Lecture

~~Chapter~~ 1

Introduction, simplices

bullets

n-1

Topics to be covered: singular homology, CW-complexes, basics of category theory, homological algebra, Künneth theorem, UCT, cohomology, cup and cap products, Poincaré duality.

Examples to keep in mind: $\mathbf{R}^n$. Inside, S^n being the n-sphere, topologized as a subspace of Euclidean space. Take quotient by $x \simeq -x$ and get $\mathbf{RP}^n$ i.e. space of lines through the origin in $\mathbf{R}^{n+1}$. Also, look at $V_k(\mathbf{R}^n)$, which is the space of orthonormal k-frames (ordered collection of k orthonormal vectors) in $\mathbf{R}^n$, called the Stiefel manifold, topologized as a subspace of $(S^{n-1})^k$. The Grassmannian, $\mathrm{Gr}_k(\mathbf{R}^n)$, the space of k-dimensional linear subspaces of $\mathbf{R}^n$. It's a quotient space of $V_k(\mathbf{R}^n)$, since a k-frame spans a k-dimensional subspace. For example, $\mathbf{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$. These are all manifolds. italic

Definition 1.1. A manifold is a Hausdorff space, locally homeomorphic to $\mathbf{R}^n$.

these manifolds Manifolds exhibit a hidden symmetry, captured by

All are compact except for $\mathbf{R}^n$. ~~A "symmetry" is exploited in~~ Poincaré duality. We'll probe general topological spaces through simplices. it.

Definition 1.2. The standard n-simplex Δ^n is the convex hull of $\{e_0, \cdots, e_n\}$ in $\mathbf{R}^{n+1}$. More explicitly,

$$\Delta^n = \{\sum t_i e_i \mid \sum t_i = 1, t_i \geq 0\} \subseteq \mathbf{R}^{n+1}$$

left

The t_is are called barycentric coordinates.

improve

Usually just drop the e_is and just write "i". There are maps between them, namely inclusions of faces. The map ~~$d^2 : \Delta^1 \to \Delta^2$~~ that misses the vertex i (~~where $0 \leq i \leq 2$)~~ is denoted d^i. They're called "face inclusions", and are maps $d^i : \Delta^{n-1} \to \Delta^n$ where $0 \leq i \leq n$; they miss the vertex i. picture.

Definition 1.3. Let X be any topological space. A singular n-simplex in X is a continuous map $\Delta^n \to X$. Denote by $\mathrm{Sin}_n(X)$ as the collection of all n-simplices of X. This seems like a "fairly insane" thing to do.

See drawing for a torus. The dir~~ection of the simplex is like an orientation given by ordering of the indices of Δ^n. The standard notation~~ is ~~$\sigma : \Delta^n \to$~~ X. picture

If $\sigma : \Delta^n \to X$. Find $(n-1)$-simplices by looking faces of simplex; get a map $d_i : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ for $0 \leq i \leq n$ by taking $\sigma \mapsto \sigma \circ d^i =: d_i\sigma$, which is the ith face of σ. This carries a lot of information about the space.

Some simplices are particularly interesting. The issue of when boundaries match up is a key aspect of simplices. If σ is a 1-simplex that goes around the hole in a torus, then $d_1\sigma = d_0\sigma$, and this means that the "boundary" is zero. So we want to understand things like $d_0\sigma - d_1\sigma$, but this isn't even a simplex anymore; we need to take formal sums and differences. We'll therefore consider the abelian group generated by simplices.

Definition 1.4. The abelian group of singular n-chains in X is the free abelian group generated by n-simplices

$$S_n(X) = \mathbf{Z}\mathrm{Sin}_n(X) .$$

Its elements are finite linear combinations, i.e. formal sums of the form display $\sum_{i \in \text{finite set}} a_i\sigma_i$ where $a_i \in \mathbf{Z}$. It's a pretty big group. If $n < 0$, say that ~~this is zero~~ $S_n(X) = 0$. Define a map $\partial : \mathrm{Sin}_n(X) \to S_{n-1}(X)$ via:

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i d_i\sigma$$

We'll extend this to $S_n(X) \to S_{n-1}(X)$. by additivity

Cycles are chains whose boundary is zero. These are the interesting chains.

