$\{ x \in \{0,1\}^* \mid x$ IN BINARY EQUALS ZERO MOD 3 $\}$

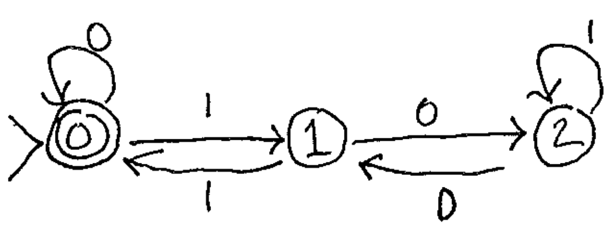
WOULD LIKE STATES TO REPRESENT STRING SEEN SO FAR MOD 3.

SO THERE ARE THREE STATES 0, 1, 2

NOTE WHEN SCAN MOVES ONE STEP RIGHT THE "NUMBER" SEEN SO FAR IS MULTIPLIED BY 2 AND THE NEW BIT SCANNED IS ADDED TO IT.

CASES

STATE	SEE	NUM FORM		$\times 2 + 1$ OR 0	
0	0	$3K$	$\rightarrow$	$3 \cdot 2K + 0$	0
0	1		$\rightarrow$	$3 \cdot 2K + 1$	1.
1	0	$3K+1$	$\rightarrow$	$3 \cdot 2K + 2 + 0$	2.
1	1		$\rightarrow$	$3 \cdot 2K + 2 + 1$ $3(2K+1) + 0$	0.
2	0	$3K+2$	$\rightarrow$	$3 \cdot 2K + 4$ $3(2K+1) + 1$	1
2	1		$\rightarrow$	$3 \cdot 2K + 4 + 1$ $3(2K+1) + 2$	2



	x	$\overset{0}{2x+0}$	$\overset{1}{2x+1}$
0 mod 3	$3k$	$6k$ $3(2k)$ 0 mod 3	$6k+1$ $3(2k)+1$ 1 mod 3
1 mod 3	$3k+1$	$6k+2$ $3(2k)+2$ 2 mod 3	$6k+2+1$ $3(2k)+3$ $3(2k+1)$ 0 mod 3
2 mod 3	$3k+2$	$6k+4$ $3(2k)+1+3$ $3(2k+1)+1$ 1 mod 3	$6k+4+1$ $3(2k)+3+2$ $3(2k+1)+2$ 2 mod 3

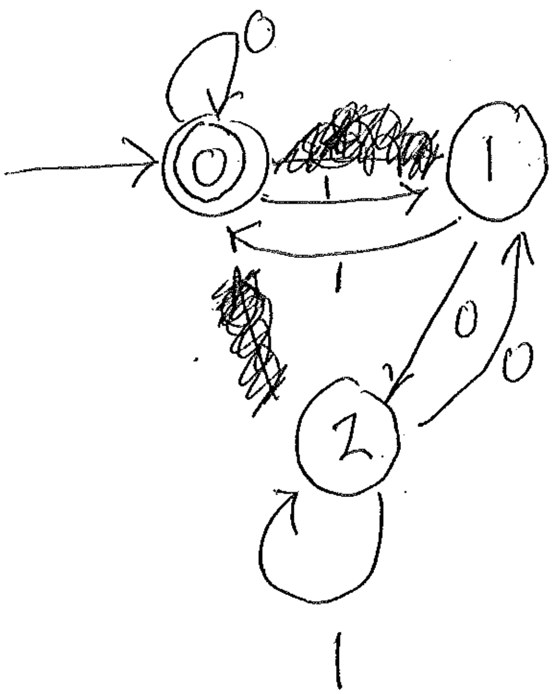
WORKS FOR x I.E. $Val(x) \bmod 3$ = state

$\Rightarrow$ WORKS FOR xc WHERE $c \in \{0, 1\}$ $Val(xc) \bmod 3$ = state

$\Sigma = \{0, 1\}$

$\{x \in \Sigma^* \mid x \text{ represents a multiple of 3 in binary}\}$

	0	1
$\to$ 0F	0	1
1	2	0
2	1	2



PRODUCT CONSTRUCTION (INTERSECTION)

THM ~~ASSUME~~ IF A AND B ARE REGULAR, THEN $A \cap B$ IS REGULAR

~~THEN~~ THERE EXISTS

$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ WITH $L(M_1) = A$

$M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ WITH $L(M_2) = B$

SINCE A AND B ARE REGULAR.

