PREPARATION FOR MYHILL-NERODE

EQUIVALENCE RELATIONS ON A SET A

IDENTITY

EQUALS

SAME AS CONDITIONS E.G. X HAS SAME BIRTHDAY AS Y

MOD

RATIONAL NUMBERS

RELATION THAT IS
- REFLEXIVE
- SYMMETRIC
- TRANSITIVE

PARTITION SET INTO DISJOINT PARTS

EQUIVALENCE CLASSES

IF R IS AN EQUIVALENCE RELATION ON A

$$\boxed{[x]_R = \{ y \mid x R y \} \text{ EQUIVALENCE CLASS OF } x \}}$$

GIVEN $[x]_R$ AND $[y]_R$ THEY ARE EITHER THE SAME OR DISJOINT.

$x R y \Rightarrow$ SAME

$x \not R y \Rightarrow$ DISJOINT

$$A = \bigcup_{x \in A} [x]_R$$

INDEX OF EQUIVALENCE RELATION IS # OF EQUIVALENCE CLASSES

EQUIVALENCE RELATION PREP MYHILL - NERODE

FOR ALL $x, y \in A$

I $x \approx x$

II $x \approx y \Rightarrow y \approx x$

III $x \approx y$ AND $y \approx z \Rightarrow x \approx z$

WHY DOESN'T

$\mathrm{II} + \mathrm{III} \Rightarrow \mathrm{I}$

THERE MAY BE x S.T. $\forall y \neq x \;\; y \not\approx x$

THATS NOT EQUIVALENT TO ANYTHING ELSE

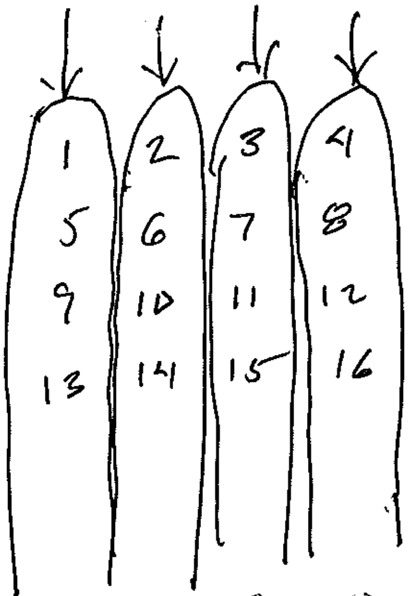
$$\boxed{x R y \text{ iff } [x]_R = [y]_R}$$

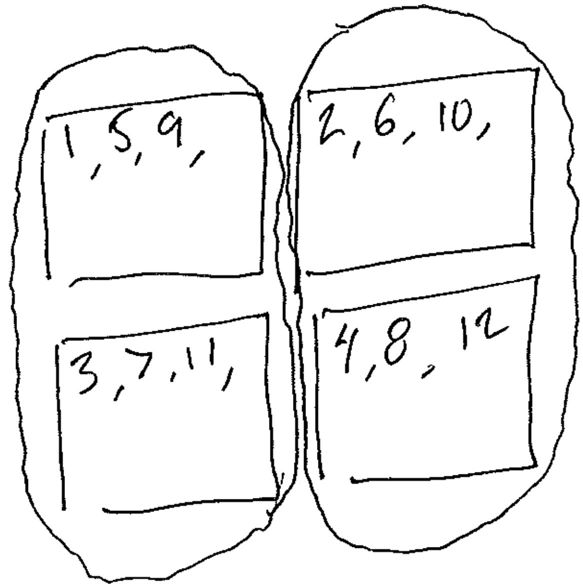
IF FOR ALL $x \in A$

$$[x]_{\sim} \subseteq [x]_{\not\sim}$$

THEN $\sim$ IS <u>FINER</u> THAN $\not\sim$

EXAMPLE:





1 2 3 0 MOD4

1 0 1 0 MOD2

$$[M]_{MOD\,4} \subseteq [M]_{MOD\,2}$$

$$\text{INDEX } R_{MOD4} \geq \text{INDEX } R_{MOD2}$$

$$4 \qquad\qquad 2$$

$$\boxed{\begin{array}{c} \forall x, y \quad x R_1 y \Rightarrow x R_2 y \\ [x]_{R_1} \subseteq [x]_{R_2} \\ \text{INDEX OF } R_1 \geq \text{INDEX OF } R_2 \end{array}}$$

