LECTURE NOTES
ON
OPERATIONS RESEARCH
CSC 408

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UNIVERSITY OF CAPE COAST, GHANA

2021 EDITION

Contents

1. Introduction

Definition 1.1 Operations research (often referred to as management science) is simply a scientific approach to decision-making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.

By a **system**, we mean an organization of interdependent components that work together to accomplish the goal of the system. For example, Ford Motor Company is a system whose goal consists of maximizing the profit that can be earned by producing quality vehicles.

The term operations research was coined during World War II. It was initiated in England, when a team of British scientists set out to assess the best utilization of war materials based on scientific principles rather than on ad hoc rules. They used it to analyze several military problems such as the deployment of radar and the management of convoy. After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

The scientific approach to decision making usually involves the use of one or more mathematical models. A **mathematical model** is a mathematical representation of an actual situation that may be used to make better decisions or simply to understand the actual situation better.

The following sections describe some model categorizations.

1.1 Prescriptive or Optimization Models

Optimization models seek to select the best element, with regard to some criterion, from some set of available alternatives. These models prescribe for an organization ways that will enable them best meet its goal(s). The components of a prescriptive model include:

1. Objective function(s): In most models, there will be a function we wish to maximize or minimize. This function is called the model's objective function.

2. Decision variables: The variables whose values are under our control and influence the

performance of the system are called decision variables. We always seek to determine the value of decision variables that maximize (or minimize) an objective function.

3. Constraints: In most situations, only certain values of decision variables are possible. Restrictions on the values of decision variables are called constraints.

In short, an optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Example 1.1.1
Maximize

$$P = 30 + 2VC - 10TFC^2 - \frac{AVC}{24} \tag{1.1}$$

Subject to

$$0 \leq VC \leq 120 \tag{1.2}$$
$$50 \leq TFC \leq 97 \tag{1.3}$$
$$AVC = 200 \tag{1.4}$$

Equation (1.1) is the objective function
VC, TFC, AVC are the decision variables
Equations (1.2) to (1.4) are the constraints.

Definition 1.2 (Feasible Region)
Any specification of the decision variables that satisfies all of the model's constraints is said to be in the feasible region.
A solution is feasible if it satisfies all the constraints.

Definition 1.3 (Optimal Solution)
An optimal solution to an optimization model is any point in the feasible region that optimizes (in this case, maximizes) the objective function.
It is optimal if, in addition to being feasible, it yields the best (maximum or minimum) value of the objective function.

1.2 Static and Dynamic Models

A static model is one in which the decision variables do not involve sequences of decisions over multiple periods. A dynamic model is a model in which the decision variables do involve sequences of decisions over multiple periods.

Basically, in a static model we solve a "one-shot" problem whose solutions prescribe optimal values of decision variables at all points in time.

Example 1.2.1
Static model

$$P = 30 + 2VC - 10TFC^2 - \frac{AVC}{24}$$

Dynamic model

$$P_{t+1} = 30 + 2VC_t - 10TFC_t^2 - \frac{AVC_{t-1}}{24}$$

1.3 Linear and Nonlinear Models

Suppose that whenever decision variables appear in the objective function and in the constraints of an optimization model, the decision variables are always multiplied by constants and added together then we have a **linear model**. However, if an optimization model is not linear, then it is a nonlinear model.

Example 1.3.1
Linear model:
Maximize

$$P = 30 + 2VC - 10TFC - \frac{AVC}{24}$$

Subject to

$$0 \leq VC \leq 120$$
$$50 \leq TFC \leq 97$$
$$AVC + TFC = 200$$

Nonlinear model:
Maximize

$$P = 30 + 2VC - 10TFC^2 - \frac{AVC}{24}$$

Subject to

$$0 \leq VC \leq 120$$
$$50 \leq TFC \leq 97$$
$$AVC * VC = 200$$

Both TFC^2 and $AVC * VC$ make it nonlinear.

1.4 Integer and Noninteger Models

If one or more decision variables must be integer, then we say that an optimization model is an integer model. If all the decision variables are free to assume **fractional values**, then the optimization model is a **noninteger model**.

When quantities such as volume, temperature and pressure are input variables are do assume fractional values, then the resultant is a noninteger model. However, if the decision variables in a model represent the number of workers starting work during each shift at a fast-food restaurant, then clearly we have an integer model.

1.5 Deterministic and Stochastic Models

Suppose that for any value of the decision variables, the value of the objective function and whether or not the constraints are satisfied is known with certainty. We then have a deterministic model. If this is not the case, then we have a stochastic model.

Example 1.5.1
Deterministic

$$P = 30 + 2VC - 10TFC - \frac{AVC}{24}$$

Stochastic

$$P = 30 + 2VC - 10TFC - \frac{AVC}{24} + \epsilon$$

where ϵ is random term.

1.6 The Seven-Step Model-Building Process

When operations research is used to solve an organization's problem, the following seven step model-building procedure should be followed:

1. Formulate the Problem: The operations researcher first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization that must be studied before the problem can be solved.

2. Observe the System: The operations researcher collects data to estimate the value of parameters that affect the organization's problem.

3. Formulate a Mathematical Model of the Problem: The operations researcher develops a mathematical model of the problem.

4. Verify the Model and Use the Model for Prediction: The operations researcher now tries to determine if the mathematical model developed in step 3 is an accurate representation of reality.

5. Select a Suitable Alternative: Given a model and a set of alternatives, the operations researcher now chooses the alternative that best meets the organization's objectives. There are often more than one alternatives.

6. Present the Results and Conclusion of the Study to the Organization: In this step, the operations researcher presents the model and recommendation from step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the one that best meets its needs.

   After presenting the results of the study, the analyst may find that the organization does not approve of the recommendation. This may result from incorrect definition of the organization's problems or from failure to involve the decision maker from the start of the project. In this case, the operations researcher should return to step 1, 2, or 3.

7. Implement and Evaluate Recommendations: If the organization has accepted the study, then the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations enable the organization to meet its objectives.

In practice, operations research does not offer a single general technique for solving all mathematical models. Instead, the type and complexity of the mathematical model dictate the nature of the solution method.

The most prominent operations research technique is linear programming. It is designed for models with linear objective and constraint functions. Other techniques include integer programming (in which the variables assume integer values), dynamic programming (in which the original model can be decomposed into smaller more manageable subproblems), network programming (in which the problem can be modeled as a network), and nonlinear programming (in which functions of the model are nonlinear).

A peculiarity of most operations research techniques is that solutions are not generally obtained in (formula-like) closed forms. Instead, they are determined by **algorithms**. An algorithm provides fixed computational rules that are applied repetitively to the problem, with each repetition (called iteration) attempting to move the solution closer to the optimum.

Because the computations in each iteration are typically tedious and voluminous, it is imperative in practice to use the computer to carry out these algorithms.

Some mathematical models may be so complex that it becomes impossible to solve them by any of the available optimization algorithms. In such cases, it may be necessary to abandon the search for the optimal solution and simply seek a good solution using **heuristics or metaheuristics**, a collection of intelligent search rules of thumb that move the solution point advantageously toward the optimum.

2. Introduction to Linear Programming

2.1 Introduction

Linear programming (LP) is a tool for solving optimization problems. The chapter focuses on how to solve graphically two variable linear programming problems. We introduces linear programming and define important terms that are used to describe linear programming problems.

Definition 2.1 (LP)
A linear programming problem (LP) is an optimization problem for which we do the following:

1. We attempt to maximize (or minimize) a linear function of the decision variables. The function that is to be maximized or minimized is called the objective function.
2. The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear inequality.
3. A sign restriction is associated with each variable. To complete the formulation of a linear programming problem, the following question must be answered for each decision variable: Can the decision variable only assume nonnegative values, or is the decision variable allowed to assume both positive and negative values?

   If a decision variable x_i can only assume nonnegative values, then we add the sign restriction $x_i \geq 0$.

   If a variable x_i can assume both positive and negative (or zero) values, then we say that x_i is unrestricted in sign (often abbreviated **urs**).

Example 2.1.1 (Maximization's Problem)
Lamiz's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for ¢27 and uses ¢10 worth of raw materials. Each soldier that is manufactured increases Lamiz's variable labor and overhead costs by ¢14.
A train sells for ¢21 and uses ¢9 worth of raw materials. Each train built increases Lamiz's variable labor and overhead costs by ¢10.
The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor.
Each week, Lamiz can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Lamiz wants to maximize weekly profit (revenues -costs).
Formulate a mathematical model of Lamiz's situation that can be used to maximize Lamiz's weekly profit.

Solution

Decision variables

We begin by defining the relevant decision variables. In any linear programming model, the decision variables should completely describe the decisions to be made (in this case, by Lamiz). Clearly, Lamiz must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

- x_1 = number of soldiers produced each week
- x_2 = number of trains produced each week

Objective function

In any linear programming problem, the decision maker wants to maximize (usually revenue or profit) or minimize (usually costs) some function of the decision variables. The function to be maximized or minimized is called the objective function.

Lamiz's weekly revenues and costs can be expressed in terms of the decision variables x_1 and x_2.

$$\begin{aligned}\text{Weekly revenues} &= \text{weekly revenues from soldiers} + \text{weekly revenues from trains} && (2.1)\\ &= 27x_1 + 21x_2 && (2.2)\end{aligned}$$

Also

$$\begin{aligned}\text{Weekly raw material costs} &= 10x_1 + 9x_2 && (2.3)\\ \text{Other weekly variable costs} &= 14x_1 + 10x_2 && (2.4)\end{aligned}$$

$$\begin{aligned}\text{Profit} &= \text{Weekly revenue} - \text{Weekly costs} && (2.5)\\ &= 27x_1 + 21x_2 - [10x_1 + 9x_2] - [14x_1 + 10x_2] && (2.6)\\ &= 3x_1 + 2x_2 && (2.7)\end{aligned}$$

Lamiz's objective is to choose x_1 and x_2 to maximize $3x_1 + 2x_2$. We use the variable z to denote the objective function value of any LP. Thus, Lamiz's objective function is

$$\text{Maximize } z = 3x_1 + 2x_2$$

Constraints

As x_1 and x_2 increase, Lamiz's objective function grows larger. This means that if Lamiz were free to choose any values for x_1 and x_2, the company could make an arbitrarily large profit by choosing x_1 and x_2 to be very large. Unfortunately, the values of x_1 and x_2 are limited by the following three restrictions (often called constraints):

1. Constraint 1: Each week, no more than 100 hours of finishing time may be used.

$$\begin{aligned} \text{Total finishing hours (FH)} &= \text{Soldier FH} + \text{Trains FH} && (2.8) \\ &= 2x_1 + x_2 && (2.9) \end{aligned}$$

   Now Constraint 1 may be expressed by

$$2x_1 + x_2 \leq 100 \tag{2.10}$$

2. Constraint 2: Each week, no more than 80 hours of carpentry time may be used.

$$\begin{aligned} \text{Total Carpentry hours (CH)} &= \text{Soldier CH} + \text{Trains CH} && (2.11) \\ &= x_1 + x_2 && (2.12) \end{aligned}$$

   Now Constraint 1 may be expressed by

$$x_1 + x_2 \leq 80 \tag{2.13}$$

3. Constraint 3: Because of limited demand, at most 40 soldiers should be produced each week.

$$x_1 \leq 40 \tag{2.14}$$

4. For the Lamiz problem, it is clear that $x_1 \geq 0$ and $x_2 \geq 0$.

Therefore, combining the sign restrictions with the objective function and constraints yields the following optimization model:

$$\text{Maximize } z = 3x_1 + 2x_2 \tag{2.15}$$

Subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 100 && (2.16) \\ x_1 + x_2 &\leq 80 && (2.17) \\ x_1 &\leq 40 && (2.18) \\ x_1 &\geq 0 && (2.19) \\ x_2 &\geq 0 && (2.20) \end{aligned}$$

2.2 Feasible Region and Optimal Solution

Two of the most basic concepts associated with a linear programming problem are feasible region and optimal solution.

Definition 2.2 (Feasible Region)
The feasible region for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions.

For example, in the Lamiz problem, the point ($x_1 = 40$, $x_2 = 20$) is in the feasible region.

On the other hand, the point ($x_1 = 15$, $x_2 = 70$) is not in the feasible region, because it fails to satisfy equation (2.17): 15 + 70 is not less than or equal to 80. Any point that is not in an LP's feasible region is said to be an **infeasible point**.

Definition 2.3 (Optimal Solution)
For a maximization problem, an optimal solution to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value.

Most LPs have only one optimal solution. However, some LPs have no optimal solution, and some LPs have an infinite number of solutions .

2.3 The Graphical Solution of Two-Variable Linear Programming Problems

The graphical solution includes two steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the points in the solution space.

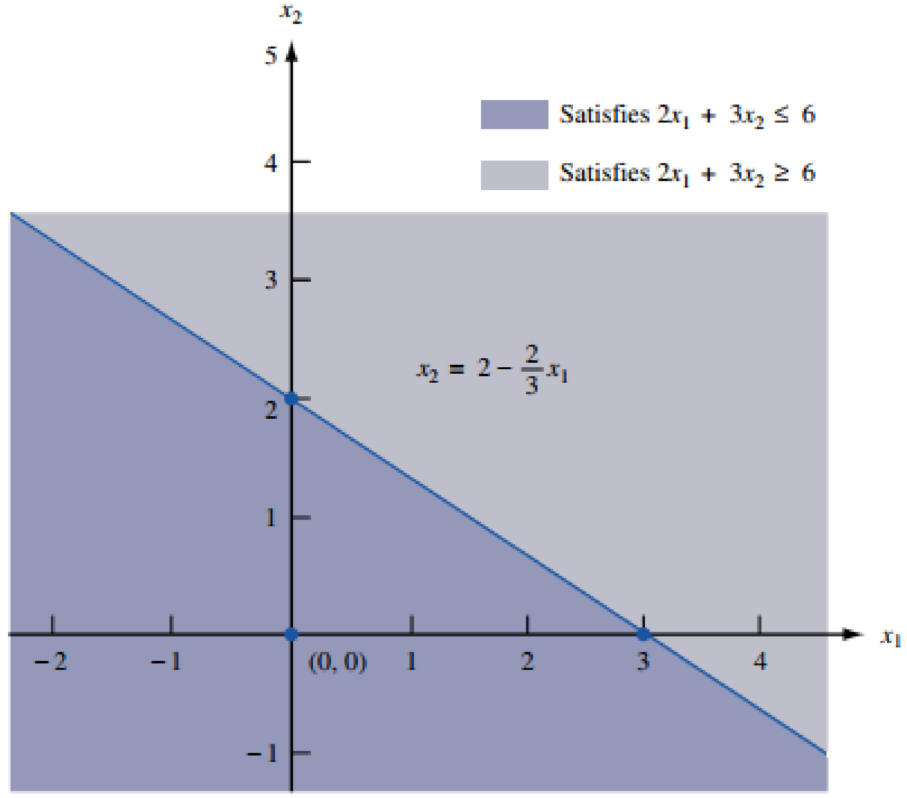
Consider a linear inequality constraint of the form $f(x_1, x_2) \geq b$ or $f(x_1, x_2) \leq b$. In general, it can be shown that in two dimensions, the set of points that satisfies a linear inequality includes the points on the line $f(x_1, x_2) = b$, defining the inequality plus all points on one side of the line.

