NONDETERMINISTIC FINITE AUTOMATON

$N = (Q, \Sigma, \delta, s, F)$

Q A FINITE SET OF STATES

Σ A FINITE SET OF SYMBOLS - ALPHABET

$\delta : Q \times \Sigma \longrightarrow 2^Q$ TRANSITION FUNCTION / RELATION ON $(Q \times \Sigma) \times Q$

$\delta(p, a) \in 2^Q$

SET OF ALL STATES N CAN MOVE TO FROM p IN ONE STEP UNDER INPUT SYMBOL a.

$p \xrightarrow{a} q$ if $q \in \delta(p, a)$

$\delta(p, a)$ CAN BE $\emptyset$

$s \in Q$ IS THE "START" STATE

$F \subseteq Q$ "ACCEPT" OR "FINAL" STATES

EXTENDED TRANSITION FUNCTION

$\hat{\delta} : Q \times \Sigma^* \rightarrow 2^Q$

$\hat{\delta}(q, \varepsilon) = \{q\}$

$\hat{\delta}(q, a) = \delta(q, a)$

$$\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q, x)} \delta(p, a)$$

$\hat{\delta}(p, w)$ SET OF ALL STATES N CAN MOVE TO FROM p UNDER INPUT STRING w.

ACCEPTED

$w \in \Sigma^*$ IS ACCEPTED BY N IF

$$\delta^*(s, w) \cap F \neq \emptyset$$

LANGUAGE N RECOGNIZES (ACCEPTS)

$$L(N) = \{ w \in \Sigma^* \mid N \text{ accepts } w \}$$

NFA Definitions

A **Non-deterministic Finite Automaton**, NFA, is a structure M such that:

$M = (Q, \Sigma, \delta, s, F)$, where
Q is a finite set of "states".
Σ is a finite set of "symbols", an "alphabet".
$\delta : Q \times \Sigma \to 2^Q$ is the "transition function".
For $p \in Q$, $A \in 2^Q$ and $a \in \Sigma$, $\delta(p, a) = A$ means
when in state p scanning symbol a, a non-deterministic transition
is made to one of the states in the set A
$s \in Q$ is the "start state".
$F \subseteq Q$ is the set of "final states".

The **extended transition function** for M is the function:

$$\hat{\delta} : Q \times \Sigma^* \to 2^Q, \quad \text{where}$$
$$\hat{\delta}(q, \epsilon) = \{q\}, \quad \text{and}$$
$$\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)$$

A string $x \in \Sigma^*$ is **accepted** by M if $\hat{\delta}(s, x) \cap F \neq \emptyset$.

The language **accepted** or **recognized** by M is
$L(M) = \{x \in \Sigma^* \mid x \text{ is accepted by } M\}$.

EVERY DFA IS ALSO A NFA

THM: LET $M = (Q, \Sigma, \delta, s, F)$ A DFA

AND $N = (Q, \Sigma, \Delta, s, F)$ A NFA

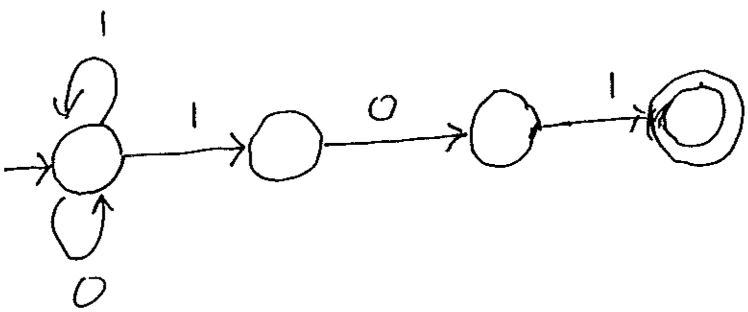
WHERE $\Delta(p, a) = \{\delta(p, a)\}$

THEN $x \in L(M)$ iff $x \in L(N)$

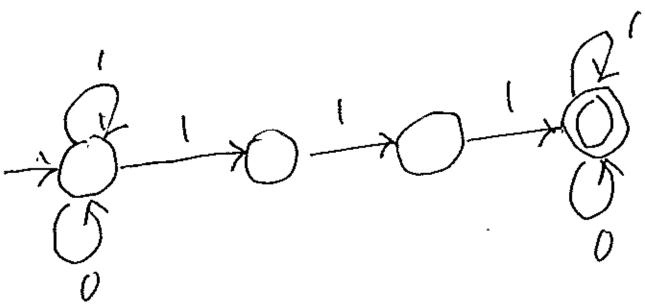
Proof: obvious.