WE WILL BUILD A ~~N~~ DFA M_3 SUCH THAT $L(M_3) = A \cap B$

<INFORMAL DESCRIPTION WITH PEBBLES>

LET

$$M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$$

$$Q_3 = Q_1 \times Q_2 = \{(p,q) \mid p \in Q_1 \text{ AND } q \in Q_2\}$$

$$F_3 = F_1 \times F_2 = \{(p,q) \mid p \in F_1 \text{ AND } q \in F_2\}$$

$$s_3 = (s_1, s_2)$$

$$\delta_3 : Q_3 \times \Sigma \to Q_3$$

$$\delta_3((p,q), a) = (\delta_1(p,a), \delta_2(q,a))$$

EXTENDED TRANSITION FUNCTION FOR M_3

$$\hat{\delta}_3((p,q),\epsilon) = (p,q)$$

$$\hat{\delta}_3((p,q),xa) = \delta_3(\hat{\delta}_3((p,q),x),a)$$

$$\boxed{\begin{aligned}\hat{\delta}(q,\epsilon) &= q \\ \hat{\delta}(q,xa) &= \delta(\hat{\delta}(q,x),a)\end{aligned}}$$

<u>LEMMA</u>

FOR ALL $x \in \Sigma^*$

$$\hat{\delta}_3((p,q),x) = (\hat{\delta}_1(p,x), \hat{\delta}_2(q,x))$$

PROOF: INDUCTION ON $|x|$

BASE CASE

$x = \epsilon$

$$\hat{\delta}_3((p,q),\epsilon) = (p,q) = (\delta_1(p,\epsilon), \delta_2(q,\epsilon))$$

~~INDUCTION STEP~~

INDUCTION STEP PROVE

$$\boxed{\begin{aligned} &\hat{\delta}_3((p,q),x) = (\hat{\delta}_1(p,x), \hat{\delta}_2(q,x)) \quad \text{I.H.} \\ &\Rightarrow \hat{\delta}_3((p,q),xa) = (\hat{\delta}_1(p,xa), \hat{\delta}_2(q,xa)) \end{aligned}}$$

WHERE $|x| \geq 0$

$$\begin{aligned} \hat{\delta}_3((p,q),xa) &= \delta_3(\hat{\delta}_3((p,q),x),a) && \text{DEF OF } \hat{\delta}_3 \\ &= \delta_3((\hat{\delta}_1(p,x), \hat{\delta}_2(q,x)),a) && \text{I.H.} \\ &= (\delta_1(\hat{\delta}_1(p,x),a), \delta_2(\hat{\delta}_2(q,x),a)) && \text{DEF OF } \delta_3 \\ &= (\hat{\delta}_1(p,xa), \hat{\delta}_2(q,xa)) && \text{DEF OF } \hat{\delta}_1 \text{ AND } \hat{\delta}_2 \end{aligned}$$

Q.E.D. FOR LEMMA

<u>THM</u> $L(M_3) = L(M_1) \cap L(M_2)$

FOR ALL $x \in L(M_3)$

$x \in L(M_3) \iff \hat{\delta}_3(s_3, x) \in F_3$ DEF OF ACC.

$\iff \hat{\delta}_3((s_1, s_2), x) \in F_1 \times F_2$ DEF OF S_3 AND F_3

$\iff (\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_1 \times F_2$ LEMMA

$\iff \hat{\delta}_1(s_1, x) \in F_1$ AND $\hat{\delta}_2(s_2, x) \in F_2$ DEF OF CART. PROD.

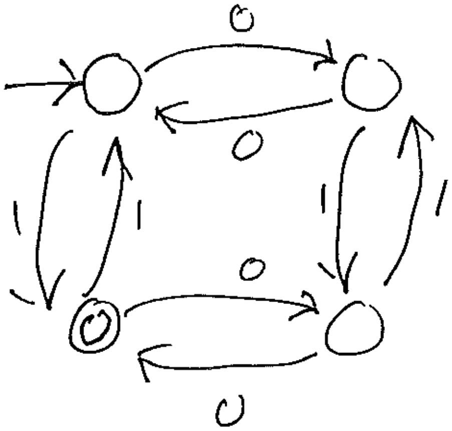
$\iff x \in L(M_1)$ AND $x \in L(M_2)$ DEF OF ACC

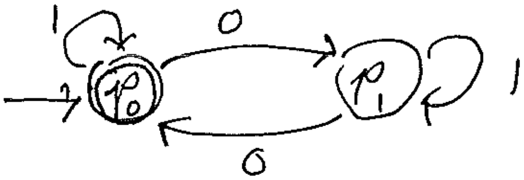
$\iff x \in (L(M_1) \cap L(M_2))$ DEF OF SET ~~INTERSECT~~ INTERSECTION

Q.E.D.

