MAT-MEK 4270 Mandatory 1

1.2.3 Exact solution

$u(t,x,y) = e^{\hat{i}(k_x x + k_y y - \omega t)}$ $\quad \hat{i} = \sqrt{-1}$

Given $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial t}$ we can set $a = \hat{i}(k_x x + k_y y - \omega t)$ so $u(t,x,y) = e^{a}$.

$\therefore u_t = e^{a}(-\hat{i}\omega) = -\omega\hat{i}e^{\hat{i}(k_x x + k_y y - \omega t)}$

Using the same technique, we can see $u_{tt} = -\omega^2 e^{\hat{i}(k_x x + k_y y - \omega t)}$

Doing the same thing for the x and y derivatives, we get

$u_{xx} = -k_x^2 e^{a}$, $u_{yy} = -k_y^2 e^{a}$

Substituting this into the wave eq, we get

$-\omega^2 e^{\hat{i}(k_x x + k_y y - \omega t)} = -c^2 e^{\hat{i}(k_x x + k_y y - \omega t)}(k_x^2 + k_y^2)$

This holds iff $-\omega^2 = -c^2(k_x^2 + k_y^2)$

We have $|k| = \sqrt{k_x^2 + k_y^2}$ and the dispersion coeff $\omega = c|k|$

So we get $-c^2|k|^2 = -c^2|k|^2$.

Hence, our solution $u(t,x,y)$ is indeed a solution to the wave equation

$u_{tt} = c^2 \Delta u$.

1.2.4 Dispersion coefficient

$u^n_{ij} = e^{\hat{i}(kh(i+j) - \tilde{\omega} n\Delta t)}$

We know that the scheme is stable when $|A| \le 1$ where $A = e^{\hat{i}\omega\Delta t}$.

We can rewrite our u^n_{ij} as $\tilde{A}E(i,j)$ where $E(i,j) = e^{\hat{i}(khi + khj)}$ and $\tilde{A} = e^{-\hat{i}\tilde{\omega}\Delta t}$.

We will sub our $\tilde{A}E(i,j)$ into our discretised 2D wave eq to get

$$\frac{(\tilde{A}^{n+1} - 2\tilde{A}^{n} - \tilde{A}^{n+1})E(i,j)}{\Delta t^2} = \tilde{A}c^2\left\{\frac{E(i+1,j) - 2E(i,j) + E(i-1,j)}{\Delta x^2} + \frac{E(i,j+1) - 2E(i,j) + E(i,j-1)}{\Delta y^2}\right\}$$

Dividing by $\tilde{A}^n E(i,j)$ and multiplying by Δt^2, we get

$$\tilde{A} - 2 + \tilde{A}^{-1} = (c\Delta t)^2\left(\frac{e^{\hat{i}khi} - 2 + e^{-\hat{i}khi}}{\Delta x^2} + \frac{e^{\hat{i}khj} - 2 + e^{-\hat{i}khj}}{\Delta y^2}\right)$$

where we have used $\frac{E(i+1,j)}{E(i,j)} = \frac{e^{\hat{i}kh(k+1)}e^{\hat{i}khj}}{e^{\hat{i}kh(i+j)}} = e^{\hat{i}khi}$

If we let $\Delta x = \Delta y = h$ and $\cos(khx) = \frac{e^{\hat{i}khi} + e^{-\hat{i}khi}}{2}$ then

$$\tilde{A} - 2 + \tilde{A}^{-1} = \left(\frac{c\Delta t}{h}\right)^2\left(2(\cos(khx) + \cos(khy) - 2)\right)$$

Since $\frac{c\Delta t}{h} = CFL = 1/\sqrt{2}$, we get

$$\tilde{A} - 2 + \tilde{A}^{-1} = \cos(khx) + \cos(khy) - 2$$

Since $|\cos(khx)| \le 1$

$|\tilde{A} + \tilde{A}^{-1}| \le |\cos(khx) + \cos(khy)|$

$2|\tilde{A}| \le |2| = 2$ so $|\tilde{A}| \le 1$ so $\tilde{A}$ makes the scheme stable.

Hence $|\tilde{A}| = |A|$ i.e. $e^{\hat{i}\tilde{\omega}\Delta t} = e^{\hat{i}\omega\Delta t} \Rightarrow \omega = \tilde{\omega}$.