Quiz (20 Questions)

① Prove that $\sqrt{3}$ is a irrational number

Sol^n

Let us assume $\sqrt{3}$ is rational

i.e, $\sqrt{3} = \frac{a}{b}$ where a & b are integers

a, b are co-prime

& HCF $(a, b) = 1$.

$$\sqrt{3} = \frac{a}{b}$$

By squaring both side

$$3 = \frac{a^2}{b^2} \text{ --- (1)}$$

$$a^2 = 3b^2$$

$$\frac{a^2}{3} = b^2$$

∴ As a^2 is divisible by 3

so, a also divisible by 3

Since a is divisible by 3

$$\therefore \frac{a}{3} = k$$

$$\boxed{a = 3k}$$

$$3 = \frac{9k^2}{b^2}$$

$$\frac{b^2}{3} = k^2$$

As b^2 is divisible by 3

so, b is also divisible by 3

$$\frac{b}{3} = \alpha$$

$$b = 3\alpha$$

$a = 3k$
$b = 3\alpha$

$\text{HCF}(a, b) = \text{HCF}(3k, 3\alpha) \geq 3$

$\therefore \sqrt{3}$ is irrational number.

(2) Prove that $\sqrt{6}$ is irrational number

(3) prove that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n} \quad n \in Z_+ \text{ and } n \geq 2$$

Sol^n:-

For $i = 2$

$$\text{LHS} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\text{RHS} = \frac{2+1}{4} = \frac{3}{4}$$

LHS = RHS

$\therefore$ For $i = 2$, it is true

~~For 3:~~

Let us assume that for $n = k$ it is true

i.e.,

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}$$

$$\left(1-\frac{1}{1^2}\right) * \left(1-\frac{1}{2^2}\right) * \cdots * \left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$$

For $n = k+1$

$$\underbrace{\left(1-\frac{1}{1^2}\right) * \left(1-\frac{1}{2^2}\right) * \cdots \left(1-\frac{1}{k^2}\right)} * \left(1-\frac{1}{(k+1)^2}\right)$$

$$\frac{k+1}{2k} * 1 - \frac{1}{(1+k)^2}$$

$$\cancel{= \frac{k+1+2k}{2k} - \frac{1}{(1+k)^2}}$$

$$\cancel{= \frac{3k+1}{2k} - \frac{1}{k^2+2k+1}}$$

$$\cancel{= \frac{(3k+1)(k^2+2k+1) - 2k}{2k(k^2+2k+1)}}$$

$$\cancel{= \frac{3k^3+6k^2+3k+k^2+2k+1-2k}{2k^3+4k^2+2k}}$$

$$\cancel{= \frac{3k^3+7k^2+3k+1}{2k^3+4k^2+2k}}$$

$$\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$\frac{\cancel{k+1}}{(2k)}\left(\frac{(k+1)^2-1}{(k+1)^{\cancel{2}}}\right)$$

$$= \frac{(k+1)^2-1}{2k(k+1)}$$

$$= \frac{k^2+2k+\cancel{1}-\cancel{1}}{2k(k+1)}$$

$$= \frac{\cancel{k}(k+2)}{2\cancel{k}(k+1)}$$

$$= \frac{(k+1)+1}{2(k+1)} = RHS.$$

$\therefore$ Hence proved

(4) Prove that :- $n! > 2^n$ for $\forall\ n \geq 4$

$\underline{\underline{sol^n}}$ For $n = 4$

LHS $= 4! = 4 \times 3 \times 2 \times 1$

$= 24$

RHS $= 2^4 = 2 \times 2 \times 2 \times 2$

$= 16$

LHS $>$ RHS

$\therefore$ For $n = 4$ it is true

Lets assume for $n = k$ it is true

i.e, $k! > 2^k$

i.e, $1 \times 2 \times 3 \times \cdots \times k > 2^k$

Now, For $n = k+1$

$(k+1)!$

$\underline{1 \times 2 \times 3 \times \cdots \times k} \times k+1$

$(\underline{k+1})\, 2^k$ (since $\underline{k!} > 2^k$)

$k!\,(k+1) > 2^k \times 2$

$k!\,(k+1) > 2^{k+1}$

$k!\,(k+1) > 2$

$(k+1)! > 2^{k+1}$ so, proved

(5) Prove that :-

Tribonacci sequence

$T_n = T_{n-1} + T_{n-2} + T_{n-3}$

for $\forall n \geq 4$

$\therefore T_1 = T_2 = T_3 = 1$

for $\forall n \leq 3$

prove that $T_n < 2^n$ for all $n \in \mathbb{Z}^+$

Sol^n

Base case

For $n = 1$

$T_1 < 2^n \Rightarrow 1 < 2$

$T_2 < 2^n \Rightarrow 1 < 4$

$T_3 < 2^n \Rightarrow 1 < 8$

$T_4 < 2^n \Rightarrow T_1 + T_2 + T_3 < 2^n$

$= \quad 3 < 2^4$

eq (1) is true

Let us assume for $n = k$

$T_k < 2^k$ —— (II)

For $n = k+1$

$T_n = T_{k+1} = T_k + T_{k-1} + T_{k-2}$

$T_k < 2^k$
$T_{k-1} < 2^{k-1}$
$T_{k-2} < 2^{k-2}$

$T_{k+1} < 2^k + 2^{k-1} + 2^{k-2}$

$T_{k+1} < 2^k \left[1 + \frac{1}{2} + \frac{1}{4} \right]$

$T_{k+1} < 2^k \left[\frac{8+2+1}{8} \right]$

$T_{k+1} < 2^k \left(\frac{11}{8} \right)$

$T_{k+1} < 2^{k+1} \left(\frac{11}{16} \right)$

$T_{k+1} < 2^{k+1}$ Hence Proved