NFA EXAMPLES

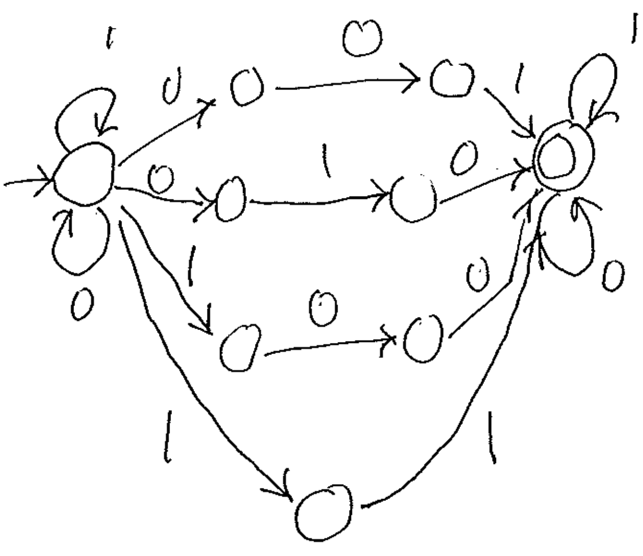
STRING ENDS WITH 101



STRING CONTAINS 111



STRING CONTAINS 001 v 010 v 100 v 11



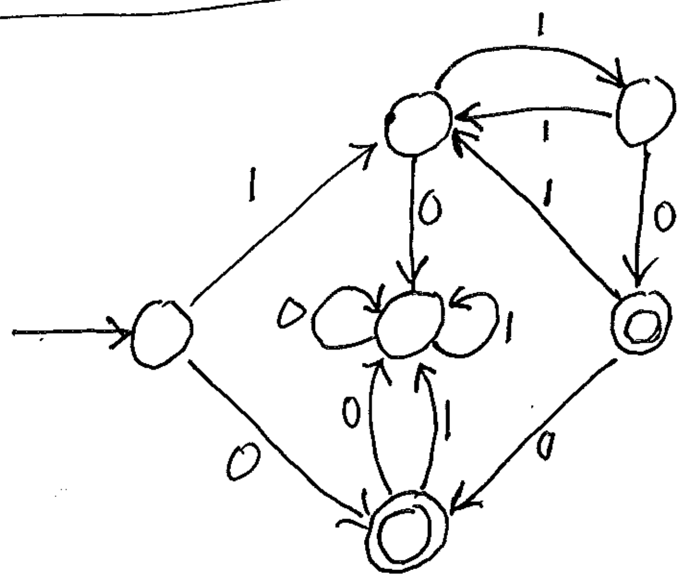
NFA EASIER TO DESIGN THAN DFA

FROM MARTIN

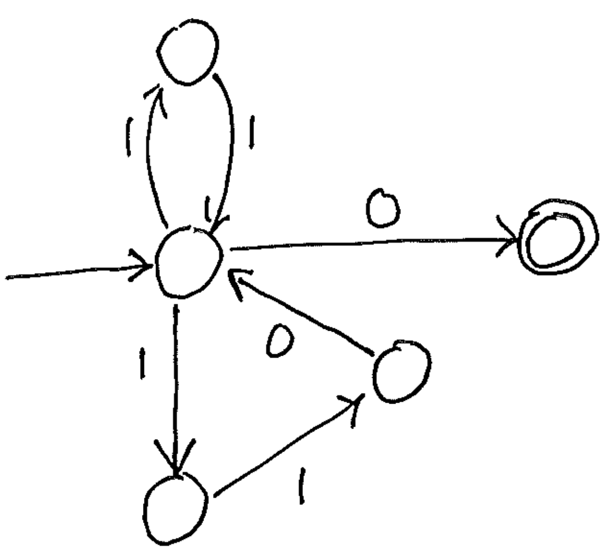
LANGUAGE: $(11 + 110)^* 0$

~~where $x \in \{11, 110\}$~~

$$\boxed{\{ x_1 x_2 \ldots x_n 0 \mid n \geq 0 \text{ AND } x_i \in \{11, 110\} \}}$$



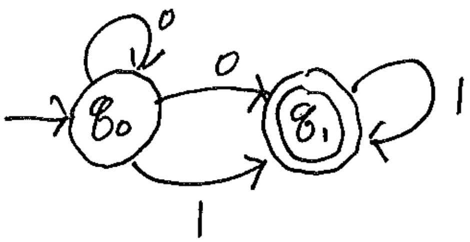
DFA

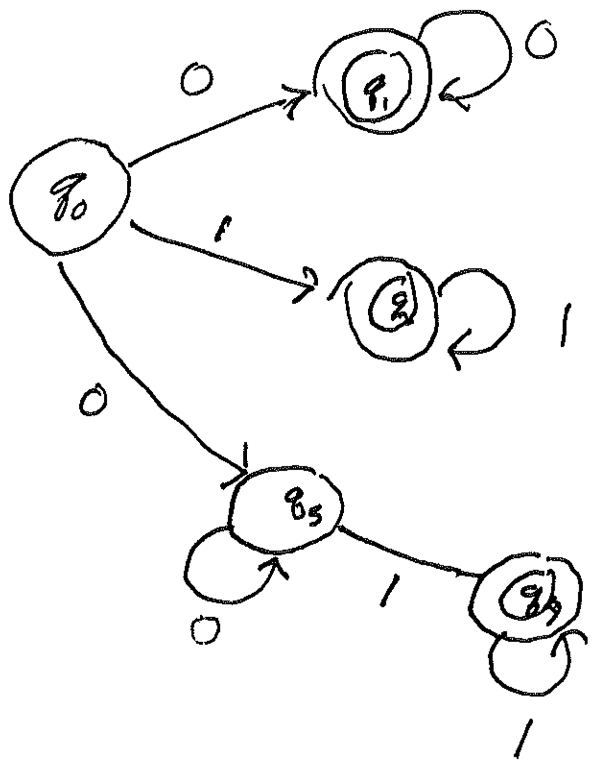


NFA

EXAMPLE OF NFA

$$L(M) = \{0^n \mid n \geq 1\} \cup \{1^n \mid n \geq 1\} \cup \{0^n 1^m \mid n \geq 1, m \geq 1\}$$





q_0
q_1
q_2
q_5
q_3
0
1
0
0
1
0
1
1

SUBSET CONSTRUCTION RABIN - SCOTT

GIVEN NFA

δ_N^* :

$$N = (Q_N, \Sigma, \delta_N, s_N, F_N)$$

$\vdots$

$$\delta_N : Q_N \times \Sigma \to 2^{Q_N}$$

$\delta_N(p, a)$ IS SET OF ALL STATES THAT N CAN MOVE TO FROM p GIVEN a AS AN INPUT.

$\hat{\delta}_N(q, \epsilon) = \{q\}$

$\hat{\delta}_N(q, xa) = \bigcup_{p \in \hat{\delta}_N(q, x)} \delta_N(p, a)$

CONSTRUCT DFA

$$M = (Q_D, \Sigma, \delta_D, s_D, F_D)$$

$$Q_D = 2^{Q_N}$$

$$s_D = \{s_N\}$$

$$F_D = \{P \subseteq Q_N \mid P \cap F_N \neq \emptyset\}$$

$$\delta_D : Q_D \times \Sigma \to Q_D \qquad 2^{Q_N} \times \Sigma \to 2^{Q_N}$$

$$\delta_D(P, a) = \bigcup_{p \in P} \delta_N(p, a) \quad \text{FOR } P \subseteq Q_N$$

$$P \in 2^{Q_N}$$

$$\hat{\delta}_D(P, \epsilon) = P$$

$$\hat{\delta}_D(P, xa) = \delta_D(\hat{\delta}_D(P, x), a)$$

THM $L(N) = L(M)$

SHOW THAT M FAITHFULLY SIMULATES N
SHOW TRANSITION FUNCTIONS ARE REALLY THE SAME.

FOR EVERY p of N and string w LEMMA

$$\overset{*}{\delta}_D(\{p\}, w) = \overset{*}{\delta}_N(p, w)$$

CAN PROVE BY INDUCTION

THIS IMPLIES $L(N) = L(M)$

PROOF
w ACCEPTED N iff $\overset{*}{\delta}_N(s_N, w) \cap F_N \neq \emptyset$ DEF.

iff
$$\overset{*}{\delta}_D(\{s_N\}, w) \cap F_N \neq \emptyset$$ LEMMA

iff
$$\overset{*}{\delta}_D(\{s_N\}, w) \in F_D$$ DEF OF F_D

BUT THIS IS ACCEPTANCE BY M
iff $w \in L(M)$

LEMMA

$$\delta_D^*(\{p\}, w) = \delta_N^*(p, w)$$

BASE $\quad w = \epsilon$

$$\delta_D^*(\{p\}, \epsilon) \qquad \{p\}$$

DEF of δ_D^*

$$\delta_N^*(p, \epsilon) = \{p\} \quad \checkmark$$

DEF δ_N^*

INDUCTIVE STEP

ASSUME $\quad \delta_D^*(\{p\}, x) = \delta_N^*(p, x) \qquad$ I.H.

