langle v, - \rangle$.

**Example 3.6.** We describe two functors $\mathcal{O}, \mathcal{C} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$

$\mathcal{O} : X \mapsto \mathcal{O}(X)$, set of open sets

$$\begin{tikzcd} X \xrightarrow{f} Y \arrow[d, mapsto, "\mathcal{O}"] \\ \mathcal{O}(Y) \xrightarrow{f^{-1}} \mathcal{O}(X) \end{tikzcd}$$

$\mathcal{C} : X \mapsto \mathcal{C}(X)$, set of closed sets

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\mathcal{C}"] \\ \mathcal{C}(Y) \arrow[r, "f^{-1}"] & \mathcal{C}(X) \end{tikzcd}$$

Then there's a natural isomorphism $\mathcal{O} \Rightarrow \mathcal{C}$ with components as $\mathcal{O}(X) \to \mathcal{C}(X) : U \mapsto X \setminus U$.

**Example 3.7.** Recall that a monoid is a set equipped with a binary product for which there exists a netural element (that is, a set that satisfies all the group axioms but the one about existence of inverses). A morphism of monoids is a function that commutes with the binary products, similar to a group homomorphism. We can then consider the category Mon of monoids, where the objects are monoids and morphisms are (monoid) morphisms.

Given any ring with unity $A$, $A$ is a monoid with respect to multiplication. So is the set of $n \times n$ matrices $M_n(A)$ with respect to multiplication. These assemble to give functors

$$M_n(-), U : \mathsf{Ring} \rightrightarrows \mathsf{Mon}$$

Consider now the determinant map $\det_A : M_n(A) \to U(A)$, this is a monoid morphism since $\det_A(XY) = \det_A(X)\det_A(Y)$. The determinant assembles to give us (that is, $\det_A$ are the components) a natural transformation

$$\det : M_n(-) \Rightarrow U$$

**Example 3.8.** Consider the category $\mathsf{HTop}_*$ and denote

$$[X,Y]_* := \mathrm{Hom}_{\mathsf{HTop}_*}((X,x_0),(Y,y_0)) = \{f : X \to Y \ : \ f \text{ is continuous}, f(x_0) = y_0\}/\text{homotopy}$$

leaving the base points implicit.

We've already seen the functor $\pi_1 : \mathsf{HTop}_* \to \mathsf{Grp}$ where $(X, x_0) \mapsto [S^1, X]_*$, the fundamental group of $X$ at basepoint $x_0$. We can similarly define, for any $n \geqslant 1$ functors

$$\pi_n : \mathsf{Htop}_* \to \mathsf{Grp},\ (X, x_0) \mapsto \pi_n(X, x_0) := [S^n, X]_*$$

$\pi_n(X, x_0)$ are called the $n^{th}$ *homotopy groups of* $X$ *at basepoint* $x_0$. It's a fact that for $n > 1$, the functor $\pi_n$ takes values in Ab.

There's another functor

$$H_n : \mathsf{HTop}_* \to \mathsf{Ab},\ (X, x_0) \mapsto H_n(X, \mathbb{Z})$$

$H_n(X, \mathbb{Z})$ are called the $n^{th}$ *singular homology group of* $X$ *with coefficients in* $\mathbb{Z}$.

For any pointed space $X$ (and $n$), there's a group homomorphism

$$h_n(X) : \pi_n(X, x_0) \to H_n(X, \mathbb{Z})$$

called the Hurewicz homomorphism, which assemble to give a natural tranformation

$$h_n : \pi_n \Rightarrow H_n$$

We have talked about categories, functors and natural transformations but we have yet to discuss or introduce the right notion of "sameness" for categories. Naively, one would hope that the following would be that notion.

**Definition 3.9.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories, then we say $\mathscr{C}$ and $\mathscr{D}$ *isomorphic* to each other if there exist functors

$$F : \mathscr{C} \leftrightarrows \mathscr{D} : G$$

such that $GF = 1_{\mathscr{C}}$ and $FG = 1_{\mathscr{D}}$, where $1_{\mathscr{C}}$ and $1_{\mathscr{D}}$ are the obvious identity functors. We then write $\mathscr{C} \cong \mathscr{D}$.

Unfortunately, turns out this notion is too strong and rarely satisfied in practice and even rarely needed in practice. Following is a weaker notion than an isomorphism of categories but is the right notion of "sameness"

**Definition 3.10.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories, then we say $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* to each other if there exist functors

$$F : \mathscr{C} \leftrightarrows \mathscr{D} : G$$

such that there exist natural isomorphisms $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$. We then write $\mathscr{C} \simeq \mathscr{D}$.

**Example 3.11.** For a field $k$, we consider the categories $\mathsf{Mat}_k$, $\mathsf{fdVec}_k$, $\mathsf{fdVec}_k^{\mathsf{basis}}$. The only category we haven't seen before if $\mathsf{fdVec}_k^{\mathsf{basis}}$, the objects of $\mathsf{bVec}_k$ are finite dimensional vector spaces with a chosen basis are morphisms are arbitrary (not necessarily basis-preserving) linear maps. These three categories are relates by a few functors

$$\begin{tikzcd} \mathsf{Mat}_k \arrow[r, shift left, "k^{(-)}"] & \mathsf{fdVec}_k^{\mathsf{basis}} \arrow[l, shift left, "D"] \arrow[r, shift left, "U"] & \mathsf{fdVec}_k \arrow[l, shift left, "C"] \end{tikzcd}$$

where

- the functor $k^{(-)}$ sends a non-negative integer $n$ to the vector space $k^n$ equipped with the standard basis, and an $m \times n$ matrix to itself as it defines a linear map $k^n \to k^m$.
- The functor $U$ is the forgetful functor.
- The functor $C$ is defined by choosing a basis for each vector space.
- The functor $D$ sends a vector space to its dimension, and a linear map between two vector spaces to its matrix representation with respect to the chosen bases.

