CMPS 101
Homework Assignment 3

1. The last exercise in the handout entitled *Some Common Functions*.

   Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

2. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

   $$\sum_{i=1}^{n} f(i) = \Theta(nf(n))$$

   (Hint: follow the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ for any positive integer k.)

3. Use the result of the preceding problem to give an alternate proof of $\log(n!) = \Theta(n\log(n))$ that does not use Stirling's formula.

4. Let $g(n)$ be an asymptotically non-negative function. Prove that $o\big(g(n)\big) \cap \Omega\big(g(n)\big) = \emptyset$.

5. Exercise 1 from the induction handout.

   Prove that for all $n \geq 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

   a. Using form IIa of the induction step.
   b. Using form IIb of the induction step.

6. Exercise 2 from the induction handout.
   Define $S(n)$ for $n \in Z^+$ by the recurrence:

   $$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

   Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

7. Let $T(n)$ be defined by the recurrence formula:

   $$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$. (Hint: follow Example 3 on page 3 of the induction handout.)

8. Let $T(n)$ be defined by the recurrence formula:

$$T(n) = \begin{cases} 2 & n = 1, 2 \\ 9T(\lfloor n/3 \rfloor) + 1 & n \geq 3 \end{cases}$$

Show that $\forall n \geq 1$: $T(n) \leq 3n^2 - 1$, and hence $T(n) = O(n^2)$. (Hint: emulate Example 4 on page 4 of the induction handout. I. Base: check the two cases $n = 1$, and $n = 2$. II. Induction step: show that for all $n \geq 3$, if for any k in the range $1 \leq k < n$ we have $T(k) \leq 3k^2 - 1$, then $T(n) \leq 3n^2 - 1$.)