PROVE

$$\delta_D^*(\{p\}, xa) = \delta_N^*(p, xa)$$

- - - - - - - - - - - - - - -

$$\delta_D^*(\{p\}, xa) = \delta_D(\delta_D^*(\{p\}, x), a) \qquad \text{DEF } \delta_D^*$$

$$= \delta_D(\delta_N^*(p, x), a) \qquad \text{I.H.}$$

$$= \bigcup_{q \in \delta_N^*(p,x)} \delta_N(q,a) \qquad \text{DEF } \delta_D$$

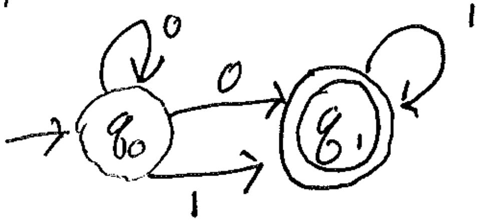
$$= \delta_N^*(p, xa) \qquad \text{DEF } \delta_N^*$$

Q.E.D.

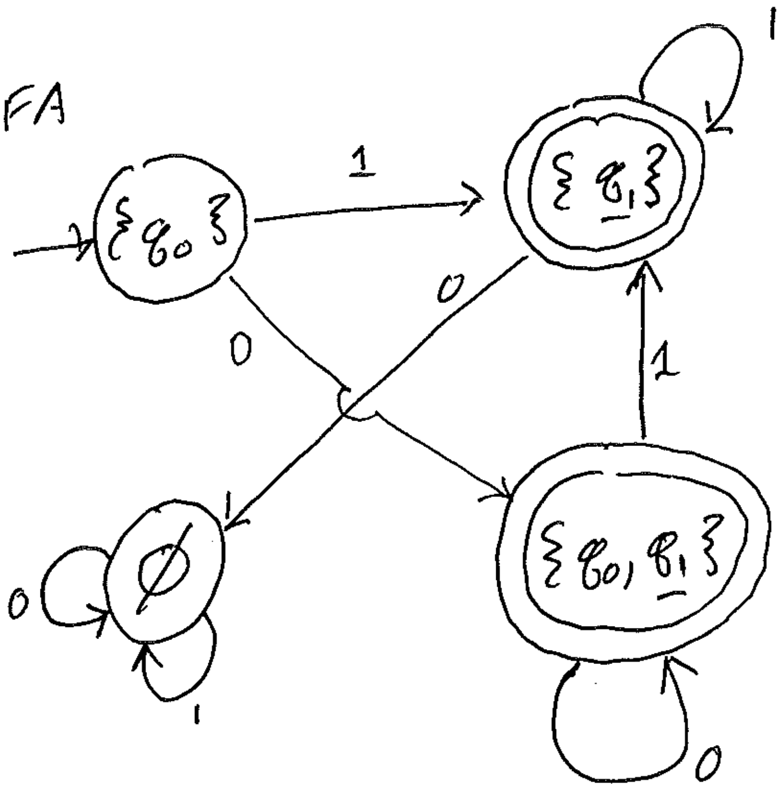
EXAMPLE OF SUBSET CONSTRUCTION I

$$L(M) = \{0^n \mid n \geq 1\} \cup \{1^n \mid n \geq 1\} \cup \{0^n 1^m \mid n \geq 1, m \geq 1\}$$

NFA



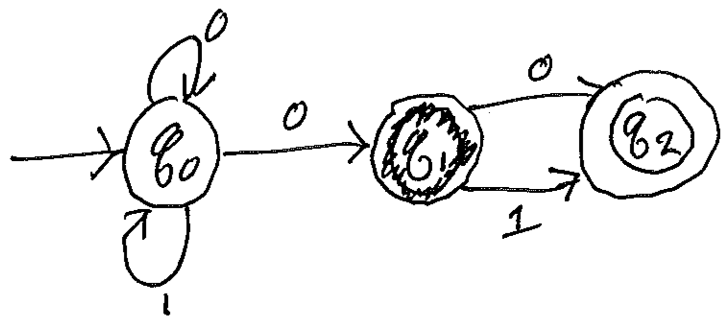
DFA



EXAMPLE OF SUBSET CONSTRUCTION II

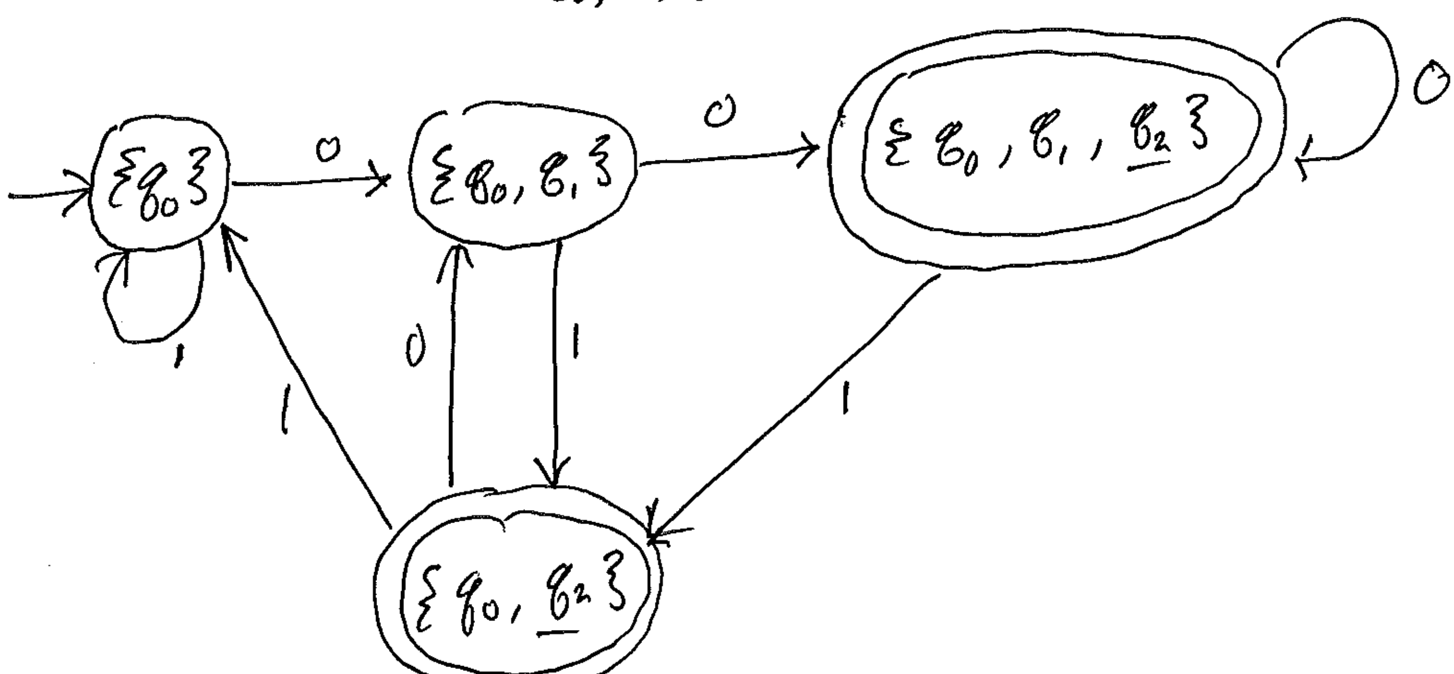
$L_2 = \{ w \in \{0,1\}^* \mid \text{2nd symbol from right is a } 0 \}$