The functors define an equivalence of categories

$$\mathsf{Mat}_k \simeq \mathsf{fdVec}_k \simeq \mathsf{fdVec}_k^{\mathsf{basis}}$$

One can prove this directly or we can prove it using a very useful characterisation of an equivalence of categories that we now give.

**Definition 3.12.** A functor $F : \mathscr{C} \to \mathscr{D}$ is

- **full** if for objects $X$, $Y$, the map

  $$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$

  is surjective.

- **faithful** if for objects $X$, $Y$, the map

  $$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$

  is injective.

- **essentially surjective (on objects)** if for every object $W$ in $\mathscr{D}$, there is some object $X$ in $\mathscr{C}$ such that $F(X) \cong W$.

**Remark 3.13.** Fullness and faithfulness are *local conditions* on morphisms, not *global* as a global condition would apply "everywhere". A full functor need not be surjective on morphisms (one reason is because such a functor may not be essentially surjective), and a faithful functor need not be injective on morphisms.

- A faithful functor that is injective on objects is called an *embedding*, and identifies the domain category as a subcategory of the codomain. In this case, faithfulness implies that the functor is (globally) injective on arrows.
- A full and faithful functor, called *fully faithful* for short, that is injective on objects defines a *full embedding* of the domain category into the codomain category. The domain then defines a *full subcategory* of the codomain.

**Theorem 3.14.** *A functor defines an equivalence of categories if and only if it is full, faithful and essentially surjective on objects.*

## 3.1. Problems

**Problem 3.1.**

- Given categories $\mathscr{C}$ and $\mathscr{D}$, describe a category (that is, verify the axioms) $[\mathscr{C}, \mathscr{D}]$ where the objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and morphisms natural tranformations.
- Show that a natural isomorphism is precisely an isomorphism in the category $[\mathscr{C}, \mathscr{D}]$.

**Problem 3.2.** One feature of "higher structures", like categories, is that they has several "levels". In particular, a category has two levels: objects and morphisms.

In contrast, a set has only one level: elements. We categorify a set with objects being its elements and the only morphisms being identity maps.

A "correct" notion of maps between such structures will have to respect the levels. So a *functor* (or a *map*/$0$-*morphism*) needs to preserve the levels; a *natural transformation* (or a $1$-*morphism*) between them will have to respect a level shifted up by 1.

(a) What should be a *functor* from a set to a category? (It has to map elements to objects as they are both at lowest level.)

(b) What should be a *natural transformation* between functors from a set to a category? (It has to map elements to morphisms as the latter are in one higher level than the former)

(c) What should be a *functor* from a category to a set? (It has to map objects to elements as they are both at lowest level.)

(d) What should be a *natural transformation* between functors from a category to a set?

**Problem 3.3.** Prove that the notion of an equivalence of categories defines an equivalence relation.

**Problem 3.4.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

(a) Prove that if $F$ is faithful then it need not be injective on morphisms.

(b) Prove that if $F$ is faithful and injective on objects (that is, if $F(X) = F(Y)$ then $X = Y$), then it is injective on morphisms.

(c) Prove that if $F$ is fully faithful then it need not be injective on objects. But show that it is "injective up to isomorphism", that is if $F(X) \cong F(Y)$ then $X \cong Y$.

**Problem 3.5.** Prove Theorem 3.14. More precisely, let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

(a) Suppose $F$ defines an equivalence, prove that $F$ is full, faithful and essentially surjective on objects. Prove faithfulness before fullness.

(b) Conversely, suppose that $F$ is full, faithful and essentially surjective on objects. For each object $W$ in $\mathscr{D}$, choose (axiom of choice is being invoked here) an object $G(W)$ of $\mathscr{C}$ and an isomorphism $\varepsilon_W : F(G(W)) \to W$. Prove that $G$ extends to a functor in such a way that $\varepsilon$, with components $\varepsilon_W$, is a natural isomorphism $FG \Rightarrow 1_{\mathscr{D}}$. Then construct a natural isomorphism $\eta : 1_{\mathscr{C}} \Rightarrow GF$, thus proving that $F$ is an equivalence.

**Problem 3.6.** Prove that the categories in Example 3.11 are equivalent.

**Problem 3.7.** It's not often that the opposite categories can be identified with familiar categories, it's a rare phenomenon. It's also rare for the categories to be equivalent to their oppposite versions. Here are some simple examples of this rare phenomena.

(a) Prove $(-)^* : \mathsf{fdVec}_k^{\mathrm{op}} \to \mathsf{fdVec}_k$ defines an equivalence of categories, see Example 2.16 for how it's defined. Through the equivalence established in Example 3.11, the functor $(-)^*$ translates to the functor $(-)^\intercal : \mathsf{Mat}_k^{\mathrm{op}} \to \mathsf{Mat}_k$ which also then defines an equivalence (isomorphism, in fact) of categories.

(b) Prove $(-)^{-1} : \mathsf{B}G^{\mathrm{op}} \to \mathsf{B}G$ defines an equivalence (isomorphism, in fact) of categories, see Example 2.16 for how it's defined.

(c) Let $X$ be any set, we can consider the poset $(\mathscr{P}(X), \subseteq)$, where $\mathscr{P}(X)$ is the power set of $X$, with respect to set containment. This then gives a category, as described previously in Example 1.4 on how posets give rise to categories. Prove that the functor

$$(-)^c : (\mathscr{P}(X), \subseteq)^{\text{op}} \to (\mathscr{P}(X), \subseteq),\ A \mapsto A^c := X \setminus A$$

defines an equivalence (isomorphism, in fact) of categories.

