An Introduction into Building Theory
(Using Abstract Simplicial Complexes and Coxeter Groups)

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Motivation and Perspectives

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Definition of a Building

A *(weak) building* is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ (called *apartments*) satisfying axioms:

(**B0**) Each apartment Σ is a Coxeter complex.

(**B1**) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

(**B2**) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Finite Reflection Groups

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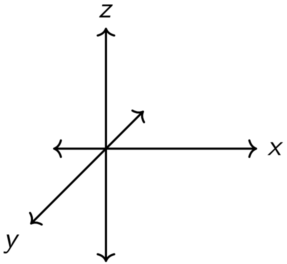
Example: Let $V = \mathbb{R}^3$ equipped with the dot product.



Inner Product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

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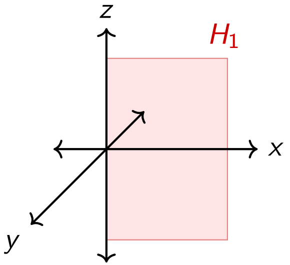
Example: Let $V = \mathbb{R}^3$ and $H_1 = \{\mathbf{u} \in \mathbb{R}^3 \ : \ \mathbf{u} = (u_1, 0, u_3)\}$.



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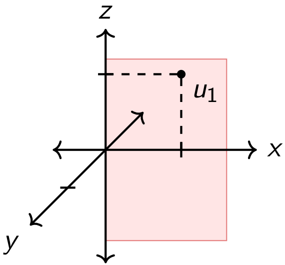
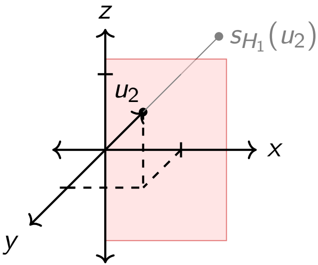
Example: Let $u_1 = (1, 0, 1)$ and $u_2 = (1, 1, 1)$.





We obtain $s_{H_1}(1, 0, 1) = (1, 0, 1)$ and $s_{H_1}(1, 1, 1) = (1, -1, 1)$.

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- $H_1 = \{\mathbf{u} \in \mathbb{R}^3 \ : \ \mathbf{u} = (u_1, 0, u_2)\}$
- $H_2 = \{\mathbf{u} \in \mathbb{R}^3 \ : \ \mathbf{u} = (u_1, \sqrt{3}u_2, u_2)\}$
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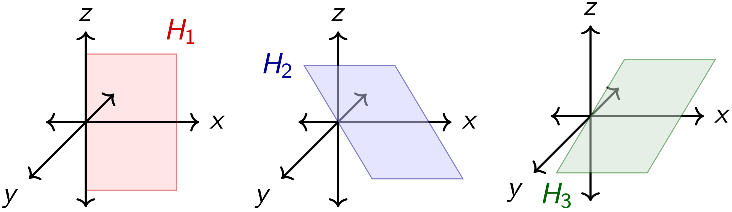

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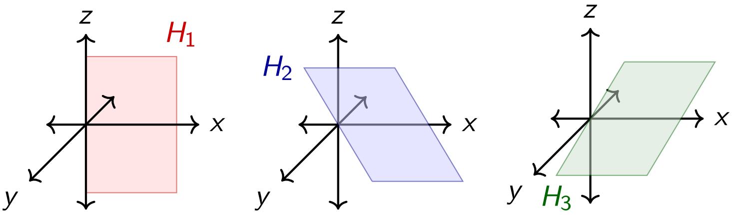

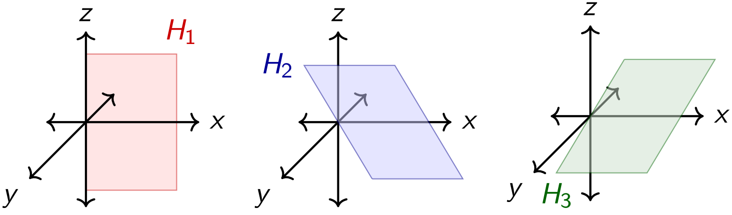

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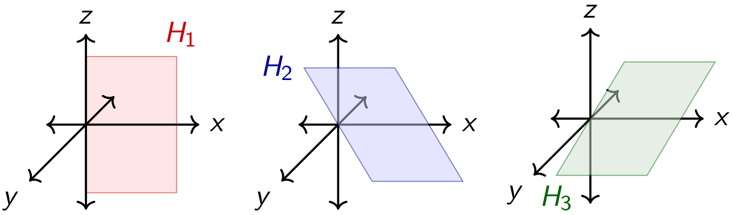

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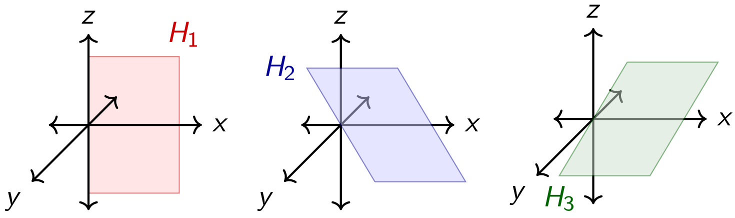