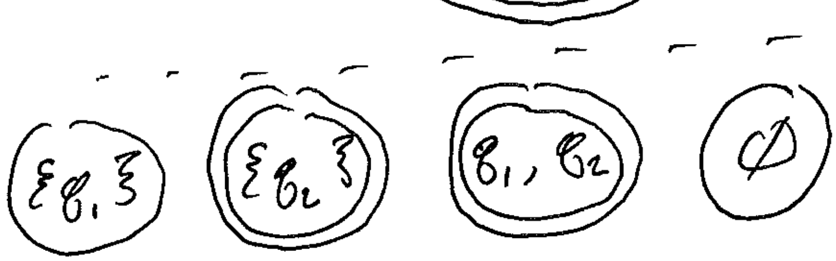
NFA



DFA

CONSIDER 8 STATES $\emptyset$, $\{q_0\}$, $\{q_1\}$, $\{q_2\}$,
$\{q_0, q_1\}$, $\{q_0, q_2\}$, $\{q_1, q_2\}$
$\{q_0, q_1, q_2\}$





THESE ARE UNREACHABLE

Exponential Blowup of Subset Construction

Definition: $L_n = \{w \in \{0,1\}^* | w \texttt{ has a 1 in the } n\texttt{th position from the end}\}$. Note an NFA with only $n+1$ states can be designed to recognized L_n.

Theorem: A DFA that recognizes L_n can not have less than 2^n states.

Proof: Toward a contradiction assume that a DFA M exists with start state s and that recognizes L_n and has less than 2^n states. Now consider strings of length n. There are 2^n of them so from the pigeon hole principle we know that at least two different ones of them, say x and y, will reach the same end state, say q, when processed by M. That is

$$|x| = |y| = 2^n$$

$$\hat{\delta}(s,x) = \hat{\delta}(s,y) = g$$

Now consider the first place from the left where x and y differ. Say the kth position. So wlg we have

$$x = u1v$$

$$y = u0w$$

and

$$|u| = k-1$$

$$|v| = |w| = n-k$$

We can construct strings

$$x' = u1v1^{|u|}$$

$$y' = u0w1^{|u|}$$

Clearly these must end in the same state, namely, $\hat{\delta}(g, 1^{|u|})$. Now $x' \in L_n$ and $y' \notin L_n$ so that state must be both an accept state and a reject state. That is a contradiction. ■

EXPONENTIAL BLOWUP

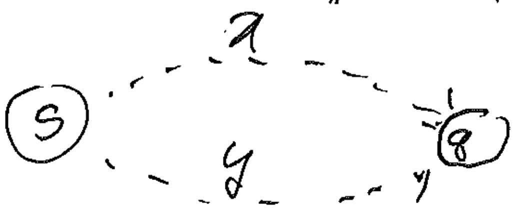
$L_n = \{ w \in \{0,1\}^* \mid w$ HAS 1 IN nTH POSITION FROM END $\}$

DFA THAT RECOG. L_n CAN NOT HAVE $< 2^n$ STATES

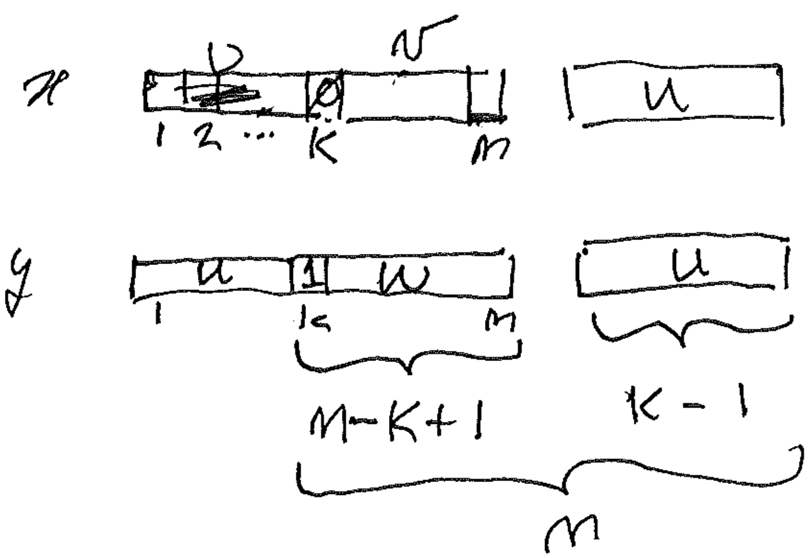
CONSIDER STRINGS OF LENGTH n

THERE 2^n DIFFERENT ONES

PIGEON HOLE AT LEAST TWO MUST END IN SAME STATE



CONSIDER FIRST POSITION WHERE THEY DIFFER SAY K-th



FOR ϵ-NFA THIS FUNCTION IS SPECIAL.

$$M = (Q, \Sigma, \delta, s, F)$$

Q

Σ

$$\delta : Q \times (\Sigma \cup \{\epsilon\}) \longrightarrow 2^Q$$

$\delta(q, a \in \Sigma) \longrightarrow P \subseteq 2^Q$ TRANSITION AND MOVE SCAN HEAD

$\delta(q, \epsilon) \longrightarrow P \subseteq 2^Q$ TRANSITION AND NOT MOVE SCAN HEAD

NOTE P MAY BE $\emptyset$.