(d) You can either take the following two statements for granted, or explore the details yourself.

- the opposite category of unital commutative ring is equivalent to the category of affine schemes.
- the opposite category of sets is equivalent to the category of complete atomic boolean algebras. When restricted to finite sets, the opposite category of finite sets is equivalent to the category of finite boolean algebras.

(e) Consider the category $\Gamma$, called *Segal's category*, described as following.

- Objects of $\Gamma$ are finite sets.
- For finite sets $S$ and $T$

$$\operatorname{Hom}_\Gamma(S, T) := \{\theta : S \to \mathscr{P}(T)\ :\ \theta(\alpha) \cap \theta(\beta) = \varnothing, \text{ whenever } \alpha \neq \beta\}$$

  where $\mathscr{P}(-)$ denotes the power set.
- The composite of $\theta : S \to \mathscr{P}(T)$ and $\phi : T \to \mathscr{P}(U)$ is the function

$$\psi : S \to \mathscr{P}(U),\ \alpha \mapsto \bigcup_{\gamma \in \theta(\alpha)} \phi(\gamma)$$

  What do the identity morphisms look like?

Prove that $\Gamma$ is equivalent to the opposite of the category $\mathsf{FinSets}_*$ of finite pointed sets (describe this to yourself).

**Problem 3.8.** A category $\mathscr{C}$ is *skeletal* if it contains just one object in each isomorphism class. A *skeleton* $\operatorname{sk}(\mathscr{C})$ of a category $\mathscr{C}$ is a full subcategory such that every object of $\mathscr{C}$ is isomorphic to precisely one object in $\operatorname{sk}(\mathscr{C})$.

(a) Prove that a skeleton of a category is skeletal.

(b) Show that any two skeletons of a category are isomorphic.

(c) Show that $\operatorname{sk}(\mathscr{C})$ of a category $\mathscr{C}$ is equivalent to $\mathscr{C}$. Therefore (a) and Problem 3.3 give us that *every* skeleton of $\mathscr{C}$ is equivalent to $\mathscr{C}$. This also shows that an equivalence need not be injective on objects.

(d) Show that two categories are equivalent if and only if they have isomorphic skeletons.

Example 3.11 and Problem 3.6 exhibit that the skeleton of $\mathsf{fdVec}_k$ is the category $\mathsf{Mat}_k$.

(e) Let $\mathsf{FinSets}_{\mathsf{bij}}$ be the maximal groupoid of the category $\mathsf{FinSets}$ of finite sets. Prove that the skeleton of $\mathsf{FinSets}_{\mathsf{bij}}$ is the category whose objects are positive integers and with $\operatorname{Hom}(n, n) = \Sigma_n$, the symmetric group on $n$ letters. The sets of morphisms between distinct positive integers are all empty.

## 4. Lecture 4 (1/28) by Vaibhav

Recall that we assumed our categories to be *locally small*. That is, we have assumed in our categories $\mathscr{C}$, for any two objects $X$, $Y$, that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a set. We introduce some more size-related notions for categories.

**Definition 4.1.** A category $\mathscr{C}$ is said to be

- *small* if $\mathscr{C}$ is locally small, and $\mathrm{obj}(\mathscr{C})$ is a set. For example, a set itself can be treated as a category with its elements as objects and the only morphisms being identity morphisms; in this way, a set is an example of a small category.
- *essentially small* if $\mathscr{C}$ is equivalent to a small category. Equivalently, if $\mathrm{sk}(\mathscr{C})$, the skeleton of $\mathscr{C}$, is small (see Problem 3.8). For example, the category of finite sets (resp. finite dimensional vector spaces) is essentially small; this follows from Problem 3.8(e) (resp. Example 3.11 and Problem 3.6).

**Remark 4.2.** We have introduced (or now know) mathematical objects with different levels of structures (assume our categories are small).

- ○ sets (treated as a small category) have
  - ▷ objects (elements) $\bullet$
- ○ categories have
  - ▷ objects $\bullet$
  - ▷ morphisms $\bullet \longrightarrow \bullet$
- ○ the category of categories has
  - ▷ objects (categories) $\bullet$ "0-dimensional".
  - ▷ morphisms (functors) $\bullet \longrightarrow \bullet$ "1-dimensional".
  - ▷ 2-morphisms (natural transformations) $\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$ "2-dimensional".
    - ◇ (identities) Given a functor $F : \mathscr{C} \to \mathscr{D}$, we can define an identity natural tranformation $1_F : F \Rightarrow F$, where the components are $1_{F(X)} : F(X) \to F(X)$.
    - ◇ (vertical composition)

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "F", ""{name=F, below}] \arrow[r, "G" description, ""{name=GU, above}, ""{name=GD, below}] \arrow[r, bend right=50, "H"', ""{name=H, above}] & \bullet \arrow[Rightarrow, from=F, to=GU, "\alpha"] \arrow[Rightarrow, from=GD, to=H, "\beta"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "F", ""{name=F, below}] \arrow[r, bend right=50, "H"', ""{name=H, above}] & \bullet \arrow[Rightarrow, from=F, to=H, "\beta\cdot\alpha"] \end{tikzcd}$$

    - ◇ (horizontal composition)

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "F", ""{name=F, below}] \arrow[r, bend right=50, "G"', ""{name=G, above}] & \bullet \arrow[r, bend left=50, "A", ""{name=A, below}] \arrow[r, bend right=50, "B"', ""{name=B, above}] & \bullet \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=A, to=B, "\beta"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "AF", ""{name=U, below}] \arrow[r, bend right=50, "BG"', ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D, "\beta*\alpha"] \end{tikzcd}$$

We can collect these to into a construction called a 2-category.

