**CMPS 101**
**Midterm 1 Review**
**Solutions to selected problems**

Problem 2

State whether the following assertions are true or false. If any statements are false, give a related statement which is true.

a. $f(n) = O(g(n))$ implies $f(n) = o(g(n))$. **False**

$f(n) = o(g(n))$ implies $f(n) = O(g(n))$

b. $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$. **True**

c. $f(n) = \Theta(g(n))$ if and only if $\lim_{n\to\infty}(f(n)/g(n)) = L$, where $0 < L < \infty$. **False**

$0 < L < \infty$ and $\lim_{n\to\infty}(f(n)/g(n)) = L$ implies $f(n) = \Theta(g(n))$

Problem 3

Prove that $\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$. In other words, if $h_1(n) = \Theta(f(n))$ and $h_2(n) = \Theta(g(n))$, then $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$.

**Proof:**

By hypothesis there exist positive constants $n_1$, $n_2$, $a_1$, $b_1$, $a_2$, and $b_2$ such that

$$\forall\, n \ge n_1: \quad 0 \le a_1 f(n) \le h_1(n) \le b_1 f(n)$$

and

$$\forall\, n \ge n_2: \quad 0 \le a_2 g(n) \le h_2(n) \le b_2 g(n)$$

If $n \ge n_0 = \max(n_1, n_2)$, then both inequalities hold. Let $c = a_1 a_2$, and $d = b_1 b_2$. Since everything in sight is non-negative, we can multiply these two inequalities to get

$$\forall\, n \ge n_0: \quad 0 \le c\, f(n)g(n) \le h_1(n)h_2(n) \le d\, f(n)g(n),$$

and hence $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$ as required. ///

Problem 4

Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for all sufficiently large $n$), and suppose that $f(n) = \Theta(g(n))$. Does it necessarily follow that $\dfrac{1}{f(n)} = \Theta\left(\dfrac{1}{g(n)}\right)$ ? Either prove this statement, or give a counter-example.

**Solution:**
The statement is <u>true</u>, as we now prove. By hypothesis there exist positive numbers $c_1$, $c_2$, and $n_0$ such that for all $n \ge n_0$: $0 < c_1 g(n) \le f(n) \le c_2 g(n)$. (Note: the strict inequality $<$ on the left follows from the fact that $f(n)$ and $g(n)$ are asymptotically positive.) Taking the reciprocals of all the positive terms in this inequality gives: $0 < \frac{1}{c_2} \cdot \frac{1}{g(n)} \le \frac{1}{f(n)} \le \frac{1}{c_1} \cdot \frac{1}{g(n)}$ for all $n \ge n_0$. Observe that both $\frac{1}{c_2} > 0$ and $\frac{1}{c_1} > 0$, whence $\frac{1}{f(n)} = \Theta\left(\frac{1}{g(n)}\right)$. ///

<u>Problem 5</u>
Give an example of functions $f(n)$ and $g(n)$ such that $f(n) = o(g(n))$ but $\log(f(n)) \ne o(\log(g(n)))$. (Hint: Consider $n!$ and $n^n$ and use the corollary to Stirling's formula in the handout on common functions.)

**Solution:**
Following the hint, we let $f(n) = n!$ and $g(n) = n^n$. Part (1) of the Corollary to Stirling's formula on page 3 of the handout on common functions showed that $f(n) = o(g(n))$. Part (3) of that same Corollary showed that $\log(n!) = \Theta(n\log(n))$, and hence $\log(f(n)) = \Theta(n\log(n)) = \Theta(\log(n^n)) = \Theta(\log(g(n))$. Since $o(\log(g(n)) \cap \Theta(\log(g(n)) = \varnothing$ by problem 6 below, we have $\log(f(n)) \ne o(\log(g(n)))$. ///

<u>Problem 6</u>
Let $g(n)$ be an asymptotically non-negative function. Prove that $o((g(n)) \cap \Omega(g(n)) = \varnothing$.