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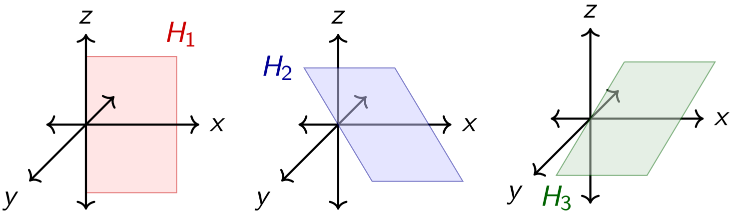

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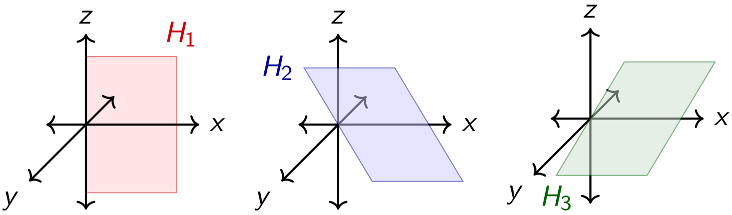

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(4) This group is isomorphic to S_3.

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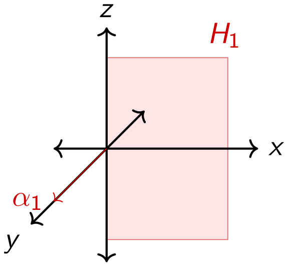
Example: For the hyperplane H_1, we can choose $\alpha_1 = (0, 1.25, 0)$.



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(4) For each $H \in \mathcal{H}$, we can explicitly write down s_H in terms of α (and thus we denote s_H by s_α):

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Example: Let $0 \neq \alpha \in H_1^{\perp}$, then

$$s_{\alpha_1}(\mathbf{u}) = (u_1, -u_2, u_3)$$

The function s_H is well-defined for any choice of $0 \neq \alpha \in H^{\perp}$.

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for all $\alpha, \beta \in \Phi$. In Lie algebra theory, this condition arises naturally. We define $\Phi^\vee := \{\alpha^\vee \ : \ \alpha \in \Phi\}$ to be the coroot system of W, in particular, a Weyl group can have multiple root systems (the choice of Φ is important to the structure).

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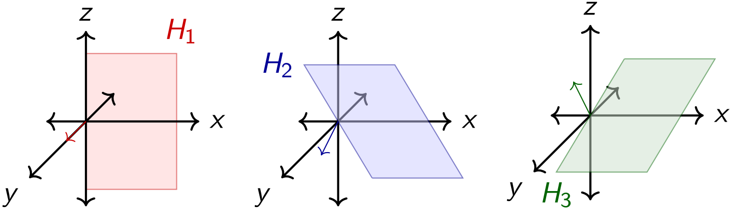
$$V = V_0 \oplus V_1$$

Example: Let $V = \mathbb{R}^3$, and H_1, H_2 and H_3 as before and choose orthogonal vectors $\alpha_1 = (0, 1, 0)$, $\alpha_2 = (0, -1, \sqrt{3})$, and $\alpha_3 = (0, 1, \sqrt{3})$, respectively.



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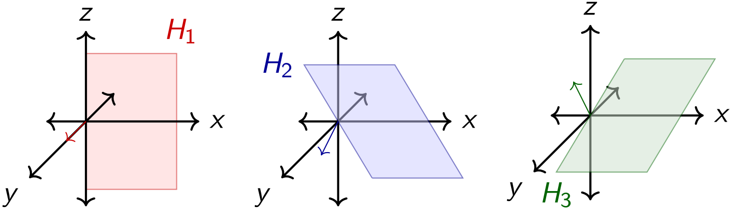

The x-axis is contained in each hyperplane and thus is orthogonal to each α_i chosen, or equivalently, is fixed by the reflections across each H_i:

$$V_0 := (0, 1, 0)^\perp \cap (0, -1, \sqrt{3})^\perp \cap (0, 1, \sqrt{3})^\perp$$

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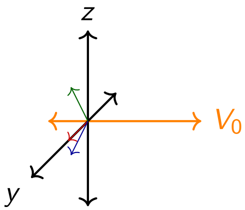
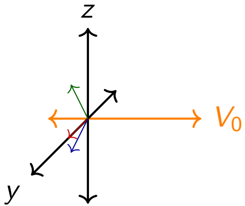
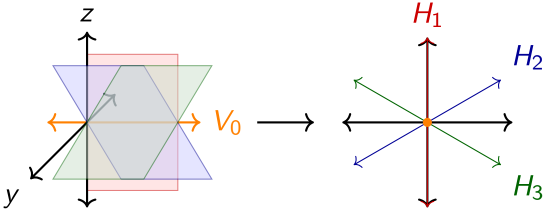




By contracting V_0 to a point, we see the essential part of (W, Φ):



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(6) One can restrict themselves to studying essential reflection groups, i.e. for $V = V_0 \oplus V_1$ as above, we require $V_0 = 0$.