MYHILL - NERODE

<u>IDEA.</u> WITH EVERY LANGUAGE $L \subseteq \Sigma^*$ WE CAN ASSOCIATE A SPECIAL RELATION R_L ON Σ^*

$$x R_L y \text{ iff for all } z \in \Sigma^* \ (xz \in L \Leftrightarrow yz \in L)$$

PROP R_L IS AN EQUIVANCE RELATION.

$$\forall x \in \Sigma^* \quad x R_L x$$

$$\forall x, y \in \Sigma^* \quad x R_L y \Rightarrow y R_L x$$

$$\forall x, y, z \in \Sigma^* \quad x R_L y \text{ AND } y R_L z \Rightarrow x R_L z$$

PROOF
STAIGHT FORWARD FROM S, R, T PROPERTIES OF "$\Leftrightarrow$"

THM

LET $L \subseteq \Sigma^*$ THEN THE FOLLOWING TWO STATEMENTS ARE EQUIVALENT.

I. L IS REGULAR.

II. THE INDEX OF R_L IS FINITE.

PROOF II $\Rightarrow$ I

ASSUME R_L HAS FINITELY MANY EQUIVALENCE CLASSES. WE WANT TO BUILD A DFA M SUCH THAT $L = L(M)$.

IDEA TAKE THE EQ. CLASSES OF R_L AS THE STATES OF M.

$$M = (Q, \Sigma, \delta, q_0, F)$$

$$Q = \{ [x]_{R_L} \mid x \in \Sigma^* \}$$

$$\delta : Q \times \Sigma \to Q$$

$$\delta([x]_{R_L}, a) = [xa]_{R_L}$$

NEED LEMMA TO SHOW WELL DEFINED.

$$q_0 = [\varepsilon]_{R_L}$$

$$F = \{ [x]_{R_L} \mid x \in L \}$$

MYHILL - NERODE

CONSTRUCTION OF DFA FOR FINITE # EQ. CLASSES.

FINESS OF EQ. RELATION

L REG IMPLIES INDEX R_L FINITE

EXAMPLES

1. $\{0^n 1^n \mid n \geq 1\}$ ON HANDOUT

2. $\{w \mid w$ IS A PALENDROME $\}$

   $\{01, 001, 0001, \ldots\}$

3. $\{1^{n!} \mid n \geq 1\}$

4. 1.54 p 91

5. Prove Reg. $L = \{0^{2n} \mid n \geq 1\}$ ON HANDOUT

2WAY AUTOMATA

PATTERN MYHILL-NERODE PROOF.

IDENTIFY A SET OF STRINGS $S \subseteq$ IN Σ^* S.T.

$|S|$ IS INFINITE

AND FOR ALL $x, y \in S$ $x \neq y$

$x \not R_L y$

SO NO TWO STRINGS ARE IN THE SAME EQUIVALENCE CLASS

~~THEREFORE THERE MUST BE A INFINITE NUMBER OF EQUIVALENCE CLASSES.~~

$|\{[x]_{R_L} \mid x \in S\}|$ IS INFINITE

AND SINCE $\{[x]_{R_L} \mid x \in S\} \subseteq \{[x]_{R_L} \mid x \in L\}$

THE INDEX OF R_L IS INFINITE.

EXAMPLE

$L = \{ w \mid w \text{ IS A PALENDROME} \}$

~~ERRORS IN THIS~~ ? YES

TAKE $x, y \in \Sigma^*$ AND $x \neq y$

NOTE $x \neq y \Rightarrow x^R \neq y^R$

~~COUNTER EXAMPLE~~

DISTINGUISHING z

$x x^R \in L$

$y x^R \notin L$

$x^R = y^R \Rightarrow x = y$ CONTRADICTION.

SO INDEX INFINITE.

INFINITE # OF SUCH x, y.

counter example.

$x = 1111$

$y = 11$

$x x^R = 11111111 \in L$

$y x^R = 111111 \in L$

CORRECTION LET $\Sigma = \{0, 1\}$

LET $x \neq y$ BE OF FORM

101

101001

1010010001

THIS WORKS

ALSO

01

001

0001

⋮

$0^j 1 0^j \in L$

$0^k 1 0^j \notin$ IF $j \neq k$

INFINITE # OF $0^j 1$

ALL PAIRWISE NOT EQUIVALENT.