There is an easy way to determine the side of the line for which an inequality such as $f(x_1, x_2) \leq b$ or $f(x_1, x_2) \geq b$ is satisfied. Just choose any point P that does not satisfy the line $f(x_1, x_2) \leq b$. Determine whether P satisfies the inequality. If it does, then all points on the same side as P of $f(x_1, x_2) = b$ will satisfy the inequality. If P does not satisfy the inequality, then all points on the other side of $f(x_1, x_2) = b$, which does not contain P, will satisfy the inequality.

For example, to determine whether

$$2x_1 + 3x_2 \geq 6$$

is satisfied by points above or below the line $2x_1 + 3x_2 = 6$, we note that $(0, 0)$ does not satisfy $2x_1 + 3x_2 \geq 6$. Because $(0, 0)$ is below the line $2x_1 + 3x_2 = 6$. This is illustrated with the figure below:



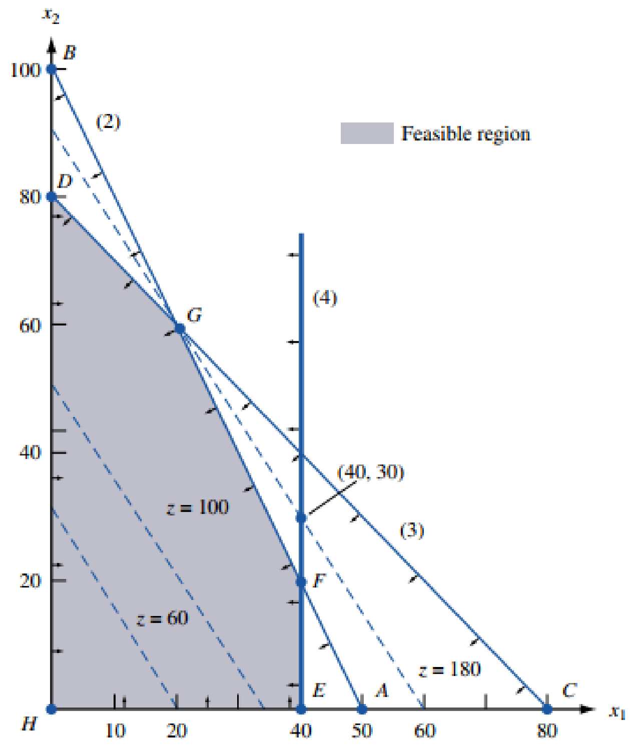
2.3.1 Finding the Feasible Solution

We illustrate how to solve two-variable LPs graphically by solving the Lamiz problem. To begin, we graphically determine the feasible region for Lamiz's problem. The feasible region for the Lamiz problem is the set of all points (x_1, x_2) satisfying

$$2x_1 + x_2 \leq 100 \tag{2.21}$$
$$x_1 + x_2 \leq 80 \tag{2.22}$$
$$x_1 \leq 40 \tag{2.23}$$
$$x_1 \geq 0 \tag{2.24}$$
$$x_2 \geq 0 \tag{2.25}$$

For a point (x_1, x_2) to be in the feasible region, (x_1, x_2) must satisfy all the inequalities equations (2.21) to (2.25).

Note that the only points satisfying equations (2.24) and (2.25) lie in the first quadrant of the $x_1 - x_2$ plane. Thus, any point that is outside the first quadrant cannot be in the feasible region. See figure below



We see that constraint equation (2.21) is satisfied by all points below or on the line AB. Inequality equation (2.22) is satisfied by all points on or below the line CD. Finally, equation (2.23) is satisfied by all points on or to the left of line EF.

We see that the set of points in the first quadrant that satisfies these inequalities is bounded by the five-sided polygon DGFEH. Any point on this polygon or in its interior is in the feasible region and infeasible otherwise. Thus, the point $(40, 30)$ is infeasible.

2.3.2 Finding the Optimal Solution

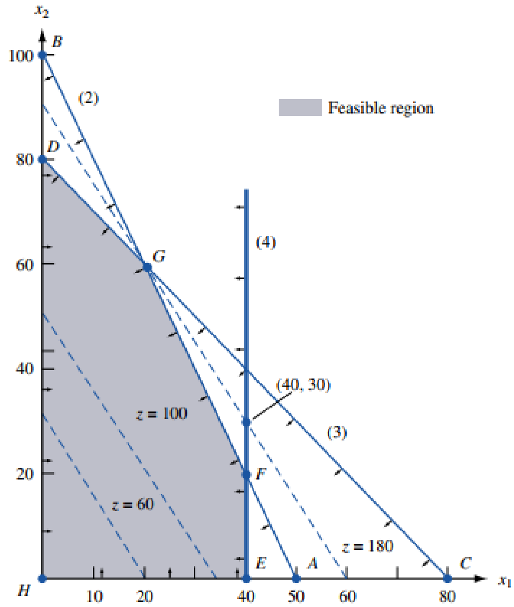
Having identified the feasible region for the Lamiz problem, we now search for the optimal solution, which will be the point in the feasible region with the largest value of $z = 3x_1 + 2x_2$.

To find the optimal solution, we need to graph a line on which all points have the same z-value. In a max problem, such a line is called an **isoprofit line** (in a min problem, an **isocost line**).

To draw an isoprofit line, we rewrite $3x_1 + 2x_2 = m$ as $x_2 = m/2 - 3/2x_1$. Hence the isoprofit line $3x_1 + 2x_2 = m$ has a slope of $-3/2$. Because all isoprofit lines are of the form $3x_1 + 2x_2 = constant$, all isoprofit lines have the same slope. This means that once we have drawn one isoprofit line, we can find all other isoprofit lines by moving parallel to the isoprofit line we have drawn. We choose any arbitrarily m value to draw the first isoprofit line. We let $m = 60$.

It is now clear how to find the optimal solution to a two-variable LP. After you have drawn a single isoprofit line, generate other isoprofit lines by moving parallel to the drawn isoprofit line in a direction that increases z (for a max problem). After a point, the isoprofit lines will no longer intersect the feasible region. The last isoprofit line intersecting (touching) the feasible region defines the largest z-value of any point in the feasible region and indicates the optimal solution to the LP.

In our problem, the objective function $z = 3x_1 + 2x_2$ will increase if we move in a direction for which both x_1 and x_2 increase. Thus, we construct additional isoprofit lines by moving parallel to $3x_1 + 2x_2 = 60$ in a northeast direction (upward and to the right). From,



we see that the isoprofit line passing through point G is the last isoprofit line to intersect the feasible region. Thus, G is the point in the feasible region with the largest z-value and is therefore the optimal solution to the Lamiz problem.

Note that point G is where the lines $2x_1 + x_2 = 100$ and $x_1 + x_2 = 80$ intersect. Solving these two equations simultaneously, we find that

$$(x_1 = 20,\ x_2 = 60)$$

is the optimal solution to the Lamiz problem. The optimal value of z may be found by substituting these values of x_1 and x_2 into the objective function. Thus, the optimal value of z is

$$z = 3(20) + 2(60) = 180$$

2.4 Binding and Nonbinding Constraints

Once the optimal solution to an LP has been found, it is useful to classify each constraint as being a binding constraint or a nonbinding constraint.

Definition 2.4 (Binding Constraint)
A constraint is binding if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint.

Thus, equations (2.21) and (2.22) are binding constraints. That is

$$2x_1 + x_2 \leq 100 \implies 2(20) + 60 \leq 100 \implies 100 = 100 \tag{2.26}$$

$$x_1 + x_2 \leq 80 \implies 20 + 60 \leq 80 \implies 80 = 80 \tag{2.27}$$

Definition 2.5 (Nonbinding Constraint)
A constraint is nonbinding if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.

Because $x_1 = 20$ is less than 40, equation (2.23) is a nonbinding constraint.

2.5 Convex Sets, and Extreme Points

Definition 2.6 (Convex Set)
A set of points S is a convex set if the line segment joining any pair of points in S is wholly contained in S.

Figure 2.1 gives four illustrations of this definition. In Figures a and b, each line segment joining two points in S contains only points in S. Thus, in both of these figures, S is convex.

In Figures c and d, S is not convex. In each figure, points A and B are in S, but there are points on the line segment AB that are not contained in S.

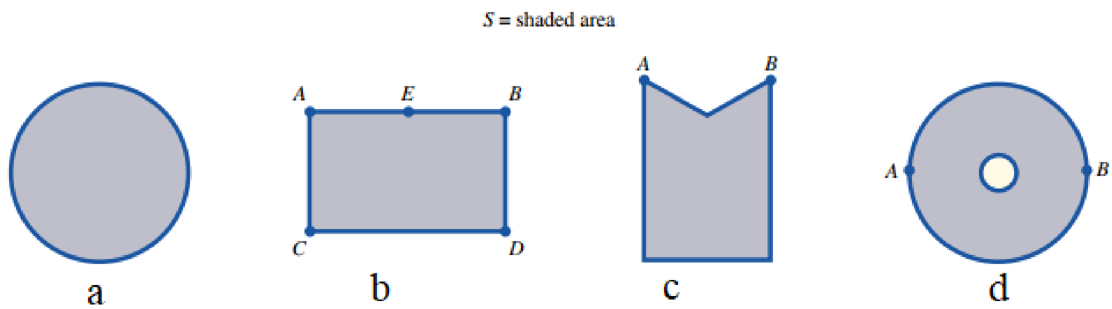


Figure 2.1:

Definition 2.7 (Extreme Point)
For any convex set S, a point P in S is an extreme point if each line segment that lies completely in S and contains the point P has P as an endpoint of the line segment.

In figure 2.1 a, each point on the circumference of the circle is an extreme point of the circle.

In b, points A, B, C, and D are extreme points of S. Although point E is on the boundary of S in b, E is not an extreme point of S. This is because E lies on the line segment AB (AB lies completely in S), and E is not an endpoint of the line segment AB. Extreme points are sometimes called **corner points**.

It can be shown that any LP that has an optimal solution has an extreme point that is optimal. This result is very important, because it reduces the set of points that yield an optimal solution from the entire feasible region (which generally contains an infinite number of points) to the set of extreme points (a finite set).

Therefore, for any LP, the largest z-value in the feasible region must be attained at an endpoint of one of the line segments forming the boundary of the feasible region.

Note 2.1.
An LP will always have an optimal extreme point when both the objective function and the constraints are linear functions. For a nonlinear objective function and constraints, the optimal solution to the optimization problem may not occur at an extreme point.

Example 2.5.1 (Minimization Problem)
Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and men. To reach these groups, Dorian Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercial spots on two types of programs: comedy shows and football games. Each comedy commercial is seen by 7 million high-income women and 2 million high income men. Each football commercial is seen by 2 million high-income women and 12 million high-income men. A 1-minute comedy ad costs ¢50,000, and a 1-minute football ad costs ¢100,000.
Dorian would like the commercials to be seen by at least 28 million high-income women and 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at minimum cost.

Solution
Dorian must decide how many comedy and football ads should be purchased, so the decision variables are

- x_1 = number of 1-minute comedy ads purchased
- x_2 = number of 1-minute football ads purchased

Then Dorian wants to minimize total advertising cost

$$\begin{aligned}\text{Total advertising cost} &= \text{cost of comedy ads} + \text{cost of football ads}\\ &= 50x_1 + 100x_2, \qquad \text{('000' dropped to be added later)}\end{aligned}$$

Dorian faces the following constraints:

Constraint 1: Commercials must reach at least 28 million high-income women (HIW).

$$\begin{aligned}\text{HIW ads} &= \text{HIW comedy ads} + \text{HIW football ads}\\ &= 7x_1 + 2x_2\end{aligned}$$

Constraint 1 may now be expressed as

$$7x_1 + 2x_2 \geq 28$$

Constraint 2: Commercials must reach at least 24 million high-income men.

$$\begin{aligned}\text{HIM ads} &= \text{HIM comedy ads} + \text{HIM football ads}\\ &= 2x_1 + 12x_2\end{aligned}$$

and Constraint 2 may be expressed as

$$2x_1 + 12x_2 \geq 24$$

Therefore Dorian LP is given by:

$$\begin{aligned}
\min\ z &= 50x_1 + 100x_2 && (2.28)\\
\text{s.t. } 7x_1 + 2x_2 &\geq 28 \quad \text{(HIW)} && (2.29)\\
2x_1 + 12x_2 &\geq 24 \quad \text{(HIM)} && (2.30)\\
x_1, x_2 &\geq 0 \quad \text{(sign restrictions)} && (2.31)
\end{aligned}$$

To solve this LP graphically, we begin by graphing the feasible region

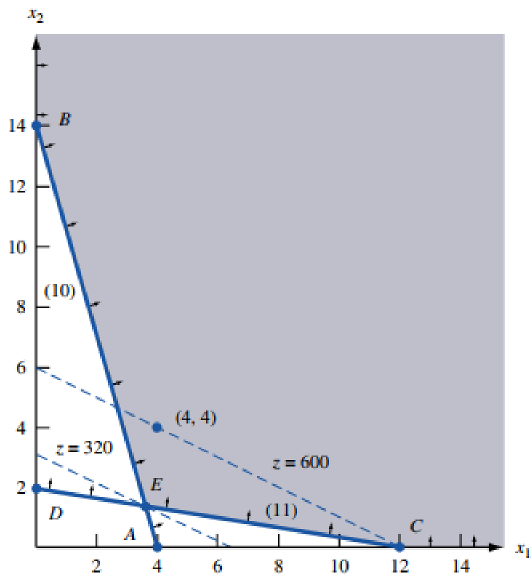


Figure 2.2:

From figure 2.2, we see that the only first-quadrant points satisfying both equations (2.29) and (2.30) are the points in the shaded region bounded by the x_1 axis, CEB, and the x_2 axis.

Like the Lamiz problem, the Dorian problem has a convex feasible region, but the feasible region for Dorian, unlike Lamiz's, contains points for which the value of at least one variable can assume arbitrarily large values. Such a feasible region is called an **unbounded feasible region**.

Because Dorian wants to minimize total advertising cost, the optimal solution to the problem is the point in the feasible region with the smallest z-value. To find the optimal solution, we need to draw an isocost line that intersects the feasible region.

We arbitrarily choose the isocost line passing through the point $(x_1 = 4, x_2 = 4)$. For this point, $z = 50(4) + 100(4) = 600$, and we graph the isocost line z= $50x_1 + 100x_2 = 600$.

We consider lines parallel to the isocost line $50x_1 + 100x_2 = 600$ in the direction of decreasing z (southwest). The last point in the feasible region that intersects an isocost line will be the point in the feasible region having the smallest z-value. From figure 2.2, we see that point E has the smallest z-value of any point in the feasible region; this is the optimal solution to the Dorian problem.