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(7) We can define a product $(W', V') \times (W'', V'') := (W' \times W'', V' \oplus V'')$. If (W, V) cannot be expressed as a product, then we say that (W, V) is irreducible. One can restrict themselves to studying irreducible reflection groups.

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Among these are: group of permutations on n letters, group of signed permutations on n letters, dihedral groups, symmetries of regular solids (see Section 1.3 in Buildings - Theory and Applications, Abramenko and Brown, 2008).

Finite Reflection Groups

We have introduced quite a bit of notation:

- V is a Euclidean vector space.
- n is the dimension of V.
- H is a hyperplane, there exists $0 \neq \alpha \in H^{\perp}$ called a root.
- $\mathcal{H}$ is a hyperplane arrangement, Φ is a generalized root system.
- s_H is the reflection across H, also denoted s_α, where α is a root (determined by H).
- W is the Weyl group.
- $V_1 = \mathrm{span}_{\mathbb{R}}(\Phi)$ is the essential part of V.
- $V_0 = V^W$ is the unessential part of V.

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Definition: A (finite) abstract simplicial complex is a (finite) set A together with a collection Δ of finite subsets of A such that if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$. An abstract simplicial complex is denoted by the collection Δ.

(1) A vertex is a $v \in A$ such that $\{v\} \in \Delta$.
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To simplify notation with subsets, I will denote subsets $X = \{a_1, a_2, \ldots, a_k\}$ by $X = a_1 a_2 \ldots a_k$.

Example: Let $A = \{1, 2, 3, 4\}$ and consider
$\Delta_1 = \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$ and $\Delta_2 = \{\varnothing, 1, 2, 4, 12, 14\}$.

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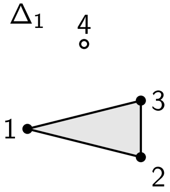

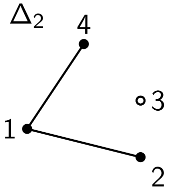

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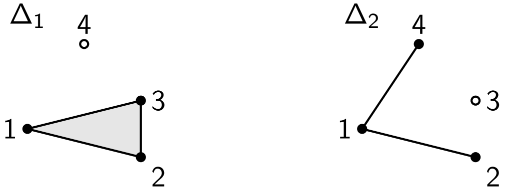
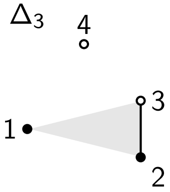


However, $\Delta_3 = \{\varnothing, 1, 2, 23, 123\}$ is not an abstract simplicial complex.



Coxeter Complex

Definition: Let Δ_1 and Δ_2 be two abstract simplicial complexes. A simplicial map is a function $f : V(\Delta_1) \to V(\Delta_2)$ such that for all $\delta \in \Delta_1$, $f(\delta) \in \Delta_2$.

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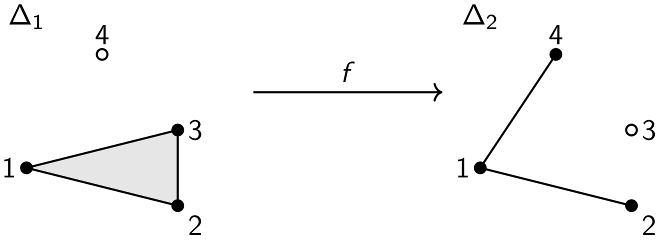

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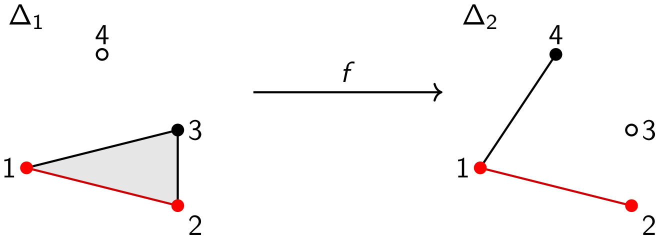

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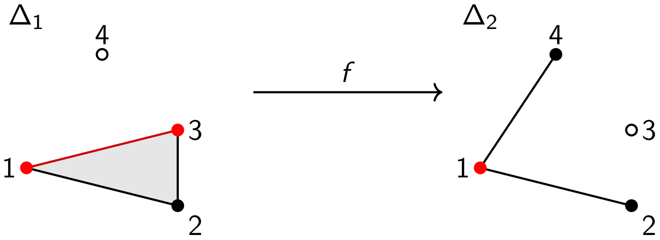

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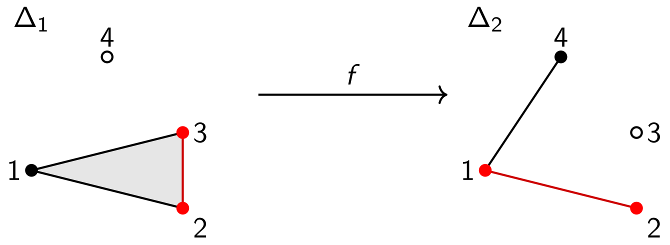

