Definition. *Given an alphabet, Σ, and a language, $L \subseteq \Sigma^*$, the **Myhill-Nerode relation,** R_L, for L on Σ^* is defined as follows:*

$$\textit{For all } x, y \textit{ in } \Sigma^*, \quad xR_Ly \quad \textit{iff} \quad \forall z \in \Sigma^*(xz \in L \Leftrightarrow yz \in L).$$

Theorem. *The Myhill-Nerode relation is an equivalance relation on Σ^*.*

Theorem. *(**Myhill-Nerode**) Given an alphabet, Σ, and a language, $L \subseteq \Sigma^*$, the following two statements are equivalent.*

1. *The language L is regular.*
2. *The index of R_L is finite.*

Proving a language is not regular with Myhill-Nerode

- Show a set $X \subseteq \Sigma^*$ that has infinite cardinality.
- Show that no two strings in X can be equivalent with respect to R_L, i.e., show that for any $x, y \in X$, there exists a $z \in \Sigma^*$ that distinguishes x and y, i.e., $xz \in L$ and $yz \notin L$ or visa versa. This proves that $[x]$ and $[y]$ are distinct equivalence classes for all $x, y \in \Sigma^*$. So the index of R_L must be greater than or equal to the cardinality of X, therefore infinite.

Example

The language, $L = \{0^n1^n \mid n \geq 0\}$, is not regular. Proof: Consider the set $X = \{0^n \mid n \geq 1\}$. It is clearly infinite. Now, any two members of X have the form, $0^i, 0^j$, where $i \neq j$. And the string $z = 1^i$ distinguishes them since, $0^i1^i \in L$ and $0^j1^i \notin L$. QED.

Proving a language is regular with Myhill-Nerode

- Show a finite number of subsets of Σ^*, say, $A_1, A_2, \ldots, A_n$.
- Show for each A_i, $\quad \forall x, y \in A_i \quad xR_Ly$.
- Show that $\bigcup A_i = \Sigma^*$.

Then the index of R_L must be less than or equal to n, therefore finite and L must be regular.

Example

The language, $L = \{0^n \mid n \text{ is even}\}$, is regular. Proof: Consider the following finite list of subsets of Σ^*: $A_1 = L, A_2 = \{0^n \mid n \text{ is odd}\}$. Claim the members of A_1 are all equivalent to each other, because for any two members, say, $0^i, 0^j$ where $i \neq j$ and both are even, it is the case that $0^iR_L0^j$, since, for any $z \in \Sigma^*$, say, 0^m, both 0^i0^m and 0^j0^m are in L if m is even and both are not in L if M is odd. The members of A_2 are all equivalent by a similar argument with the roles of even and odd switched. Now clearly every string in Σ^* is either even or odd in length, so $A_1 \cup A_2 = \Sigma^*$. QED.