Definition 1.5. An n-cycle in X is an n-chain c with $\partial c = 0$. Denote $Z_n(X) = \ker(S_n(X) \xrightarrow{\partial} S_{n-1}(X))$.

For example, with the σ on the torus described before, $\partial c = d_0\sigma - d_1\sigma$ and this is zero.

Theorem 1.6. *Any boundary is a cycle, i.e.,* $B_n(X) := \mathrm{Im}(\partial : S_{n+1}(X) \to S_n(X)) \subseteq Z_n(X)$. *This is homework.*

Definition 1.7. The nth singular homology group of X is:

$$H_n(X) = Z_n(X)/B_n(X) = \frac{\ker(\partial : S_n(X) \to S_{n-1}(X))}{\mathrm{Im}(\partial : S_{n+1}(X) \to S_n(X))}$$

The kernel is a free abelian group, and so is the image because they're both subgroups of free abelian groups; but the quotient isn't necessarily free abelian. The quotient is finitely generated for all the spaces we are talking about although the kernel and image are uncountably generated.

Chapter 2

Simplices, more about homology

the standard n-simplex — They are the elements of the set

Recall ~~that~~ $\Delta^n \subseteq \mathbf{R}^{n+1}$. A singular simplex is a map $\sigma : \Delta^n \to X$. ~~which forms~~ $\mathrm{Sin}_n(X)$. For example, $\mathrm{Sin}_0(X)$ consists of points of X. You have a huge collection of maps $d^i : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ and $s^i : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n+1}(X)$, and the collection $\{\mathrm{Sin}_n(X), d^i, s^i\}$ forms a simplicial set. You therefore get a functor **Top** $\to$ {simplicial sets}. Simplicial sets are really cool, because they're combinatorial models for topological spaces.

Now, you get induced maps on ~~the~~ free abelian groups. This ~~extends the functor from **Top** to semi-simplicial sets to the collection of~~ produce, for each space, a semi-simplicial abelian group~~s~~. Using the d^is ~~and s^is~~, you get a boundary map ∂, and therefore a chain complex because $\partial^2 = 0$ (see homework). ~~In other words, a huge diagram (of categories and functors, which we'll discuss more next week)~~): we capture this process in a diagram:

$$
\begin{tikzcd}
\mathbf{Top} \arrow[r] \arrow[d] & \{\text{semi-simplicial sets}\} \arrow[r] & \{\text{semi-simplicial abelian groups}\} \arrow[d] \\
\{\text{simplicial sets}\} \arrow[ur] & & \{\text{chain complexes}\} \arrow[d, "\text{take homology}"] \\
& & \{\text{graded abelian group}\}
\end{tikzcd}
$$

If you have a chain complex $\partial : A_n \to A_{n-1}$, ~~then~~ you have its homology $H_n(A, \partial) = \ker \partial / \mathrm{Im}\, \partial$. ← display, with n.

Let's practice with simplices. Suppose we have $\sigma : \Delta^1 \to X$. Construct a map $\phi : \Delta^1 \to \Delta^1$ via $(t, 1-t) \mapsto (1-t, t)$. This reverses the orientation of σ. Composing σ with ϕ gives another singular simplex $\overline{\sigma}$. It is *not* true that $\overline{\sigma} = -\sigma$ in $S_1(X)$. are "homologous" cycle

Claim: $\overline{\sigma} \equiv -\sigma \bmod B_1(X) = \mathrm{Im}(\partial)$, i.e., they ~~define the same homology class~~. That is, if $d_0\sigma = d_1\sigma$, so $\sigma \in Z_1(X)$, then $[\overline{\sigma}] = -[\sigma]$ in $H_1(X)$. In other words, $\overline{\sigma} + \sigma$ is a boundary. We have to come up with a 2-simplex in X whose boundary is $\overline{\sigma} + \sigma$.