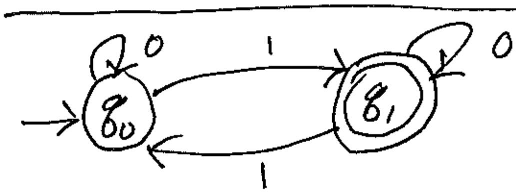
EXAMPLE PRODUCT CONSTRUCTION

EVEN # 0s
ODD # 1s



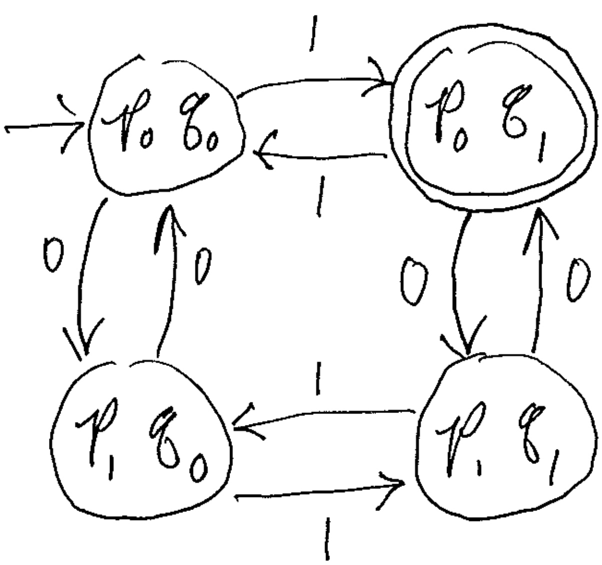


EVEN # 0s



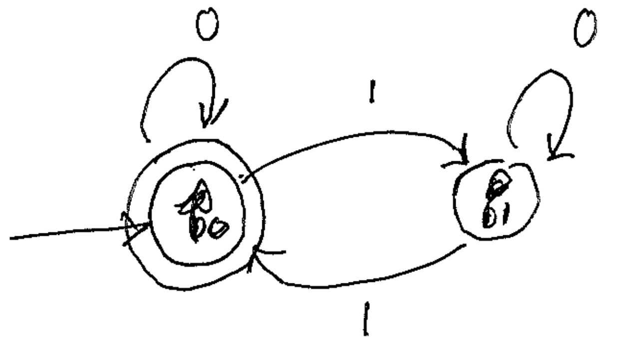
ODD # 1s

PRODUCT CONSTRUCTION.



