18.906: Problem Set III

Due Wednesday, March 22, 2017, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

Extra credit for finding mistakes and telling me about them early!

A solution to extra credit part of 2(c): Minjae Park points out that the image of a compact Hausdorff space in a weak Hausdorff space is in fact Hausdorff. So any example of a compact space that is weak-Hausdorff but not Hausdorff will do.

10. Let $\omega \in \pi_1(S^1 \vee S^2)$ and $\alpha \in \pi_2(S^1 \vee S^2)$ be represented by the inclusion of the two spheres into the wedge. Form a new CW complex X by attaching a 3-cell by means of a map representing the homotopy class $2\alpha - \omega \cdot \alpha \in \pi_2(S^1 \vee S^2)$. Show that the inclusion of S^1 into X induces isomorphisms in π_1 and in homology. [So no simple adjustment to the Whitehead theorem will work. Notice however that the map on universal covers is not an isomorphism in homology.]

11. (Extra credit) Is there a relative Hurewicz theorem in the "0-connected" case? That is: Suppose that A meets every path component of X. Is there an expression for $H_1(X, A)$ in terms of $\pi_1(X, A)$ (with some structure)? A case in point: Let A be the disjoint union of two circles, and let X be A plus a line segment joining the two circles together.

12. (a) Use the cellular approximation theorem to show that for any CW complex X, the pair (X, X_n) is n-connected. (Check of numbers: (X, X_0) is 0-connected; i.e., every path component contains a 0-cell, which seems right!)

(b) Find an example of a relative CW complex (X, A), a simple space Y, and a map $f : X_n \to Y$, such that the obstruction cocycle $\theta(f) \in C^{n+1}(X, A; \pi_n(Y))$ is nonzero but the obstruction class $[\theta(f)] \in H^{n+1}(X, A; \pi_n(Y))$ is zero. Describe a extension of $f|X_{n-1}$ to X_{n+1}.

13. (a) [The original version of this problem was wrong: thank you Luke and others.] Let X be a simple space of finite type (i.e. such that every homology group is finitely generated). Let r_n and t_n be the Betti numbers and torsion numbers (as described below, for the singular complex of X). Construct a CW complex K with $r_n + t_n + t_{n-1}$ cells in dimension n, and a weak equivalence $K \to X$.

(b) Let $C_\bullet$ be a chain complex such that (1) C_n is a free abelian group for each n and (2) $H_n(C_\bullet)$ is a finitely generated abelian group for each n. Write r_n for the rank of $H_n(C_\bullet)$ and t_n for the minimal number of elements required to generate the torsion subgroup of $H_n(C_\bullet)$. Construct a chain complex $D_\bullet$ with

$$\operatorname{rank}(D_n) = r_n + t_n + t_{n-1}$$

and a quasi-isomorphism $f : D_\bullet \to C_\bullet$.

14. [The original version of this problem was wrong too! Thank you Oron.] Let Y be a simple space and N an integer, and suppose that $N\pi_*(Y) = 0$ Let (X, A) be a relative CW complex and assume that $H_*(X, A; \mathbb{F}_p) = 0$ whenever the prime p divides N. Show that the restriction map $[X, Y] \to [A, Y]$ is bijective.

15. Let $\mathrm{Gr}_k(\mathbf{R}^n)$ denote the Grassmannian manifold of k-dimensional subspaces of $\mathbf{R}^n$. The trivial n-plane bundle $\mathrm{Gr}_k(\mathbf{R}^n) \times \mathbf{R}^n \downarrow \mathrm{Gr}_k(\mathbf{R}^n)$ contains a "tautologous" sub-bundle γ whose fiber over $V \in \mathrm{Gr}_k(\mathbf{R}^n)$ is $\{(V, x) : x \in V\}$. Using the standard inner product on $\mathbf{R}^n$, we also have the orthogonal complement $\gamma^\perp$, whose fiber over V is $\{(V, x) : x \in V^\perp\}$. Express the tangent bundle of $\mathrm{Gr}_k(\mathbf{R}^n)$ in terms of γ and $\gamma^\perp$.