Let π denote the projection map from $[0, 1, 2]$ to $[0, 1]$ (MUST UPLOAD PICTURE HERE). Then, $\partial(\sigma \circ \pi) = \sigma\pi d^0 - \sigma\pi d^1 + \sigma\pi d^2 = \overline{\sigma} - c^1_{\sigma(0)} + \sigma$ where $c^1_{\sigma(0)}$ is the constant 1-simplex at $\sigma(0)$ (similarly for $c^n_{\sigma(0)}$). The $c^1_{\sigma(0)}$ is an error term. How do we correct this? Consider the constant 2-simplex $c^2_{\sigma(0)}$ at $\sigma(0)$; then the boundary is $c^1_{\sigma(0)} - c^1_{\sigma(0)} + c^1_{\sigma(0)}$, which is $c^1_{\sigma(0)}$. So, $\overline{\sigma} + \sigma = \partial(\sigma \circ \pi + c^2_{\sigma(0)})$. picture

Let's compute the homologies of $\emptyset$ and $*$. Well, $\mathrm{Sin}_n(\emptyset) = \emptyset$, so $S_*(\emptyset) = 0$. So, $\cdots \to S_2 \to S_1 \to S_0$ is the zero chain complex. This means that $Z_*(\emptyset) = 0$ and similarly for boundaries. The homology in all dimensions is therefore 0.

Now for $*$. Clearly $\mathrm{Sin}_n(*) = *$, and this generates $S_n(*)$, which is ~~hence~~ thus $\mathbf{Z}$. The chain complex is $S_0(*) \leftarrow S_1(*) \leftarrow S_2(*) \leftarrow \cdots$. What are the boundary maps? $\partial(c^1_*) = 0$, ~~but~~ $\partial(c^2_*) = c^1_*$, ~~but~~ $\partial(c^3_*) = 0$, because you have two pluses and two minuses. As a chain complex: say more

$$\cdots \to \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z}$$

reverse

The cycles therefore alternate between 0s and $\mathbf{Z}$s, namely as $\mathbf{Z}, \mathbf{Z}, 0, \mathbf{Z}, 0, \cdots$. The boundaries are the same as the cycles except for dimension zero, namely as $0, \mathbf{Z}, 0, \mathbf{Z}, 0, \cdots$. This means that $H_0(*) = \mathbf{Z}$ but $H_i(*) = 0$ for $i > 0$. do these constructions do

What ~~happens~~ to maps of spaces? Suppose you have $f : X \to Y$. You have a map $f_* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$ induced by composition, namely $\sigma \mapsto f \circ \sigma =: f_*\sigma$. What about face maps; does the following diagram commute?

$$\begin{tikzcd}
\mathrm{Sin}_n(X) \arrow[r, "f_*"] \arrow[d, "d_i"] & \mathrm{Sin}_n(Y) \arrow[d, "d_i"] \\
\mathrm{Sin}_{n-1}(X) \arrow[r, "f_*"] & \mathrm{Sin}_{n-1}(Y)
\end{tikzcd}$$

so the answer is yes!

We see that $d_i f_* \sigma = (f_*\sigma) \circ d^i = f \circ \sigma \circ d^i$, and $f_*(d_i\sigma) = f_*(\sigma \circ d^i) = f \circ \sigma \circ d^i$, This also holds for the free abelian groups. You therefore get a map of chain complexes. SA

A chain map $f : C_* \to D_*$ is a map $C_n \to D_n$ ~~with an appropriate commutative diagram~~. Does this induce a map in homology $f_* : H_n(C) \to H_n(D)$? ~~From diagram chasing~~, we get a map $Z_n(C_\bullet)(C) \to Z_n(C_\bullet)(D)$ and a map $B_n(C) \to B_n(D)$, which gives a map on homology. ~~Rather simple diagram chasing~~. This means that we get a map $f_* : H_n(X) \to H_n(Y)$. spell out