PUMPING LEMMA

IF L IS REGULAR THEN

(P) THERE EXISTS A $p \geq 0$ SUCH THAT FOR ANY STRING $s \in L$ WITH $|s| \geq p$, THERE EXIST STRINGS xyz SUCH THAT $s = xyz$, $y \neq \epsilon$, $|xy| \leq p$, AND FOR ALL $i \geq 0$, THE STRING $xy^iz \in L$

CONTRAPOSITIVE – USED TO PROVE LANGS. NOT REGULAR

$\neg P \Rightarrow$ L IS NOT REGULAR.

($\neg P$) FOR ALL $p \geq 0$ THERE EXISTS A STRING $s \in L$ WITH $|s| \geq p$, AND FOR ALL x, y, z. SUCH THAT $xyz = s$, $y \neq \epsilon$, $|xy| \leq p$ THERE EXISTS AN $i \geq 0$ SUCH THAT $xy^iz \notin L$.

DEAMON GAME

DEAMON PICKS p

WE PICK $s \in L$ AND $|s| \geq p$

DEAMON PICKS x, y, z $= s$ $xy \leq p$ $y \neq \varepsilon$

WE PICK i

& PROVE, REGARDLESS OF DEAMON x, y, z PICK

$xy^iz \notin L$

L REGULAR IMPLIES

$$(\exists p)(\forall s)\Big[s \in L \text{ and } |s| \geq p \Rightarrow$$

$$(\exists x, y, z)\big(s = xyz,\ |xy| \leq p,\ |y| \geq 1 \text{ and}$$

$$(\forall i \geq 0)(x y^i z \in L)\big)\Big]$$

PUMPING LEMMA PROOF

LET $M = (Q, \Sigma, \delta, q_0, F)$ BE DFA RECOG. $A = L(M)$

LET $p = |Q|$ NUMBER OF STATES

LET $w = a_1 a_2 \ldots a_m$ BE A STRING IN A
AND $m \geq p$

NOTE $|w| = m$

LET $q_0, s_1, s_2, \ldots, s_m \in F$ BE SEQ OF
STATES THAT PROCESSES w

I.E. $s_{i+1} = \delta(s_i, a_{i+1})$

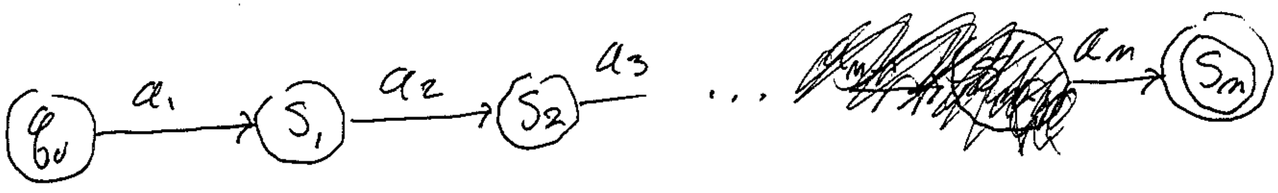
IN THE FIRST $p+1$ STATES OF SEQ ~~STATES I.E. $\geq p+1$ STATES~~
~~S SEQ HAS $m+1$~~

AT LEAST TWO MUST BE THE SAME. PIGEON HOLE PRINCIPLE

CALL FIRST PAIR s_j AND s_l NOTE $l \leq p$

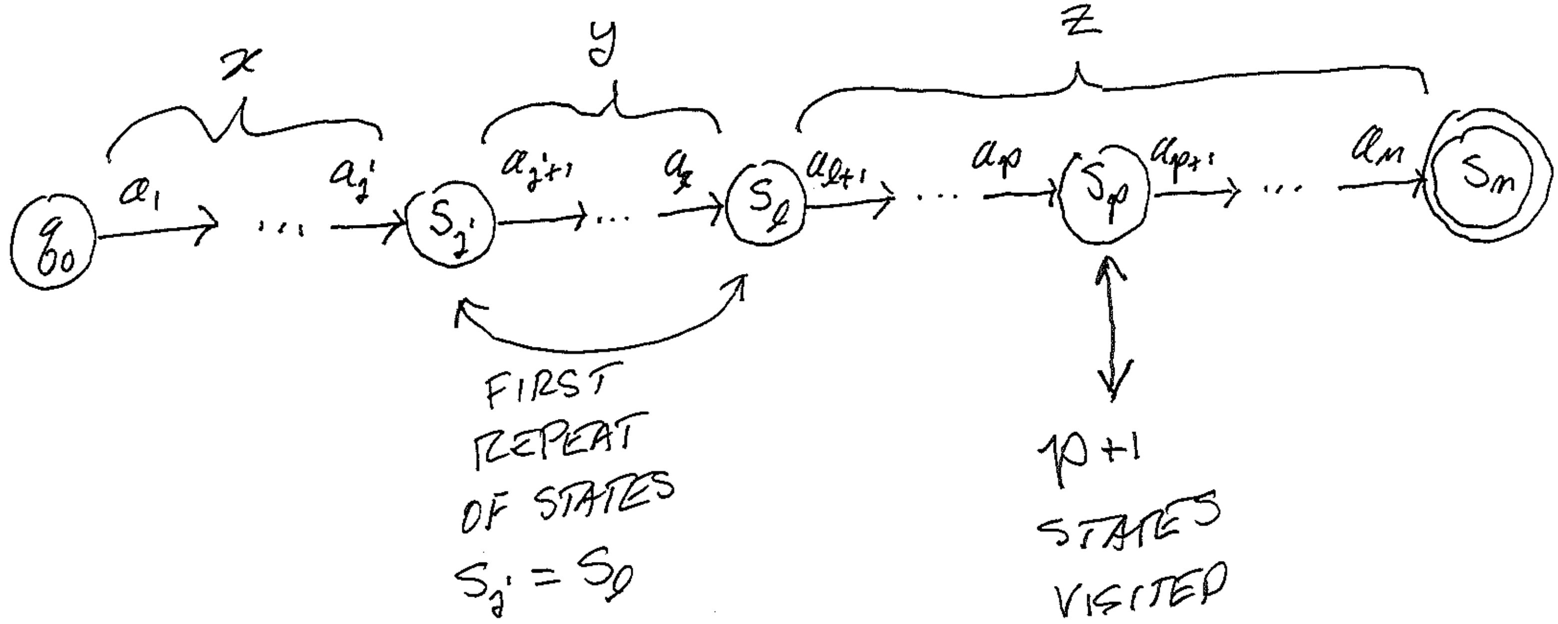
NOW LET $x = a_1 \ldots a_j$
$y = a_{j+1} \ldots a_l$
$z = a_{l+1} \ldots a_m$