	a_1	a_2	$\dots$	a_m	ϵ
q_0					
q_1		$\{$	$\}$		
q_2					
$\vdots$					

$s \in Q$

$F \subseteq Q$

THEOREM For every ε-NFA M there is an NFA $\hat{M}$ such that $L(M)=L(\hat{M})$

$$M = (Q, \Sigma, \delta, s, F) \qquad \varepsilon\text{-NFA}$$

$$\hat{M} = (\hat{Q}, \Sigma, \hat{\delta}, \hat{s}, \hat{F}) \qquad \text{NFA}$$

$$\hat{Q} = Q$$

$$\hat{s} = s$$

$$\hat{\delta} : \hat{Q} \times \Sigma \to 2^{\hat{Q}}$$

$$\hat{\delta}(p, a) = \{ q \mid \text{THERE ARE STATES } p_1, p_2 \text{ S.T.}$$

$$p_1 \in E(p) \qquad p \overset{\varepsilon}{\rightsquigarrow} p_1 \xrightarrow{a} p_2 \overset{\varepsilon}{\rightsquigarrow} q$$

$$p_2 \in \delta(p_1, a)$$

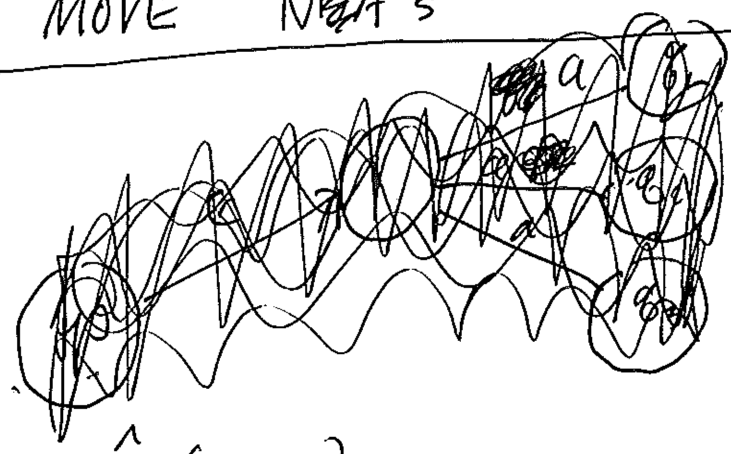
$$q \in E(p_2) \}$$

$E(p)$ = SET OF ALL STATES REACHABLE FROM p BY ε-MOVES ONLY
ε - CLOSURE OF p.

$$\hat{F} = \begin{cases} F \cup \{s\} & \text{IF } F \cap E(s) \neq \emptyset \\ F & \text{OTHERWISE} \end{cases}$$

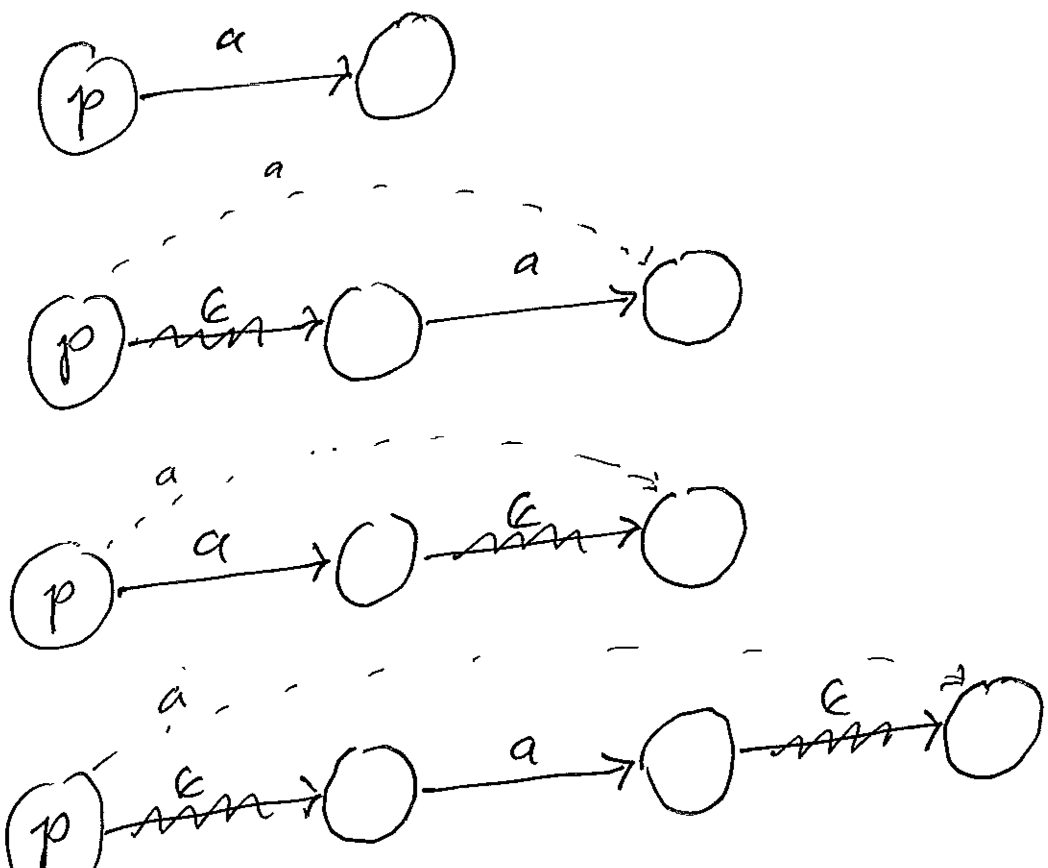
EXERCISE: SHOW WITH INDUCTION ON $|x|$ THEY SIMULATE EACH OTHER.

ε MOVE NFA's SEE NOTE BOOK



$\hat{\delta}(p, a)$

ALL PLACES CAN GET TO FROM p WITH INPUT a



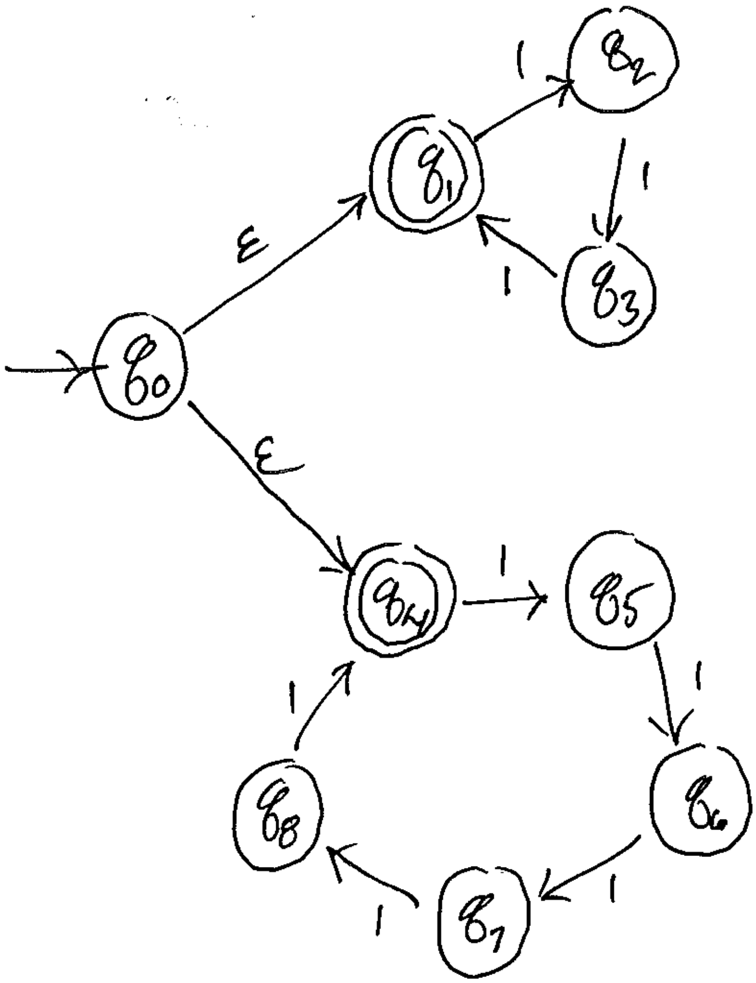
if $\overset{a}{- - \rightarrow}$ transition are included