**Definition 4.3.** For a category $\mathscr{C}$, an object $I$ is called an *initial object* if there exists a unique morphism $I \to X$ for any object $X$ of $\mathscr{C}$.

Dually, an object $T$ is called a *final* or *terminal object* if there exists a unique morphism $X \to T$ for any object $X$ of $\mathscr{C}$ (i.e., it's an initial object in $\mathscr{C}^{\mathrm{op}}$).

In other words, $I$ is an initial object if and only if $\mathrm{Hom}_{\mathscr{C}}(I, X) \cong *$ (the singleton set).

**Example 4.4.** An initial object in Set is $\varnothing$, the unique map $\varnothing \hookrightarrow X$ can be understood to be the inclusion of the empty set as a subset of any set $X$. A terminal object in Set is the singleton set $\{1\}$, the unique map $X \to \{1\}$ is the constant function for any set $X$. See Problem 4.1 for more examples and ideas.

**Remark 4.5.** We can interpret the notion of an initial object in terms of functors.

For an object $Z$ in a category $\mathscr{C}$, recall the functor $h^Z = \mathrm{Hom}_{\mathscr{C}}(Z, -) : \mathscr{C} \to \mathrm{Set}$ where $X \mapsto \mathrm{Hom}_{\mathscr{C}}(Z, X)$ and

$$\begin{tikzcd}
X \arrow[d, "f"] & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(Z,X) \arrow[d, "f\circ-"] \\
Y & & \mathrm{Hom}_{\mathscr{C}}(Z,Y)
\end{tikzcd}$$

For any category $\mathscr{C}$ and any singleton set $*$, we can define the constant functor $* : \mathscr{C} \to \mathrm{Set}$ where any object $X \mapsto *$ and any morphism $f \mapsto 1_*$.

Then, an object $I$ in $\mathscr{C}$ is an initial object if and only if there exists a natural isomorphism

$$\rho : \mathrm{Hom}_{\mathscr{C}}(I, -) \overset{\sim}{\Longrightarrow} *$$

That is, for any object $X$ we have a bijection $\rho_X : \mathrm{Hom}_{\mathscr{C}}(I, X) \overset{\sim}{\to} *$ and for any morphism $X \xrightarrow{f} Y$ the following square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(I,X) \arrow[r, "\rho_X"] \arrow[d, "f\circ-"'] & * \arrow[d, "1_*"] \\
\mathrm{Hom}_{\mathscr{C}}(I,Y) \arrow[r, "\rho_Y"'] & *
\end{tikzcd}$$

commutes.

We can think of this as saying $I$ witnesses the structure of the constant functor $*$; we say that $I$ *represents* the functor $*$.

**Definition 4.6.** A functor $F : \mathscr{C} \to \mathrm{Set}$ is *representable* if there exists an object $X$ of $\mathscr{C}$ and a natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X, -) \overset{\sim}{\Longrightarrow} F$$

Note: if $F$ were a contravariant functor, that is a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$, then for it to be a representable functor we would want a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(-, X) = \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, -) \overset{\sim}{\Longrightarrow} F$.

We then say *$F$ is represented by $X$*.

Therefore, a category $\mathscr{C}$ has an initial object if and only if the constant functor $*$ is representable.

**Definition 4.7.** Given a functor $F : \mathscr{C} \to \mathsf{Set}$, a *representation of $F$* or a *universal object for $F$* is a pair $(X, \alpha)$, where $X$ is an object in $\mathscr{C}$ and $\alpha$ is a choice of a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(X, -) \overset{\sim}{\Longrightarrow} F$.

**Example 4.8.** We give an example and a non-example of representable functors.

- Recall the functor $\mathscr{P}^* : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ which sends a set $A \mapsto \mathscr{P}(A)$ to its power set, and a function $f : A \to B$ is sent to

  $$f^* : \mathscr{P}(B) \to \mathscr{P}(A),\ S \mapsto f^{-1}(S)$$

  The functor is representable, and is represented by the set $\Omega = \{\mathrm{T}, \mathrm{F}\}$.

  We claim that we obtain a natural isomorphism $C : \mathrm{Hom}_{\mathsf{Set}}(-, \Omega) \Longrightarrow \mathscr{P}^*$ with the components as the bijections

  $$\begin{aligned} C_A : \mathrm{Hom}_{\mathsf{Set}}(A, \Omega) &\longrightarrow \mathscr{P}^*(A) \\ \chi &\longmapsto \chi^{-1}(\mathrm{T}) \\ \left(\chi_S : a \mapsto \begin{cases} \mathrm{T} & \text{if } a \in S \\ \mathrm{F} & \text{if } a \notin S \end{cases}\right) &\longleftarrow\!\shortmid\ S \end{aligned}$$

  for any set $A$. We now verify that the components $(C_A)_A$ do assemble to give a natural isomorphism; that is, given any function $f : A \to B$, we verify the following diagram commutes.

  $$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Set}}(B, \Omega) \arrow[r, "C_B"] \arrow[d, "- \circ f"'] & \mathscr{P}(B) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathsf{Set}}(A, \Omega) \arrow[r, "C_A"'] & \mathscr{P}(A) \end{tikzcd}$$

  Consider any function $\chi \in \mathrm{Hom}_{\mathsf{Set}}(B, \Omega)$, then

  $$f^* \circ C_B(\chi) = f^*(\chi^{-1}(\mathrm{T})) = f^{-1}(\chi^{-1}(\mathrm{T})) = f^{-1} \circ \chi^{-1}(\mathrm{T})$$
  $$C_A \circ (- \circ f)(\chi) = C_A(\chi \circ f) = (\chi \circ f)^{-1}(\mathrm{T})$$

  We know from properties of inverse image that $(\chi \circ f)^{-1}(\mathrm{T}) = f^{-1} \circ \chi^{-1}(\mathrm{T})$, hence the diagram indeed commutes. Thus, we have proven that $C$ is a natural isomorphism and therefore $(\Omega, C)$ is a representation of $\mathscr{P}^*$.