**Proof:**
Assume to get a contradiction that $f(n) \in o((g(n)) \cap \Omega(g(n))$. Then since $f(n) = \Omega(g(n))$ we have

(1) $\qquad \exists c_1 > 0,\ \exists n_1 > 0,\ \forall n \ge n_1:\quad 0 \le c_1 g(n) \le f(n)$

Also, since $f(n) = o(g(n))$ we have

(2) $\qquad \forall c_2 > 0,\ \exists n_2 > 0,\ \forall n \ge n_2:\quad 0 \le f(n) < c_2 g(n)$

Let $c_2 = c_1$. Then $c_2 > 0$, and by (2) there exists $n_2 > 0$ such that $0 \le f(n) < c_1 g(n)$ for all $n \ge n_2$. Pick any $m \ge \max(n_1, n_2)$. Then by (1) we have $0 \le c_1 g(m) \le f(m) < c_1 g(m)$, and hence $c_1 g(m) < c_1 g(m)$, a contradiction. Our assumption was therefore false, and no such function $f(n)$ can exist. We conclude that $o((g(n)) \cap \Omega(g(n)) = \varnothing$. ///

Problem 7 (d)

Use limits to prove the following: $f(n)+o(f(n))=\Theta(f(n))$

**Proof:**

In this equation, the term $o(f(n))$ stands for some function $h(n)$ satisfying $\lim_{n\to\infty}\left(\frac{h(n)}{f(n)}\right)=0$. Therefore $\lim_{n\to\infty}\left(\frac{f(n)+h(n)}{f(n)}\right)=\lim_{n\to\infty}\left(1+\frac{h(n)}{f(n)}\right)=1+\lim_{n\to\infty}\left(\frac{h(n)}{f(n)}\right)=1$, showing that $f(n)+h(n)=\Theta(f(n))$. Note that this result justifies the practice of dropping low order terms when finding the asymptotic growth rate of a function. ///

Problem 8

Let $g(n)=n$ and $f(n)=n+\frac{1}{2}n^2(\sin(n)+1)$. Show that

a. $f(n)=\Omega(g(n))$
b. $f(n)\neq O(g(n))$
c. $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$ does not exist, even in the sense of being infinite.

Note: this is the 'Example C' mentioned in the handout on asymptotic growth rates.

**Proof of (a):**

For any $n\geq 1$ we have $-1\leq \sin(n)\leq 1$ and hence $\sin(n)+1\geq 0$. Thus

$$f(n)=n+\frac{1}{2}n^2(\sin(n)+1)\geq n=g(n).$$

Thus $0\leq 1\cdot g(n)\leq f(n)$ for all $n\geq 1$, whence $f(n)=\Omega(g(n))$. ///

**Proof of (b):**

We must show that the sentence '$\exists c>0$, $\exists n_0>0$, $\forall n\geq n_0$: $0\leq f(n)\leq c\cdot g(n)$' is false. We do this by showing that it's negation '$\forall c>0$, $\forall n_0>0$, $\exists n\geq n_0$: $c\cdot g(n)<f(n)$' is true. Pick $c>0$ and $n_0>0$ arbitrarily. Define $n=\frac{\pi}{2}+2\pi\cdot k$ where the integer $k$ is chosen so large as to guarantee that $n\geq \max(c, n_0)$. (This is possible since $\frac{\pi}{2}+2\pi\cdot k\to\infty$ as $k\to\infty$.) Then $n\geq n_0$ and $n\geq c>c-1$, whence $n+1>c$. Observe also that $\sin(n)=1$, and therefore

$$f(n)=n+\frac{1}{2}n^2(\sin(n)+1)=n+n^2=n(1+n)>cn=c\cdot g(n)$$

as required. ///

**Proof of (c):**
Observe that

$$\frac{f(n)}{g(n)} = \frac{n+\frac{1}{2}n^2(\sin(n)+1)}{n} = 1+\frac{1}{2}n(\sin(n)+1),$$

which oscillates with increasing amplitude between 1 and $1+n$ as $n\to\infty$, and therefore has no limit, even in the sense of being infinite. ///

Problem 10

Use Stirling's formula to prove that $\binom{2n}{n}=\Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

**Proof:**
By Stirling's formula

$$\binom{2n}{n}=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{(n!)^2}=\frac{\sqrt{2\pi\cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta(1/2n)\right)}{\left(\sqrt{2\pi\, n}\cdot\left(\frac{n}{e}\right)^{n}\cdot\left(1+\Theta(1/n)\right)\right)^2}$$

$$=\frac{2^{2n}}{\sqrt{\pi\, n}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2}=\frac{1}{\sqrt{\pi}}\cdot\frac{4^n}{\sqrt{n}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2}$$

so that

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}}=\frac{1}{\sqrt{\pi}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2}\to\frac{1}{\sqrt{\pi}}\quad\text{as}\quad n\to\infty$$

The result now follows since $0<\frac{1}{\sqrt{\pi}}<\infty$. ///

Problem 11

Consider the following *sketch* of an algorithm called ProcessArray which performs some unspecified operation on a subarray $A[p\cdots r]$.

