EXAMPLES OF FAULTY REASONING

2, 4, 6, 8, 10, 12, 14, ARE EVEN
ALL EVEN NUMBERS ARE LESS THAN 100.
OR EVEN ALL EVEN NUMBERS ARE LESS THAN $10^{1000000}$.

VERY EASY TO FIND COUNTER EXAMPLE

FERMAT 17th Century
NUMBERS OF FORM $2^{2^n} + 1$ ARE PRIME

0	3
1	5
2	17
3	257
4	65,537

TRUE FOR THESE

EULER 18th century

$$2^{2^5} + 1 = 4,294,967,297 = 641 \cdot 6700417$$

NUMBERS OF FORM $991n^2 + 1$ ARE NEVER PERFECT SQUARES.

COUNTER EXAMPLE: FIRST ONE

$$n = 12,055,735,790,331,359,447,442,538,767.$$

~~SKIP~~

PROOFS

DEDUCTIVE - SEQUENCE OF JUSTIFIED STEPS

INDUCTIVE - RECURSIVE PROOF OF PARAMETERIZED STATEMENT.

GIVENS DEFINITIONS
AXIOMS
ALREADY PROVEN THEOREMS / LEMMAS

$\downarrow$

CONCLUSION

Proofs about SETS

Proofs by CONTRADICTION

Proofs by COUNTER EXAMPLE

CONTRAPOSITIVE May be easier

$p \to q \qquad \neg q \to \neg p$

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p implies q   Variants of implies

p only if q

q if p

when ever p holds q follows.

VOCABULARY

THEOREM
PROPOSITION
LEMMA
} STATEMENT TO BE PROVED OR ALREADY PROVED

AXIOMS
POSTULATES
} STATEMENTS WE ASSUME TO BE TRUE

COROLLARY

IMMEDIATELY FOLLOWS FROM PROOF OF THEOREM

CONJECTURE

ODD NUMBERS DIFF OF 2 SQUARES

$$n = 2i + 1 \qquad i \in \{0, 1, 2, 3, \ldots\}$$

$$= i^2 + 2i + 1 - i^2$$

$$= (i+1)^2 - i^2$$

Q.E.D.

$(2K)^2 = 4K^2 = 2\cdot(2K^2)$ EVEN

CASES PROOF

$(2K+1)^2 = 4K^2+4K+1$

$\cancel{= 4K(K+1)+1}$

$\cancel{= 2(2K}$

$= 2(2K^2+2K)+1$

i.e. ~~EVEN~~ ODD

---

p^2 EVEN $\Rightarrow$ p EVEN

p NOT EVEN $\Rightarrow$ p^2 NOT EVEN <u>CONTRAPOSITIVE</u>

$p = 2m+1$ FOR SOME m

$p^2 = 4m^2+2m+1$

$= 2(2m^2+m)+1$

NOT EVEN

$\sqrt{2}$ IRRATIONAL

ASSUME $\sqrt{2} = \frac{p}{q}$

AND p AND q ARE POSITIVE INTEGERS

AND p AND q HAVE NO COMMON DIVISOR

$p^2 = 2q^2 \Rightarrow p^2$ IS EVEN

p^2 EVEN $\Rightarrow p$ EVEN

$p = 2m$

$(2m)^2 = 2q^2$

$q^2 = 2m^2 \Rightarrow q^2$ IS EVEN

$\Rightarrow q$ IS EVEN

$\Rightarrow$ BOTH p AND q DIVISIBLE BY 2

CONTRADICTION

COUNTEREXAMPLE

EVERY POS INTEGER IS SUM OF SQUARES OF TWO INTEGERS

CONSIDER 3

ONLY SQUARES ≤ 3 0, 1

$0 + 1 \neq 3$

Q.E.D.

$$\left.\begin{array}{l} 0+0 \\ 0+1 \\ 1+0 \\ 1+1 \end{array}\right\} \neq 3$$

ALL PRIMES ARE ODD.

CONSIDER 2

DIRECT PROOF

IF n & m ARE PERFECT SQUARES THEN nm IS PERFECT SQUARE

$n = s^2$

$m = t^2$

$$\begin{aligned} nm &= sstt \\ &= stst \\ &= (st)^2 \end{aligned}$$

Q.E.D.