- Consider the abelianisation functor $(-)^{\mathrm{ab}} : \mathsf{Grp} \to \mathsf{Ab}$ which sends a group $G \mapsto G^{\mathrm{ab}}$ to its abelianisation where $G^{\mathrm{ab}} := G/[G, G]$. A group homomorphism $f : G \to H$ is sent to the induced group homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. We prove that this functor is not representable, and for that we use Problem 4.5. Recall that monomorphisms in these categories are simply injective functions (see Problem 2.8).

Consider the non-abelian group $S_3$, and its subgroup $A_3$, which is abelian. Therefore $A_3^{\mathrm{ab}} \cong A_3$ and one can prove that $S_3^{\mathrm{ab}} \cong C_2$, the cyclic group of order 2. Consider the inclusion group homomorphism $\iota : A_3 \hookrightarrow S_3$, which is indeed injective. But the induced map $\iota^{\mathrm{ab}} : A_3 \to C_2$ is necessarily the trivial homomorphism and hence not injective. Thus $(-)^{\mathrm{ab}}$ is not a representable functor.

**Remark 4.9.** Representable functors are rare and form a very special class of functors. Some of the biggest questions in algebraic geometry, for example, in the past century were if certain nice geometric functors were representable, and what to do if they were found to be not.

**Definition 4.10** (a first attempt)**.** A *universal property* of an object $X$ is expressed by a representable functor $F$ together with a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(X, -) \cong F$ (or $\mathrm{Hom}_{\mathscr{C}}(-, X) \cong F$ if $F$ is contravariant).

We will give a slightly better description once we have discussed the Yoneda lemma, and we can then also interpret this as an initial object in some category.

**Example 4.11.** Let $R$ be a commutative ring with unity and $M$ and $N$ are $R$-modules, we consider the functor

$$\mathrm{Bil}_R(M \times N, -) : \mathsf{Mod}_R \to \mathsf{Set},$$

where any $R$-module $T$ is sent to the set of $R$-bilinear maps $\mathrm{Bil}_R(M \times N, R)$. These are maps

$$f : M \times N \to T$$

such for any $m \in M$ and $n \in N$, the maps $f(m, -) : N \to T$ and $f(-, n) : M \to T$ are $R$-linear.

This functor has as a representation $(M \otimes_R N, \iota)$, where $M \otimes_R N$ is an $R$-module called the *tensor product* and $\iota : M \times N \to M \otimes_R N,\ (m, n) \mapsto m \otimes n$ is an $R$-bilinear map. The map $\iota$ affords the natural isomorphism

$$\mathrm{Hom}_R(M \otimes_R N, T) \cong \mathrm{Bil}_R(M \times N, T)$$

which we illustrate by exhibiting the universal property of $M \otimes_R N$.

The universal property is as follows: for any $R$-module $T$ and $R$-bilinear map $f : M \times N \to T$ there exists a unique $R$-linear map $\tilde{f} : M \otimes_R N \to T$ such that the diagram

$$\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr, "\iota"'] & & T \\
& M \otimes_R N \arrow[ur, dashed, "\exists ! \tilde{f}"'] &
\end{tikzcd}$$

commutes. That is, we have the bijection

$$\mathrm{Hom}_R(M \otimes_R N, T) \xrightarrow{\sim} \mathrm{Bil}_R(M \times N, T) : g \mapsto g \circ \iota$$

**Example 4.12.** Let $G$ be a group and $N$ a normal subgroup of $G$, we consider the functor

$$\mathrm{Kil}_N(G,-) : \mathsf{Grp} \to \mathsf{Set},$$

where any group $H$ is sent to the set

$$\mathrm{Kil}_N(G,H) = \{\phi : G \to H \ : \ N \subseteq \ker\phi\} \subseteq \mathrm{Hom}_{\mathsf{Grp}}(G,H)$$

This functor has as a representation $(G/N, \pi)$, where $G/N$ is the usual quotient group (of left cosets) and $\pi : G \to G/N,\ g \mapsto gN$ is the natural projection map. The map $\pi$ affords the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Grp}}(G/N,H) \cong \mathrm{Kil}_N(G,H)$$

which we illustrate by exhibiting the universal property of $G/N$.

The universal property is as follows: for any module $H$ and group homomorphism $\phi : G \to H$ such that $N \subseteq \ker\phi$ there exists a unique group homomorphism $\tilde{\phi} : G/N \to H$ such that the diagram

$$\begin{tikzcd}
G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & N \\
& G/N \arrow[ur, dashed, "\exists!\tilde{\phi}"'] &
\end{tikzcd}$$

commutes. That is, we have the bijection

$$\mathrm{Hom}_{\mathsf{Grp}}(G/N,H) \xrightarrow{\sim} \mathrm{Kil}_N(G,H) : \psi \mapsto \psi \circ \pi$$

## 4.1. Problems

**Problem 4.1.**

(a) Look at the categories in Examples 1.2, 1.3 and 1.4, and try determining or describing the initial and terminal objects in those categories.

(b) Prove that an initial object of a category is *unique up to unique isomorphism*. Precisely put, suppose $I$ and $I'$ are two initial objects of a category $\mathscr{C}$, prove that there exists a unique isomorphism $f : I \to I'$.

Applying this to $\mathscr{C}^{\mathrm{op}}$ gives us that terminal objects are also unique upto unique isomorphism, this again is an example of a duality argument.