Note that point E is where the lines $7x_1+2x_2 = 28$ and $2x_1+12x_2 = 24$ intersect. Simultaneously solving these equations yields the optimal solution

$$(x_1 = 3.6,\ x_2 = 1.4)$$

The optimal z-value can then be found by substituting these values of x_1 and x_2 into the objective function. Thus, the optimal z-value is

$$z = 50(3.6) + 100(1.4) = 320 = 320,000$$

Because at point E both the HIW and HIM constraints are satisfied with equality, both constraints are binding.

Example 2.5.2 (Alternative or Multiple Optimal Solutions)
An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and body assembly shop. If the paint shop were only painting trucks, then 40 per day could be painted. If the paint shop were only painting cars, then 60 per day could be painted. If the body shop were only producing cars, then it could process 50 per day. If the body shop were only producing trucks, then it could process 50 per day.
Each truck contributes ¢300 to profit, and each car contributes ¢200 to profit. Use linear programming to determine a daily production schedule that will maximize the company's profits.

Solution
The company must decide how many cars and trucks should be produced daily. This leads us to define the following decision variables:

- x_1 = number of trucks produced daily
- x_2 = number of cars produced daily

The company's daily profit (in hundreds of cedis) is $3x_1 + 2x_2$, so the company's objective function may be written as

$$max\ z = 3x_1 + 2x_2$$

Constraint 1: The fraction of the day during which the paint shop is busy is less than or equal to 1. That is

$$\frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1$$

Constraint 2: The fraction of the day during which the body shop is busy is less than or equal to 1. That is

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1$$

Because $x_1 \geq 0$ and $x_2 \geq 0$ must hold, the relevant LP is

$$max\ z = 3x_1 + 2x_2 \tag{2.32}$$

$$st.\ \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1 \tag{2.33}$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1 \tag{2.34}$$

$$x_1, x_2 \geq 0 \tag{2.35}$$

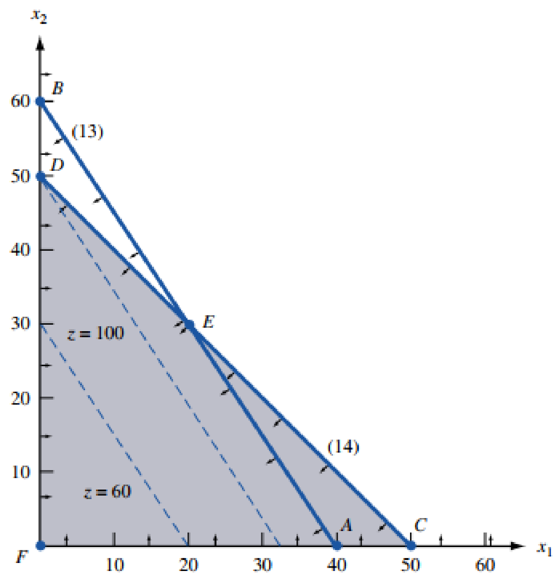


Figure 2.3:

The feasible region for this LP is the shaded region bounded by AEDF.

For our isoprofit line, we choose the line passing through the point $(20, 0)$. Because $(20, 0)$ has a z-value of $3(20) + 2(0) = 60$, this yields the isoprofit line $z = 3x_1 + 2x_2 = 60$.

Examining lines parallel to this isoprofit line in the direction of increasing z (northeast), we find that the last "point" in the feasible region to intersect an isoprofit line is the entire line segment AE. This means that any point on the line segment AE is optimal. We can use any point on AE to determine the optimal z-value. For example, point A, $(40, 0)$, gives $z = 3(40) = 120$.

In summary, the auto company's LP has an infinite number of optimal solutions, or multiple or alternative optimal solutions. The technique of **goal programming** is often used to choose among alternative optimal solutions.

Example 2.5.3 (Infeasible LP)
Suppose that auto dealers require that the auto company in example (2.5.2) produce at least 30 trucks and 20 cars. Find the optimal solution to the new LP.

Solution
After adding the constraints $x_1 \geq 30$ and $x_2 \geq 20$ to the LP of example (2.5.2), we obtain the following LP:

$$max\ z = 3x_1 + 2x_2 \tag{2.36}$$
$$st.\ \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1 \tag{2.37}$$
$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1 \tag{2.38}$$
$$x_1 \geq 30 \tag{2.39}$$
$$x_2 \geq 20 \tag{2.40}$$
$$x_1, x_2 \geq 0 \tag{2.41}$$

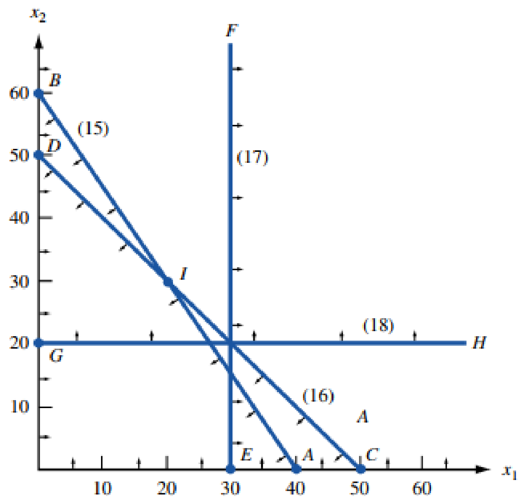


Figure 2.4:

Constraint (2.37) is satisfied by all points on or below AB
Constraint (2.38) is satisfied by all points on or below CD
Constraint (2.39) is satisfied by all points on or to the right of EF
Constraint (2.40) is satisfied by all points on or above GH.

From figure 2.4 it is clear that no point satisfies all of equations (2.37) to (2.40). Hence, we have an empty feasible region and is an infeasible LP. The LP is infeasible because producing 30 trucks and 20 cars requires more paint shop time than is available.

Exercise 2.1

1. Graphically find all optimal solutions to the following LP:

   (a)

$$\begin{aligned} min\ z &= x_1 - x_2 \\ \text{s.t. } x_1 + x_2 &\leq 6 \\ x_1 - x_2 &\geq 0 \\ x_2 - x_1 &\geq 3 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

   (b)

$$\begin{aligned} \min\ z &= 3x_1 + 5x_2 \\ \text{s.t. } 3x_1 + 2x_2 &\geq 36 \\ 3x_1 + 5x_2 &\geq 45 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

2. Leary Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs ¢4 and yields 3 units of A, 1 of B, and 1 of C. Running process 2 for an hour costs ¢1 and produces 1 unit of A and 1 of B. To meet customer demands, at least 10 units of A, 5 of B, and 3 of C must be produced daily. Graphically determine a daily production plan that minimizes the cost of meeting Leary Chemical's daily demands.

3. If an LP's feasible region is not unbounded, we say the LP's feasible region is bounded. Suppose an LP has a bounded feasible region. Explain why you can find the optimal solution to the LP (without an isoprofit or isocost line) by simply checking the z-values at each of the feasible region's extreme points. Why might this method fail if the LP's feasible region is unbounded?

4. Money manager Boris Milkem deals with French currency (the franc) and American currency (the dollar). At 12 midnight, he can buy francs by paying .25 dollars per franc and dollars by paying 3 francs per dollar. Let x_1= number of dollars bought (by paying francs) and x_2 = number of francs bought (by paying dollars). Assume that both types of transactions take place simultaneously, and the only constraint is that at 12:01 A.M. Boris must have a nonnegative number of francs and dollars.

   (a) Formulate an LP that enables Boris to maximize the number of dollars he has after all transactions are completed.

   (b) Graphically solve the LP and comment on the answer.

3. Simplex Algorithm

In Chapter 2, we saw how to solve two-variable linear programming problems graphically. Unfortunately, most real-life LPs have many variables, so a method is needed to solve LPs with more than two variables. We devote most of this chapter to a discussion of the simplex algorithm, which is used to solve even very large LPs. In many industrial applications, the simplex algorithm is used to solve LPs with thousands of constraints and variables.

In 1947, George Dantzig developed the simplex algorithm for solving linear programming problems. Since the development of the simplex algorithm, LP has been used to solve optimization problems in industries as diverse as banking, education, forestry, petroleum, and trucking.

3.1 How to Convert an LP to Standard Form

We have seen that an LP can have both equality and inequality constraints. It also can have variables that are required to be nonnegative as well as those allowed to be unrestricted in sign (urs). Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which all constraints are equations and all variables are nonnegative. An LP in this form is said to be in **standard form**.

To convert an LP into standard form, each inequality constraint must be replaced by an equality constraint. We illustrate this procedure using the following problem.

Definition 3.1 (Slack vrs Excess Variable)
If constraint i of an LP is a $\leq$ constraint, then we convert it to an equality constraint by adding a slack variable s_i to the ith constraint and adding the sign restriction $s_i \geq 0$
On the other hand, if the ith constraint of an LP is a $\geq$ constraint, then it can be converted to an equality constraint by subtracting an excess variable e_i from the ith constraint and adding the sign restriction $e_i \geq 0$

Example 3.1.1 (Slack Variable)

Leather Limited manufactures two types of belts: the deluxe model and the regular model. Each type requires 1 sq yd of leather. A regular belt requires 1 hour of skilled labor, and a deluxe belt requires 2 hours. Each week, 40 sq yd of leather and 60 hours of skilled labor are available. Each regular belt contributes ¢3 to profit and each deluxe belt, ¢4. If we define

- x_1 = number of deluxe belts produced weekly
- x_2= number of regular belts produced weekly

The appropriate LP is

$$
\begin{aligned}
max\ z &= 4x_1 + 3x_2 && (3.1)\\
s.t.\ x_1 + x_2 &\le 40 \qquad (Leather\ constraint) && (3.2)\\
2x_1 + x_2 &\le 60 \qquad (Labor\ constraint) && (3.3)\\
x_1, x_2 &\ge 0 && (3.4)
\end{aligned}
$$

How can we convert equations (3.2) and (3.3) to equality constraints?

We define for each $\le$ constraint a slack variable s_i, which is the amount of the resource unused in the ith constraint. Then the standard form is given as

$$
\begin{aligned}
max\ z &= 4x_1 + 3x_2 && (3.5)\\
s.t.\ x_1 + x_2 + s_1 &= 40 && (3.6)\\
2x_1 + x_2 + s_2 &= 60 && (3.7)\\
x_1, x_2, s_1, s_2 &\ge 0 && (3.8)
\end{aligned}
$$

Example 3.1.2 (Excess variable)

Convert the following LP problem to its standard form

$$min\ z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$
\begin{aligned}
s.t.\ 400x_1 + 200x_2 + 150x_3 + 500x_4 &\ge 500 && (3.9)\\
3x_1 + 2x_2 &\ge 6 && (3.10)\\
2x_1 + 2x_2 + 4x_3 + 4x_4 &\ge 10 && (3.11)\\
2x_1 + 4x_2 + x_3 + 5x_4 &\ge 8 && (3.12)\\
x_1, x_2, x_3, x_4 &\ge 0 && (3.13)
\end{aligned}
$$

Solution

To convert the ith $\ge$ constraint to an equality constraint, we define an excess variable (sometimes called a surplus variable) e_i. e_i is the amount by which the ith constraint is oversatisfied. Thus, the standard form is given as

$$min\ z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$
\begin{aligned}
s.t.\ 400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 &= 500 && (3.14)\\
3x_1 + 2x_2 - e_2 &= 6 && (3.15)\\
2x_1 + 2x_2 + 4x_3 + 4x_4 - e_3 &= 10 && (3.16)\\
2x_1 + 4x_2 + x_3 + 5x_4 - e_4 &= 8 && (3.17)\\
x_i, e_i \geq 0 \qquad i &= 1, 2, 3, 4. && (3.18)
\end{aligned}
$$

Note 3.1.
If an LP has both $\leq$ and $\geq$ constraints, then simply apply the procedures we have described to the individual constraints.

3.2 Basic and Nonbasic Variables

Consider a system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n &= b_3\\
\vdots \qquad\qquad\qquad \vdots \qquad & \quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n &= b_m
\end{aligned}
\tag{3.19}
$$

Equation (3.19) can be recast as

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots a_{1n}\\
a_{21} & a_{22} & a_{23} & \cdots a_{2n}\\
a_{31} & a_{32} & a_{33} & \cdots a_{3n}\\
\vdots & \vdots & \vdots & \vdots\\
a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn}
\end{bmatrix}
\begin{bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1\\ b_2\\ b_3\\ \vdots\\ b_m \end{bmatrix}
\tag{3.20}
$$

The $x_i's$ are the unknown to be determined. Thus, equation (3.20) is of the form

$$Ax = b \tag{3.21}$$

where

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots a_{1n}\\
a_{21} & a_{22} & a_{23} & \cdots a_{2n}\\
a_{31} & a_{32} & a_{33} & \cdots a_{3n}\\
\vdots & \vdots & \vdots & \vdots\\
a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn}
\end{bmatrix}, \qquad
x = \begin{bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n \end{bmatrix}, \qquad
\text{and } b = \begin{bmatrix} b_1\\ b_2\\ b_3\\ \vdots\\ b_m \end{bmatrix}
$$

Definition 3.2
A basic solution to $Ax = b$ is obtained by setting $n - m$ variables equal to 0 and solving for the values of the remaining m variables.

Given m linear equations in n variables and assume $n \geq m$

To find a basic solution to $Ax = b$, we choose a set of $n-m$ variables (the **nonbasic variables, or NBV**) and set each of these variables equal to 0. Then we solve for the values of the remaining $n-(n-m) = m$ variables (**the basic variables, or BV**) that satisfy $Ax = b$.

Of course, the different choices of nonbasic variables will lead to different basic solutions. To illustrate, we find all the basic solutions to the following system of two equations in three variables (3 variables, 2 equations):

$$x_1 + x_2 = 3 \tag{3.22}$$
$$-x_2 + x_3 = -1 \tag{3.23}$$

We begin by choosing a set of $3-2=1$ nonbasic variables. For example, if NBV is x_3, then BV are x_1, x_2. We obtain the values of the basic variables by setting $x_3 = 0$ and solving we find that $x_1 = 2,\ x_2 = 1$. Thus,

$$x_1 = 2,\ x_2 = 1,\ x_3 = 0$$

is a basic solution to equations (3.22) and (3.23).

However, if we choose NBV x_1 and BV x_2, x_3, we obtain the basic solution

$$x_1 = 0,\ x_2 = 3,\ x_3 = 2$$

If we choose NBV x_2, we obtain the basic solution

$$x_1 = 3,\ x_2 = 0,\ x_3 = -1$$

3.3 Feasible Solutions

Definition 3.3 (BFS)
Any basic solution to $Ax = b$ in which all variables are nonnegative is a basic feasible solution (or bfs).

Thus, the basic solution $x_1 = 2,\ x_2 = 1,\ x_3 = 0$ and $x_1 = 0,\ x_2 = 3,\ x_3 = 2$ are basic feasible solutions, while $x_1 = 3,\ x_2 = 0,\ x_3 = -1$ fails to be a basic solution (because $x_3 \leq 0$).

Theorem 3.1
A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

Definition 3.4
For any LP with m constraints, two basic feasible solutions are said to be adjacent if their sets of basic variables have $m-1$ basic variables in common.