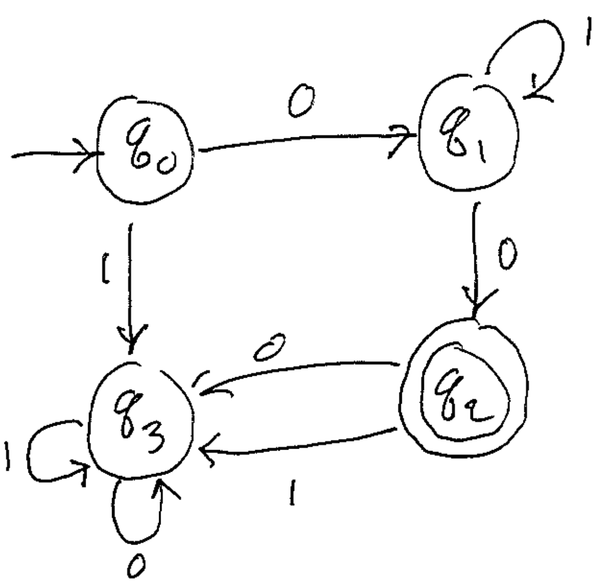
EXAMPLE OF PRODUCT CONSTRUCTION

$A \cap B$



A

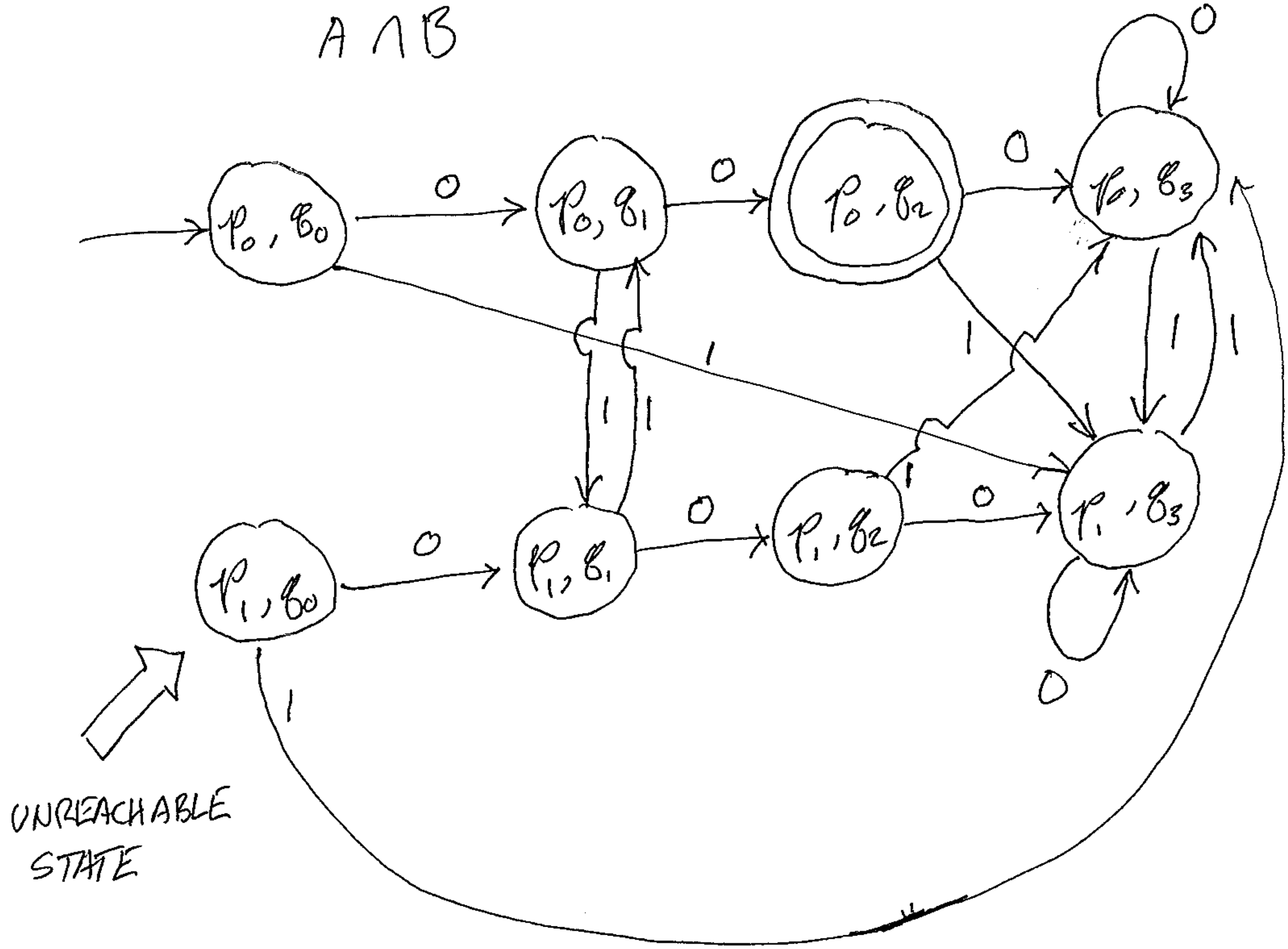
EVEN # OF 1s



B

$\{ 01^m0 \in \Sigma^* \mid m \geq 0 \}$

$A \cap B$



p0 , q0
p0, q1
p0 , q2
p0, q3
p1 , q0
p1, q1
p1 , q2
p1 , q3
0
1
UNREACHABLE
STATE

~~PROOF~~

THEOREM $A \cup B$ IS REGULAR

IF A IS REGULAR AND B IS REGULAR.

A REG $\Rightarrow$ $\sim A$ REG

B REG $\Rightarrow$ $\sim B$ REG

$\sim A$ REG AND $\sim B$ REG $\Rightarrow$ $\sim A \cap \sim B$ REG

$\sim A \cap \sim B$ REG $\Rightarrow$ $\sim(\sim A \cap \sim B)$ REG

$\sim(\sim A \cap \sim B)$ REG $\Rightarrow$ $A \cup B$ REG DE MORGAN.

Q.E.D.

NONDETERMINISTIC FINITE AUTOMATON

$N = (Q, \Sigma, \delta, s, F)$

Q A FINITE SET OF STATES

Σ A FINITE SET OF SYMBOLS - ALPHABET

$\delta : Q \times \Sigma \longrightarrow 2^Q$ TRANSITION FUNCTION
~~RELATION~~ ON $(Q \times \Sigma) \times Q$

$\delta(p, a) \in 2^Q$

SET OF ALL STATES N CAN MOVE TO FROM p IN ONE STEP UNDER INPUT SYMBOL a.

