Homework 5

CMPS130 Computational Models, Spring 2015

1.29

a.

Proof by Contradiction: Assuming that $A_1 = \{0^n\ 1^n\ 2^n \mid n \geq 0\}$ is regular, then the pumping lemma holds. Therefore, there exists a pumping length p, such that any string longer than p can be pumped. The string $s = 0^p\ 1^p\ 2^p$ is in A_1 and longer than p, therefore it can be split into xyz such that $xy^iz \in A_1$. If y contains only 0's, 1's or 2's, then $xyyz$ does not have an equal number of 0's, 1's and 2. If y contains more than one kind of symbol (e.g. 01), then the symbols in yy will not be grouped together anymore (e.g. 0101), so $xyyz$ cannot have the form $0^n\ 1^n\ 2^n$.

Due to these contradictions, the pumping lemma does not hold for A_1, so it is not regular.

b.

Proof by Contradiction: Assuming that $A_2 = \{www \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$ is regular, then the pumping lemma holds. Therefore, there exists a pumping length p, such that any string longer than p can be pumped. The string $s = \mathtt{a}\ \mathtt{b}^p\ \mathtt{a}\ \mathtt{b}^p\ \mathtt{a}\ \mathtt{b}^p$ is in A_2 and longer than p, therefore it can be split into xyz such that $|xy| \leq p$, $|y| > 0$ and $xy^iz \in A_2$. Since $|xy| \leq p$ and the a are always p symbols apart, y can have at most one a. If y contains exactly one a's, then $xyyz$ has exactly four a's, so it cannot have the form www anymore. If y contains no a's, then $xyyz$ contains three a's but one of these will be followed by more b's than the other two a's, since $|y| > 0$, so $xyyz$ cannot have the form www.

Due to these contradictions, the pumping lemma does not hold for A_2, so it is not regular.

c.

Proof by Contradiction: Assuming that $A_3 = \{\mathtt{a}^{2^n} \mid n \geq 0\}$ is regular, then the pumping lemma holds. Therefore, there exists a pumping length $p \geq 1$, such that any string longer than p can be pumped. The string $s = \mathtt{a}^{2^p}$ is in A_2 and $\forall p \geq 1.\ 2^p > p$, so s is longer than p, and therefore can be split into xyz such that $|xy| \leq p$, $|y| > 0$ and $xy^iz \in A_2$. From $|y| > 0$ follows that $xyyz$ is longer than $\mathtt{a}^{2^p}$. Similarly, from $|xy| \leq p$ and $2^p > p$ follows that $|y| < 2^p$, so the string $xyyz$ is shorter than $\mathtt{a}^{2^p+2^p} = \mathtt{a}^{2^{p+1}}$, so $xyyz$ cannot have the form $\mathtt{a}^{2^n}$.

Due to these contradictions, the pumping lemma does not hold for A_3, so it is not regular.

1.30

The "proof" considers $s = \texttt{0}^p\texttt{1}^p = xyz$. If y contains only 0's, then $xyyz$ will be of the form $\texttt{0}^{p+|y|}\texttt{1}^p$ and similarly if y contains only 1's, then $xyyz$ will be of the form $\texttt{0}^p\texttt{1}^{p+|y|}$. However, both of these strings are in the language $\texttt{0}^*\texttt{1}^*$ which does not require an equal number of 0's and 1's, therefore there is no contradiction.

1.42

The shuffle S of two regular languages A and B is also regular.

$$S(A,B) = \{a_1b_1a_2b_2...a_kb_k \mid a_1a_2..a_k \in A \ \wedge \ b_1b_2...b_k \in B \ \wedge \ a_i \in \Sigma^*\}$$

For all regular languages A and B, we can show that $S(A,B)$ is also regular by constructing an NFA that accepts the language $S(A,B)$. Since A and B are regular languages, we can assume there are DFAs $D_A = (Q_A, \Sigma_A, \delta_A, q_{A,0}, F_A)$ and $D_B = (Q_B, \Sigma_B, \delta_B, q_{B,0}, F_B)$.

$$\begin{array}{lll}
N = (Q, \Sigma, \delta, q_0, F) & \Sigma = \Sigma_A \cup \Sigma_B & \delta(\langle q_A, q_B, A\rangle, s) = \{\langle \delta_A(q_A, s), q_B, A\rangle\} \\
Q = Q_A \times Q_B \times \{A, B\} & & \delta(\langle q_A, q_B, B\rangle, s) = \{\langle q_A, \delta_B(q_B, s), B\rangle\} \\
q_0 = \langle q_{A,0}, q_{B,0}, A\rangle & & \delta(\langle q_A, q_B, A\rangle, \epsilon) = \{\langle q_A, q_B, B\rangle\} \\
F = \{\langle q_a, q_b, A\rangle \mid q_a \in F_A \ \wedge \ q_b \in F_B\} & & \delta(\langle q_A, q_B, B\rangle, \epsilon) = \{\langle q_A, q_B, A\rangle\}
\end{array}$$

The NFA N keeps track of the DFA state in D_A, the DFA state in D_B and whether the next symbol is expected to be for D_A or D_B. Here, $< q_a, q_b, M >$ denotes the tuple of state $q_a \in Q_A$, $q_b \in Q_B$ and $M \in \{A, B\}$. The language described by the DFA D is regular but we still need to show that it actually accepts the shuffle $S(A,B)$ for each regular language A and B.

Proof.