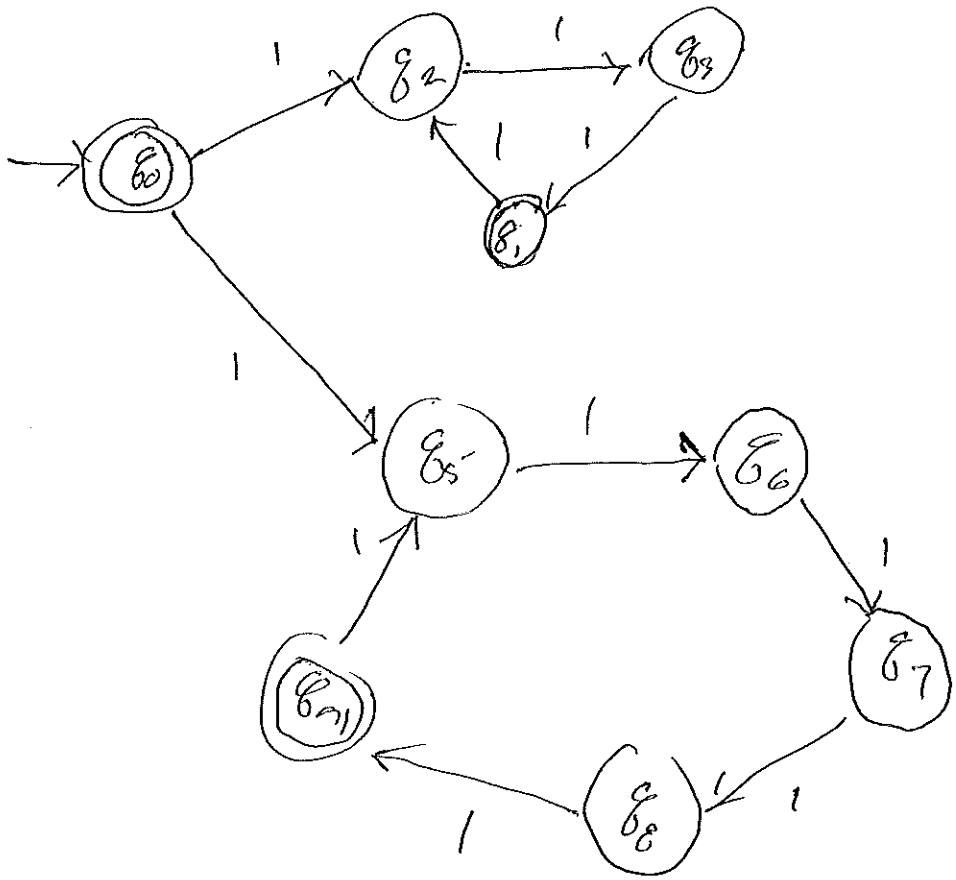
CAN GET ANY WHERE ε ALLOWED WITHOUT USING ε's.

NOTE IF THERE IS ε MOVE(S) FROM START STATE TO AN ACCEPT STATE, THE START STATE MUST BE MADE AN ACCEPT STATE

ε-NFA EXAMPLE (DEF. IN BOOK)

$L = \{ 1^n \mid n \text{ is multiple of } 3 \text{ or of } 5 \}$





q0
q2
q3
q1
q5
q6
q7
q8
qn1
1
1
1
1
1
1
1
1
1
1

CLOSURE

REGULAR LANGUAGES ROBUST ARE CLOSED UNDER

COMPLEMENT

UNION

INTERSECTION

DIFFERENCE

CONCATENATION

KLEENE STAR

QUOTIENT

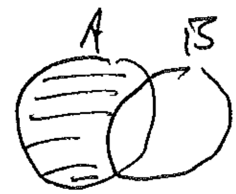
REVERSAL

QUOTIENT

$$L_1 / L_2 = \{ x \mid \text{THERE IS A } y \in L_2 \text{ S.T. } xy \in L_1 \}$$

(PREFIXES OF STRINGS IN L_1 WHOSE SUFFIX IS IN L_2)

FACT: IF L_1 REGULAR L_1/L_2 IS REGULAR
EVEN IF L_2 IS NOT REGULAR.

