Exponential Blowup of Subset Construction

Definition: $L_n = \{w \in \{0,1\}^* | w \texttt{ has a 1 in the } n\texttt{th position from the end}\}$. Note an NFA with only $n+1$ states can be designed to recognized L_n.

Theorem: A DFA that recognizes L_n can not have less than 2^n states.

Proof: Toward a contradiction assume that a DFA M exists with start state s and that recognizes L_n and has less than 2^n states. Now consider strings of length n. There are 2^n of them so from the pigeon hole principle we know that at least two different ones of them, say x and y, will reach the same end state, say q, when processed by M. That is

$$|x| = |y| = 2^n$$

$$\hat{\delta}(s, x) = \hat{\delta}(s, y) = g$$

Now consider the first place from the left where x and y differ. Say the kth position. So wlg we have

$$x = u1v$$

$$y = u0w$$

and

$$|u| = k - 1$$

$$|v| = |w| = n - k$$

We can construct strings

$$x' = u1v1^{|u|}$$

$$y' = u0w1^{|u|}$$

Clearly these must end in the same state, namely, $\hat{\delta}(g, 1^{|u|})$. Now $x' \in L_n$ and $y' \notin L_n$ so that state must be both an accept state and a reject state. That is a contradiction. ∎