(c) Let $I$ and $T$ be initial and terminal objects in a category $\mathscr{C}$, prove that if there exists a morphism $f : T \to I$, then $f$ is necessarily an isomorphism.

(d) An object that's both an initial and terminal object is called a *zero object*. Did you encounter any zero objects while solving (a)?

(e) Prove that a zero object exists in a category $\mathscr{C}$ if and only if $\mathscr{C}$ has an initial object, a terminal object and a morphism from the terminal to the initial object.

**Problem 4.2.** Formulate a similar statement for terminal objects to the one that's been formulated for initial objects in Remark 4.5.

**Problem 4.3.** For objects $X$ in a category $\mathscr{C}$, consider the slice category $\mathscr{C}/X$. Prove that $1_X$ is a terminal object in $\mathscr{C}/X$. We think of this as "$X$ is the terminal object of $\mathscr{C}/X$".

Using Problem 2.2 (or not), what can you say about the initial object of the slice category $X/\mathscr{C}$.

**Problem 4.4.** Prove that if $I$ is an initial object in a category $\mathscr{C}$, then the slice category $I/\mathscr{C}$ is isomorphic to $\mathscr{C}$.

Using Problem 2.2 (or not), prove that if $T$ is a terminal object in a category $\mathscr{C}$, then $\mathscr{C}/T \cong \mathscr{C}$.

**Problem 4.5.** Prove that if $F : \mathscr{C} \to \mathsf{Set}$ is representable, then $F$ preserves monomorphisms, i.e., sends every monomorphism in $\mathscr{C}$ to an injective function. What would be the statement if $F$ was contravariant?

Hint: it's enough to prove this for the functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ for some object $X$ of $\mathscr{C}$ (why?).

Using this, produce an example of a non-representable functor (covariant or contravariant), different from the one seen in Example 4.8.

**Problem 4.6.** Prove that

(a) the forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ is represented by the group $\mathbb{Z}$.

(b) the forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$ is represented by the ring $R$ treated as an $R$-module.

(c) the functor $U(-)^n : \mathsf{Ring} \to \mathsf{Set}$, where any ring $R$ is sent to the set $R^n$, is represented by the ring $\mathbb{Z}[t_1, \ldots, t_n]$.

(d) the functor $\mathcal{O} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ defined in Example 3.6, where we send a space to its set of open sets, is represented by the *Sierpinski space* $\mathcal{S} = \{0, 1\}$ where the open sets are $\varnothing$, $\{0\}$ and $\mathcal{S}$.

(e) the functor $\mathrm{Hom}_{\mathsf{Set}}(-, X) \times \mathrm{Hom}_{\mathsf{Set}}(-, Y) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$, $T \mapsto \mathrm{Hom}_{\mathsf{Set}}(T, X) \times \mathrm{Hom}_{\mathsf{Set}}(T, Y)$ is represented by the cartesian product $X \times Y$.

**Problem 4.7.** Given a group $G$, prove that a functor $E : \mathsf{B}G \to \mathsf{Set}$, i.e., a left $G$-set $E$ (see Example 3.1), is representable if and only if there is an isomorphism $G \cong E$ of left $G$-sets.

This implies that the action of $G$ on $E$ is free (every stabilizer group is trivial) and transitive (the orbit of any point is the entire set), and that $E$ is non-empty. One thinks of this $E$ being a group that has forgotten its identity element. Such a $G$-set is called a *$G$-torsor*.

## 5. Lecture 5 (2/04) by David

Recall the notion of a representable functor: given a functor $F : \mathscr{C} \to \mathsf{Set}$, we say $F$ is representable if there exists an object $X$ in $\mathscr{C}$ such that we have a natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X, -) \cong F.$$

We may then ask the following questions, that were either directly or implicitly brought up last time.

- Is this $X$ unique?
- How does this relate to initial (or terminal) objects?
- How does this relate to universal properties?

First, recall the notation $\mathscr{D}^{\mathscr{C}}$ from Problem 3.1.

**Theorem 5.1** (Yoneda Lemma). *Let $\mathscr{C}$ be locally small and $F : \mathscr{C} \to \mathsf{Set}$ a functor. For any object $X$ in $\mathscr{C}$ there's a bijection*

$$\mathrm{Nat}(h^X, F) \cong F(X),$$

*where the former is the set of natural transformations from $h^X = \mathrm{Hom}_{\mathscr{C}}(X, -)$ to $F$, that is, the set of morphisms from $h^X$ to $F$ in $\mathsf{Set}^{\mathscr{C}}$.*

*Similarly, if $G : \mathscr{C}^{op} \to \mathsf{Set}$ is a contravariant functor, then for any object $X$ in $\mathscr{C}$ there's a bijection*

$$\mathrm{Nat}(h_X, G) \cong G(X),$$

*where the former is the set of morphisms from $h_X = \mathrm{Hom}_{\mathscr{C}}(-, X)$ to $G$ in $\mathsf{Set}^{\mathscr{C}^{op}}$.*

We will see a proof soon.

**Corollary 5.2** (Yoneda Embedding). *The functors*

$$\mathscr{Y}^* : \mathscr{C}^{op} \to \mathsf{Set}^{\mathscr{C}},\ X \mapsto h^X$$

*and*

$$\mathscr{Y}_* : \mathscr{C} \to \mathsf{Set}^{\mathscr{C}^{op}},\ X \mapsto h_X$$

*are fully faithful.*

*Proof.* Recall that $\mathscr{Y}^*$ is fully faithful if

$$\mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Nat}(h^X, h^Y) : f \mapsto \mathscr{Y}^*(f) = - \circ f$$

is a bijection for any objects $X$ and $Y$. Yoneda Lemma (Theorem 5.1) tells us that when applied to $F = h^Y = \mathrm{Hom}_{\mathscr{C}}(Y, -)$ we get

$$\mathrm{Hom}_{\mathscr{C}}(Y, X) = \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) \cong \mathrm{Hom}_{\mathsf{Set}^{\mathscr{C}}}(\mathscr{Y}^*(X), \mathscr{Y}^*(Y)) = \mathrm{Nat}(h^X, h^Y).$$

The proof of Yoneda Lemma (see end) tells us that this bijection in this case is indeed the map needed. Therefore $\mathscr{Y}^*$ is fully faithful.