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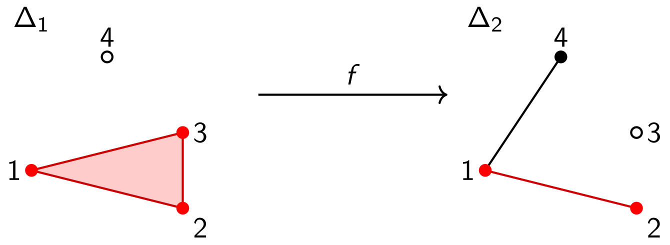

Coxeter Complex

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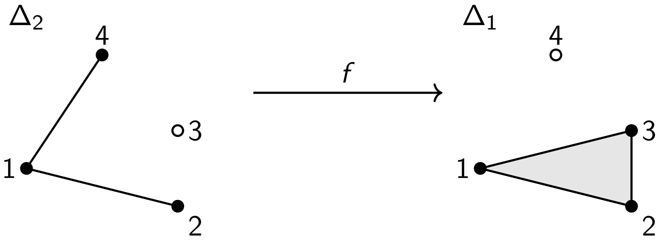

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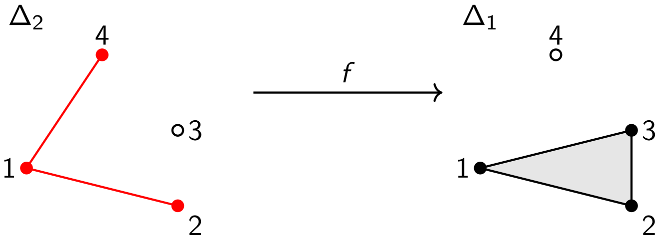

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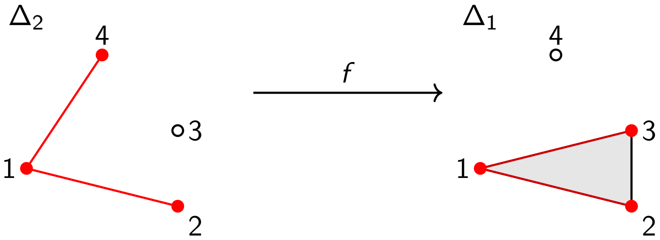

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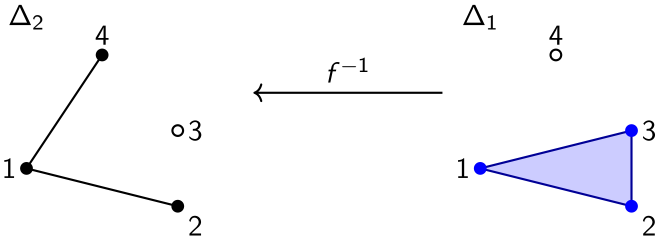

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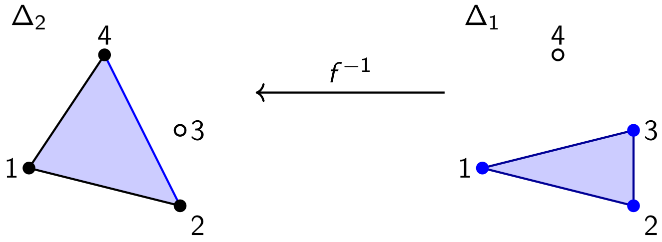

Coxeter Complex

Our next goal is to construct to a chamber complex associated to an essential finite reflection group (W, V), which we will denote by $\Sigma := \Sigma(W, S)$.

Coxeter Complex

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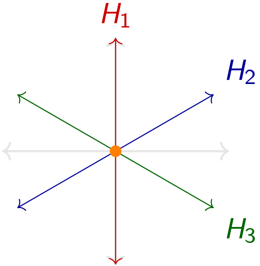

where $V = \mathbb{R}^3$ with the usual inner product. We can reformulate this example in terms of the essential part of the action.

Coxeter Complex

Let $V = \mathbb{R}^2$, with the usual inner product. Let $\mathcal{H} = \{H_1, H_2, H_3\}$.

- $H_1 = \left\{\mathbf{u} \in \mathbb{R}^2 \ : \ \mathbf{u} = (0, u_2)\right\}$
- $H_2 = \left\{\mathbf{u} \in \mathbb{R}^2 \ : \ \mathbf{u} = (\sqrt{3}u_2, u_2)\right\}$
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Set $s \coloneqq s_{H_1}$ and $t \coloneqq s_{H_2}$, then $s_{H_3} = s \circ t \circ s$. We will denote the *set of fundamental reflections* by $S = \{s, t\}$.

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Set $s := s_{H_1}$ and $t := s_{H_2}$, then $s_{H_3} = s \circ t \circ s$. We will denote the *set of fundamental reflections* by $S = \{s, t\}$. To each hyperplane, we will associate a linear function $f_i : V \to \mathbb{R}$ such that $f_i|_{H_i} = 0$ and $f_i \neq 0$.