PIGEON HOLE PRINCIPLE PROOF

FOR ANY n
THERE EXISTS A MULTIPLE THAT IS ONLY
0's & 1's IN DECIMAL NOTATION.

CONSIDER:

$$1 \quad 11 \quad 111 \quad \ldots \quad \underbrace{111\ 1}_{n+1}$$

DIVIDE BY n ONLY $\underline{n}$ REMAINDERS POSSIBLE $0, \; , n-1$

SO BY PHP TWO OF 1^n NUMBERS
$\underline{n+1 \text{ OF THESE}}$

$$a = k_1 n + r$$

$$b = k_2 n + r$$

$$a - b = \underbrace{(K_a - K_b)\, n}_{\text{multiple of } n}$$

$111\ldots0$

INDUCTION

IF A SET OF NATURAL NUMBERS

1. CONTAINS 1

2. CONTAINS THE SUCCESSOR OF EACH OF ITS MEMBERS

THEN IT CONTAINS ALL THE NATURAL NUMBERS

CONSIDER THE SET OF NUMBERS FOR WHICH A STATEMENT $S(n)$ IS TRUE.

IF $S(1)$ IS TRUE AND

$S(K)$ IS TRUE IMPLIES $S(K+1)$ IS TRUE FOR ALL $K \geq 1$

THEN THE SET OF NUMBERS FOR WHICH $S(n)$ IS TRUE MUST BE THE SET OF ALL NATURAL NUMBERS.

INDUCTION

THEOREM

$(\forall n \in \mathbb{N})\ P(n)$

PROOF:

I. BASE CASE(S)

PROVE $P(1)$

PROVE $P(2)$

$\vdots$

PROVE $P(b)$

II. INDUCTIVE STEP:

PROVE $(\forall k \geq b)$ $P(k)$ IMPLIES $P(k+1)$

PROVE $(\forall (b \leq j \leq k)\ P(j))$ IMPLIES $P(k+1)$

INDUCTION

CIR

FIND SUM OF FIRST m ODD NUMBERS

$$1 + 3 + \dots + (2m - 1) = S(m)$$

$S(1) = 1$		1
$S(2) = 1 + 3$		4
$S(3) = 1 + 3 + 5$		9
$S(4) = 1 + 3 + 5 + 7$		16
$S(5) = 1 + 3 + 5 + 7 + 9$		25

SINCE $1, 4, 9, 16, 25$
ARE OBVIOUSLY $1^2, 2^2, 3^2, 4^2, 5^2$
CAN WE CONCLUDE

$$S(m) = m^2 \quad ?$$

NO

EXAMPLES ARE EVIDENCE THAT SUGGESTS GENERAL STATEMENT
BUT ARE NOT <u>CONVINCING</u> EVIDENCE OF GENERAL STATEMENT.

THEY ARE GOOD FOR FORMING CONJECTURES
BUT DO NOT <u>PROVE</u> THEM

(HOW EVER NOTE THAT ONE EXAMPLE (COUNTER) IS ENOUGH TO DISPROVE A GENERAL STATE MENT).

2h

INDUCTIVE PROOF

FOR $n = 1$

$S(1) = 1$ TRUE

ASSUME

$S(K) = K^2$ FOR SOME $K \geq 1$ I. H.

DOES THIS IMPLY $S(K+1) = (K+1)^2$?

$$S(K+1) = S(K) + \underbrace{2(K+1) - 1}_{(K+1)\text{TH ODD NUMBER}}$$

$$= K^2 + 2K + 2 - 1$$

$$= K^2 + 2K + 1$$

$$= (K+1)^2 \quad \text{TRUE.}$$

Q.E.D.

STRONG INDUCTION

$\forall n > 1$ n IS DIVISIBLE BY A PRIME

CONSIDER ANY $n > 1$

I n IS A PRIME DONE

II n IS NOT A PRIME THEN

$n = a \cdot b$ WHERE $a, b < n$

$a, b > 1$

BUT I.H. a IS DIVISIBLE BY PRIME SAY p

WE HAVE $p \mid a$ AND $a \mid n$

THEREFORE $p \mid n$

Q.E.D.