3.4 The Simplex Algorithm

Here we describe how the simplex algorithm can be used to solve LPs in which the goal is to maximize the objective function. The simplex algorithm proceeds as follows:

1. Convert the LP to standard form.
2. Obtain a basic feasible solution **bfs** (if possible) from the standard form.
3. Determine whether the current bfs is optimal.

   For maximization problem, if all nonbasic variables have nonnegative coefficients in row 0, then the current bfs is optimal.

   For minimization problem, if all nonbasic variables have nonpositive coefficients in row 0, then the current bfs is optimal.
4. If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable.

   For maximization problem, choose the variable with the most negative coefficient in row 0 to enter the basis.

   For minimization problem, choose the variable with the most positive coefficient in row 0 to enter the basis.
5. Determine which basic variable should become a nonbasic variable using elementary row operations (ERO) and the ratio test.
6. Find a new bfs with a better objective function value. Go back to step 3

In performing the simplex algorithm, write the objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

in the form

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0$$

We call this format the row 0 version of the objective function (row 0 for short)

Example 3.4.1
The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given the Table below:

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for ¢60, a table for ¢30, and a chair for ¢20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, Dakota wants to maximize total revenue.

Solution
Decision variables
x_1 = number of desks produced
x_2 = number of tables produced

x_3 = number of chairs produced

Then the LP problem is

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$
\begin{aligned}
s.t. \quad & 8x_1 + 6x_2 + x_3 \leq 48 && \text{(Lumber constraint)} && (3.24)\\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 && \text{(Finishing constraint)} && (3.25)\\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 && \text{(Carpentry constraint)} && (3.26)\\
& x_2 \leq 5 && \text{(Limitation on table demand)} && (3.27)\\
& x_1, x_2, x_3 \geq 0 && && (3.28)
\end{aligned}
$$

Iteration 0

Step 1

Now we convert the constraints of the LP to the standard form and convert the LP's objective function to the row 0 format.

$$z - 60x_1 - 30x_2 - 20x_3 = 0 \tag{3.29}$$

Putting equations (3.24) to (3.27) and (3.29) and the sign restrictions yields the equations and basic variables shown below

Canonical Form 0

Row		Basic Variable
0	$z - 60x_1 - 30x_2 - 20x_3 = 0$	$z = 0$
1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1 = 48$
2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2 = 20$
3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3 = 8$
4	$x_2 + s_4 = 5$	$s_4 = 5$

Step 2

After obtaining a canonical form, we therefore search for the initial bfs. By inspection, we see that if we set $x_1 = x_2 = x_3 = 0$, we can solve for the values of s_1, s_2, s_3, and s_4 by setting s_i equal to the right-hand side of row i.

Observe that each basic variable may be associated with the row of the canonical form in which the basic variable has a coefficient of 1.

With this convention, the basic feasible solution for our initial canonical form has

$$BV = \{z, s_1, s_2, s_3, s_4\} \quad \text{and} \quad NBV = \{x_1, x_2, x_3\}$$

For this basic feasible solution

$$z = 0, \ s_1 = 48, \ s_2 = 20, \ s_3 = 8, \ s_4 = 5, \ x_1 = x_2 = x_3 = 0$$

Step 3: Is the Current Basic Feasible Solution Optimal?

Once we have obtained a basic feasible solution, we need to determine whether it is optimal; if the bfs is not optimal, then we try to find a bfs adjacent to the initial bfs with a larger z-value. To do this, we try to determine whether there is any way that z can be increased by increasing some nonbasic variable from its current value of zero while holding all other nonbasic variables at their current values of zero. From equation (3.30) $z = 0$ can be increased by increasing either $x_1,\ x_2$ or x_3. Hence the current solution is not optimal.

$$z = 60x_1 + 30x_2 + 20x_3 \tag{3.30}$$

Similarly, the current bfs is not optimal because the variables in row 0 (3.31) have negative coefficients.

$$\text{row } 0 \implies z - 60x_1 - 30x_2 - 20x_3 = 0 \tag{3.31}$$

Step 4: Determine the Entering Variable
We choose the entering variable (in a max problem) to be the nonbasic variable with the **most negative coefficient in row 0**.

$$z - 60x_1 - 30x_2 - 20x_3 = 0 \tag{3.32}$$

We could observe that x_1 has the most negative coefficient in row 0. Thus, x_1 is the entering variable.

Step 5: In Which Row Does the Entering Variable Become Basic?
When entering a variable into the basis, compute the ratio for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called **the winner of the ratio test**. The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. That is

$$\begin{aligned}
&\text{Row 1 limit on } x_1 = \frac{48}{8} = 6, && \text{so } s_1 \geq 0 \text{ when } x_1 \leq 6\\
&\text{Row 2 limit on } x_1 = \frac{20}{4} = 5 && \text{so } s_2 \geq 0 \text{ when } x_1 \leq 5\\
&\text{Row 3 limit on } x_1 = \frac{8}{2} = 4 && \text{so } s_3 \geq 0 \text{ when } x_1 \leq 4\\
&\text{Row 4 limit on } x_1 = \text{no limit} && \text{so } s_4 \geq 0 \text{ for all values of } x_1
\end{aligned}$$

Now, row 3 is the winner of the ratio test for entering x_1 into the basis.

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

To make x_1 a basic variable in row 3, we use elementary row operations to make x_1 have a coefficient of 1 in row 3 and a coefficient of 0 in all other rows. This procedure is called pivoting on row 3; and row 3 is the pivot row. The final result is that x_1 replaces s_3 as the basic variable for row 3.

Step 6
This yields the new canonical form 1 as below

Canonical Form 1

Row		Basic Variable
Row 0′	$z + 15x_2 - 5x_3 + 30s_3 = 240$	$z = 240$
Row 1′	$-x_3 + s_1 - 4s_3 = 16$	$s_1 = 16$
Row 2′	$-x_2 + 0.5x_3 + s_2 - 2s_3 = 4$	$s_2 = 4$
Row 3′	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4$	$x_1 = 4$
Row 4′	$x_2 + s_4 = 5$	$s_4 = 5$

Looking for a basic variable in each row of the current canonical form, we find that

$$BV = \{z, s_1, s_2, x_1, s_4\} \quad \text{and} \quad NBV = \{s_3, x_2, x_3\}$$

For this basic feasible solution

$$z = 240,\ s_1 = 16,\ s_2 = 4,\ x_1 = 4,\ s_4 = 5,\ s_3 = x_2 = x_3 = 0$$

Step 3: Is new bfs optimal?

We now try to find a bfs that has a still larger z-value. We begin by examining canonical form 1 to see if we can increase z by increasing the value of some nonbasic variable (while holding all other nonbasic variables equal to zero). Rearranging row 0' to solve for z yields

$$z = 240 - 15x_2 + 5x_3 - 30s_3 \tag{3.33}$$

Clearly z can be increased by increasing x_3. Hence the current solution is not optimal.

Similarly, the current bfs is not optimal because a variable x_3 in row $0'$ (3.34) have negative coefficients.

$$\text{row } 0' \implies z + 15x_2 - 5x_3 + 30s_3 = 240 \tag{3.34}$$

Iteration 1

Step 4: Determine the Entering Variable

From equation (3.33), we see that increasing the nonbasic variable x_2 by 1 with other variables as zeros will decrease z by 15. We don't want to do that.

Increasing the nonbasic variable s_3 by 1 with other variables as zeros will decrease z by 30. Again, we don't want to do that.

On the other hand, increasing x_3 by 1 with other variables as zeros will increase z by 5. Thus, we choose to enter x_3 into the basis.

Again x_3 is the only variable with negative coefficient from the equation

$$z + 15x_2 - 5x_3 + 30s_3 = 240$$

Step 5

$$\begin{array}{rll}
s_1 = 16 + x_3 \implies & \text{Row 1' limit on } x_3 = \text{no limit} & \text{so } s_1 \geq 0 \text{ for all values of } x_3 \\
s_2 = 4 - 0.5x_3 \implies & \text{Row 2' limit on } x_3 = \dfrac{4}{0.5} = 8 & \text{so } s_2 \geq 0 \text{ when } x_3 \leq 8 \\
x_1 = 4 - 0.25x_3 \implies & \text{Row 3' limit on } x_3 = \dfrac{4}{0.25} = 16 & \text{so } x_1 \geq 0 \text{ when } x_3 \leq 16 \\
s_4 = 5 \implies & \text{Row 4' limit on } x_3 = \text{no limit} & \text{so } s_4 \geq 0 \text{ for all values of } x_1
\end{array}$$

Now, row 2' is the winner of the ratio test for entering x_3 into the basis.

This means that we should use EROs to make x_3 a basic variable in row 2'. This implies that s_2 will leave the basis.

Step 6

This yields the new canonical form 2 as below

Canonical Form 2

Row		Basic Variable
0″	$z + 5x_2 + 10s_2 + 10s_3 = 280$	$z = 280$
1″	$-2x_2 + s_1 + 2s_2 - 8s_3 = 24$	$s_1 = 24$
2″	$-2x_2 + x_3 + 2s_2 - 4s_3 = 8$	$x_3 = 8$
3″	$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$	$x_1 = 2$
4″	$x_2 + s_4 = 5$	$s_4 = 5$

Looking for a basic variable in each row of the current canonical form, we find that

$$BV = \{z, s_1, x_3, x_1, s_4\} \quad \text{and} \quad NBV = \{s_2, s_3, x_2\}$$

For this basic feasible solution

$$z = 280,\ s_1 = 24,\ x_3 = 8,\ x_1 = 2,\ s_4 = 5,\ s_2 = s_3 = x_2 = 0$$

Step 3

We now try to find a bfs that has a still larger z-value. Rearranging row 0" to solve for z yields

$$z = 280 - 5x_2 - 10s_2 - 10s_3 \tag{3.35}$$

From equation (3.35), we see that increasing x_2 by 1 will decrease z by 5; increasing s_2 by 1 will decrease z by 10; increasing s_3 by 1 will decrease z by 10. Thus, increasing any nonbasic variable will cause z to decrease. This might lead us to believe that our current bfs from canonical form 2 is an optimal solution.

Our current bfs from canonical form 2 is

$$\begin{aligned} z &= 60x_1 + 30x_2 + 20x_3 && (3.36)\\ &= 60(2) + 30(0) + 20(8) && (3.37)\\ &= 120 + 160 && (3.38)\\ &= 280 && (3.39) \end{aligned}$$

3.5 Using the Simplex Algorithm to Solve Minimization Problems

There are two different ways that the simplex algorithm can be used to solve minimization problems. We illustrate these methods by solving the following LP:

$$\min z = 2x_1 - 3x_2 \tag{3.40}$$
$$\text{s.t. } x_1 + x_2 \leq 4 \tag{3.41}$$
$$x_1 - x_2 \leq 6 \tag{3.42}$$
$$x_1,\ x_2 \geq 0 \tag{3.43}$$

Solution Method 1: Converting to a maximum problem
This method multiplies the objective function for the min problem by -1 and solves the problem as a maximization problem with objective function $-z$. The optimal solution to the max problem will give you the optimal solution to the min problem.

The optimal solution to this LP problem is the point (x_1, x_2) in the feasible region for the LP that makes $z = 2x_1 - 3x_2$ the smallest.

Equivalently, we may say that the optimal solution to this LP is the point in the feasible region that makes $-z = -2x_1 + 3x_2$ the largest. This means that we can find the optimal solution to the new LP as

$$\max\ -z = -2x_1 + 3x_2 \tag{3.44}$$
$$\text{s.t. } x_1 + x_2 \leq 4 \tag{3.45}$$
$$x_1 - x_2 \leq 6 \tag{3.46}$$
$$x_1,\ x_2 \geq 0 \tag{3.47}$$

After adding slack variables s_1 and s_2 to the two constraints, we obtain the initial tableau as

$-z$	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	2	-3	0	0	0	$-z = 0$	
0	1	1	1	0	4	$s_1 = 4$	$\frac{4}{1} = 4^*$
0	1	-1	0	1	6	$s_2 = 6$	None

Because x_2 is the only variable with a negative coefficient in row 0, we enter x_2 into the basis.

The ratio test indicates that x_2 should enter the basis in row 1. The resulting tableau is shown below

$-z$	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	5	0	3	0	12	$-z = 12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Because each variable in row 0 has a nonnegative coefficient, this is an optimal tableau. Thus, the optimal solution is

$$-z = 12,\ x_2 = 4,\ s_2 = 10,\ x_1 = s_1 = 0$$

That is

$$-z = -2x_1 + 3x_2 \tag{3.48}$$
$$= -2(0) + 3(4) \tag{3.49}$$
$$= 12 \tag{3.50}$$

Method 2

A simple modification of the maximum simplex algorithm can be used to solve minimum problems.

In a minimum case the current bfs is optimal if all nonbasic variables in row 0 have nonpositive coefficients.

Here if any nonbasic variable in row 0 has a positive coefficient, choose the variable with the 'most positive' coefficient in row 0 to enter the basis.

If we use this method to solve this LP, then our initial tableau is given as

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−2	3	0	0	0	$z = 0$	
0	1	1	1	0	4	$s_1 = 4$	$\frac{4}{1} = 4$*
0	1	−1	0	1	6	$s_2 = 6$	None

The pivot term is encircled and the winner of the ratio test is denoted by *.

Because x_2 has the most positive coefficient in row 0, we enter x_2 into the basis.

The ratio test indicates that x_2 should enter the basis in the first constraint, row 1. The resulting tableau is shown below

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	−5	0	−3	0	−12	$z = -12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Because each variable in row 0 has a nonpositive coefficient, this is an optimal tableau. Thus, the optimal solution is

$$z = -12,\ x_2 = 4,\ s_2 = 10,\ x_1 = s_1 = 0$$

That is

$$z = 2x_1 - 3x_2 \tag{3.51}$$
$$= 2(0) - 3(4) \tag{3.52}$$
$$= -12 \tag{3.53}$$

3.6 LP with Alternative Optimal Solutions

We indicated earlier that some LPs have more than one optimal extreme point. If an LP has more than one optimal solution, then we say that it has multiple or alternative optimal solutions. We show now how the simplex algorithm can be used to determine whether an LP has alternative optimal solutions.

Example 3.6.1
The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given the Table below:

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for ¢60, a table for ¢35, and a chair for ¢20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, Dakota wants to maximize total revenue.

Solution
The only modification is that tables sell for ¢35 instead of ¢30

The new tableau is as below

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	−60	−35	−20	0	0	0	0	0	$z=0$	
0	8	6	1	1	0	0	0	48	$s_1=48$	$\frac{48}{8}=6$
0	4	2	1.5	0	1	0	0	20	$s_2=20$	$\frac{20}{4}=5$
0	2	1.5	0.5	0	0	1	0	8	$s_3=8$	$\frac{8}{2}=4$*
0	0	1	0	0	0	0	1	5	$s_4=5$	None

Because x_1 has the most negative coefficient in row 0, we enter x_1 into the basis. The ratio test indicates that x_1 should be entered in row 3.