x TAKES M FROM q_0 TO s_j
y s_j TO s_l (NOTE $s_j = s_l$ SO y LOOPS)
z s_l TO s_m



$m \geq p$

FOR $m = p$
THERE ARE $p+1$
STATES VISITED



S_i INDICATES i-th STATE VISITED
NOT NECESSARILY THE STATE LABEL.

CAN INSERT ARBITRARY NUMBER OF y'S AND STRING MUST STILL BE ACCEPTED.

M MUST ACCEPT xy^iz FOR ALL $i \geq 0$

NOTE $j \neq l$ SO $|y| > 0$

NOTE $l \leq p$ SO $|xy| \leq p$

Q.E.D.

EXAMPLE OF USING PUMPING LEMMA

$\boxed{A = \{ 0^n 1^n \mid n \geq 0 \} \text{ is NOT REGULAR.}}$

DEAMON PICKS p

WE PICK s s.t. $|s| \geq p$

$s = 0^p 1^p$

THEN FOR ANY PARTITION $s = xyz$

s.t. $|xy| \leq p$

$|y| > 0$

WE CAN FIND AN $i \geq 0$ s.t.

$xy^i z \notin A$

LET $i = 2$

NOTE $|xy| \leq p \Rightarrow xy$ CONSISTS OF ONLY 0's

$|y| > 0 \Rightarrow y$ ONE OR MORE 0's

SO xy^2z HAS ~~MORE~~ ~~1 OR~~ MORE 0s THAN 1s SO $\notin A$.

USING CLOSURE PROPERTIES.

$L'' = \{ \omega \mid \#\text{ of } 0s = \#\text{ of } 1s \}$

p

$s = 0^p 1^p$ ← WE CAN PICK ANY STRING THAT FITS CRITERIA, DON'T HAVE TO PICK ONE THATS DIFFICULT.

$s = xyz$ $\quad \underbrace{|y| \geq 1, \ |xy| \leq p}_{\Rightarrow y \text{ all } 0s}$ AND $\forall i \geq 0$ $\ xy^iz \in L$

$i = 2 \Rightarrow$ MORE 0s THAN 1s CONTRADICTION

ANOTHER PROOF ~~FOR~~ LET $L = \{ 0^n 1^n \mid n \geq 1 \}$ KNOW NOT REG

$$\underset{\substack{\uparrow \\ \text{NOT REG}}}{L} = L'' \cap \underset{\substack{\uparrow \\ \text{REG}}}{0^* 1^*} \Rightarrow L'' \text{ NOT REG.}$$

EXAMPLE 2 PROVING NOT REG.

LET $B = \{ w \mid w$ HAS EQUAL # OF 1's AND 0's $\}$

B IS NOT REGULAR.

WE KNOW OR CAN PROVE

$$C = \{ 0^n 1^m \mid n \geq 0,\ m \geq 0 \}$$

IS REGULAR.

$$B \cap C = A = \{ 0^n 1^m \mid \geq m \geq 0 \}$$ NOT REG

TOWARD CONTRADICTION ASSUME B REG.

THEN $B \cap C$ MUST BE REG. CONTRADICTION.

Q.E.D

$L = \{ 1^j 0^i \mid j < i \}$ IS NOT REGULAR

DEAMON PICKS $p \geq 0$

WE PICK $S = 1^p 0^{p+1}$

NOTE $S \in L$ AND $|S| \geq p$

DEAMON PICKS $x\, y\, z$ SUCH THAT

$xyz = s$

$|xy| \leq p$

$|y| > 0$

WE PICK $i = 2$

THEN $x y^2 z \notin L$ BECAUSE

$|xy| \leq p$ AND $|y| > 0 \Rightarrow y$ IS ONE OR MORE 1s

AND $xy^2z = 1^{\geq p+1} 0^{p+1} \notin L$.

