18.906: Problem Set VI

Due Wednesday, May 10, 2017, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

Extra credit for finding mistakes and telling me about them early!

26. $\varnothing$

27. What happens at the very bottom of the Serre exact sequence – in dimensions 0, 1, 2, ..., till you stop getting an exact sequence? For this, don't make any additional assumptions on the fibration $\pi : E \to B$. (So you will use the notation of homology with local coefficients.)

28. $\varnothing$

29. **(a)** Embed the composable pair of homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ into the big commutative diagram below, and show that the perimeter is an exact sequence.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & & & 0 & & \\
 & & & \ker\beta \arrow[rr] \arrow[dr] & & \operatorname{coker}\alpha \arrow[ur] \arrow[ddrr] & & & \\
 & & & & B \arrow[ur] \arrow[dr] & & & & \\
0 \arrow[r] & \ker\beta\alpha \arrow[uurr] \arrow[rr] & & A \arrow[ur] \arrow[rr] & & C \arrow[rr] \arrow[dr] & & \operatorname{coker}\beta\alpha \arrow[r] \arrow[dl] & 0 \\
 & & \ker\alpha \arrow[ul] \arrow[ur] & & & & \operatorname{coker}\beta \arrow[dl] \arrow[dr] & & \\
 & 0 \arrow[ur] & & 0 \arrow[ul] & & 0 & & 0 &
\end{tikzcd}$$

(b) Let $\mathcal{C}$ be a Serre class of abelian groups. Show that $\mathcal{C}$-isomorphisms satisfy "2 out of 3": In $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, if two of $\alpha, \beta, \beta\alpha$ are $\mathcal{C}$-isomorphisms then so is the third.

(c) Verify the mod $\mathcal{C}$ 5-lemma.

(d) We have the "standard" Serre classes,

- $\mathcal{C}_{tor}$ = torsion abelian groups

• $\mathcal{C}_{fg}$ = finitely generated abelian groups
• $\mathcal{C}_P$ = torsion abelian groups A such that $l : A \to A$ is an isomorphism for all primes l not in the set P. [This differs from the notation used by Davis and Kirk.]
Verify that each one satisfies the additional axiom:

$$\text{If } A \in \mathcal{C} \text{ then so is } H_q(K(A,1)) \text{ for all } q > 0.$$

Random facts that may be useful in this problem: Any module over any ring is the direct limit of the direct system of its finitely generated submodules. $\mathbb{Z}_{(P)} = \mathbb{Z}[1/l : l \text{ prime not in } P] \subseteq \mathbb{Q}$ is a PID with cyclic modules $\mathbb{Z}_{(P)}$ and $\mathbb{Z}/n$ where all the prime divisors of n lie in P. As an abelian group, $\mathbb{Z}_{(P)}$ itself is a direct limit of a direct system with values $\mathbb{Z}$.

(e) Let $\mathcal{C}$ be a Serre class satisfying the axiom

$$\text{If } A, B \in \mathcal{C} \text{ then so are } A \otimes B \text{ and } \mathrm{Tor}_1(A, B).$$

Let A be an abelian group such that $H_q(K(A,1)) \in \mathcal{C}$ for all $q > 0$. Show that $H_q(K(A,n)) \in \mathcal{C}$ for all $q > 0$ and $n > 1$.

So while $H_q(K(A,n))$ may be hard to compute, we know for example that it's finitely generated if A is; finite (for $q > 0$) if A is; p-torsion (for $q > 0$) if A is; etc.

30. Let π be an abelian group and $n \geq 1$. The Hurewicz theorem tells us that the lowest nonzero reduced homology group of the space $K(\pi, n)$ is $H_n(K(\pi, n)) = \pi$, functorially in π. What is the next value of s for which $\pi \mapsto H_s(K(\pi, n))$ is a nonzero functor? Give an example of a group π for which that first group is in fact nonzero.

Let p be a prime number. Then $H_n(K(\pi, n); \mathbb{Z}_{(p)}) = \pi \otimes \mathbb{Z}_{(p)}$ as a functor of π. What is the next value of s for which $\pi \mapsto H_s(K(\pi, n); \mathbb{Z}_{(p)})$ is a nonzero functor? Give an example of a group π for which that first group is in fact nonzero.

In both cases, a big part of the question involves finitely generated abelian groups or $\mathbb{Z}_{(p)}$-modules.

31. Use the work of **22 (a)** to produce a fibration sequence of the form $\mathbb{R}P^3 \to S^2 \to \mathbb{C}P^\infty$. (Think of $\mathbb{C}P^\infty$ as $BSO(2)$ and $\mathbb{R}P^3$ as $SO(3)$.) Determine the behavior of the resulting cohomology spectral sequence (with integer coefficients), including any additive extensions that occur in the passage to $H^*(S^2)$. Do the same for the homology spectral sequence.

32. Let ξ be a fibration with fibers homotopy equivalent to S^{n-1}. Show that if n is odd then the Euler class has order 2. Give an example (for some odd n) in which the Euler class is nevertheless nonzero.

33. We gave two definitions of the Euler class for an R-oriented n-plane bundle ξ over a path connected space X. (There are many others!)

(1) It's the image of a generator of $H^{n-1}(\mathbb{S}(\xi)) = E_2^{0,n-1}$ under the transgression.

(2) It's the image of the Thom class under restriction $H^n(\mathbb{D}(\xi), \mathbb{S}(\xi)) \to H^n(\mathbb{D}(\xi)) = H^n(X)$.

Show that they coincide (up to sign, perhaps!).