$p \xrightarrow{a} q$ if $q \in \delta(p, a)$

$\delta(p, a)$ CAN BE $\emptyset$

$s \in Q$ IS THE "START" STATE

$F \subseteq Q$ "ACCEPT" OR "FINAL" STATES

EXTENDED TRANSITION FUNCTION

$\delta^* : Q \times \Sigma^* \rightarrow 2^Q$

$\delta^*(q, \varepsilon) = \{q\}$

$\delta^*(q, a) = \delta(q, a)$

$$\delta^*(q, xa) = \bigcup_{p \in \delta^*(q, x)} \delta(p, a)$$

$\delta^*(p, w)$ SET OF ALL STATES N CAN MOVE TO FROM p UNDER INPUT STRING w.

ACCEPTED

$w \in \Sigma^*$ IS ACCEPTED BY N IF

$$\overset{*}{\delta}(s, w) \cap F \neq \emptyset$$

LANGUAGE N RECOGNIZES (ACCEPTS)

$$L(N) = \{ w \in \Sigma^* \mid N \text{ accepts } w \}$$

EVERY DFA IS ALSO A NFA

THM: LET $M = (Q, \Sigma, \delta, s, F)$ A DFA

AND $N = (Q, \Sigma, \Delta, \{s\}, F)$ A NFA

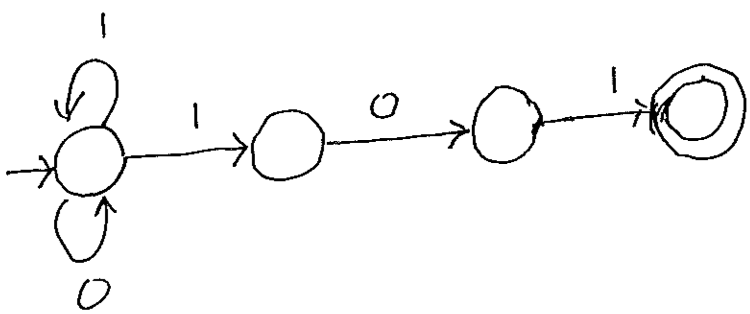
WHERE $\Delta(p, a) = \{\delta(p, a)\}$

THEN $x \in L(M)$ iff $x \in L(N)$

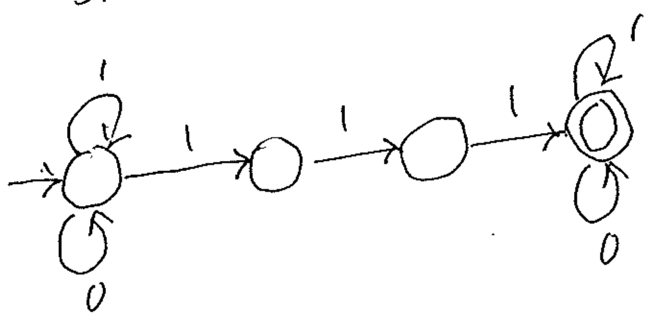
Proof: obvious.