- $f_1(x, y) = x$
- $f_2(x, y) = -x + \sqrt{3}y$
- $f_3(x, y) = x + \sqrt{3}y$

I will refer to the pair (W, S) as a Coxeter group.

Definition: A *cell* in V with respect to $\mathcal{H} = \{H_i\}_{i \in I}$ is a nonempty set A obtained by choosing for each i a sign $\sigma_i \in \{+, -, 0\}$ and specifying $f_i = \sigma_i$.

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$$A = \bigcap_{i \in I} U_i$$

for U_i is either H_i or an open half-space with H_i as a boundary. We denote the set of cells $\Sigma(\mathcal{H})$.

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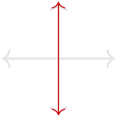
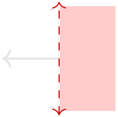
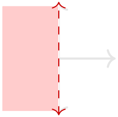
Definition: If $\sigma_i = 0$ for <u>exactly one</u> $i \in I$, then we call the cell a *panel*.



$f_1 = 0$



$f_1 = +$



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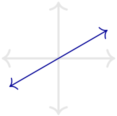
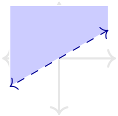


$f_2 = 0$



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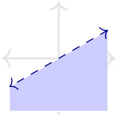

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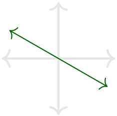
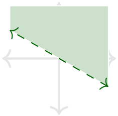


$f_3 = 0$



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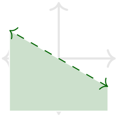

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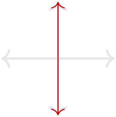
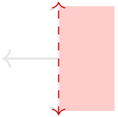
Coxeter Complex



$f_1 = 0$



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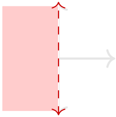

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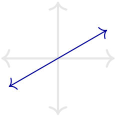
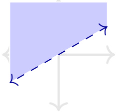


$f_2 = 0$



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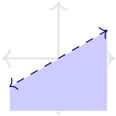

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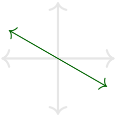

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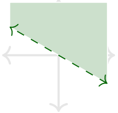

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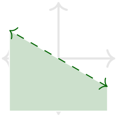
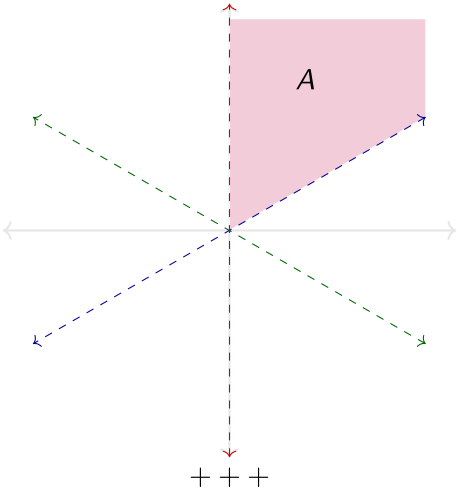


$f_3 = -$

Coxeter Complex



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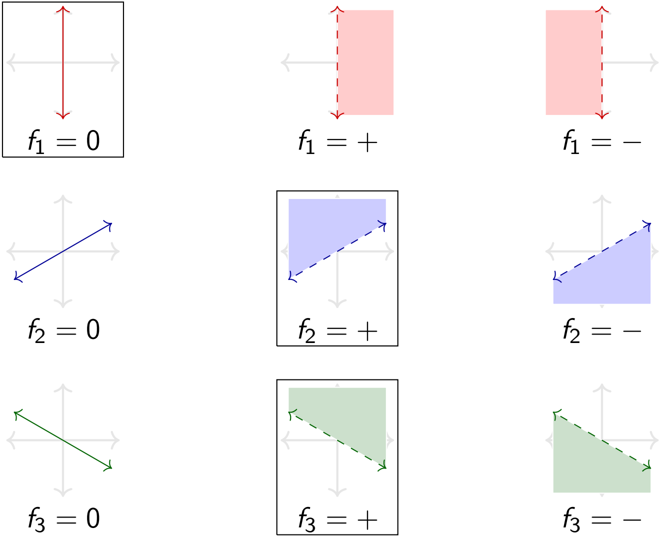

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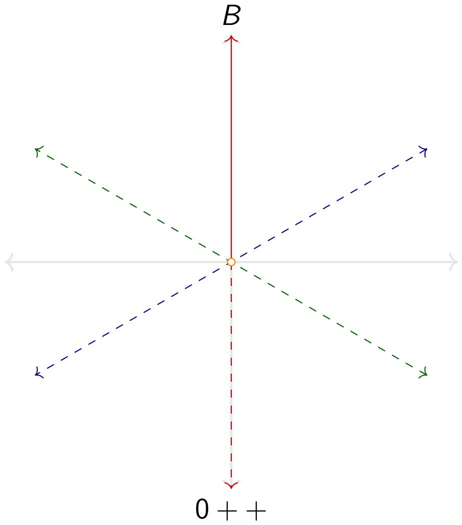

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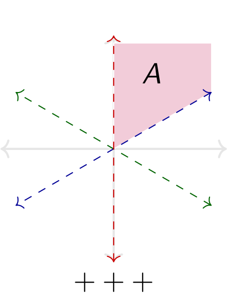
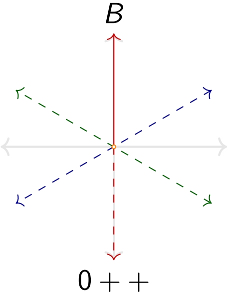
Example: In our previous examples of cells, $B \leq A$.