ProcessArray($A$, $p$, $r$) (Preconditions: $1\le p$ and $r\le \text{length}[A]$ )
1. Perform 1 basic operation
2. if $p<r$
3. $\quad q\leftarrow\left\lfloor\frac{p+r}{2}\right\rfloor$
4. $\quad$ ProcessArray(A, p, q)
5. $\quad$ ProcessArray(A, q+1, r)

a. Write a recurrence formula for the number $T(n)$ of basic operations performed by this algorithm when called on the full array $A[1\cdots n]$, i.e. by ProcessArray($A$, 1, $n$). (Hint: recall our analysis of MergeSort.)

**Solution:**

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 & n \geq 2 \end{cases}$$

b. Show that the solution to this recurrence is $T(n) = 2n - 1$, whence $T(n) = \Theta(n)$.

**Proof:**
Observe that when $n = 1$ we have $T(1) = 2 \cdot 1 - 1 = 1$. When $n \geq 2$ we have

$$\begin{aligned} \text{RHS} &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \\ &= (2\lfloor n/2 \rfloor - 1) + (2\lceil n/2 \rceil - 1) + 1 \\ &= 2(\lfloor n/2 \rfloor + \lceil n/2 \rceil) - 1 \\ &= 2n - 1 \\ &= T(n) \\ &= \text{LHS} \end{aligned}$$

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Problem 12

Consider the following algorithm which does nothing but waste time:

WasteTime($n$) (pre: $n \geq 1$)
1. if $n > 1$
2. for $i \leftarrow 1$ to $n^3$
3. waste 2 units of time
4. for $i \leftarrow 1$ to 7
5. WasteTime$\left(\lceil n/2 \rceil\right)$
6. waste 3 units of time

a. Write a recurrence formula for the amount of time $T(n)$ wasted by this algorithm.

**Solution:**

$$T(n) = \begin{cases} 0 & n = 1 \\ 7T(\lceil n/2 \rceil) + 2n^3 + 3 & n \geq 2 \end{cases}$$

b. Show that when $n$ is an exact power of 2, the solution to this recurrence relation is given by $T(n) = 16n^3 - \dfrac{1}{2} - \dfrac{31}{2} n^{\lg 7}$, and hence $T(n) = \Theta(n^3)$.

**Proof:**

If $n=1$ then $T(1)=16\cdot 1^3 - \frac{1}{2} - \frac{31}{2}1^{\lg 7} = 16-\frac{32}{2}=0$. When $n\geq 2$ is an exact power of 2 we have

$$\begin{aligned}
\text{RHS} &= 7T(n/2)+2n^3+3 \\
&= 7\left(16\left(\frac{n}{2}\right)^3 - \frac{1}{2} - \frac{31}{2}\left(\frac{n}{2}\right)^{\lg 7}\right) + 2n^3 + 3 \\
&= 7\left(\frac{16}{8}n^3 - \frac{1}{2} - \frac{31}{2}\left(\frac{n^{\lg 7}}{7}\right)\right) + 2n^3 + 3 \\
&= 14n^3 - \frac{7}{2} - \frac{31}{2}n^{\lg 7} + 2n^3 + \frac{6}{2} \\
&= 16n^3 - \frac{1}{2} - \frac{31}{2}n^{\lg 7} \\
&= T(n) \\
&= \text{LHS}
\end{aligned}$$

///

<u>Problem 13</u>

Define $T(n)$ by the recurrence formula

$$T(n)=\begin{cases} 1 & 1\leq n<3 \\ 2T(\lfloor n/3\rfloor) + 4n & n\geq 3 \end{cases}$$

Use Induction to show that $\forall\ n\geq 1$: $T(n)\leq 12n$, and hence $T(n)=O(n)$.