NFA EXAMPLES

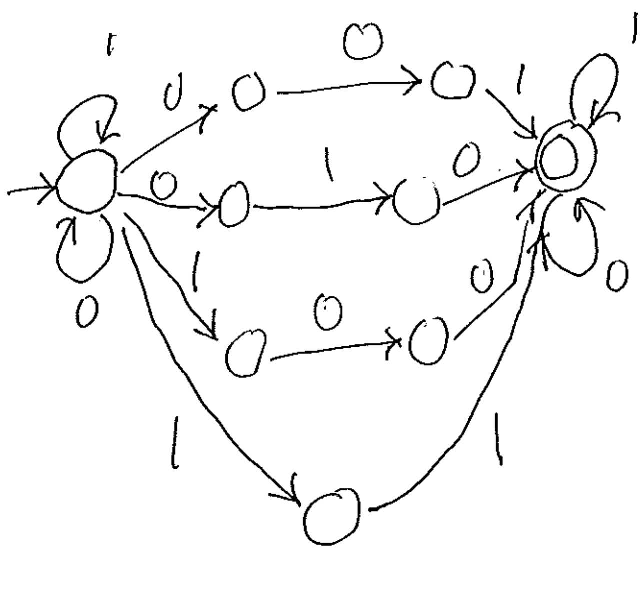
STRING ENDS WITH 101



STRING CONTAINS 111



STRING CONTAINS 001 v 010 v 100 v 11



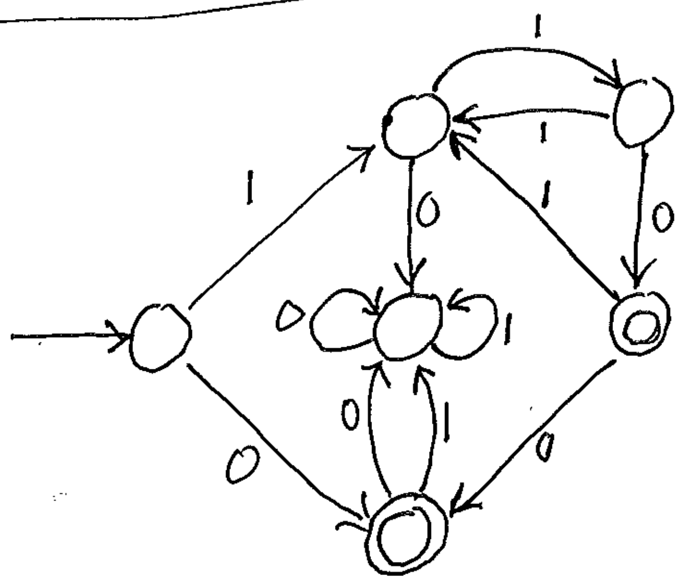
NFA EASIER TO DESIGN THAN DFA

FROM MARTIN

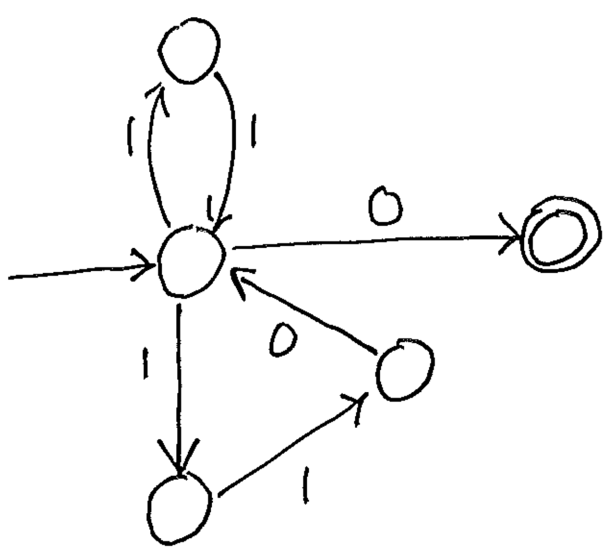
LANGUAGE: $(11 + 110)^* 0$

~~where $x \in \{11, 110\}$~~

$$\boxed{\{ x_1 x_2 \ldots x_n 0 \mid n \geq 0 \text{ AND } x_i \in \{11, 110\} \}}$$



DFA



NFA

SUBSET CONSTRUCTION RABIN - SCOTT 1959

GIVEN NFA

δ_N^*:

$$N = (Q_N, \Sigma, \delta_N, s_N, F_N)$$

$\vdots$

$$\delta_N : Q_N \times \Sigma \to 2^{Q_N}$$

$\delta_N(p, a)$ IS SET OF ALL STATES THAT N CAN MOVE TO FROM p GIVEN a AS AN INPUT.

$$\hat{\delta}_N(q, \epsilon) = \{q\}$$

$$\hat{\delta}_N(q, xa) = \bigcup_{p \in \hat{\delta}_N(q, x)} \delta_N(p, a)$$

CONSTRUCT DFA

$$M = (Q_D, \Sigma, \delta_D, s_D, F_D)$$

$$Q_D = 2^{Q_N}$$

$$s_D = \{s_N\}$$

$$F_D = \{P \subseteq Q_N \mid P \cap F_N \neq \emptyset\}$$

$$\delta_D : Q_D \times \Sigma \to Q_D \qquad 2^{Q_N} \times \Sigma \to 2^{Q_N}$$

$$\delta_D(P, a) = \bigcup_{p \in P} \delta_N(p, a) \quad \text{FOR } P \subseteq Q_N$$

$$P \in 2^{Q_N}$$

$$\hat{\delta}_D(P, \epsilon) = P$$

$$\hat{\delta}_D(P, xa) = \delta_D(\hat{\delta}_D(P, x), a)$$

THM $L(N) = L(M)$

SHOW THAT M FAITHFULLY SIMULATES N
SHOW TRANSITION FUNCTIONS ARE REALLY THE SAME.

FOR EVERY p of N and string w LEMMA

$$\overset{*}{\delta}_D(\{p\}, w) = \overset{*}{\delta}_N(p, w)$$ CAN PROVE BY INDUCTION

THIS IMPLIES $L(N) = L(M)$

PROOF
w ACCEPTED N iff $\overset{*}{\delta}_N(s_N, w) \cap F_N \neq \emptyset$ DEF.

iff
$\overset{*}{\delta}_D(\{s_N\}, w) \cap F_N \neq \emptyset$ LEMMA

iff
$\overset{*}{\delta}_D(\{s_N\}, w) \in F_D$ DEF OF F_D

BUT THIS IS ACCEPTANCE BY M
iff $w \in L(M)$

LEMMA

$$\delta_D^*(\{p\}, w) = \delta_N^*(p, w)$$

BASE $\quad w = \epsilon$

$$\delta_D^*(\{p\}, \epsilon) = \{p\} \qquad \text{DEF of } \delta_D^*$$

$$\delta_N^*(p, \epsilon) = \{p\} \quad \checkmark \qquad \text{DEF } \delta_N^*$$

INDUCTIVE STEP

ASSUME $\delta_D^*(\{p\}, x) = \delta_N^*(p, x)$ $\qquad$ I.H.