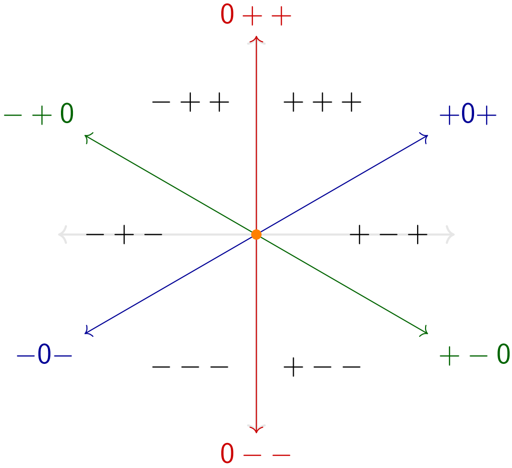




We say that B is a panel of A, and the hyperplane H_1 containing B is the *wall* of A.

Coxeter Complex

In this example, there are 13 cells: 6 chambers, 6 open rays and the origin.



Coxeter Complex

We distinguish an arbitrary chamber C in the cell decomposition.

Coxeter Complex

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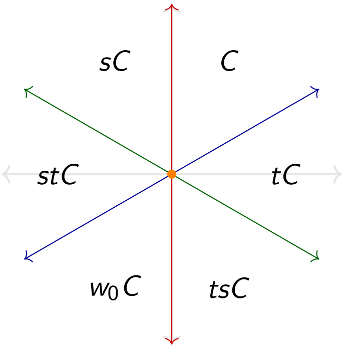

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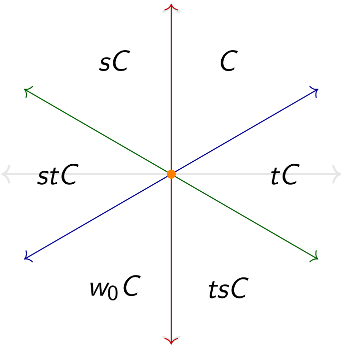

Note $w_0 = sts = tst$. By this definition, it is clear W acts transitively on the chambers and $|\mathcal{C}(\mathcal{H})| = |W|$.

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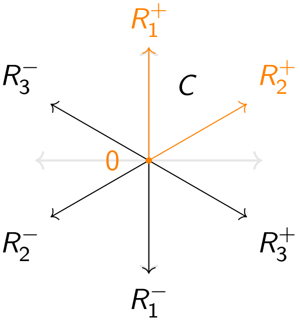
Example: Let C be the fundamental chamber, C, R_1^+, R_2^+, and 0 are the faces of C.

$$W_C = \{e\}$$

$$W_{R_1^+} = \{e, s\}$$

$$W_{R_2^+} = \{e, t\}$$

$$W_0 = W$$



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$$\phi : \Sigma_{\leq C} \xrightarrow{\sim} (\text{parabolic subgroups})^{\text{op}}$$

$$\begin{tikzcd}
 & C & & & & \{e\} & \\
R_1^+ \arrow[ur] & & R_2^+ \arrow[ul] \arrow[rr, "\phi", "\sim"'] & & \langle s \rangle \arrow[ur] & & \langle t \rangle \arrow[ul] \\
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We can extend this isomorphism (of posets) to the whole poset Σ by including cosets of parabolic subgroups $\hat{\phi} : \Sigma \xrightarrow{\sim} (\text{parabolic cosets})^{\text{op}}$.

Coxeter Complex

$$\begin{tikzcd}[column sep=tiny]
& stC & & sC & & C & & tC & & tsC & & w_0C \\
R_2^- \arrow[ur] & & R_3^- \arrow[ul] \arrow[ur] & & R_1^+ \arrow[ul] \arrow[ur] & & R_2^+ \arrow[ul] \arrow[ur] & & R_3^+ \arrow[ul] \arrow[ur] & & R_1^- \arrow[ul] \arrow[ur] & \\
& & & & & & 0 \arrow[ullllll] \arrow[ullll] \arrow[ull] \arrow[u] \arrow[urr] \arrow[urrrr] & & & & &
\end{tikzcd}$$

$$\Big\downarrow \hat{\phi} \; \wr$$

$$\begin{tikzcd}[column sep=tiny]
& \{st\} & & \{s\} & & \{e\} & & \{t\} & & \{ts\} & & \{w_0\} \\
st\langle s\rangle \arrow[ur] & & st\langle t\rangle \arrow[ul] \arrow[ur] & & \langle s\rangle \arrow[ul] \arrow[ur] & & \langle t\rangle \arrow[ul] \arrow[ur] & & ts\langle s\rangle \arrow[ul] \arrow[ur] & & ts\langle t\rangle \arrow[ul] \arrow[ur] & \\
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Therefore, using the isomorphism of posets $\hat{\phi}$, we can identify

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Definition: Any simplicial complex Δ isomorphic to $\Sigma(W, S)$ for some (W, S) Coxeter group (i.e. a finite reflection group) is a Coxeter complex.