The new[2] tableau is as below

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	0	10	−5	0	0	30	0	240	$z=240$	
0	0	0	−1	1	0	−4	0	16	$s_1=16$	None
0	0	−1	0.5	0	1	−2	0	4	$s_2=4$	$\frac{4}{0.5}=8$*
0	1	0.75	0.25	0	0	0.5	0	4	$x_1=4$	$\frac{4}{0.25}=16$
0	0	1	0	0	0	0	1	5	$s_4=5$	None

Now only x_3 has a negative coefficient in row 0, so we enter x_3 into the basis. The ratio test indicates that x_3 should enter the basis in row 2

The new[3] tableau is as below

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	0	−2	0	1	2	−8	0	24	$s_1 = 24$
0	0	−2	1	0	2	−4	0	8	$x_3 = 8$
0	1	1.25	0	0	−0.5	1.5	0	2	$x_1 = 2$*
0	0	1	0	0	0	0	1	5	$s_4 = 5$

The resulting is optimal.

$$z = 280,\ s_1 = 24,\ x_3 = 8,\ x_1 = 2, s_4 = 5,\ \text{ and }\ x_2 = s_2 = s_3 = 0$$

Recall that all basic variables must have a zero coefficient in row 0 (or else they wouldn't be basic variables). However, in our optimal tableau, there is a nonbasic variable, x_2, which also has a zero coefficient in row 0. Let us see what happens if we enter x_2 into the basis.

The ratio test indicates that x_2 should enter the basis in row 3.

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	1.6	0	0	1	1.2	−5.6	0	27.2	$s_1 = 27.2$
0	1.6	0	1	0	1.2	−1.6	0	11.2	$x_3 = 11.2$
0	0.8	1	0	0	−0.4	1.2	0	1.6	$x_2 = 1.6$
0	−0.8	0	0	0	0.4	−1.2	1	3.4	$s_4 = 3.4$

The important thing to notice is that because x_2 has a zero coefficient in the optimal tableau's row 0, the pivot that enters x_2 into the basis does not change row 0. This means that all variables in our new row 0 will still have nonnegative coefficients. Thus, our new tableau is also optimal. Because the pivot has not changed the value of z, an alternative optimal solution for the Dakota example is

$$z = 280,\ s_1 = 27.2,\ x_3 = 11.2,\ x_2 = 1.6,\ s_4 = 3.4,\ \text{and}\ x_1 = s_3 = s_2 = 0$$

Thus, Dakota has multiple (or alternative) optimal extreme points.

Definition 3.5 (Degeneracy)
An LP is degenerate if it has at least one bfs in which a basic variable is equal to zero.

A Degenerate LP

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−5	−2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	−1	0	1	0	$s_2 = 0$	0*

In this bfs, the basic variable $s_2 = 0$. Thus, the LP that generated this tableau is a degenerate LP.
Degeneracy can cause the simplex iterations to cycle indefinitely, thus never terminating the algorithm. The condition also reveals the possibility of at least one redundant constraint.

Exercise 3.1
Use the simplex algorithm to find the optimal solution to the following LP:

1.

$$\begin{aligned} \max\ z &= 2x_1 + 3x_2 \\ \text{s t } x_1 + 2x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 8 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

2.

$$\begin{aligned} \max\ z &= x_1 + x_2 \\ \text{s t } 4x_1 + x_2 &\leq 100 \\ x_1 + x_2 &\leq 80 \\ x_1 &\leq 40 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

3.

$$\begin{aligned} \min\ z &= 4x_1 - x_2 \\ \text{s t } 2x_1 + x_2 &\leq 8 \\ x_2 &\leq 5 \\ x_1 - x_2 &\leq 4 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

4.

$$\begin{aligned} \min\ z &= -x_1 - x_2 \\ \text{s t } x_1 - x_2 &\leq 1 \\ x_1 + x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

5. Find alternative optimal solutions to the following LP:

$$\begin{aligned} \max\ z = x_1 + x_2 \\ \text{s.t. } x_1 + x_2 + x3 \leq 1 \\ x_1 + 2x_3 \leq 1 \\ x_i \geq 0 \end{aligned}$$

6. How many optimal basic feasible solutions does the following LP have?

$$\begin{aligned} \max\ z\ 2x_1 + 2x_2 \\ \text{s.t. } x_1 + x_2 \leq 6 \\ 2x_1 + x_2 \leq 13 \\ x_i \geq 0 \end{aligned}$$

3.7 Big M Method

Recall that the simplex algorithm requires a starting bfs. In all the problems we have solved so far, we found a starting bfs by using the slack variables as our basic variables. If an LP has any $\geq$ or equality constraints, however, a starting bfs may not be readily apparent. When a bfs is not readily apparent, the Big M method (or the two-phase simplex method) may be used to solve the problem.

3.7.1 Description of Big M Method

1. Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed.

2. Convert each inequality constraint to standard form. This means that if constraint i is a $\leq$ constraint, we add a slack variable s_i, and if constraint i is a $\geq$ constraint, we subtract an excess variable e_i.

3. After step 1 has been completed, if constraint i is a $\geq$ or $=$ constraint, add an **artificial variable** a_i. Also add the sign restriction $a_i \geq 0$.

4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function.

   If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.

   When an artificial variable leaves the basis, its column may be dropped from future tableaus because the purpose of an artificial variable is only to get a starting basic feasible solution.

5. Because each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. This ensures that we begin with a canonical form. In choosing the entering variable, remember that M is a very large

positive number. For example, $4M - 2$ is more positive than $3M - 900$, and $-6M - 5$ is more negative than $-5M - 40$.

Now solve the transformed problem by the simplex method. If all artificial variables are equal to zero in the optimal solution, then we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, then the original problem is infeasible.

Example 3.7.1
Bevco manufactures an orange-flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 ounce(oz) of sugar and 1 milligram(mg) of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco ¢2 to produce an ounce of orange soda and ¢3 to produce an ounce of orange juice. Bevco's marketing department has decided that each 10-oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use linear programming to determine how Bevco can meet the marketing department's requirements at minimum cost.

Solution
Let
$x_1 =$ number of ounces of orange soda in a bottle of Oranj
$x_2 =$ number of ounces of orange juice in a bottle of Oranj

Then the appropriate LP is

$$\begin{aligned}
\min z &= 2x_1 + 3x_2 & \\
\text{such that } \frac{1}{2}x_1 + \frac{1}{4}x_2 &\leq 4 & \text{Sugar constraint} \\
x_1 + 3x_2 &\geq 20 & \text{Vitamin C constraint} \\
x_1 + x_2 &= 10 & \text{10 oz in bottle of Oranj} \\
x_1, x_2 &\geq 0 &
\end{aligned}$$

Step 1
Because none of the constraints has a negative right-hand side, we don't have to multiply any constraint through by -1.

Step 2
Add a slack variable s_1 to row 1 and subtract an excess variable e_2 from row 2. The result is

$$\begin{aligned}
\min z &= 2x_1 + 3x_2 \\
\text{Row 1: } &\frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4 \\
\text{Row 2: } &x_1 + 3x_2 - e_2 = 20 \\
\text{Row 3: } &x_1 + x_2 = 10
\end{aligned}$$

Step 3
In searching for a bfs, we see that $s_1 = 4$ could be used as a basic (and feasible) variable for row 1, and $e_1 = -20$ could be used as a basic variable for row 2. Unfortunately, $e_2 = -20$ violates the sign restriction $e_2 = 0$. Finally, in row 3 there is no readily apparent basic variable.

To remedy this problem, we simply 'invent' a basic feasible variable for each constraint that needs one. Because these variables are created by us and are not real variables, we call them artificial variables.

So we add an artificial variable a_2 to row 2 and an artificial variable a_3 to row 3. The result is

$$\min\ z = 2x_1 + 3x_2 \tag{3.54}$$

$$\text{Row 1: } \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4 \tag{3.55}$$

$$\text{Row 2: } x_1 + 3x_2 - e_2 + a_2 = 20 \tag{3.56}$$

$$\text{Row 3: } x_1 + x_2 + a_3 = 10 \tag{3.57}$$

From this tableau, we see that our initial bfs will be $s_1 = 4,\ a_2 = 20,$ and $a_3 = 10$.

Step 4

Because we are solving a min problem, we add $Ma_2 + Ma_3$ to the objective function (if we were solving a max problem, we would add $-Ma_2 - Ma_3$). This makes a_2 and a_3 very unattractive, and the act of minimizing z will cause a_2 and a_3 to be zero.

The objective function is now

$$\min\ z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Step 5

Because a_2 and a_3 are in our starting bfs (that's why we introduced them), they must be eliminated from row 0.

To eliminate a_2 and a_3 from row 0, simply replace row 0 by $row0 + M * (row2) + M * (row3)$. This yields

$$\begin{array}{rr}
\text{Row 0:} & z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0 \\
\text{Row 2:} & Mx_1 + 3Mx_2 - Me_2 + Ma_2 = 20M \\
\text{Row 3:} & Mx_1 + Mx_2 + Ma_3 = 10M \\
\text{New Row 0:} & z + (2M-2)x_1 + (4M-3)x_2 - Me_2 = 30M
\end{array}$$

Combining the new row 0 with rows equations (3.55) to (3.57) yields the initial tableau below:

Initial Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$2M-2$	$4M-3$	0	$-M$	0	0	$30M$	$z = 30M$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1 = 4$	16
0	1	3	0	-1	1	0	20	$a_2 = 20$	$\frac{20}{3}$*
0	1	1	0	0	0	1	10	$a_3 = 10$	10

We are solving a min problem, so the variable with the most positive coefficient in row 0 should enter the basis. Because $4M - 3 > 2M - 2$, variable x_2 should enter the basis. The ratio test indicates that x_2 should enter the basis in row 2, which means the artificial variable a_2 will leave the basis.

Let new row 2 be $1/3 * (row2)$. Then we can eliminate x_2 from row 0 by adding $-(4M-3) * (newrow2)$ to row 0.

After using EROs to eliminate x_2 from row 1 and row 3, we obtain the tableau below

First Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$\frac{60+10M}{3}$	$z = \frac{60+10M}{3}$	
0	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	$s_1 = \frac{7}{3}$	$\frac{28}{5}$
0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	$x_2 = \frac{20}{3}$	20
0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	$a_3 = \frac{10}{3}$	5*

The ratio test indicates that x_1 should enter the basis in the third row of the current tableau. Then a_3 will leave the basis.

New row 1 and new row 2 are computed as usual using ERO yielding the tableau below:

Optimal Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25	$z = 25$
0	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$	$s_1 = \frac{1}{4}$
0	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5	$x_2 = 5$
0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5	$x_1 = 5$

Because all variables in row 0 have nonpositive coefficients, this is an optimal tableau; all artificial variables are equal to zero in this tableau, so we have found the optimal solution to the Bevco problem:

$$z = 25, \ x_1 = x_2 = 5, \ s_1 = 1/4, \ e_2 = 0$$

This means that Bevco can hold the cost of producing a 10-oz bottle of Oranj to ¢25 by mixing 5 oz of orange soda and 5 oz of orange juice.

Note 3.2.
If any artificial variable is positive in the optimal Big M tableau, then the original LP has no feasible solution.
Considering the tableau

z	x_1	s_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	$1 - 2M$	0	0	$-M$	0	$3 - 4M$	$30 + 6M$	$z = 6M + 30$
0	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$	$s_1 = \frac{3}{2}$
0	-2	0	0	-1	1	-3	6	$a_2 = 6$
0	1	1	0	0	0	1	10	$x_2 = 10$

An artificial variable a_2 is positive in the optimal tableau, this shows that the original LP has no feasible solution.

Exercise 3.2 Use the Big M method to solve the following LPs:

1.

$$
\begin{aligned}
\min\ z &= 4x_1 + 4x_2 + x_3 \\
\text{s.t. } x_1 + x_2 + x_3 &\leq 2 \\
2x_1 + x_2 &\leq 3 \\
2x_1 + x_2 + 3x_3 &\geq 3 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

2.

$$
\begin{aligned}
\max\ z &= 3x_1 + x_2 \\
\text{s.t. } x_1 + x_2 &\geq 3 \\
2x_1 + x_2 &\leq 4 \\
x_1 + x_2 &= 3 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

3.

$$
\begin{aligned}
\min\ z &= x_1 + x_2 \\
\text{s.t. } 2x_1 + x_2 + 2x_3 &= 4 \\
x_1 + x_2 + 2x_3 &= 2 \\
x_1,\ x_2,\ x_3 &\geq 0
\end{aligned}
$$

3.8 Finding the Dual of an LP

Associated with any LP is another LP, called the dual. Knowing the relation between an LP and its dual is vital to understanding advanced topics in linear and nonlinear programming.

When taking the dual of a given LP, we refer to the given LP as the **primal**. If the primal is a max problem, then the dual will be a min problem, and vice versa.

For convenience, we define the variables for the max problem to be $z, x_1, x_2, \cdots, x_n$ and the variables for the min problem to be $w, y_1, y_2, \cdots, y_m$.

We begin by explaining how to find the dual of a max problem in which all variables are required to be nonnegative and all constraints are $\le$ constraints (called a normal max problem). A normal max problem may be written as

$$\begin{aligned} \max \ z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t. } & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{aligned} \tag{3.58}$$

$x_j \ge 0 \quad (j = 1, 2, \cdots, n)$

The dual of a normal max problem such as (3.58) is defined to be

$$\begin{aligned} \min \ w &= b_1y_1 + b_2y_2 + \cdots + b_my_m \\ \text{s.t. } & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\ & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\ & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \end{aligned} \tag{3.59}$$

$y_i \ge 0 \quad (i = 1, 2, \cdots, m)$

A min problem such as (3.59) that has all $\ge$ constraints and all variables nonnegative is called a **normal min problem**.

Similarly, if the primal is a normal min problem, then we define the dual of (3.59) to be (3.58).

A tabular approach makes it easy to find the dual of an LP. If the primal is a normal max problem, then it can be read across the simplex tableau; then the dual is found by reading downwards.

Similarly, if the primal is a normal min problem, we find it by reading down; the dual is found by reading across in the table.