PROVE

$$\delta_D^*(\{p\}, xa) = \delta_N^*(p, xa)$$

- - - - - - - - - - - - - - - - - - - -

$$\delta_D^*(\{p\}, xa) = \delta_D(\delta_D^*(\{p\}, x), a) \qquad \text{DEF } \delta_D^*$$

$$= \delta_D(\delta_N^*(p, x), a) \qquad \text{I.H.}$$

$$= \bigcup_{q \in \delta_N^*(p, x)} \delta_N(q, a) \qquad \text{DEF } \delta_D$$

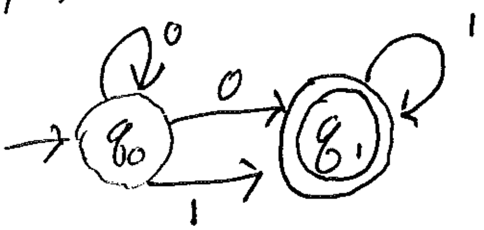
$$= \delta_N^*(p, xa) \qquad \text{DEF } \delta_N^*$$

Q.E.D.

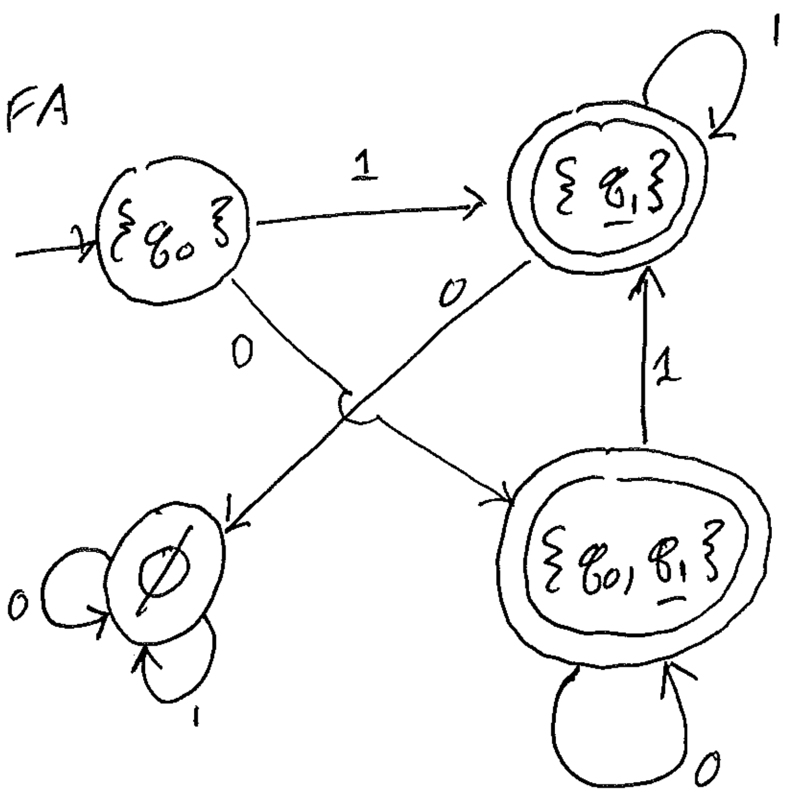
EXAMPLE OF SUBSET CONSTRUCTION I

$$L(M) = \{0^n \mid n \geq 1\} \cup \{1^n \mid n \geq 1\} \cup \{0^n 1^m \mid n \geq 1, m \geq 1\}$$