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Definition of a Building

A *(weak) building* is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ (called *apartments*) satisfying axioms:

(**B0**) Each apartment Σ is a Coxeter complex.

(**B1**) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

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Remark: Coxeter complexes characterize thin buildings with a single apartment.

Buildings

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(4) All apartments are Coxeter complexes for the same Coxeter group.

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Note: If P is a poset, then a flag is a linearly ordered subset of P.

We will assume that P is partitioned into nonempty subsets P_0, P_1, ..., P_{n-1}, where $p \in P_i$ is said to have type i.

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Proof: S_n acts on each $\Sigma(\mathcal{F})$, each chain is contained in a composition series, and apply Jordan-Hölder theorem to obtain canonical isomorphisms and projections.

Buildings

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Therefore, we have seven 2-dimensional subspaces, which we will call "type 2" vertices. Note that each V_i contains exactly three subspaces.

$$V_1 = L_1 + L_2 = L_1 + L_4 = L_2 + L_4$$

Buildings

For each $\Sigma(\{L_i, L_j, L_k\})$ such that $V = L_i \oplus L_j \oplus L_k$, we obtain an apartment:

$$\begin{tikzcd}
 & L_i & \\
L_i + L_j \arrow[ur, dash] & & L_i + L_k \arrow[ul, dash] \\
L_j \arrow[u, dash] & & L_k \arrow[u, dash] \\
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Chambers are edges with a type 1 and type 2 vertex, S_3 acts by permuting the labels $\{i, j, k\}$.

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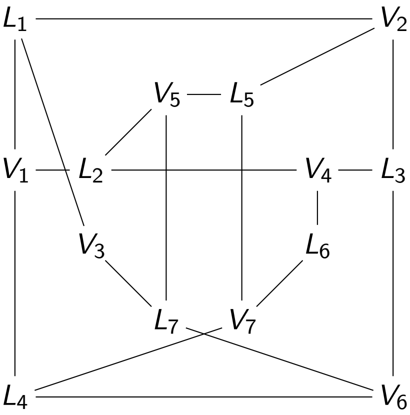

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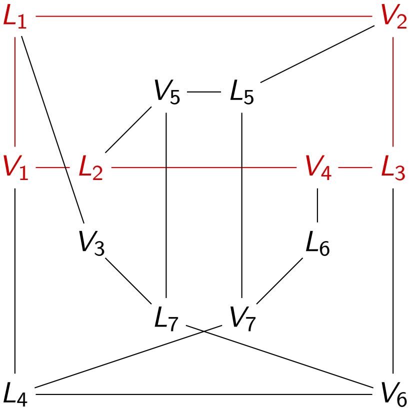

The apartment $\Sigma(\{L_1, L_2, L_3\})$ is shown above.

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Note: Any element of $\mathbb{Q}_p$ is of the form:

$$x = \sum_{i=k}^{\infty} a_i p^i$$

for $0 \leq a_i < p$. The valuation of x, is $\nu(x) = k$, where k is the smallest index such that $a_k \neq 0$.

Definition: A non-archimedean absolute value on a field F is a map

$$|-|: F \to \mathbb{R}$$

such that

(1) $|x| \geq 0$ for all $x \in F$, and $|x| = 0$ if and only if $x = 0$.

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such that

(1) $|x| \geq 0$ for all $x \in F$, and $|x| = 0$ if and only if $x = 0$.

(2) $|xy| = |x||y|$ for all $x, y \in F$

(3) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in F$.

Example: For $\mathbb{Q}_p$, the absolute value is defined by $|x|_p = p^{-\nu(x)}$.

Important Note: The image of $\nu(x)$ is an integer and thus $\nu(\mathbb{Q}_p^\times) = \mathbb{Z}$ is discrete. Or, in particular importance for us, $\log(|\mathbb{Q}_p^\times|)$ is discrete. And, $\mathcal{O} = \{x \in \mathbb{Q}_p \ : \ |x|_p \leq 1\}$.

Buildings

Instead of choosing the maximal compact subgroup, we look at tori. The torus will be

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;:\; a \in \mathbb{Q}_p^\times \right\}$$

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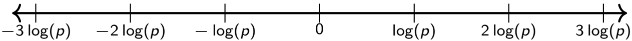
Let $A(T)$ be the apartment corresponding to T, it is an $\mathbb{R}$ affine space with simplicial structure defined by the map

$$\begin{aligned} T &\to \mathbb{R} \\ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} &\mapsto \log(|a|_p) \end{aligned}$$



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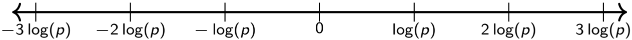

Our collection of vertices are of the form: $n\log(p)$ for some $n \in \mathbb{Z}$.