$A - B =$ 

$A \cap \overline{B}$

A REG

B REG

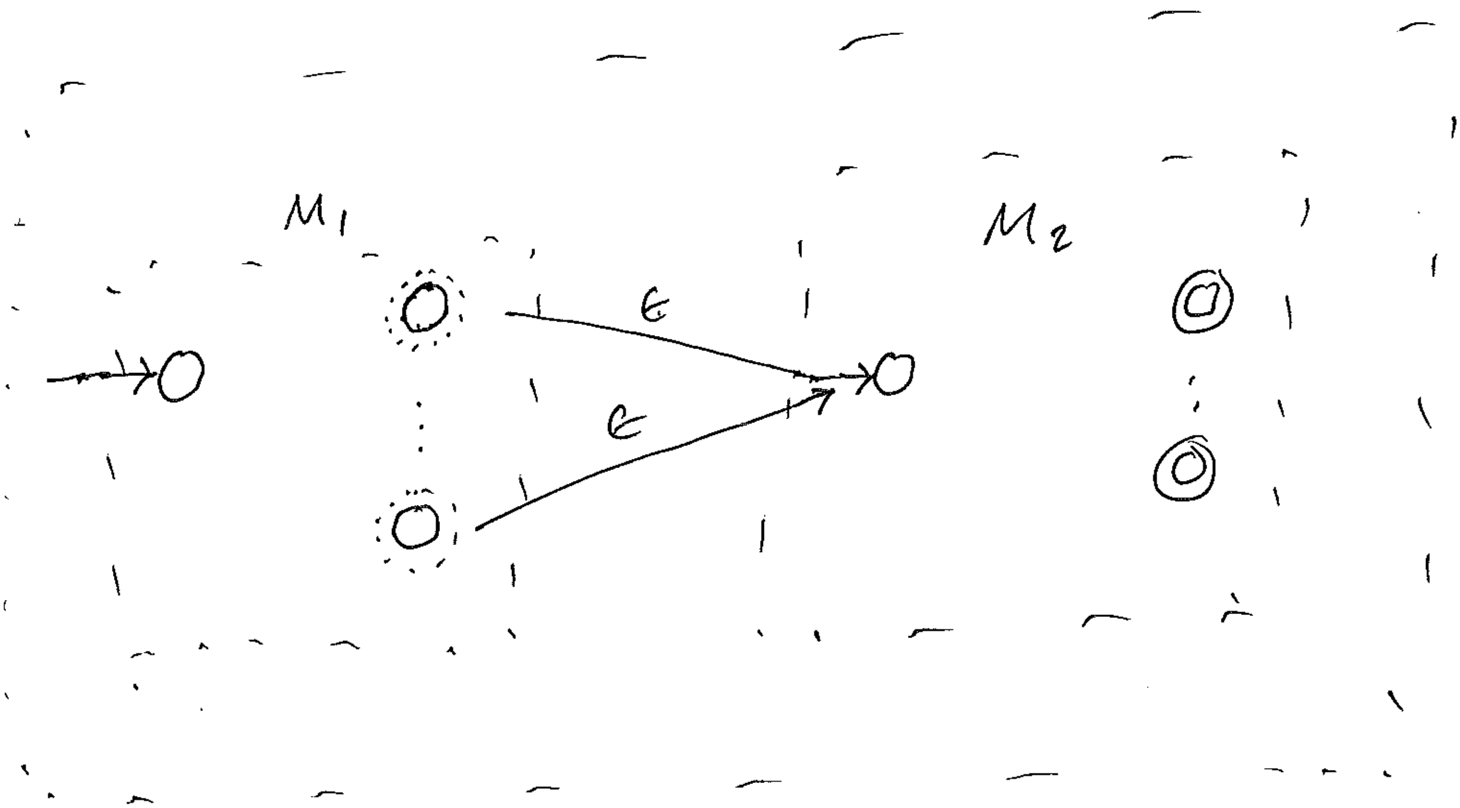
~~A-B~~ $A \cap \overline{B}$

B REG $\Rightarrow \overline{B}$ REG INTERSECT REG.

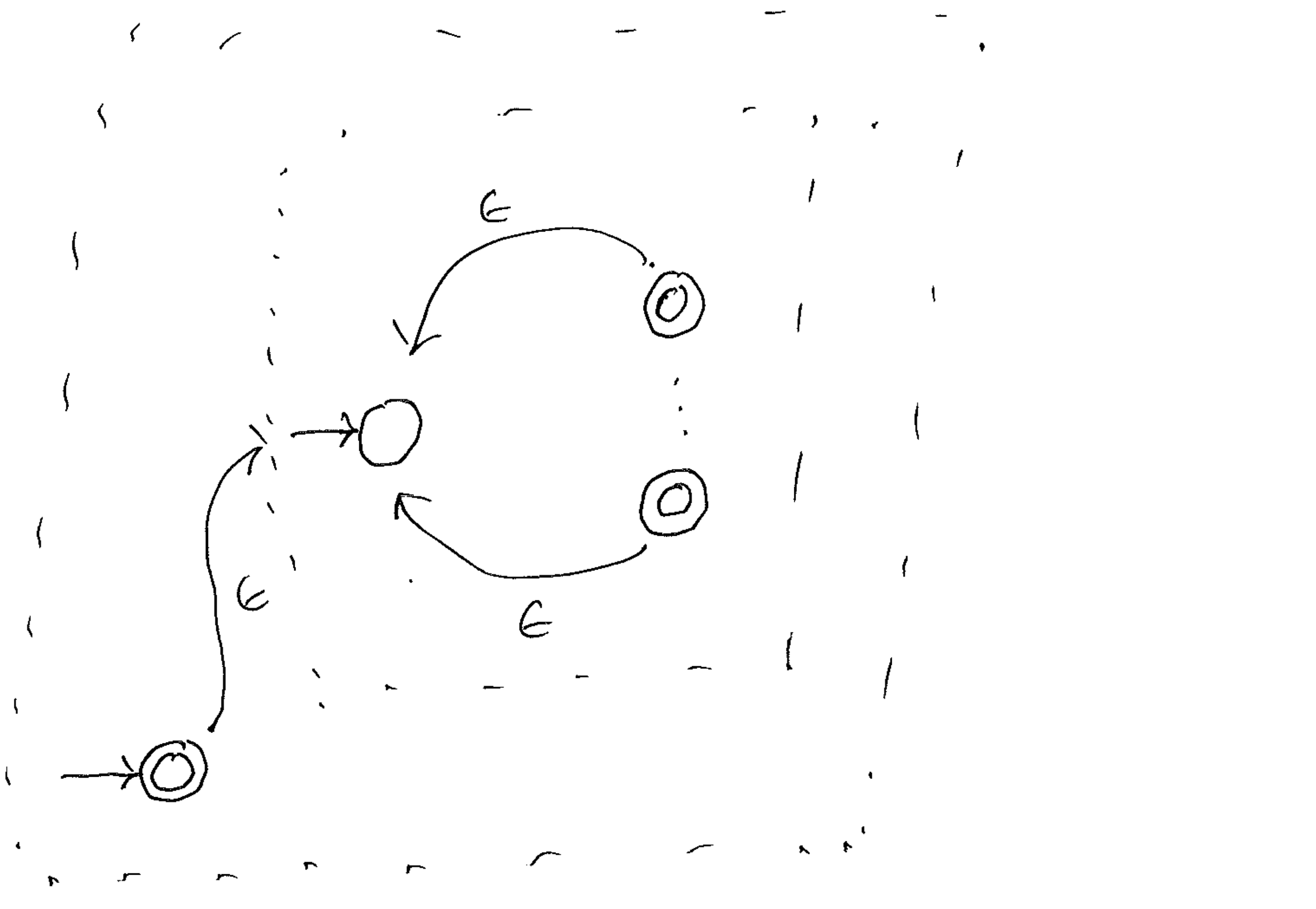
$\Rightarrow A \cap \overline{B}$ REG

$\Rightarrow A - B$ REG

CLOSED UNDER CONCATENATION.

