18.906: Problem Set II

Due Wednesday, March 8, 2017, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

Extra credit for finding mistakes and telling me about them early!

5. (a) Give an example (in **CG**) of a degenerate basepoint (i.e. a compactly generated space with a point $*$ such that $\{*\} \hookrightarrow X$ is not a cofibration).

(b) Show that any map $f : X \to Y$ factors (functorially) as a composite $X \to M \to Y$ where $X \to M$ is a cofibration and $M \to Y$ is a homotopy equivalence.

6. (a) Suppose $g : X \to B$ lifts across the fibration $p : E \to B$ up to homotopy: we are given $f : X \to E$ such that $p \circ f \simeq g$. Show that there is $f' \simeq f$ such that $p \circ f' = g$. In particular, if p has a section-up-to-homotopy, then it has a section.

(b) Show that if $p : E_0 \to B_0$ is a fibration and $f : B \to B_0$ is a homotopy equivalence, then the induced map $B \times_{B_0} E_0 \to E_0$ is again a homotopy equivalence.

(c) Show that if $p : E \to B$ and $p' : E' \to B$ are fibrations and $f : E \to E'$ is a homotopy equivalence such that $p' \circ f = p$, then f is a "fiber homotopy equivalence"; that is, there is a homotopy inverse g such that it and the two homotopies to respective identity maps are all fiber-preserving (e.g. $p \circ g = p'$).

In particular, if $p : E \to B$ is a fibration, then for each $b \in B$ the fiber $p^{-1}(b)$ and the homotopy fiber of p over b are homotopy equivalent (by equivalences that depend continuously on b).

7. (a) Let $f : X \to Y$ and fix $* \in Y$. We've defined the homotopy fiber of f over $*$ to be the space

$$F(f, *) = \{(x, \omega) \in X \times Y^I : \omega(0) = *, \omega(1) = f(x)\}.$$

It comes equipped with a fibration $p : F(f, *) \to X$ sending (x, ω) to x. The loop space $\Omega(Y, *)$ "acts" on the homotopy fiber $F(f, *)$ from the right: let $\omega \in \Omega(Y, *)$ and $(x, \sigma) \in F(f, *)$, and define

$$(x, \sigma) \cdot \omega = (x, \sigma \cdot \omega)$$

where

$$(\sigma \cdot \omega)(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \sigma(2t - 1) & 1/2 \le t \le 1\,. \end{cases}$$

In particular, taking $X = *$, we get the usual "multiplication" $\Omega Y \times \Omega Y \to \Omega Y$, which is known to be associative and unital up to homotopy (and to admit a homotopy inverse, sending ω to $\overline{\omega} : t \mapsto \omega(1 - t)$). The same proof shows that the action of $\Omega(Y, *)$ on $F(f, *)$ is associative and unital up to homotopy.

Suppose a group G acts on a set S (from the right) with orbit space X. The fiber product $S \times_X S$ consists of pairs of elements in the same orbit. The action is free exactly when the map $S \times G \to S \times_X S$, sending (s, g) to (s, sg), is bijective.

Note that $p((x, \sigma) \cdot \omega) = x = p(x, \sigma)$. We get a map

$$F(f, *) \times \Omega(Y, *) \to F(f, *) \times_X F(f, *)$$

to the fiber product by sending $((x, \sigma), \omega)$ to $((x, \sigma), (x, \sigma) \cdot \omega)$.

Finally, the problem: Show that this map is a homotopy equivalence.

So in a sense the action of $\Omega(Y, *)$ on $F(f, *)$ is free (a better term is "principal"), with orbit space X.

(b) Let $f : X \to Y$ and $g : W \to X$ be pointed maps. Establish a natural homeomorphism between the space of pointed maps $h : W \to F(f, *)$ such that $ph = g$ and the space of (pointed, of course) null-homotopies of the composite fg.

8. **(a)** Let π be a group, and suppose we are given a group homomorphism $\mu : \pi \times \pi \to \pi$ such that $\mu(g, 1) = g = \mu(1, g)$ (where $1 \in \pi$ is the identity element). Show that $\mu(g, h) = gh$ and that π is abelian. Explain why this gives another proof that $\pi_2(X)$ is abelian (or maybe it's the same proof).

(b) Now consider the category of pairs (π, G), where π is a group and G is another group with π acting on it (from the left, say). Morphisms are pairs of homomorphisms compatible with the group actions. This category has finite products, given by forming the product in each factor. Suppose we have a "unital multiplication" on an object (π, G). First explain what this must mean, and then show that both groups must be abelian and the action of π on G must be trivial (i.e. each element of π acts by the identity automorphism of G).

Conclude that any path connected H-space X is *simple* as a space: the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all n. (An H-space is a pointed space X together with a map $\mu : X \times X \to X$ such that $x \mapsto \mu(x, *)$ and $x \mapsto \mu(*, x)$ are both pointed homotopic to the identity map on X.)

9. **(a)** By passing to π_0, the action described in **7.** provides a right action of the group $\pi_1(Y, *)$ on $\pi_0(F(f, *))$.

(1) Show that two elements in $\pi_0(F(f, *))$ map to the same element of $\pi_0(X)$ if and only if they are in the same orbit under this action.

(2) Suppose ω is a path in Y from $*$ to y. Write $\omega_\# : \pi_1(Y, *) \to \pi_1(Y, y)$ for the group isomorphism sending σ to $\omega\sigma\omega^{-1}$. Show that the isotropy group of the component of (x, ω) in $F(f, *)$ is

$$\omega_\#^{-1} \mathrm{im}\,(\pi_1(X, x) \to \pi_1(Y, f(x))) \subseteq \pi_1(Y, *)\,.$$

(b) Suppose that X is path connected, and pick $* \in X$. Conclude from (1) that the evident surjection $\pi_n(X, *) \to [S^n, X]$ can be identified with the orbit projection for the action of $\pi_1(X, *)$ on $\pi_n(X, *)$.