- $w \in S(A,B) \ \Rightarrow \ \hat{\delta}(q_0, w) \in F$.
  Given a word $w = a_1b_1a_2b_2...a_kb_k$, the NFA will update q_A based on δ_A and q_B based on δ_B. Because $a_1a_2..a_k \in A$, it follows that $\hat{\delta}(q_{A,0}, a_1a_2..a_k) \in F_A$ and, $\hat{\delta}(q_{B,0}, b_1b_2..b_k) \in F_B$, therefore $\hat{\delta}(q_0, w) \in F$.
- $\hat{\delta}(q_0, w) \in F \ \Rightarrow \ w \in S(A,B)$.
  Given a word $w = a_1b_1a_2b_2...a_kb_k$ such that the NFA reaches an accepting state, all the symbols that updated the component q_A of the state in N form the word $a_1a_2...a_k$ which is in A and similarly $b_1b_2...b_k \in B$, so $w \in S(A,B)$.

1.46

a.

The class of regular languages is closed under intersection, therefore it is sufficient to show that $\{\texttt{0}^n\texttt{1}^m\texttt{0}^n \mid m, n \geq 0\} \cap 0^*10^0 = \{\texttt{0}^n\texttt{1}\texttt{0}^n \mid n \geq 0\}$ is not regular, to follow that $\{\texttt{0}^n\texttt{1}^m\texttt{0}^n \mid m, n \geq 0\}$ cannot be regular.

Proof by Contradiction: Assuming that $\{\texttt{0}^n\texttt{1}^m\texttt{0}^n \mid m, n \geq 0\}$ is regular, then the pumping lemma holds. Therefore, there exists a pumping length $p \geq 1$, such that any string longer than p can be pumped. The string $s = 0^p 1 0^p$ is longer than p, and therefore can be split into xyz such that $|xy| \leq p$, $|y| > 0$ and $xy^i z$ is in the language. Since $|xy| \leq p$, both x and y cannot contain any 1's, so $xyyz$ will have the form $0^{p+|y|} 1 0^p$ and is therefore not in the language. Due to these contradictions, the pumping lemma does not hold for $\{0^n 1^m 0^n \mid m, n \geq 0\}$, so it is not regular and therefore $\{0^n 1^m 0^n \mid m, n \geq 0\}$ cannot be regular.

b.

The class of regular languages is closed under intersection and complement and we know that $\{0^n 1^n \mid n \geq 0\} = \overline{\{0^n 1^m \mid n \neq m\}} \cap 0^* 1^*$ is not regular. Therefore, $\{0^n 1^m \mid n \neq m\}$ cannot be regular.

c.

The class of regular languages is closed under intersection and complement and we know (from **a.**) that $\{0^n 1^m 0^n \mid m, n \geq 0\} = \overline{\{w \mid w \in \{0,1\}^* \text{ is not a palindrome}\}} \cap 0^* 1^* 0^*$ is not regular. Therefore, $\{w \mid w \in \{0,1\}^* \text{ is not a palindrome}\}$ also cannot be regular.

d.

Proof by Contradiction: Assuming that $\{wtw \mid w, t \in \{0,1\}^+\}$ is regular, then the pumping lemma holds. Therefore, there exists a pumping length $p \geq 1$, such that any string longer than p can be pumped. The string $s = 0^p 1 0^p 1$ is longer than p, and therefore can be split into xyz such that $|xy| \leq p$, $|y| > 0$ and $xy^i z$ is in the language. Since $|xy| \leq p$, both x and y cannot contain any 1's, so $xyyz$ will have the form $0^{p+|y|} 1 0^p 1$ and is therefore not in the language. Due to these contradictions, the pumping lemma does not hold and therefore $\{wtw \mid w, t \in \{0,1\}^+\}$ cannot be regular.

1.47

Proof by Contradiction: Assuming that $Y = \{x_1 \# x_2 ... \# x_k \mid k \geq 0 \ \wedge \ \forall i \neq j.\ \ x_i \neq x_j\}$ is regular. Since the class of regular languages is closed under complement, $\overline{Y} = \{x_1 \# x_2 ... \# x_k \mid k \geq 0 \ \wedge \ \exists i, j . i \neq j \ \wedge \ x_i = x_j\}$ will also be regular and thus the pumping lemma holds. This means there exists a pumping length $p \geq 1$, such that any string longer than p can be pumped. The string $s = 1^p \# 1^p$ is in $\overline{Y}$ and longer than p, and therefore can be split into xyz such that $|xy| \leq p$, $|y| > 0$ and $xy^i z \in Y$.

Due to $|xy| \leq p$, y cannot contain the # symbol which is at the $p + 1$'th position in the string. Therefore the string $xyyz$ will have the form $1^{p+|y|} \# 1^p$ which does not repeat a sequence will therefore not be in $\overline{Y}$.

Due to these contradictions, the pumping lemma does not hold for Y, so it is not regular.

1.55

e.

Minimum pumping length $p = 2$. For strings of length 2 (`01`), 4 (`0101`), etc. you can always pump a `01` group. For pumping length 1, however, you would have to pump a single symbol, so either `0` or `1` which would result in sequences of `0`'s or `1`'s which are not in the language.

f.

Minimum pumping length $p = 1$. There are no words of length 1 or more, so the pumping lemma is trivially true as there are no such strings cannot be pumped.

i.

Minimum pumping length $p = 5$. There are no words of length 5 or more, so the pumping lemma is trivially true as there are no such strings cannot be pumped. For values below 5, pumping with $xyyyyyz$ would always result strings that are longer than 5 and therefore not in the language.

j.

Minimum pumping length $p = 1$. For any string of length 1 or more, it is possible to split $s = xyz$ into $x = \epsilon,\ \ y = w_0,\ \ z = w_1 w_2 \ldots$ and any $xy^i z = w_0^i w_1 w_2 \ldots$ will always be in the language.