CMPS 101
Homework Assignment 5 Solutions

1. (This is the 2nd exercise on page 1 of the handout on recurrence relations.) Define the function $S(n)$ by the recurrence

$$S(n) = \begin{cases} 0 & n = 1 \\ S(\lceil n/2 \rceil) + 1 & n \ge 2 \end{cases}$$

Use the iteration method to show that $S(n) = \lceil \lg(n) \rceil$, and hence $S(n) = \Theta(\log(n))$.

Proof:
We have

$$\begin{aligned} S(n) &= 1 + S(\lceil n/2 \rceil) \\ &= 1 + 1 + S\left(\left\lceil \frac{\lceil n/2 \rceil}{2} \right\rceil\right) = 2 + S(\lceil n/2^2 \rceil) \\ &= 2 + 1 + S\left(\left\lceil \frac{\lceil n/2^2 \rceil}{2} \right\rceil\right) = 3 + S(\lceil n/2^3 \rceil) \\ &\vdots \\ &= k + S(\lceil n/2^k \rceil) \end{aligned}$$

The recurrence terminates at the first (i.e. smallest) recursion depth k for which $\lceil n/2^k \rceil = 1$. This is equivalent to

$$\begin{aligned} & 0 < n/2^k \le 1 \\ \therefore\ & 0 < n \le 2^k \\ \therefore\ & \log_2(n) \le k \end{aligned}$$

Since we seek the smallest such k, we have $k - 1 < \log_2(n) \le k$ and hence $k = \lceil \log_2(n) \rceil$. For this value of k then, $S(\lceil n/2^k \rceil) = 0$, and therefore $S(n) = \lceil \lg(n) \rceil$. ■

2. Consider the function $T(n)$ defined by the recurrence formula

$$T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}$$

   a. Use the iteration method to write a summation formula for $T(n)$.
   Solution:

$$\begin{aligned} T(n) &= n + 2T(\lfloor n/3 \rfloor) \\ &= n + 2(\lfloor n/3 \rfloor + 2T(\lfloor \lfloor n/3 \rfloor /3 \rfloor)) \\ &= n + 2\lfloor n/3 \rfloor + 2^2 T(\lfloor n/3^2 \rfloor) \\ &= n + 2\lfloor n/3 \rfloor + 2^2 \lfloor n/3^2 \rfloor + 2^3 T(\lfloor n/3^3 \rfloor) \quad \text{etc..} \end{aligned}$$

   After substituting the recurrence into itself k times, we get

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k T(\lfloor n/3^k \rfloor).$$

This process terminates when the recursion depth k is chosen so that $1 \le \lfloor n/3^k \rfloor < 3$, which is equivalent to $1 \le n/3^k < 3$, whence $3^k \le n < 3^{k+1}$, so $k \le \log_3(n) < k+1$, and hence $k = \lfloor \log_3(n) \rfloor$. With this value of k we have $T(\lfloor n/3^k \rfloor) = T(1 \text{ or } 2) = 6$. Therefore

$$T(n) = \sum_{i=0}^{\lfloor \log_3(n) \rfloor - 1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 6 \cdot 2^{\lfloor \log_3(n) \rfloor}.$$

■

b. Use the summation in (a) to show that $T(n) = O(n)$

Solution:

Using the above summation, we have

$$\begin{aligned}
T(n) &\le n \left(\sum_{i=0}^{\lfloor \log_3(n) \rfloor - 1} (2/3)^i \right) + 6 \cdot 2^{\log_3(n)} && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\
&\le n \left(\sum_{i=0}^{\infty} (2/3)^i \right) + 6 n^{\log_3(2)} && \text{adding } \infty\text{-many positive terms} \\
&= n \left(\frac{1}{1-(2/3)} \right) + 6 n^{\log_3(2)} && \text{by a formula for geometric series} \\
&= 3n + 6 n^{\log_3(2)} = O(n) && 2 < 3 \Rightarrow \log_3(2) < 1 \Rightarrow n^{\log_3(2)} = o(n)
\end{aligned}$$