**Proof:** (Multiple base cases, strong version)

I. Observe $T(1)=1\leq 12\cdot 1$ and $T(2)=1\leq 12\cdot 2$, so the base cases are satisfied.

IId. Let $n\geq 3$ and assume for all $k$ in the range $1\leq k<n$ that $T(k)\leq 12k$. In particular, since $1\leq\lfloor n/3\rfloor<n$, we have $T(\lfloor n/3\rfloor)\leq 12\lfloor n/3\rfloor$. We must show that $T(n)\leq 12n$. Observe

$$\begin{aligned}
T(n) &= 2T(\lfloor n/3\rfloor) + 4n && \text{by the recurrence formula for } T(n) \\
&\leq 2\cdot 12\lfloor n/3\rfloor + 4n && \text{by the induction hypothesis} \\
&\leq 2\cdot 12(n/3) + 4n && \text{since } \lfloor x\rfloor\leq x \text{ for any real number } x \\
&= 8n + 4n \\
&= 12n
\end{aligned}$$

The result now holds for all $n\geq 3$. ///

Problem 15

Define $S(n)$ for $n \in Z^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Use induction to prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

**Proof:** Let $P(n)$ be the inequality $S(n) \geq \lg(n)$.

I. The inequality $S(1) \geq \lg(1)$ reduces to $0 \geq 0$, which is obviously true, so $P(1)$ holds.

IId. Let $n > 1$ and assume for all $k$ in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. Then

$$\begin{aligned} S(n) &= S(\lceil n/2 \rceil) + 1 && \text{by the definition of } S(n) \\ &\geq \lg\lceil n/2 \rceil + 1 && \text{by the induction hypothesis with } k = \lceil n/2 \rceil \\ &\geq \lg(n/2) + 1 && \text{since } \lceil x \rceil \geq x \text{ for any } x \\ &= \lg(n) - \lg(2) + 1 \\ &= \lg(n) \end{aligned}$$

showing that $P(n)$ holds. Therefore $S(n) \geq \lg(n)$ for all $n \geq 1$, as claimed. ///

Problem 16

Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i) = \Theta(nf(n))$$

(Hint: Emulate the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ for any positive integer *k*.)

**Proof:**

Since $f(n)$ is increasing we have $\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} f(n) = nf(n) = O(nf(n))$. Note also that

$$\begin{aligned} \sum_{i=1}^{n} f(i) &\geq \sum_{i=\lceil n/2 \rceil}^{n} f(i) && \text{by discarding some positive terms} \\ &\geq \sum_{i=\lceil n/2 \rceil}^{n} f(\lceil n/2 \rceil) && \text{since } f(n) \text{ is increasing} \\ &= (n - \lceil n/2 \rceil + 1) f(\lceil n/2 \rceil) && \text{by counting terms} \\ &= (\lfloor n/2 \rfloor + 1) f(\lceil n/2 \rceil) && \text{since } n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \\ &> ((n/2) - 1 + 1) f(n/2) && \text{since } f(n) \text{ is increasing, } \lceil x \rceil \geq x \text{, and } \lfloor x \rfloor > x - 1 \\ &= (n/2) f(n/2) \\ &= \Omega(nf(n)) && \text{since } f(n/2) = \Theta(f(n)) \text{, whence } f(n/2) = \Omega(f(n)) \end{aligned}$$

It follows from an exercise in the handout on Asymptotic Growth rates that $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$, as claimed.

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Problem 17

Use the result of the preceding problem to give an alternate proof of $\log(n!) = \Theta(n\log(n))$ that does not use Stirling's formula.

**Proof:**
Observe that $\log(n)$ is a positive increasing function, and that $\log(n/2) = \log(n) - \log(2) = \Theta(\log(n))$. We may therefore apply the result of problem 17 with $f(n) = \log(n)$, and properties of logarithms to get

$$\log(n!) = \sum_{i=1}^{n} \log(i) = \Theta(n\log(n))$$

as claimed. ///

Problem 18

Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$.

**Proof:** Let $P(n)$ be the statement $T(n) \leq (4/3)n^2$. Then $P(1)$ is true, since $T(1) = 1 \leq 4/3 = (4/3)\cdot 1^2$, and the base case is satisfied. Let $n > 1$ be chosen arbitrarily, and suppose for all $k$ in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. We must show as a consequence that $T(n) \leq (4/3)n^2$. Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. ///