Finding the Dual of a Normal Max or Min Problem

		max z				
min w		$(x_1 \ge 0)$	$(x_2 \ge 0)$	$\cdots$	$(x_n \ge 0)$	
		x_1	x_2		x_n	
$(y_1 \ge 0)$	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\le b_1$
$(y_2 \ge 0)$	y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\le b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$(y_m \ge 0)$	y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\le b_m$
		$\ge c_1$	$\ge c_2$		$\ge c_n$	

Example 3.8.1 We illustrate the use of the table by finding the dual of the Dakota problem. The Dakota problem is

The LP problem is

$$\begin{aligned} \max\ z &= 60x_1 + 30x_2 + 20x_3 \\ s.t.\quad 8x_1 + 6x_2 + x_3 &\le 48 && \text{(Lumber constraint)} \\ 4x_1 + 2x_2 + 1.5x_3 &\le 20 && \text{(Finishing constraint)} \\ 2x_1 + 1.5x_2 + 0.5x_3 &\le 8 && \text{(Carpentry constraint)} \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

where
x_1 = number of desks manufactured
x_2 = number of tables manufactured
x_3 = number of chairs manufactured

Using the tabular approach, and reading downwards, we find the Dakota dual to be

Finding the Dual of the Dakota Problem

		max z			
min w		$(x_1 \ge 0)$	$(x_2 \ge 0)$	$(x_3 \ge 0)$	
		x_1	x_2	x_3	
$(y_1 \ge 0)$	y_1	8	6	1	≤ 48
$(y_2 \ge 0)$	y_2	4	2	1.5	≤ 20
$(y_3 \ge 0)$	y_3	2	1.5	0.5	≤ 8
		≥ 60	≥ 30	≥ 20	

$$\begin{aligned} \min\ w &= 48y_1 + 20y_2 + 8y_3 \\ s.t.\quad 8y_1 + 4y_2 + 2y_3 &\ge 60 \\ 6y_1 + 2y_2 + 1.5y_3 &\ge 30 \\ y_1 + 1.5y_2 + 0.5y_3 &\ge 20 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

3.8.1 Finding the Dual of a Nonnormal LP

Unfortunately, many LPs are not normal max or min problems. For example

$$\begin{aligned} \max\ z &= 2x_1 + x_2 \\ \text{s.t.}\ x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\ge 3 \\ x_1 - x_2 &\le 1 \\ x_1 \ge 0,\ x_2 &\text{ urs} \end{aligned}$$

is not a normal max problem because it has a $\ge$ constraint, an equality constraint, and an unrestricted(urs) variable.

Fortunately, the LP can be transformed into normal form. To place a max problem into normal form, we proceed as follows:

1. Multiply each $\geq$ constraint by -1, converting it into a $\leq$ constraint.

   For example, $2x_1 - x_2 \geq 3$ would be transformed into $-2x_1 + x_2 \leq -3$.

2. Replace each equality constraint by two inequality constraints (a $\leq$ constraint and a $\geq$ constraint). Then convert the $\geq$ constraint to a $\leq$ constraint.

   For example, we would replace $x_1 + x_2 = 2$ by the two inequalities $x_1 + x_2 \leq 2$ and $x_1 + x_2 \geq 2$. Then we would convert $x_1 + x_2 \geq 2$ to $-x_1 - x_2 \leq -2$.

3. For unrestricted-in-sign variable, we replace each

$$x_i = x_i' - x_i'' \; ; \quad x_i' \geq 0 \text{ and } x_i'' \geq 0$$

   Thus x_2 should be replaced by $x_2 = x_2' - x_2''$

After these transformations we have

$$\begin{aligned}
\max z &= 2x_1 + x_2' - x_2'' \\
\text{s.t. } x_1 + x_2' - x_2'' &\leq 2 \\
-x_1 - x_2' + x_2'' &\leq -2 \\
-2x_1 + x_2' - x_2'' &\leq -3 \\
x_1 - x_2' + x_2'' &\leq 1 \\
x_1, x_2', x_2'' &\geq 0
\end{aligned}$$

The above is a normal max problem, so we could find the dual as:

$$\min z = 2y_1 - 2y_2 - 3y_3 + y_4 \tag{3.60}$$
$$\text{s.t } y_1 - y_2 - 2y_3 + y_4 \geq 2 \tag{3.61}$$
$$y_1 - y_2 + y_3 - y_4 \geq 1 \tag{3.62}$$
$$-y_1 + y_2 - y_3 + y_4 \geq -1 \tag{3.63}$$
$$y_i \geq 0 \tag{3.64}$$

Equations (3.61) and (3.62) could be combined. Which further simplify the LP as

$$\min z = 2y_1 - 2y_2 - 3y_3 + y_4 \tag{3.65}$$
$$\text{s.t } y_1 - y_2 - 2y_3 + y_4 \geq 2 \tag{3.66}$$
$$y_1 - y_2 + y_3 - y_4 = 1 \tag{3.67}$$
$$y_i \geq 0 \tag{3.68}$$

Exercise 3.3
Find the duals of the following LPs:

1.

$$\begin{aligned} \max\ \ z = 2x_1 + x_2 \\ \text{s.t.}\ \ -x_1 + x_2 \le 1 \\ x_1 + x_2 \le 3 \\ x_1 - 2x_2 \le 4 \\ x_1, x_2 \ge 0 \end{aligned}$$

2.

$$\begin{aligned} \max\ \ z = 4x_1 - x_2 + 2x_3 \\ \text{s.t.}\ \ x_1 + x_2 \le 5 \\ 2x_1 + x_2 \le 7 \\ 2x_2 + x_3 \ge 6 \\ x_1 + x_3 = 4 \\ x_1 \ge 0,\ \ x_2, x_3 \text{ urs} \end{aligned}$$

4. Transportation, and Assignment Problems

In this chapter, we discuss two special types of linear programming problems: transportation, and assignment. Each of these can be solved by the simplex algorithm, but specialized algorithms for each type of problem are much more efficient.

4.1 General Description of a Transportation Problem

1. A set of m supply points from which a good is shipped. Supply point i can supply at most s_i units.
2. A set of n demand points to which the good is shipped. Demand point j must receive at least d_j units of the shipped good.
3. Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}.

   Let x_{ij}= number of units shipped from supply point i to demand point j, then the general formulation of a transportation problem is

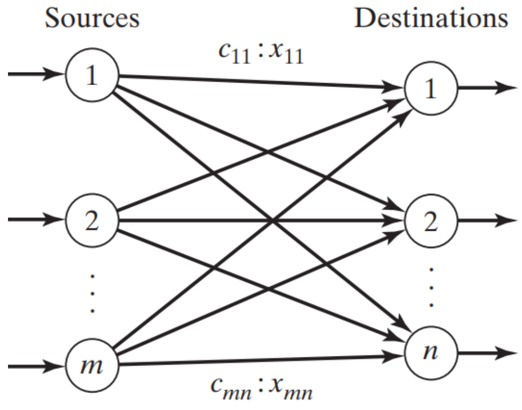
$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{4.1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \leq s_i \ ; \quad i = 1,2,\cdots m \qquad \text{(Supply constraint)} \tag{4.2}$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j \ ; \quad j = 1,2,\cdots n \qquad \text{(Demand constraint)} \tag{4.3}$$

$$x_{ij} \geq 0$$

If a problem has the constraints given in above and is a maximization problem, then it is still a transportation problem.



If the total supply equals total demand, then the problem is said to be a **balanced transportation problem**. That is

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

In a balanced transportation problem, all the constraints must be binding. For a balanced transportation problem, equations (4.1) to (4.3) may be written as

$$\begin{aligned}
&\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
&\text{s.t. } \sum_{j=1}^{n} x_{ij} = s_i \ ; \quad i = 1, 2, \cdots m && \text{(Supply constraint)} \\
&\sum_{i=1}^{m} x_{ij} = d_j \ ; \quad j = 1, 2, \cdots n && \text{(Demand constraint)} \\
&x_{ij} \geq 0
\end{aligned}$$

4.1.1 Balancing the transportation model

The transportation tableau representation assumes that model is balanced, meaning that the total demand equals to the total supply If the model is unbalanced, a dummy source or a dummy destination must be added to restore balance.

If total supply exceeds total demand, we can balance a transportation problem by creating a dummy demand point that has a demand equal to the amount of excess supply. Because shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. Shipments to the dummy demand point indicate unused supply capacity.

If a transportation problem has a total supply that is strictly less than total demand, then the problem has no feasible solution. When total supply is less than total demand, it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, a penalty is often associated with unmet demand.

4.1.2 Transportation Tableau

A transportation problem is specified by the supply s, the demand d, and the shipping costs c_{ij}, so the relevant data can be summarized in a transportation tableau below as:

					Supply
	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand	d_1	d_2	$\cdots$	d_n	

The square, or cell, in row i and column j of a transportation tableau corresponds to the variable x_{ij}. If x_{ij} is a basic variable, its value is placed in the lower left-hand corner of the ijth cell of the tableau.

Below is an illustrative example

Example 4.1.1 Powerco has **three** electric power plants that supply the needs of **four** cities.
Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1–35 million; plant 2–50 million; plant 3–40 million.
The peak power demands in these cities, which occur at the same time (2P.M.), are as follows (in kwh): city 1–45 million; city 2–20 million; city 3–30 million; city 4–30 million.
The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

Shipping Costs, Supply, and Demand for Powerco

From	To: City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Demand (million kwh)	45	20	30	30	

Solution

The objective function, supply constraints, demand constraints, and sign restrictions for the Powerco's problem is as follows:

$$\min \; z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \tag{4.4}$$

Supply constraints

$$\text{s.t. } x_{11} + x_{12} + x_{13} + x_{14} \leq 35 \tag{4.5}$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50 \tag{4.6}$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40 \tag{4.7}$$

Demand constraints

$$x_{11} + x_{21} + x_{31} \geq 45 \tag{4.8}$$
$$x_{12} + x_{22} + x_{32} \geq 20 \tag{4.9}$$
$$x_{13} + x_{23} + x_{33} \geq 30 \tag{4.10}$$
$$x_{14} + x_{24} + x_{34} \geq 30 \tag{4.11}$$

$x_{ij} \geq 0; \quad (i = 1, 2, 3; \quad j = 1, 2, 3, 4)$

The transportation tableau is

	City 1	City 2	City 3	City 4	Supply
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Demand	45	20	30	30	

4.2 Finding Basic Feasible Solutions for Transportation Problems

We discuss three methods that can be used to find a basic feasible (bfs) solution for a balanced transportation problem:

1. Northwest corner method
2. Minimum-cost or Least-cost method
3. Vogel's Approximation method (VAM)

4.2.1 Northwest corner method

To find a bfs by the northwest corner method, we begin in the upper left (or northwest) corner of the transportation tableau and set x_{11} as large as possible. Clearly, x_{11} can be no larger than the smaller of s_1 and d_1.

If $x_{11} = s_1$, cross out the first row of the transportation tableau; this indicates that no more basic variables will come from row 1. Also change d_1 to $d_1 - s_1$.

If $x_{11} = d_1$, cross out the first column of the transportation tableau; this indicates that no more basic variables will come from column 1. Also change s_1 to $s_1 - d_1$.

If $x_{11} = s_1 = d_1$, cross out either row 1 or column 1 (but not both). If you cross out row 1, change d_1 to 0; if you cross out column 1, change s_1 to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed-out row or column. Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row and column. A basic feasible solution has now been obtained.

We illustrate the use of the northwest corner method by finding a bfs for the balanced transportation problem below

				5
				1
				3
2	4	2	1	

To begin, we set $x_{11} = \min\{5, 2\} = 2$. Then we cross out column 1 and change s_1 to $5 - 2 = 3$. This yields the tableau below

2				3
				1
				3
×	4	2	1	

The most northwest remaining variable is x_{12}. We set $x_{12} = \min\{3, 4\} = 3$. Then we cross out row 1 and change d_2 to $4 - 3 = 1$. This yields the tableau

2	3			×
				1
				3
×	1	2	1	

The most northwest available variable is now x_{22}. We set $x_{22} = \min\{1, 1\} = 1$. Because both the supply and demand corresponding to the cell are equal, we may cross out either row 2 or column 2 (but not both). For no particular reason, we choose to cross out row 2. Then d_2 must be changed to $1 - 1 = 0$. The resulting tableau is

2	3			×
	1			×
				3
×	0	2	1	

The most northwest available cell is now x_{32}, so we set $x_{32} = \min\{3, 0\} = 0$. Then we cross out column 2 and change s_3 to $3 - 0 = 3$. The resulting tableau is

2	3			×
	1			×
	0			3
×	×	2	1	

We now set $x_{33} = \min\{3, 2\} = 2$. Then we cross out column 3 and reduce s_3 to $3 - 2 = 1$. The resulting tableau is

2	3			×
	1			×
	0	2		1
×	×	×	1	

The only available cell is x_{34}. We set $x_{34} = \min\{1, 1\} = 1$. Then we cross out row 3 and column 4. No cells are available, so we are finished. We have obtained the bfs

$$x_{11} = 2, \; x_{12} = 3, \; x_{22} = 1, \; x_{32} = 0, \; x_{33} = 2, \; x_{34} = 1$$

The method ensures that no basic variable will be assigned a negative value and also that each supply and demand constraint is satisfied.

4.2.2 Minimum-Cost Method for Finding a Basic Feasible Solution

The northwest corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. The minimum-cost method uses the shipping costs in an effort to produce a bfs that has a lower total cost.

To begin the minimum-cost method, find the variable with the smallest shipping cost (call it x_{ij}). Then assign x_{ij} its largest possible value, $\min\{s_i, d_j\}$. As in the northwest corner method, cross out row i or column j and reduce the supply or demand of the noncrossed-out row or column by the value of x_{ij}. Then choose from the cells that do not lie in a crossed-out row or column the cell with the minimum shipping cost and repeat the procedure. Continue until there is only one cell that can be chosen. In this case, cross out both the cell's row and column. Remember that (with the exception of the last variable) if a variable satisfies both a supply and demand constraint, only cross out a row or column, not both.

To illustrate the minimum cost method, we find a bfs for the balanced transportation problem in the table below:

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	

The variable with the minimum shipping cost is x_{22}. We set $x_{22} = \min\{10, 8\} = 8$. Then we cross out column 2 and reduce s_2 to $10 - 8 = 2$. The resulting tableau is

$\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	5
$\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	2
$\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	15
12	×	4	6	

We could now choose either x_{11} or x_{21} (both having shipping costs of 2). We arbitrarily choose x_{21}. We set $x_{21} = \min\{2, 12\} = 2$. Then we cross out row 2 and change d_1 to $12 - 2 = 10$. The resulting tableau is

$\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	5
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
$\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	15
10	×	4	6	

We set $x_{11} = \min\{5, 10\} = 5$. Then we cross out row 1 and change d_1 to $10 - 5 = 5$. The resulting tableau is

5 $\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	×
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
$\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	15
5	×	4	6	

We set $x_{31} = \min\{15, 5\} = 5$. Then we cross out column 1 and reduce s_3 to $15 - 5 = 10$. The resulting tableau is

5 $\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	×
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
5 $\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	10
×	×	4	6	

We set $x_{33} = \min\{10, 4\} = 4$. Then we cross out column 3 and reduce s_3 to $10 - 4 = 6$. The resulting tableau is

5 $\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	×
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
5 $\boxed{3}$	$\boxed{8}$	4 $\boxed{4}$	$\boxed{6}$	6
×	×	×	6	

The only cell that we can choose is x_{34}. We set $x_{34} = \min\{6, 6\}$ and cross out both row 3 and column 4. We have now obtained the bfs:

$$x_{11} = 5,\ x_{21} = 2,\ x_{22} = 8,\ x_{31} = 5,\ x_{33} = 4,\ x_{34} = 6$$

Again the minimum-cost method may sometimes yield a relatively high-cost bfs. When this arises, we resort to the Vogel's method.

4.2.3 Vogel's Method for Finding a Basic Feasible Solution

Begin by computing for each row and column the **penalty**. The penalty equals the difference between the two smallest costs in the row or column.

Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest shipping cost.