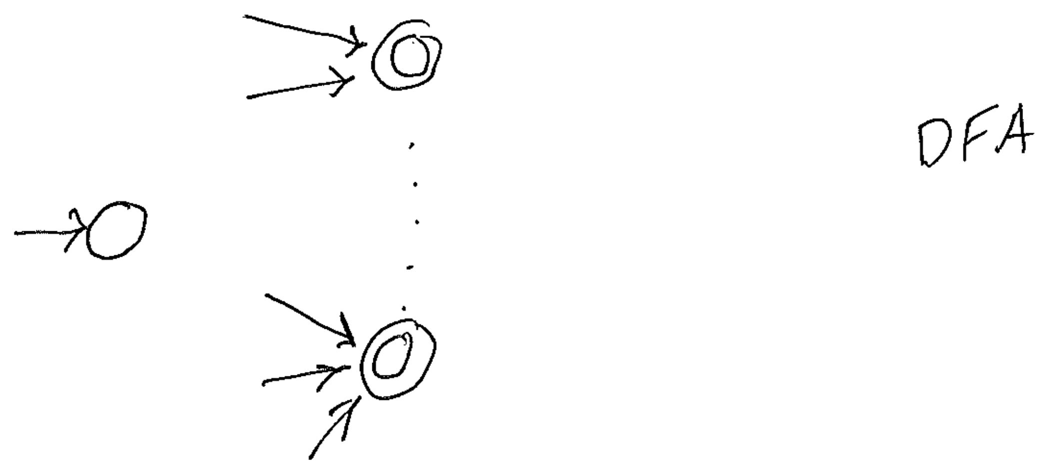


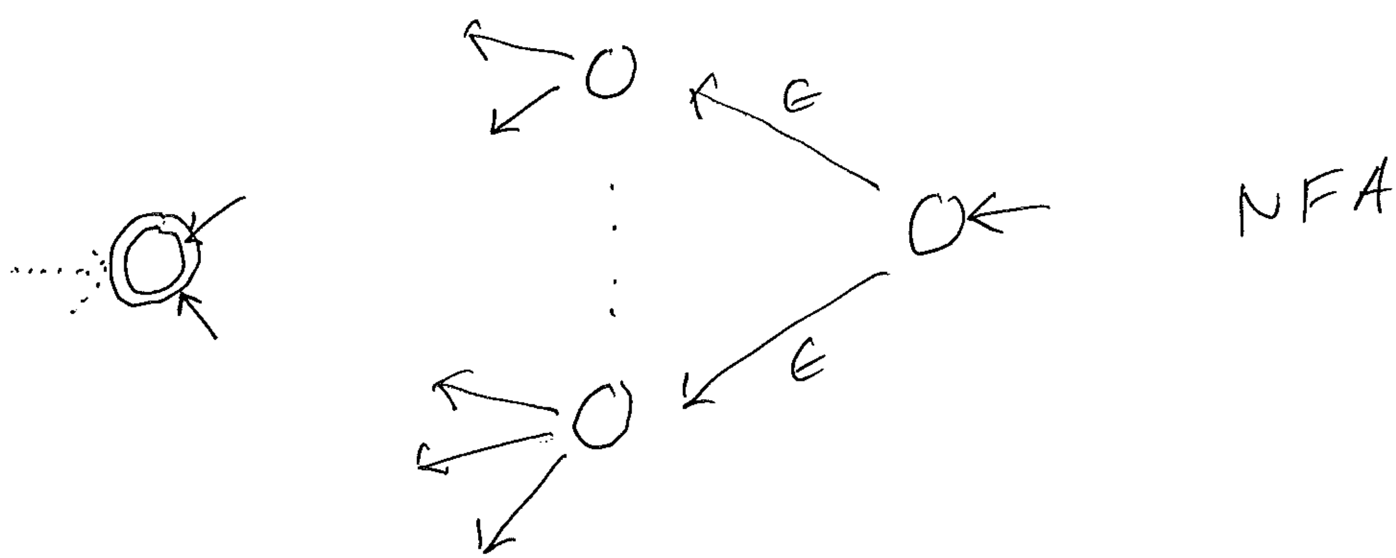
CLOSED UNDER KLEENE STAR



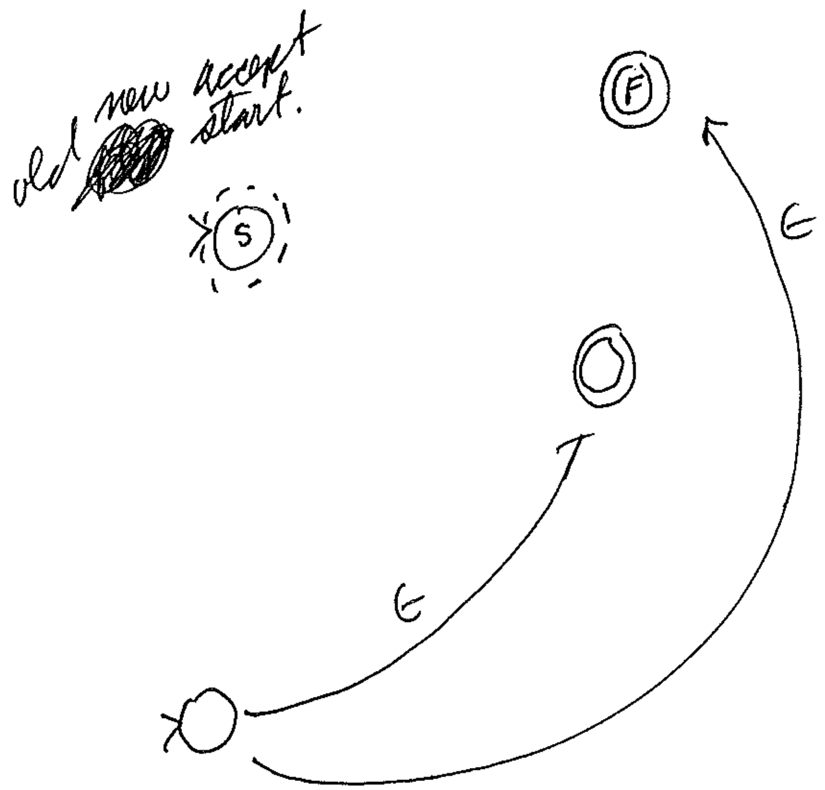
CLOSED UNDER

REVERSAL

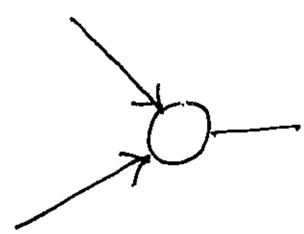


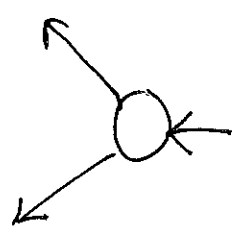


REVERSAL IS REG.



NEW START
ε MOVES TO OLD ACCEPTS





IF A PATH FROM S → F MUST BE PATH FROM F → S IN NEW AND OF COURSE ACCEPTS.

IF A PATH FROM F → S IN NEW MUST BE PATH IN OLD FROM S → F.

1.31

IF A IS REG. THAN THERE MUST EXIST A DFA M_A THAT RECOGNIZES A.

M_A CAN BE MODIFIED TO BE A NFA N_A THAT RECOGIZES A^R.

I ADD A NEW START STATE WITH ϵ MOVES TO ALL THE ACCEPT STATES OF M_A.

II CHANGE THOSE ACCEPT STATES TO REJECT STATES.

III CHANGE M_A'S START STATE TO AN ACCEPT STATE.

~~[illegible]~~

IV REVERSE ALL TRANSITION ARROWS IN M_A.