NFA



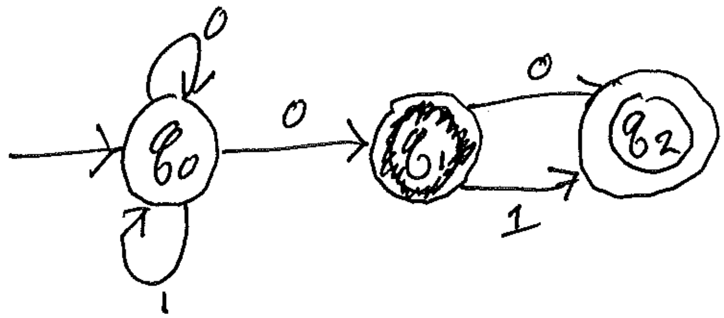
DFA



EXAMPLE OF SUBSET CONSTRUCTION II

$L_2 = \{ w \in \{0,1\}^* \mid \text{2nd symbol from right is a } 0 \}$

NFA

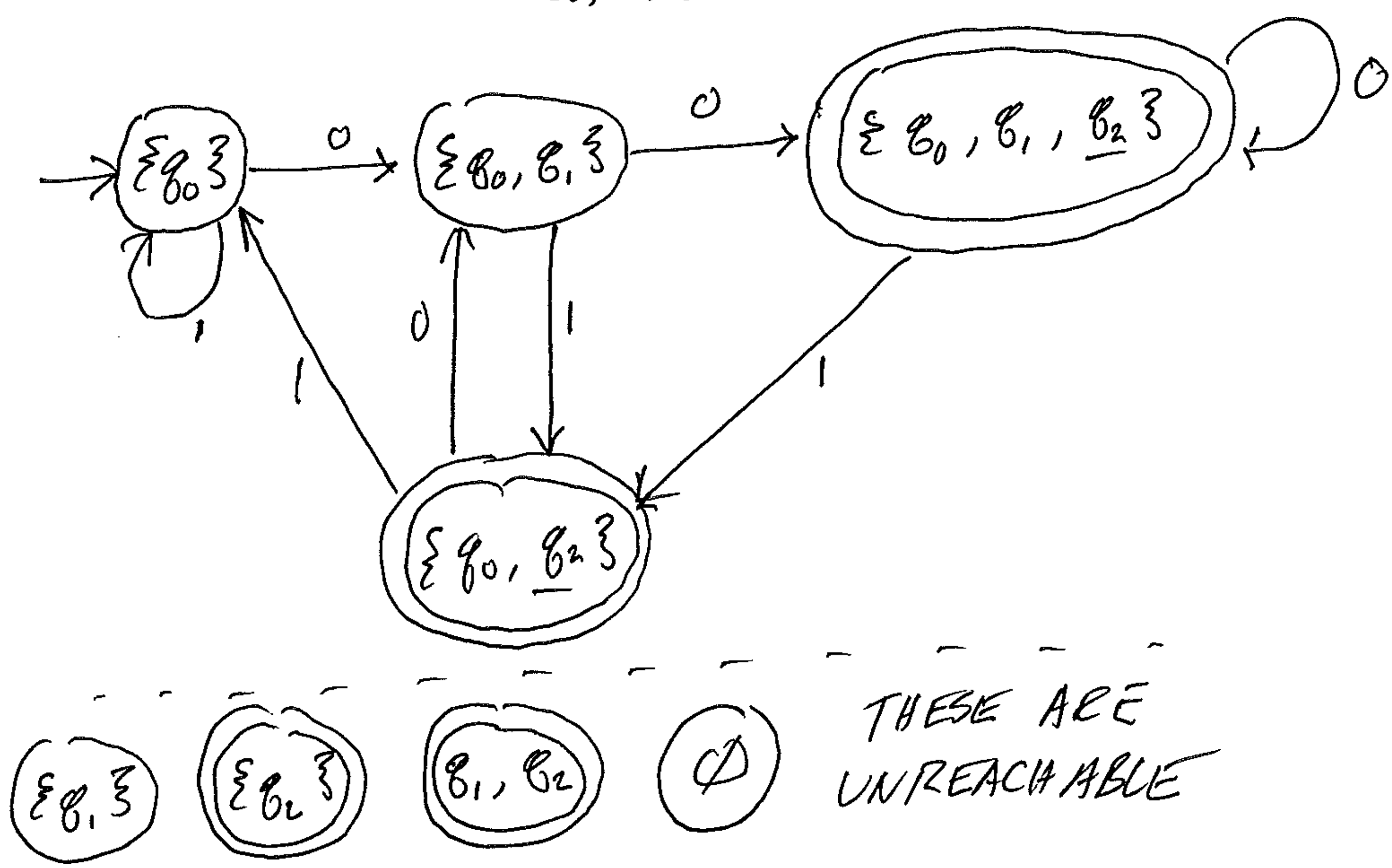


DFA

CONSIDER 8 STATES $\emptyset$, $\{q_0\}$, $\{q_1\}$, $\{q_2\}$,

$\{q_0, q_1\}$, $\{q_0, q_2\}$, $\{q_1, q_2\}$

$\{q_0, q_1, q_2\}$



THESE ARE UNREACHABLE