Therefore $T(n) = O(n)$. ■

c. Use the Master Theorem to show that $T(n) = \Theta(n)$

Solution:

Let $\varepsilon = 1 - \log_3(2) > 0$. Then $\log_3(2) + \varepsilon = 1$, and $n = n^{\log_3(2)+\varepsilon} = \Omega(n^{\log_3(2)+\varepsilon})$. Also for any c in the range $2/3 \le c < 1$, and any positive n, we have $2(n/3) = (2/3)n \le cn$, so the regularity condition holds. By case (3) of the Master Theorem $T(n) = \Theta(n)$. ■

3. Use the Master theorem to find asymptotic solutions to the following recurrences.

a. $T(n) = 7T(n/4) + n$

Solution:

$4 < 7 \Rightarrow 1 < \log_4(7) \Rightarrow \log_4(7) - 1 > 0$. Let $\varepsilon = \log_4(7) - 1$. Then $\varepsilon > 0$, and $1 = \log_4(7) - \varepsilon$, whence $n = n^{\log_4(7)-\varepsilon} = O(n^{\log_4(7)-\varepsilon})$. By case (1) we have $T(n) = \Theta(n^{\log_4(7)})$. ■

b. $T(n) = 9T(n/3) + n^2$

Solution:

Observe that $n^2 = n^{\log_3(9)} = \Theta(n^{\log_3(9)})$, and therefore $T(n) = \Theta(n^2 \log(n))$ by case (2). ■

c. $T(n) = 6T(n/5) + n^2$

Solution:

Observe $6 < 25 \Rightarrow \log_5(6) < 2 \Rightarrow 2 - \log_5(6) > 0$. Let $\varepsilon = 2 - \log_5(6)$. Then $\log_5(6) + \varepsilon = 2$, and $n^2 = \Omega(n^{\log_5(6)+\varepsilon})$. Also for any c in the range $6/25 \le c < 1$, and for any positive n, we have $6(n/5)^2 = (6/25)n^2 \le cn^2$, so the regularity condition holds. Therefore $T(n) = \Theta(n^2)$ by case (3) of the Master Theorem. ■

d. $T(n) = 6T(n/5) + n\log(n)$

Solution:

Observe $\log_5(6) > 1$, so letting $\varepsilon = \dfrac{\log_5(6) - 1}{2}$, we have $\varepsilon > 0$ and $1 + \varepsilon = \log_5(6) - \varepsilon$. Therefore by l'Hopital's rule

$$\lim_{n\to\infty} \frac{n\log(n)}{n^{\log_5(6)-\varepsilon}} = \lim_{n\to\infty} \frac{n\log(n)}{n^{1+\varepsilon}} = \lim_{n\to\infty} \frac{\log(n)}{n^{\varepsilon}} = 0,$$

showing that $n\log(n) = o(n^{\log_5(6)-\varepsilon}) \subseteq O(n^{\log_5(6)-\varepsilon})$. Case (1) now gives $T(n) = \Theta(n^{\log_5(6)})$. ■

e. $T(n) = 7T(n/2) + n^2$

Solution:

Observe that $7 > 4 \Rightarrow \log_2(7) > 2$, so upon setting $\varepsilon = \log_2(7) - 2$ we have $\varepsilon > 0$. It follows that $2 = \log_2(7) - \varepsilon$, whence $n^2 = n^{\log_2(7)-\varepsilon} = O(n^{\log_2(7)-\varepsilon})$. Case 1 now gives $T(n) = \Theta(n^{\log_2(7)})$. ■

f. $S(n) = aS(n/4) + n^2$ (Note: your answer will depend on the parameter a.)

Solution:

We have three cases to consider corresponding to the three cases of the Master Theorem:

Case 1:

$a > 16 \Rightarrow \log_4(a) > 2 \Rightarrow \varepsilon = \log_4(a) - 2 > 0 \Rightarrow n^2 = O(n^{\log_4(a)-\varepsilon})$, so $S(n) = \Theta(n^{\log_4(a)})$.