As described in the northwest corner and minimum-cost methods, make this variable as large as possible, cross out a row or column, and change the supply or demand associated with the basic variable. Now recompute new penalties (using only cells that do not lie in a crossed-out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column. A bfs has now been obtained.

We illustrate Vogel's method by finding a bfs to the table below:

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	10	7 – 6 = 1
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 – 15 = 63
Demand	15	5	5		
Column Penalty	15 – 6 = 9	80 – 7 = 73	78 – 8 = 70		

Column 2 has the largest penalty, so we set $x_{12} = \min\{10, 5\} = 5$. Then we cross out column 2 and reduce s_1 to $10 - 5 = 5$ After recomputing the new penalties (observe that after a column is crossed out, the column penalties will remain unchanged). The resulting tableau is

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$ 5	$\boxed{8}$	5	8 – 6 = 2
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 – 15 = 63
Demand	15	×	5		
Column Penalty	9	—	70		

The largest penalty now occurs in column 3, so we set $x_{13} = \min\{5, 5\}$. We may cross out either row 1 or column 3. We arbitrarily choose to cross out column 3, and we reduce s_1 to $5 - 5 = 0$. Because each row has only one cell that is not crossed out, there are no row penalties. The resulting tableau is

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$ 5	$\boxed{8}$ 5	0	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	9	—	—		

Column 1 has the only (and, of course, the largest) penalty. We set $x_{11} = \min\{0, 15\} = 0$, cross out row 1, and change d_1 to $15 - 0 = 15$. The result is

				Supply	Row Penalty
	$\boxed{6}$ 0	$\boxed{7}$ 5	$\boxed{8}$ 5	×	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	—	—	—		

No penalties can be computed, and the only cell that is not in a crossed-out row or column is x_{21}. Therefore, we set $x_{21} = 15$ and cross out both column 1 and row 2

$\boxed{6}$ 0	$\boxed{7}$ 5	$\boxed{8}$ 5	10
$\boxed{15}$ 15	$\boxed{80}$	$\boxed{78}$	15
15	5	5	

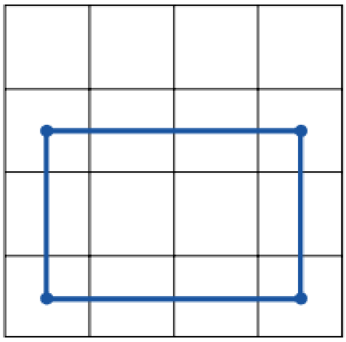
The obtained bfs is :

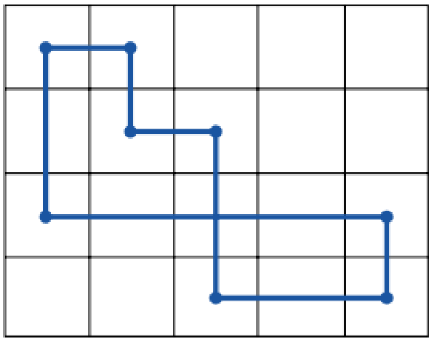
$$x_{11} = 0,\ x_{12} = 5,\ x_{13} = 5,\ x_{21} = 15$$

4.3 How to Pivot in a Transportation Problem

Definition 4.1 (Loop)
An ordered sequence of at least four different cells is called a loop if

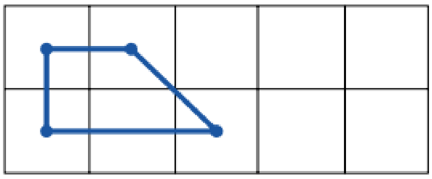
1. Any two consecutive cells lie in either the same row or same column
2. No three consecutive cells lie in the same row or column
3. The last cell in the sequence has a row or column in common with the first cell in the sequence

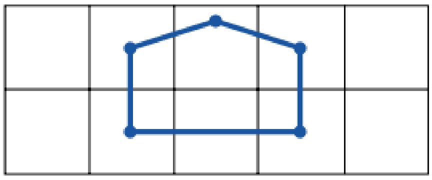




Left loop (2, 1)–(2, 4)–(4, 4)–(4, 1)

Right loop (1, 1)–(1, 2)–(2, 2)–(2, 3)–(4, 3)–(4, 5)–(3, 5)–(3, 1)





Left is not a loop because (1, 2) and (2, 3) do not lie in the same row or column.

Right is not a loop because (1, 2), (1, 3), and (1, 4) all lie in the same row.

Theorem 4.1
In a balanced transportation problem with m supply points and n demand points, the cells corresponding to a set of $m+n-1$ variables contain no loop if and only if the $m+n-1$ variables yield a basic solution.

Now the steps to find a pivot for a transportation problem is as follows:

1. Determine the variable that should enter the basis.

2. Find the loop (it can be shown that there is only one loop) involving the entering variable and some of the basic variables.

3. Counting only cells in the loop, label those found in step 2 that are an even number (0, 2, 4, and so on) of cells away from the entering variable as even cells. Also label those that are an odd number of cells away from the entering variable as odd cells.

4. Find the odd cell whose variable assumes the smallest value. Call this value θ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by θ and increase the value of each even cell by θ. The values of variables not in the loop remain unchanged. The pivot is now complete.

   If $\theta = 0$, then the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case, a degenerate bfs existed before and will result after the pivot. If more than one odd cell in the loop equals θ, you may arbitrarily choose one of these odd cells to leave the basis; again, a degenerate bfs will result.

We illustrate the pivoting procedure on the Powerco example. When the northwest corner method is applied to the Powerco example, the bfs in the table below is found.

35				35
10	20	20		50
		10	30	40
45	20	30	30	

For this bfs, the basic variables are

$$x_{11} = 35,\ x_{21} = 10,\ x_{22} = 20,\ x_{23} = 20,\ x_{33} = 10,\ x_{34} = 30$$

Suppose we want to find the bfs that would result if x_{14} was to enter the basis. The loop involving x_{14} and some of the basic variables is

$$\begin{array}{cccccc} \text{E} & \text{O} & \text{E} & \text{O} & \text{E} & \text{O} \\ (1,4)– & (3,4)– & (3,3)– & (2,3)– & (2,1)– & (1,1) \end{array}$$

In this loop, (1, 4), (3, 3), and (2, 1) are the even cells, and (1, 1), (3, 4), and (2, 3) are the odd cells. The odd cell with the smallest value is $x_{23} = 20$. Thus, after the pivot, x_{23} will have left the basis. We now add 20 to each of the even cells and subtract 20 from each of the odd cells. The results is the bfs:

35 – 20			0 + 20	35
10 + 20	20	20 – 20 (nonbasic)		50
		10 + 20	30 – 20	40
45	20	30	30	

There is no loop involving the cells (1, 1), (1, 4), (2, 1), (2, 2), (3, 3), and (3, 4), so the new solution is a bfs. After the pivot, the new bfs is

$$x_{11} = 15,\ x_{14} = 20,\ x_{21} = 30,\ x_{22} = 20,\ x_{33} = 30,\ x_{34} = 10,$$

and all other variables equal 0.

The preceding illustration of the pivoting procedure makes it clear that each pivot in a transportation problem involves only additions and subtractions.

4.4 How to Determine the Entering Nonbasic Variable

Let u_i be the shadow price of the ith supply constraint, and v_j be the shadow price of the jth demand constraint. The shadow price of the ith constraint of a linear programming problem is the amount by which the optimal z-value is improved if the right-hand side is increased by 1.

We assume that the first supply constraint has been dropped, so we may set $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1\ u_2\ \cdots\ u_m \quad v_1\ v_2\ \cdots\ v_n]$ for the current bfs.

In the case of a minimum problem, if $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal.

If this is not the case, then we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure. This yields a new bfs.

For a maximization problem if $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure.

To illustrate the previous discussion, consider the bfs for the Powerco problem shown below:

$\boxed{8}$ 35	$\boxed{6}$	$\boxed{10}$	$\boxed{9}$	35
$\boxed{9}$ 10	$\boxed{12}$ 20	$\boxed{13}$ 20	$\boxed{7}$	50
$\boxed{14}$	$\boxed{9}$	$\boxed{16}$ 10	$\boxed{5}$ 30	40
45	20	30	30	

The resulting equations are

$$@\ x_{00} \Longrightarrow \qquad u_1 = 0 \tag{4.12}$$
$$@\ x_{11} \Longrightarrow \qquad u_1 + v_1 = 8 \tag{4.13}$$
$$@\ x_{21} \Longrightarrow \qquad u_2 + v_1 = 9 \tag{4.14}$$
$$@\ x_{22} \Longrightarrow \qquad u_2 + v_2 = 12 \tag{4.15}$$
$$@\ x_{23} \Longrightarrow \qquad u_2 + v_3 = 13 \tag{4.16}$$
$$@\ x_{33} \Longrightarrow \qquad u_3 + v_3 = 16 \tag{4.17}$$
$$@\ x_{34} \Longrightarrow \qquad u_3 + v_4 = 5 \tag{4.18}$$

Solving

$$v_1 = 8,\ u_2 = 1,\ v_2 = 11,\ v_3 = 12,\ u_3 = 4,\ v_4 = 1$$

For each nonbasic variable, we now compute $\bar{c}_{ij} = u_i + v_j - c_{ij}$. We obtain

$$\bar{c}_{12} = 0 + 11 - 6 = 5 \qquad \bar{c}_{13} = 0 + 12 - 10 = 2$$
$$\bar{c}_{14} = 0 + 1 - 9 = -8 \qquad \bar{c}_{24} = 1 + 1 - 7 = -5$$
$$\bar{c}_{31} = 4 + 8 - 14 = -2 \qquad \bar{c}_{32} = 4 + 11 - 9 = 6$$

Because $\bar{c}_{32}$ is the most positive, we would next enter x_{32} into the basis.

4.5 Transportation Simplex Method

The transportation simplex method is used to find an optimal solution for a transportation problem. The two were known methods are

1. Stepping Stone Method
2. Modified distribution method (MoDi)

MoDi is an improvement over the stepping stone method. We will therefore focus on the MoDi.

The steps involved in using the MoDi to find an optimal solution are as follows:

1. If the problem is unbalanced, balance it
2. Use one of the methods described above (northwest corner, minimum-cost, Vogel's) to find a bfs.
3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1\ u_2\ \cdots\ u_m \quad v_1\ v_2\ \cdots\ v_n]$ for the current bfs.
4. In the case of a minimum problem, if $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal.

   If this is not the case, then we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure. This yields a new bfs.

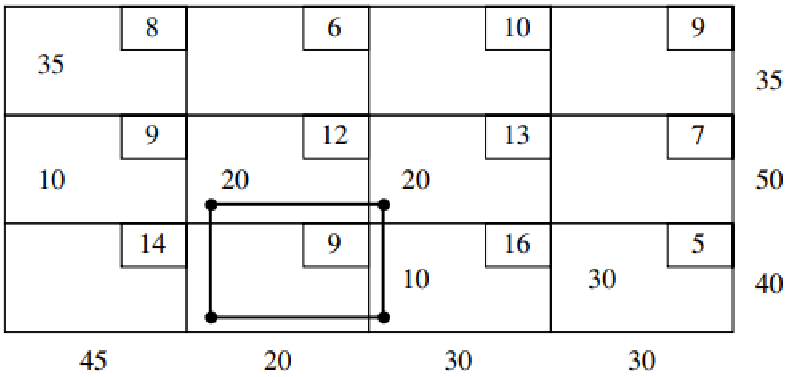
   For a maximization problem if $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure.
5. Using the new bfs, return to steps 3 and 4.

We illustrate the procedure for solving a transportation problem by solving the Powerco problem.

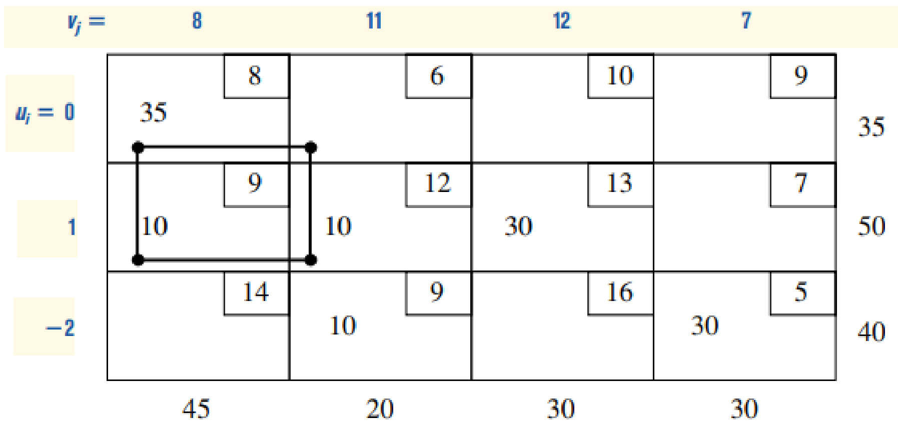
We begin with the bfs below.

35 $\boxed{8}$	$\boxed{6}$	$\boxed{10}$	$\boxed{9}$	35
10 $\boxed{9}$	20 $\boxed{12}$	20 $\boxed{13}$	$\boxed{7}$	50
$\boxed{14}$	$\boxed{9}$	10 $\boxed{16}$	30 $\boxed{5}$	40
45	20	30	30	

We have already determined that x_{32} should enter the basis. As shown below:



The loop involving x_{32} and some of the basic variables is (3, 2)–(3, 3)–(2, 3)–(2, 2). The odd cells in this loop are (3, 3) and (2, 2). Because $x_{33} = 10$ and $x_{22} = 20$, the pivot will decrease the value of x_{33} and x_{22} by 10 and increase the value of x_{32} and x_{23} by 10. The resulting bfs is



The $u_i's$ and $v_j's$ for the new bfs are

$$
\begin{aligned}
u_1 &= 0 \\
u_1 + v_1 &= 8 \\
u_2 + v_1 &= 9 \\
u_2 + v_2 &= 12 \\
u_2 + v_3 &= 13 \\
u_3 + v_2 &= 9 \\
u_3 + v_4 &= 5
\end{aligned}
$$

We find that $\bar{c}_{12} = 5,\ \bar{c}_{24} = 1,\ \bar{c}_{13} = 2$ are the only positive $\bar{c}_{ij}$. Thus, we next enter x_{12} into the basis.