The second statement for $\mathscr{Y}_*$ similarly follows from the contravariant version of the Yoneda Lemma.

In particular,

$$\mathrm{Nat}(h^X, h^Y) \cong \mathrm{Hom}_{\mathscr{C}}(Y, X) \quad \text{and} \quad \mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

That is, natural transformations between representable functors correspond to maps between the representing objects. □

**Example 5.3** (Revisiting Example 4.11)**.** For a field $k$, consider two vector spaces $V$ and $W$. Recall the functor

$$\mathrm{Bil}_k(V \times W, -) : \mathsf{Vec}_k \to \mathsf{Set},$$

where any $k$-vector space $U$ is sent to the set of $k$-bilinear maps $\mathrm{Bil}_R(V \times W, U)$. A representation of this functor is $V \otimes_k W$, that is

$$\mathrm{Hom}_k(V \otimes_k W, U) \cong \mathrm{Bil}_k(V \times W, U)$$

Yoneda Lemma (rather, the proof) tells us that the object $V \otimes_k W$ representing $\mathrm{Bil}_k(V \times W, -)$ is uniquely determined by a "universal element" of $\mathrm{Bil}_k(V \times W, V \otimes_k W)$, which is the bilinear map

$$\iota : V \times W \to V \otimes_k W,\ (v, w) \mapsto v \otimes w$$

We can use Problem 5.4, a consequence of the Yoneda embedding, to prove the following facts about tensor products.

1. $k \otimes_k V \cong V$ for any vector space $V$;
2. $V \otimes_k W \cong W \otimes_k V$ for any pair of vector spaces $V$, $W$;
3. $(U \otimes_k V) \otimes_k W \cong U \otimes_k (V \otimes_k W)$ for any triple of vector spaces $U$, $V$ and $W$.

We sketch the proof of (2) and leave the remaining two as exercises. We show that we have a natural isomorphism $\mathrm{Bil}_k(V \times W, -) \cong \mathrm{Bil}_k(W \times V, -)$, then Problem 5.4 immediately gives us $V \otimes_k W \cong W \otimes_k V$.

The components for this natural isomorphism are given as, for any vector space $U$

$$\mathrm{Bil}_k(V \times W, U) \to \mathrm{Bil}_k(W \times V, U)$$

$$f \mapsto \tilde{f}, \text{ where } \tilde{f}(w, v) := f(v, w)$$

One checks this is a bijection and natural in $U$, and thus we have $V \otimes_k W \cong W \otimes_k V$.

*Sketch of Proof of Theorem 5.1 (Yoneda Lemma).* We want to establish a bijection

$$\mathrm{Nat}(h^X, F) \cong F(X)$$

Consider a natural transformation $\eta : h^X \to F$, then consider its component at $X$, it is a function $\eta_X : \mathrm{Hom}_{\mathscr{C}}(X, X) \to F(X)$. Then $x_\eta := \eta_X(1_X) \in F(X)$. So we have a function,

$$\Phi : \mathrm{Nat}(h^X, F) \to F(X),\ \eta \mapsto x_\eta$$

Furthermore, given a morphism $u : X \to A$ for some object $A$ in $\mathscr{C}$, by naturality of $\eta$ we have a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(X,X) \arrow[r, "\eta_X"] \arrow[d, "u\circ -"'] & F(X) \arrow[d, "F(u)"] \\
\operatorname{Hom}_{\mathscr{C}}(X,A) \arrow[r, "\eta_A"'] & F(A)
\end{tikzcd}$$

Taking $1_X \in \operatorname{Hom}_{\mathscr{C}}(X,X)$ along the two sides of the square we get

$$\eta_A(u) = \eta_A(u \circ 1_X) = F(u) \circ \eta_X(1_X) = F(u)(x_\eta)$$

This motivates us to give the following function

$$\Psi : F(X) \to \operatorname{Nat}(h^X, F),\ x \mapsto \eta_x$$

where $\eta_x$ is the natural transformation with components, for any object $A$ in $\mathscr{C}$, given by

$$\eta_{x,A} : \operatorname{Hom}_{\mathscr{C}}(X,A) \to F(A),\ u \mapsto F(u)(x)$$

One readily checks that this is indeed natural in $A$, and that $\Phi$ and $\Psi$ are inverses of each other. $\square$

**Remark 5.4.** Suppose $F : \mathscr{C} \to \mathsf{Set}$ is representable with representation $(X, \alpha)$, that is $\alpha : h^X \Rightarrow F$ is a natural isomorphism. Now, the Yoneda Lemma tells us that $\alpha$ necessarily corresponds to an element of $F(X)$, which is precisely $\xi := \alpha_X(1_X)$ (the map $\Phi$). We can, in fact, construct $\alpha$ from just $\xi$ (the map $\Psi$). $\xi$ is then called a *universal element* and a representation of $F$ is just given to be $(X, \xi)$. More precisely, $\xi \in F(X)$ is an element such that

$$\alpha_{\xi,A} : \operatorname{Hom}_{\mathscr{C}}(X,A) \to F(A),\ u \mapsto F(u)(x)$$

is a bijection, i.e., for every $a \in F(A)$ there exists a unique $u \in \operatorname{Hom}_{\mathscr{C}}(X,A)$ such that $a = F(u)(\xi)$.