Case 2:

$a = 16 \Rightarrow \log_4(a) = 2 \Rightarrow n^2 = \Theta(n^{\log_4(a)})$, whence $S(n) = \Theta(n^2\log(n))$.

Case 3:

$1 \le a < 16 \Rightarrow \log_4(a) < 2 \Rightarrow \varepsilon = 2 - \log_4(a) > 0 \Rightarrow n^2 = \Omega(n^{\log_4(a)+\varepsilon})$. Further, for any c in the range $a/16 \le c < 1$ we have $a(n/4)^2 = (a/16)n^2 \le cn^2$, showing that the regularity condition holds. Therefore $S(n) = \Theta(n^2)$. ■

4. p.75: 4.3-2

The recurrence $T(n) = 7T(n/2) + n^2$ describes the running time of an algorithm *A*. A competing algorithm *B* has a running time given by $S(n) = aS(n/4) + n^2$. What is the largest integer value for a such that *B* is a faster algorithm than *A* (asymptotically speaking)? In other words, find the largest integer a such that $S(n) = o(T(n))$.

Solution:
We seek the largest integer a for which $S(n) = o(T(n))$. Using parts (e) and (f) of the previous problem, we find that $S(n) = o(T(n))$ in cases 2 and 3 since $4 < 7 \Rightarrow 2 < \log_2(7)$ and hence $n^2 = o(n^{\log_2(7)})$, and $n^2 \log(n) = o(n^{\log_2(7)})$. In case 1 we have $S(n) = o(T(n))$ if and only if $n^{\log_4(a)} = o(n^{\log_2(7)})$, i.e. if and only if $\log_4(a) < \log_2(7)$. This is equivalent to $a < 4^{\log_2(7)} = 7^{\log_2(4)} = 7^2 = 49$. The largest such integer is $a = 48$. ■

5. Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + f(n)$, where $f(n)$ is a polynomial satisfying $\deg(f) > \log_b(a)$. Prove that case (3) of the Master Theorem applies, and in particular that the regularity condition necessarily holds.

Proof:
Let $d = \deg(f)$ and replace $f(n)$ by the asymptotically equivalent function n^d. The Master Theorem tells us to compare the polynomials n^d and $n^{\log_b(a)}$. Let $\varepsilon = d - \log_b(a)$, which is positive since $d > \log_b(a)$. Therfore $d = \log_b(a) + \varepsilon$, and hence

$$n^d = \Omega(n^d) = \Omega\left(n^{\log_b(a)+\varepsilon}\right)$$

verifying the first hypothesis of case (3). To prove that the regularity condition holds, observe that $d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1$. Pick any c in the range $a/b^d \le c < 1$. Then for any $n \ge 1$ we have $a(n/b)^d = (a/b^d)n^d \le cn^d$, which verifies the regularity condition. ■

6. Show that the number vertices of odd degree in any graph must be even. (Hint: Use the Handshake Lemma mentioned in the Graph Theory handout.)

Proof:
The handshake lemma says: $\sum_{x \in V(G)} \deg(x) = 2|E(G)|$. Let $E = \{x \in V(G) \mid \deg(x) \text{ is even}\}$ and $O = \{y \in V(G) \mid \deg(y) \text{ is odd}\}$. The handshake lemma can then be written as:

$$\sum_{x \in E} \deg(x) + \sum_{y \in O} \deg(y) = 2|E(G)|$$

The right hand side of the above equation is obviously even and the first term on the left hand side is also even, being the sum of even numbers. Therefore the second term on the left hand side is even as well. Observe that the sum of an odd number of odd numbers is necessarily odd, while the sum of an even number of odd numbers is even. It follows that the sum

$$\sum_{y \in O} \deg(y)$$

contains an even number of terms. Therefore G contains an even number of odd-degree vertices. ■