$\therefore$ INFINITE # EQUIV. CLASSES

$\therefore$ L NOT REGULAR.

EXAMPLE

$L = \{ 1^{m!} \mid m \geq 1 \}$

TAKE i, j WITH $i \neq j$

CONSIDER $1^{i!}$ AND $1^{j!}$

CLAIM $1^{i!} \not R_L \; 1^{j!}$

NOT TRUE! BAD ALGEBRA.
SEE NEXT PAGE FOR GOOD PROOF.

PROOF

$\cancel{1^{i!} \, 1^{i+1} = 1^{(i+1)!} \in L}$ $\quad j! + i + 1$

$\cancel{1^{j!} \, 1^{i+1} \notin L}$

BECAUSE $\cancel{j!(i+1)}$ CAN NOT BE A FACTORIAL IF IS NOT EQUAL TO j.

SO $\{ 1^{i!} \mid i \geq 1 \}$ IS INFINITE SET ALL OF WHICH ARE PAIR WISE NON EQUIVALENT.

~~SO INDEX IS INFINITE.~~

$i < j$

$$1^{i!} \, 1^{i \, i!} = 1^{(1+i)\, i!} = 1^{(i+1)!} \in L$$

$$1^{j!} \, 1^{i \, i!} =$$

$$1^{\frac{j!}{i!} i!} \, 1^{i \, i!} = 1^{\left(\frac{j!}{i!} + i\right) i!} \notin L$$

PROOF

$$q! \overset{?}{=} \Big(\underbrace{\tfrac{j!}{i!} + i}_{\text{CAN THIS BE A FACTORIAL NUMBER}}\Big)\, i!$$

$\Downarrow$ DIV BY $i!$

$$q(q-1)\cdots(i+1) = \frac{j!}{i!} + i$$

$$= j(j-1)\cdots(i+1) + i$$

CAN'T BE BECAUSE

LEFT SIDE remainder of 0 when divided by $i+1$

RIGHT SIDE leave remainder of i when divided by $i+1$

PROB 1.54 page 91. EXAMPLE M-N WORKS PUMPING LEMMA DOESN'T.

$L = \{ a^i b^j c^k \mid i, j, k \geq 0 \quad i = 1 \Rightarrow j = k \}$

(aka F)

if $i = 0$ (no a's)

any string $|s| \geq 1$ can be pumped.

just make y either b or c.

first letter

if $i = 1$

$|s| \geq 1$

$y = a$

~~if~~

if $i \geq 2$ $\left\{ \begin{array}{l} i = 2 \\ y = aa \end{array} \right.$ $\underbrace{aa}\, b \ldots c \ldots$

$|s| \geq 1$

$y = \text{first } a$

~~$a b^{j} b^{j} \in L$~~ ?

~~$a b^{j'} b^{j'} \notin L$~~

$a b^{j} c^{j} \in L$

$a b^{k} c^{j} \notin L$ BECAUSE $j \neq k$

FOR ALL $j \neq k$

$A = \{ a b^i \mid i \geq 1 \}$

$L = \{ ww \mid w \in \Sigma^* \}$

$S = \{ 0^i 1 \mid i \geq 1 \}$

$0^i 1 0^i 1 \in L$

$0^j 1 0^i 1 \notin L$ FOR $j \neq i$

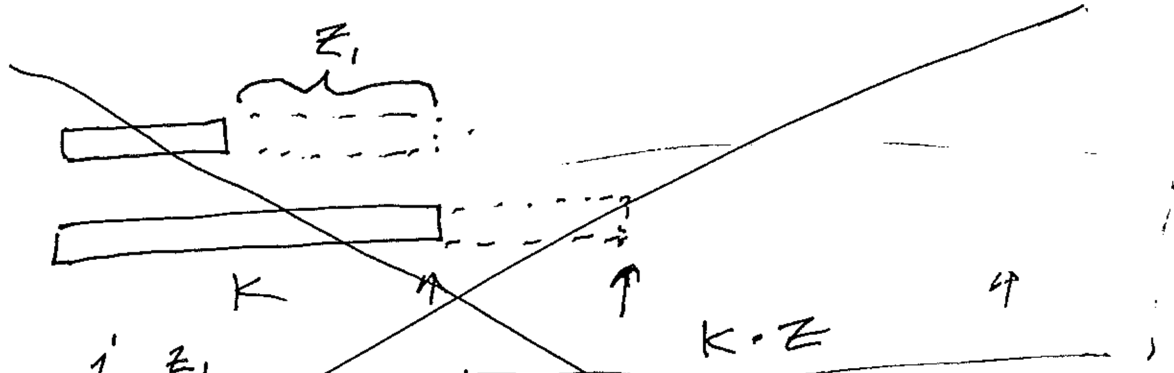
$L = \{ 0^n \mid n \text{ IS A PRIME NUMBER} \}$