For a functor $F$ with a representation $(X, \xi)$, a *universal property* is the description of the natural isomorphism $h^X \Rightarrow F$ given by $\xi$.

One notes in Examples 4.11 and 4.12, $\iota$ and $\pi$ precisely the image of the identity maps under the given bijections towards the end; they are indeed the universal elements.

## 5.1. Problems

**Problem 5.1.** Consider the category $\mathsf{Set}^{\mathscr{C}^{\mathrm{op}}}$, prove that the subcategory of representable functors is equivalent to $\mathscr{C}$.

**Problem 5.2.** Using the Yoneda embedding (Corollary 5.2) with respect to $\mathscr{C} = \mathsf{Mat}_A$ (see Example 1.4) prove that *every row operation on matrices with n rows is defined by left multiplication by some $n \times n$ matrix*.

**Problem 5.3.** Using the Yoneda embedding (Corollary 5.2) with respect to $\mathscr{C} = \mathsf{B}G$ (see Example 1.4) prove *Cayley's Theorem: any group is isomorphic to a subgroup of a permutation group.*

**Problem 5.4.** Suppose $F : \mathscr{C} \to \mathsf{Set}$ is representable by objects $X$ and $Y$ in $\mathscr{C}$, that is

$$\mathrm{Hom}_{\mathscr{C}}(X, -) \cong F \cong \mathrm{Hom}_{\mathscr{C}}(Y, -),$$

prove that $X \cong Y$ using Yoneda embedding (Corollary 5.2) and Problem 3.4.

Use this to prove that any two initial objects are isomorphic.

**Problem 5.5.** Prove statements (1) and (3) in Example 5.3.

**Problem 5.6.** Given an object $X$ in a category $\mathscr{C}$, what's the universal element (see Remark 5.4) of the functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ (or $\mathrm{Hom}_{\mathscr{C}}(-, X)$, for that matter)?

**Problem 5.7.** Consider Example 5.3. Let $V$ and $W$ be $k$-vector spaces.

(a) Construct a *"category of bilinear maps"* out of $V \times W$.

(b) Prove that $M \otimes_k N$ is an initial object in this category.

More generally, this refers to the *category of elements*, see Problem 5.8 (f), (g), (h).

**Problem 5.8.** Given functors $F : \mathscr{D} \to \mathscr{C}$ and $G : \mathscr{E} \to \mathscr{C}$, we describe the *comma category* $F \downarrow G$. It has

- as objects triples $(D, E, f)$, where $D$ is an object in $\mathscr{D}$ and $E$ in $\mathscr{E}$, and $f : F(D) \to G(E)$ is a morphism in $\mathscr{C}$.
- as morphisms $(D, E, f) \to (D', E', f')$ pairs of morphisms $(h, k)$ where $h : D \to D'$ is a morphism in $\mathscr{D}$ and $k : E \to E'$ in $\mathscr{E}$ such that the diagram

$$\begin{tikzcd}
F(D) \arrow[r, "f"] \arrow[d, "F(h)"'] & G(E) \arrow[d, "G(k)"] \\
F(D') \arrow[r, "f'"'] & G(E')
\end{tikzcd}$$

  commutes.

Convince yourself this is indeed a category. What are the identity morphisms? How's composition defined?

Consider the following questions.

(a) Describe two canonical projection functors $H_{\mathscr{D}} : F \downarrow G \to \mathscr{D}$ and $H_{\mathscr{E}} : F \downarrow G \to \mathscr{E}$.

(b) Let $\mathscr{D} = \mathscr{C}$ and $F = 1_{\mathscr{C}}$, and let $\mathscr{E} = *$, the singleton set treated as a category. Let $X = G(*)$, then prove that $F \downarrow G$ is the slice category $\mathscr{C}/X$, which is therefore sometimes denoted as $\mathscr{C} \downarrow X$.

(c) Similarly, provide a description of $X/\mathscr{C}$ as a comma category. That is, make sense of the alternate notation $X \downarrow \mathscr{C}$.

(d) Describe the projection functors given in (a) in (b) and (c).

(e) If we let $\mathscr{D} = \mathscr{E} = \mathscr{C}$ and $F = G = 1_{\mathscr{C}}$, the resulting category is called the *arrow category of* $\mathscr{C}$ and denoted as either $\mathscr{C}^{\to}$ or $\mathrm{Arr}(\mathscr{C})$ or $\mathscr{C}^2$ or, of course, $1_{\mathscr{C}} \downarrow 1_{\mathscr{C}}$. Describe this category, i.e., its objects and morphisms.

(f) Let $F : \mathscr{C} \to \mathsf{Set}$ be a functor and consider $* \to \mathsf{Set}$ which we also denote by $*$. The category $* \downarrow F$ is called the *category of elements of* $F$ and sometimes denoted as $\int F$ or $\mathrm{el}(F)$. Prove that $\int F$ has the following description: it has

- as objects pairs $(C, x)$ where $C$ is an object in $\mathscr{C}$ and $x \in F(C)$.
- as morphisms $(C, x) \to (C', x')$ a morphism $f : C \to C'$ such that $F(f)(x) = x'$.

(g) Prove that if $F$ is representable, that is $F \cong \mathrm{Hom}_{\mathscr{C}}(X, -)$, then $\int F$ is equivalent to the category $X/\mathscr{C}$. In particular, it has an initial element, namely $1_X$.

(h) Conversely, prove that if $\int F$ has an initial object, then $F$ is representable.

(i) Describe the functor $\int F \to \mathscr{C}$ as given in (a). Given an object $C$ in $\mathscr{C}$ describe the objects in $\int F$ that get sent to $C$ under this functor, this is called the fiber over $C$.

## References

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