Homework 1

CMPS130 Computational Models, Spring 2015

0.1

a. The set of odd natural numbers greater than zero.

b. The set of even integers.

c. The set of even natural numbers great than one.

d. The set of natural numbers divisible by six great than one.

e. The set of palindromes over the alphabet $\{\mathtt{0}, \mathtt{1}\}$.

f. The empty set.

0.2

a. $\{1, 10, 100\}$

b. $\{n \mid n \in \mathcal{Z} \ \land \ n > 5\}$

c. $\{n \mid n \in \mathcal{N} \ \land \ n < 5\}$

d. $\{\mathtt{aba}\}$

e. $\{\epsilon\}$

f. $\{\}$

0.3

a. No

b. Yes

c. $\{\mathtt{x,\ y,\ z}\}$

d. $\{\mathtt{x,\ y}\}$

e. $\{\mathtt{(x,\ x),\ (x,\ y),\ (y,\ x),\ (y,\ y),\ (z,\ x),\ (z,y)}\}$

f. $\{\{\}, \{\mathtt{x}\}, \{\mathtt{y}\}, \{\mathtt{x}, \mathtt{y}\}\}$

0.4

$$|A \times B| = |\{(a,b) \mid a \in A, b \in B\}| = \sum_{a \in A} \sum_{b \in B} 1 = |A| * |B|$$

0.5

$$|\mathcal{P}(C)| = |\{D \cup \{c \cup d \mid d \in D\} \mid c \in C \ \wedge \ D = \mathcal{P}(C \setminus c)\}| = 2 * |\mathcal{P}(C \setminus c)| = 2^{|C|}$$

0.6

a. $f(2) = 7$

b. $dom(f) = \{1,2,3,4,5\} \qquad range(f) = \{6,7\}$

c. $g(2,10) = 6$

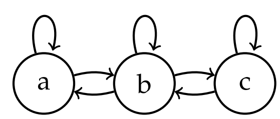
d. $dom(g) = \left\{ \begin{matrix} (1,6), & (1,7), & (1,8), & (1,9), & (1,10), \\ (2,6), & (2,7), & (2,8), & (2,9), & (2,10), \\ (3,6), & (3,7), & (3,8), & (3,9), & (3,10), \\ (4,6), & (4,7), & (4,8), & (4,9), & (4,10), \\ (5,6), & (5,7), & (5,8), & (5,9), & (5,10) \end{matrix} \right\} \qquad range(g) = \{6,7,8,9,10\}$

e. $g(4, f(4)) = g(4,7) = 8$

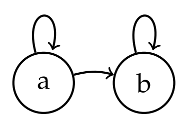
0.7

Using a finite set S and a relation $R \subseteq S \times S$.

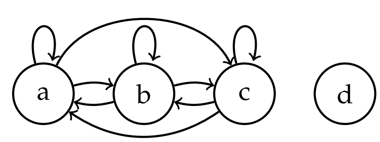
a. $S = \{a,b,c\}$
$R = \{(a,a),(a,b),(b,a),(b,b),(b,c),(c,b),(c,c)\}$



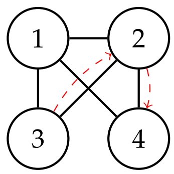
b. $S = \{a,b\}$
$R = \{(a,a),(a,b),(b,b)\}$



c. $S = \{a,b,c,d\}$
$R = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$



0.8



$deg(1) = 3 \quad deg(3) = 2$

0.9

$$G = (V, E) = (\{1, 2, 3, 4, 5, 6\}, \{(x, y) \mid x \in \{1, 2, 3\} \ \wedge y \in \{4, 5, 6\}\})$$

0.10

You cannot divide by $(a - b)$ unless you know that $a \neq b$. However, this contradicts with the assumption that $a = b$.

0.11

In the induction step, the proof assumes that H_1 and H_2 must have the same color if because all the horses in the set H_1 and H_2 have the same color and there is only one horse difference between H_1 and H_2. However, this does not work for the base case $k = 2$ in which H_1 and H_2 will both only include a single horse. These sets are therefore trivially one-colored but might have different colors, therefore the set H might have more than one color.

0.12

Here, we only consider graphs that do *not* have self-loops: $\forall v \in V : (v, v) \notin E$. Otherwise, there is a simple counter-example with two nodes and exactly one edge: $G = (\{a, b\}, \{(a, a)\})$.

Let $G = (V, E)$ be a graph with $|V| \geq 2$ nodes, such that v_{min} has the minimum degree of all nodes and v_{max} has the maximum degree of all nodes.

Case 1: If $v_{min} = 0$, then the node v_{max} can at most be connected to all nodes other than v_{min} in the graph, so $v_{max} \leq |V| - 2$ and $v_{max} - v_{min} \leq |V| - 2$.

Case 2: If $v_{min} \geq 1$, then the node v_{max} can at most be connected to all nodes in the graph, so $v_{max} \leq |V| - 1$ and $v_{max} - v_{min} \leq |V| - 1 - 1$.

In both cases, there are $|V|$ nodes but only $v_{max} - v_{min} + 1 = |V| - 1$ possible degrees, so by the pigeonhole principle, at least two nodes have the same degree.

Set intersection distribution $\overline{(A \cap B)} = (\overline{A} \cup \overline{B})$

If an element a is in $a \in \overline{(A \cap B)}$ then it is not in both A and B, so either it is not in A ($a \in \overline{A}$) or it is not in B ($a \in \overline{B}$) – in any case, it is in the union of both complements ($\overline{A} \cup \overline{B}$).

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Countable Sets

According to the definition, a set is countable if it either finite or has the same size as $\mathcal{N}$.

Taking the set $\mathcal{N}$, the resulting set of multiplying all elements by two ($\{2n \mid n \in \mathcal{N}\}$) has the same size as $\mathcal{N}$ and it therefore countable. Subtracting 1 from each element of this set does not change its size, so the resulting set $\{2n - 1 \mid n \in \mathcal{N}\}$ includes all odd numbers but is countable.

Induction Example

Basis: For $n = 1$,

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{1} i^2 = 1^2 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{1 \cdot (1+1)(2*1+1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

Induction Step: For $n = k+1$, assuming that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{k+1} i^2 = (k+1)^2 + \sum_{i=1}^{k} i^2 \\
&= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} && \text{(I.H.)} \\
&= \frac{6(k^2+2k+1) + (k^2+k)(2k+1)}{6} \\
&= \frac{6k^2+12k+6+2k^3+k^2+2k^2+k}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$