Buildings

We then take the collection of tori $\{gTg^{-1} \ : \ g \in \mathrm{SL}_2(\mathbb{Q}_p)\}$ and glue together the corresponding system of apartments

$$\mathcal{A} = \{A(gTg^{-1}) \ : \ g \in \mathrm{SL}_2(\mathbb{Q}_p)\}$$

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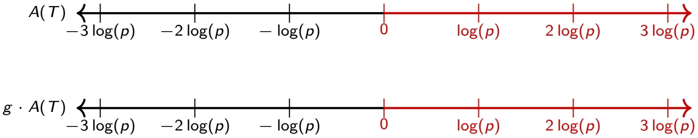
The stabilizers of each vertex is of the form

$$\mathrm{Stab}(\log(|a|_p)) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \mathrm{SL}_2(\mathcal{O}) \cdot \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

and for any two apartments $A(T)$ and $g \cdot A(T) = A(gTg^{-1})$, we glue at $n\log(p)$ if and only if $g \in \mathrm{Stab}(n\log(p))$.

Buildings

Example: Let $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For $a = up^{-n}$, note that $g \in \mathrm{Stab}(n\log(p))$ if and only if $|a|_p \geq 1$ if and only if $n\log(p) \geq 0$.



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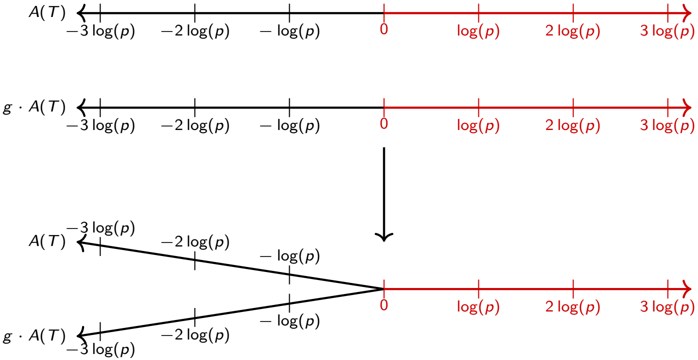
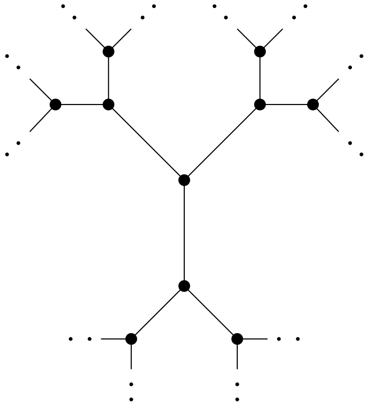


Assume that $p = 2$, then the building is given by:



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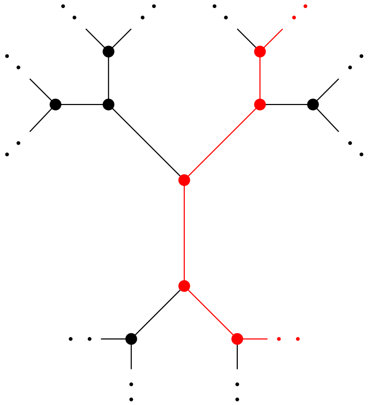
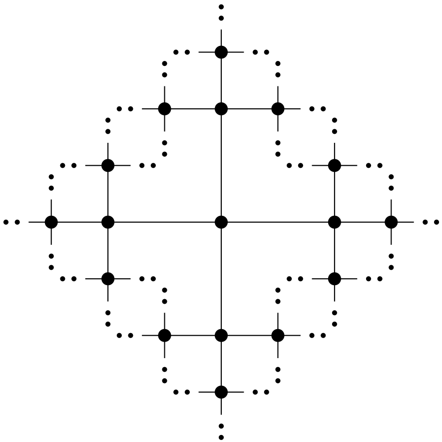
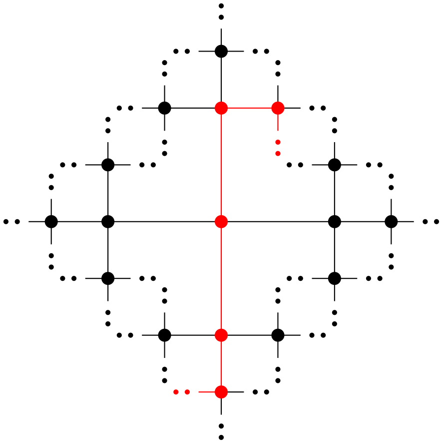


An apartment $A(gTg^{-1})$ is given by an infinite path along the tree.

Similarly for $p = 3$, then the building is given by:



Similarly for $p = 3$, then the building is given by:



Again, an apartment $A(gTg^{-1})$ is given by an infinite path along the tree.

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