$L = \{ 0^i 1^j \mid i > j \}$ IS NOT REGULAR

NOTE L REG $\Rightarrow$ L^R IS REG

SO L^R NOT REG $\Rightarrow$ L IS NOT REGULAR.

$L^R = \{ 1^j 0^i \mid j < i \}$

ALSO PUMPING DOWN

p

$s = 0^{p+1} 1^p$

$xyz = 0^{p+1} 1^p$

$|xy| \leq p$

$|y| > 0$

PICK $i = 0$

THEN

$xy^0z = 0^{\leq p} 1^p \notin L$

BOOK SIPSER

$F = \{ ww \mid w \in \{0,1\}^* \}$ IS NOT REGULAR

p

$S = 0^p 1 0^p 1$

FOR ANY $xyz = 0^p 1 0^p 1$

WITH $|xy| \leq p$

WE HAVE x AND y ARE ONLY 0s

SO WITH $|y| > 0$ y MUST BE ONLY 0s

SO LET $i = 2$ THEN

A $xy^2z \notin F$

BECAUSE THERE WILL BE MORE 0s BEFORE THE FIRST 1 THAN BEFORE THE SECOND ONE.

NOTE 1 MUST END THE STRING W

PUMPING LEMMA FOR PALINDROMES

$L = \{ w \mid w = w^R \}$

L IS NOT REGULAR

PROOF:

DRAGON PICKS p

WE PICK $S = 0^p 1 0^p$ CLEARLY $\in L$

DRAGON PICKS SOME xyz SUCH THAT

$xyz = S$

$|xy| \leq p$

$|y| > 0$

WE SHOW THAT FOR ANY SUCH CHOICE WE CAN PICK AN i SUCH THAT

$xy^iz \notin L$

NOTE xy CONSISTS ONLY OF 0's
SO $i \geq 2$ PRODUCES

$xy^iz = 0^{>p} 1 0^p \notin L$

$0^p 1 1 0^p$ WORKS ALSO

$L = \{0^m 1^n \mid m \neq n\}$

1.46 (b)

$s = 0^p 1^{p+p!}$

$xyz = 0^p 1^{p+p!}$

$\left.\begin{matrix} |xy| \leq p \\ |y| > 0 \end{matrix}\right\} \Rightarrow$ y IS ONE OR MORE 0s
SAY $1 \leq k \leq p$

PICK $i = \frac{p!}{k} + 1$ SO $|y| = k$ AND $|y^i| = k \cdot i = k \cdot \frac{p!}{k} + k = p! + k$

THEN
$xyz = 0^{p-k} 0^{p!+k} 1^{p+p!}$

$= 0^{p+p!} 1^{p+p!} \notin L$

TOWARD CONTRADICTION ASSUME L REGULAR
THEN $\overline{L}$ IS REGULAR
$\overline{L} \cap 0^*1^*$ REGULAR
I.E. $\{0^m 1^n \mid m = n\}$ REG   CONTRADICTION!

WHY $p!$?

FIND STRING IN L THAT WHEN PUMPED BECOMES 0^n1^n

$$0^p 1^{p+m} \quad \& \ m > 0$$

WANT #1s TO BE DIFF. THAN #0s

$$0^{p-k}0^{k})^{i}$$

WANT PUMPING TO MAKE EQUAL TO $p+m$

$$p-k+ki = p+m$$

$$k(i-1) = m$$

$$(i-1) = \frac{m}{k}$$

$$i = \frac{p!}{k} + 1$$

SO WANT $k \mid m$

BUT K CAN BE 1 TO p

SO let $m = p!$

SIPSER EX 1.76

$$L = \{ 1^{n^2} \mid n \geq 0 \}$$

DRAGON PICKS p

WE PICK $s = 1^{p^2}$ NOTE $s \in L$ AND $|s| \geq p$

DRAGON PICKS x, y, z SUCH THAT

$xyz = s$

$|xy| \leq p$

$|y| \geq 1$

WE PICK $i = 2$ THEN

NOTE $y \leq p$

$$|xy^2z| - |xyz| = |y| \leq p$$

$$p^2$$

$$|xy^2z| \leq p^2 + p$$ ← ADDING p^2 TO BOTH SIDES

CLAIM CANNOT BE PERFECT SQUARE

BECAUSE $p^2 < p^2 + p < (p+1)^2$ INBETWEEN CONSEQUTIVE SQS.

$$p^2 + 2p + 1$$

FOR FINITE SET p MUST BE GREATER THAN LONGEST STRING OTHERWISE THE SET COULD NOT BE FINITE

THESE ARE ALL REGULAR SETS

	max length string	MINIMUM p
{ [illegible] }	[illegible]	0
{ ε }	0	1
{ 1 }	1	2

NO STRINGS LENGTH p OR GREATER