$\{ 0^i \mid i \text{ is prime} \}$ infinite

TAKE ANY TWO DIFFERENT ONES
TOWARD CONTRADICTION ASSUME THEY ARE EQUIVALENT

0^i

0^k $\qquad k > i$

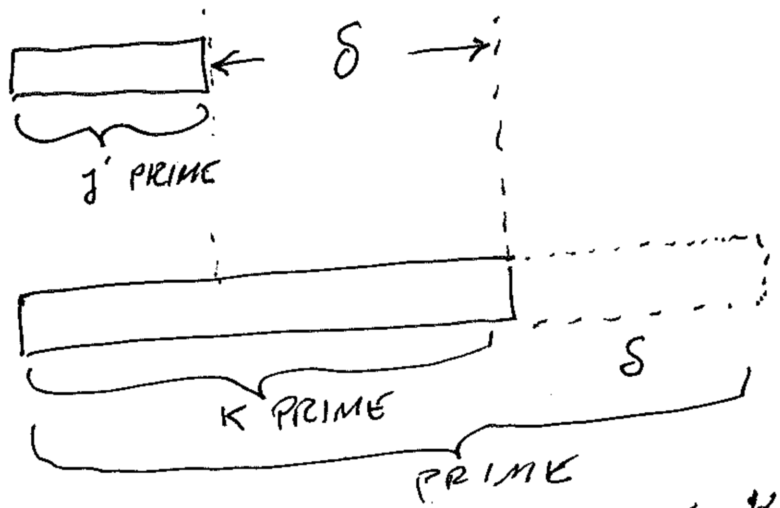


$0^i 0^{z_1} \in L$

$0^k 0^{z_1} \in L$

SEE NEXT PAGE

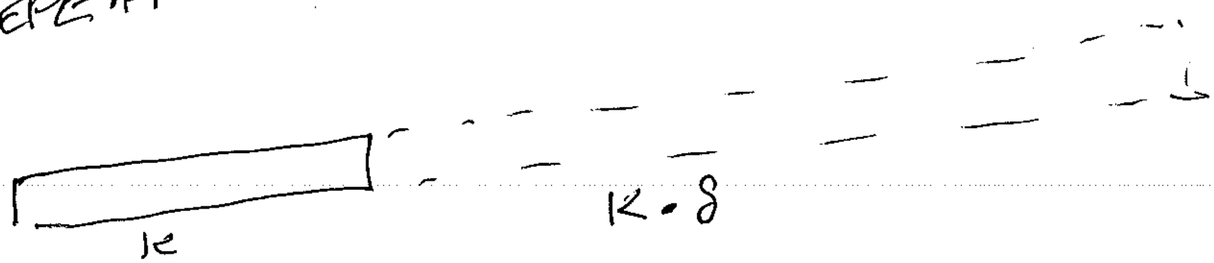
$L = \{0^p \mid p \text{ IS A PRIME NUMBER}\}$



$0^j 0^\delta$ IS PRIME BECAUSE K IS GIVEN PRIME
BECAUSE OF ASSUMPTION $0^j R_L 0^k$

$\Rightarrow 0^k 0^\delta$ IS PRIME

REPEAT SAME ARGUMENT K TIMES



$K(1+\delta)$ IS PRIME

CONTRADICTION!

EXAMPLE

$$L = \{ w \in \{0,1\}^* \mid w \text{ HAS EVEN \# OF 0s AND EVEN \# OF 1s} \}$$

4 EQ. CLASSES FOR R_L

$$[\varepsilon]_{R_L} = \{ w \mid \text{EVEN \# OF 0s AND EVEN \# OF 1s} \}$$

$$[0]_{R_L} = \{ w \mid \text{ODD} \quad \text{EVEN} \}$$

$$[1]_{R_L} = \{ w \mid \text{EVEN} \quad \text{ODD} \}$$

$$[01]_{R_L} = \{ w \mid \text{ODD} \quad \text{ODD} \}$$