The loop involving x_{12} and some of the basic variables is (1, 2)–(2, 2)–(2, 1)–(1, 1). The odd cells are (2, 2) and (1, 1). Because $x_{22} = 10$ is the smallest entry in an odd cell, we decrease x_{22} and x_{11} by 10 and increase x_{12} and x_{21} by 10. The resulting bfs is

	$v_j =$ 8	6	12	2	
$u_i = 0$	25 $\boxed{8}$	10 $\boxed{6}$	$\boxed{10}$	$\boxed{9}$	35
1	20 $\boxed{9}$	$\boxed{12}$	30 $\boxed{13}$	$\boxed{7}$	50
3	$\boxed{14}$	10 $\boxed{9}$	$\boxed{16}$	30 $\boxed{5}$	40
	45	20	30	30	

The $u_i's$ and $v_j's$ for the new bfs are

$$
\begin{aligned}
u_1 &= 0\\
u_1 + v_1 &= 8\\
u_2 + v_1 &= 9\\
u_1 + v_2 &= 6\\
u_2 + v_3 &= 13\\
u_3 + v_2 &= 9\\
u_3 + v_4 &= 5
\end{aligned}
$$

In computing $\bar{c}_{ij}$ for each nonbasic variable, we find that the only positive is $\bar{c}_{13} = 2$ Thus, x_{13} enters the basis. The loop involving x_{13} and some of the basic variables is (1, 3)–(2, 3)–(2, 1)–(1, 1). The odd cells are x_{23} and x_{11}. Because $x_{11} = 25$ is the smallest entry in an odd cell, we decrease x_{23} and x_{11} by 25 and increase x_{13} and x_{21} by 25. The resulting bfs is

	$v_j =$ 6	6	10	2	
$u_i = 0$	$\boxed{8}$	10 $\boxed{6}$	25 $\boxed{10}$	$\boxed{9}$	35
3	45 $\boxed{9}$	$\boxed{12}$	5 $\boxed{13}$	$\boxed{7}$	50
3	$\boxed{14}$	10 $\boxed{9}$	$\boxed{16}$	30 $\boxed{5}$	40
	45	20	30	30	

The $u_i's$ and $v_j's$ for the new bfs are

$$
\begin{aligned}
u_1 &= 0\\
u_1 + v_3 &= 10\\
u_2 + v_1 &= 9\\
u_1 + v_2 &= 6\\
u_2 + v_3 &= 13\\
u_3 + v_2 &= 9\\
u_3 + v_4 &= 5
\end{aligned}
$$

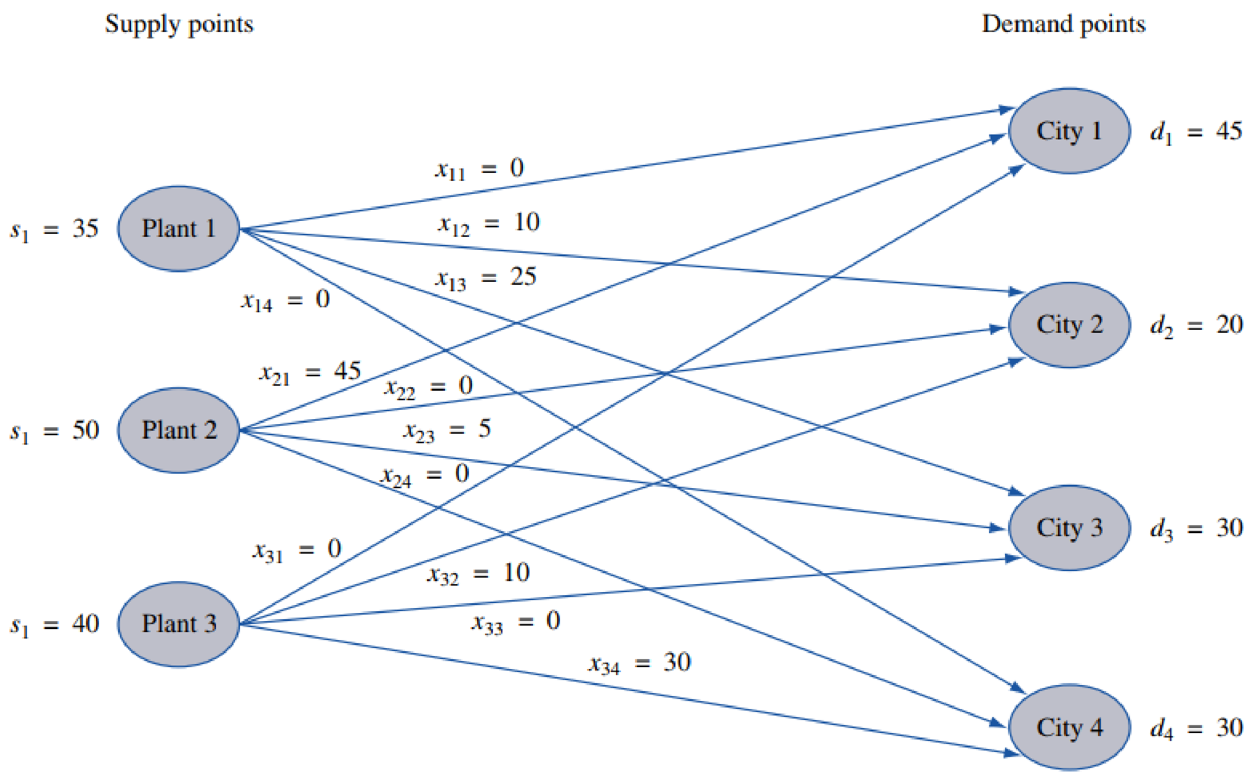
At this point all $\bar{c}_{ij} \leq 0$, so an optimal solution has been obtained. Thus, the optimal solution to the Powerco problem is

$$x_{12} = 10,\ x_{13} = 25,\ x_{21} = 45,\ x_{23} = 5,\ x_{32} = 10,\ x_{34} = 30$$

and

$$z = 6(10) + 10(25) + 9(45) + 13(5) + 9(10) + 5(30) = ¢1,020$$

The graphical representation of Powerco problem and its optimal solution is



Exercise 4.1

1. (a) A company supplies goods to three customers, who each require 30 units. The company has two warehouses. Warehouse 1 has 40 units available, and warehouse 2 has 30 units available. The costs of shipping 1 unit from warehouse to customer are shown below.

	To		
From	Customer 1	Customer 2	Customer 3
Warehouse 1	\$15	\$35	\$25
Warehouse 2	\$10	\$50	\$40

There is a penalty for each unmet customer unit of demand: With customer 1, a penalty cost of ¢90 is incurred; with customer 2, ¢80; and with customer 3, ¢110. Formulate a balanced transportation problem to minimize the sum of shortage and shipping costs.

(b) A shoe company forecasts the following demands during the next six months: month 1—200; month 2—260; month 3—240; month 4—340; month 5—190;

month 6—150. It costs ¢7 to produce a pair of shoes with regular-time labor (RT) and ¢11 with overtime labor (OT). During each month, regular production is limited to 200 pairs of shoes, and overtime production is limited to 100 pairs. It costs ¢1 per month to hold a pair of shoes in inventory. Formulate a balanced transportation problem to minimize the total cost of meeting the next six months of demand on time.

(c) A hospital needs to purchase 3 gallons of a perishable medicine for use during the current month and 4 gallons for use during the next month. Because the medicine is perishable, it can only be used during the month of purchase. Two companies (Daisy and Laroach) sell the medicine. The medicine is in short supply. Thus, during the next two months, the hospital is limited to buying at most 5 gallons from each company. The companies charge the prices shown below.

Company	Current Month's Price per Gallon ($)	Next Month's Price per Gallon ($)
Daisy	800	720
Laroach	710	750

Formulate a balanced transportation model to minimize the cost of purchasing the needed medicine.

2. Use the following methods to find the bfs of problems in (1) :
   (a) Northwest corner method
   (b) Minimum-cost method
   (c) Vogel's method
3. Use the transportation simplex to solve these problems ((1)). Begin with the bfs found in question (2).

4.6 Assignment Problems

Although the transportation simplex appears to be very efficient, there is a certain class of transportation problems, called assignment problems, for which the transportation simplex is often very inefficient. In this section, we define assignment problems and discuss an efficient method that can be used to solve them.

In general, an assignment problem is a balanced transportation problem in which all supplies and demands are equal to 1.

The assignment problem's matrix of costs is its **cost matrix.**

Example 4.6.1
Machineco has four machines and four jobs to be completed. Each machine must be assigned to complete one job. The time required to set up each machine for completing each job is shown below.

Setup Times for Machineco

	Time (Hours)			
Machine	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

Machineco wants to minimize the total setup time needed to complete the four jobs. Use linear programming to solve this problem.

Solution

Machineco must determine which machine should be assigned to each job. We define (for $i, j = 1, 2, 3, 4$)
$x_{ij} = 1$ if machine i is assigned to meet the demands of job j
$x_{ij} = 0$ if machine i is not assigned to meet the demands of job j

Then Machineco's problem may be formulated as

$$\begin{aligned} \min = \quad z = &14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\ &+7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44} \end{aligned}$$

Machine constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 1 \tag{4.19}$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1 \tag{4.20}$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1 \tag{4.21}$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1 \tag{4.22}$$

Job constraints

$$x_{11} + x_{21} + x_{31} + x_{41} = 1 \tag{4.23}$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1 \tag{4.24}$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1 \tag{4.25}$$
$$x_{14} + x_{24} + x_{34} + x_{44} = 1 \tag{4.26}$$

The objective function will not pick up the time required when $x_{ij} = 0$ and will pick up the time required to set up machine i for job j when $x_{ij} = 1$.

By the minimum cost method, we obtain the bfs below

		Job 1	Job 2	Job 3	Job 4	
	$v_j =$	3	4	8	7	
Machine 1	$u_i = 0$	$\boxed{14}$	1 $\boxed{5}$	0 $\boxed{8}$	0 $\boxed{7}$	1
Machine 2	−2	$\boxed{2}$	$\boxed{12}$	$\boxed{6}$	1 $\boxed{5}$	1
Machine 3	−5	$\boxed{7}$	$\boxed{8}$	1 $\boxed{3}$	$\boxed{9}$	1
Machine 4	−1	1 $\boxed{2}$	0 $\boxed{4}$	$\boxed{6}$	$\boxed{10}$	1
		1	1	1	1	

We find that $\bar{c}_{43} = 1$ is the only positive $\bar{c}_{ij}$. We therefore enter x_{43} into the basis. The loop involving x_{43} and some of the basic variables is (4, 3)–(1, 3)–(1, 2)–(4, 2). The odd variables in the loop are x_{13} and x_{42}. Because $x_{13} = x_{42} = 0$, either x_{13} or x_{42} will leave the basis. We arbitrarily choose x_{13} to leave the basis. After performing the pivot, we obtain the bfs below

		Job 1	Job 2	Job 3	Job 4	
	$v_j =$	3	5	7	7	
Machine 1	$u_i = 0$	$\boxed{14}$	1 $\boxed{5}$	$\boxed{8}$	0 $\boxed{7}$	1
Machine 2	−2	$\boxed{2}$	$\boxed{12}$	$\boxed{6}$	1 $\boxed{5}$	1
Machine 3	−4	$\boxed{7}$	$\boxed{8}$	1 $\boxed{3}$	$\boxed{9}$	1
Machine 4	−1	1 $\boxed{2}$	0 $\boxed{4}$	0 $\boxed{6}$	$\boxed{10}$	1
		1	1	1	1	

All $\bar{c}_{ij}$ are now nonpositive, so we have obtained an optimal assignment:

$$x_{12} = 1, \; x_{24} = 1, \; x_{33} = 1, \; x_{41} = 1$$

Thus, machine 1 is assigned to job 2, machine 2 is assigned to job 4, machine 3 is assigned to job 3, and machine 4 is assigned to job 1. A total setup time of

$$5 + 5 + 3 + 2 = 15$$

hours is required.

4.6.1 The Hungarian Method

The Hungarian method is an algorithm that is usually used to solve assignment (min) problems.

The steps involved are as follows:

1. Find the minimum element in each **row** of the $m \times m$ cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row.

   For this new matrix, find the minimum cost in each **column**. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost the minimum cost in its column.

2. Draw the minimum number of lines (horizontal, vertical, or both) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

3. Find the smallest nonzero element (call its value k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to step 2.

Example 4.6.2 We illustrate the Hungarian method by solving the Machineco problem

Solution
The row minimum is given as

				Row Minimum
14	5	8	7	5
2	12	6	5	2
7	8	3	9	3
2	4	6	10	2

Step 1
For each row, we subtract the row minimum from each element in the row, obtaining

9	0	3	2	
0	10	4	3	
4	5	0	6	
0	2	4	8	
0	0	0	2	Column Minimum

We now subtract 2 from each cost in column 4, obtaining

9	0	3	0
0	10	4	1
4	5	0	4
0	2	4	6

Step 2
As shown, lines through row 1, row 3, and column 1 cover all the zeros in the reduced cost matrix. Since fewer than four lines are required to cover all the zeros, so we proceed to step 3.

Step 3
The smallest uncovered element equals 1, so we now subtract 1 from each uncovered element in the reduced cost matrix and add 1 to each twice-covered element. The resulting matrix is

10	0	3	0
0	9	3	0
5	5	0	4
0	1	3	5

Four lines are now required to cover all the zeros. Thus, an optimal solution is available.

To find an optimal assignment, observe that the only covered 0 in column 3 is x_{33}, so we must have $x_{33} = 1$.

Also, the only available covered zero in column 2 is x_{12}, so we set $x_{12} = 1$ and observe that neither row 1 nor column 2 can be used again.

Now the only available covered zero in column 4 is x_{24}. Thus, we choose $x_{24} = 1$ (which now excludes both row 2 and column 4 from further use).

Finally, we choose $x_{41} = 1$.

Thus, we have found the optimal assignment

$$x_{12} = 1,\ x_{24} = 1,\ x_{33} = 1,\ x_{41} = 1$$

This agrees with the result obtained by the transportation simplex.

Note 4.1.

- *To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by -1 and solve the problem as a* *minimization problem.*

- *If the number of rows and columns in the cost matrix are unequal, then the assignment problem is unbalanced. The Hungarian method may yield an incorrect solution if the problem is unbalanced. Thus, any assignment problem should be balanced (by the addition of one or more dummy points) before it is solved by the Hungarian method.*

Exercise 4.2

1. Five employees are available to perform four jobs. The time it takes each person to perform each job is given in the table below.

	Time (hours)			
Person	**Job 1**	**Job 2**	**Job 3**	**Job 4**
1	22	18	30	18
2	18	—	27	22
3	26	20	28	28
4	16	22	—	14
5	21	—	25	28

Note: Dashes indicate person cannot do that particular job.

Determine the assignment of employees to jobs that minimizes the total time required to perform the four jobs

2. Greydog Bus Company operates buses between Boston and Washington, D.C. A bus trip between these two cities takes 6 hours. Federal law requires that a driver rest for four or more hours between trips. A driver's workday consists of two trips: one from Boston to Washington and one from Washington to Boston. The Table below gives the departure times for the buses.

Trip	Departure Time	Trip	Departure Time
Boston 1	6 A.M.	Washington 1	5:30 A.M.
Boston 2	7:30 A.M.	Washington 2	9 A.M.
Boston 3	11.30 A.M.	Washington 3	3 P.M.
Boston 4	7 P.M.	Washington 4	6:30 P.M.
Boston 5	12:30 A.M.	Washington 5	12 midnight

Greydog's goal is to minimize the total downtime for all drivers. How should Greydog assign crews to trips? Note: It is permissible for a driver's "day" to overlap midnight. For example, a Washington-based driver can be assigned to the Washington–Boston 3 P.M. trip and the Boston–Washington 6 A.M. trip.

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