

## 18.906: Problem Set I

Due Wednesday, February 22, 2017, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

Extra credit for finding mistakes and telling me about them early!

1. (a) Show that any limit can be expressed as an equalizer of two maps between products.

(b) Let  $\mathbf{C}$  and  $\mathbf{D}$  be two categories and  $F : \mathbf{C} \rightarrow \mathbf{D}$  and  $G : \mathbf{D} \rightarrow \mathbf{C}$  two functors. In class we said that an adjunction between  $F$  and  $G$  is an isomorphism

$$\mathbf{D}(FX, Y) \cong \mathbf{C}(X, GY)$$

that is natural in both variables. Show that this is equivalent to giving natural transformations

$$\alpha_X : X \rightarrow GFX, \quad \beta_Y : FGY \rightarrow Y,$$

such that

$$\beta_{FX} \circ F\alpha_X = 1_{FX}, \quad G\beta_Y \circ \alpha_{GY} = 1_{GY}.$$

(c) Suppose that  $F$  and  $F'$  are both left adjoint to  $G : \mathbf{D} \rightarrow \mathbf{C}$ . Show that there is a unique natural isomorphism  $F \rightarrow F'$  that is compatible with the adjunction.

2. (a) Construct left and right adjoints to the forgetful functor

$$u : \mathbf{Top} \rightarrow \mathbf{Set},$$

and conclude that for any small category  $I$ , the limit and the colimit of a functor  $X : I \rightarrow \mathbf{Top}$  consists of a topology on the corresponding limit or colimit of underlying sets.

(b) Show that the colimit (in  $\mathbf{Top}$ ) of any diagram of  $k$ -spaces is again a  $k$ -space, and serves as the colimit in  $k\text{-}\mathbf{Top}$ . (Suggestion: Show that in  $\mathbf{Top}$  any coproduct of  $k$ -spaces is a  $k$ -space and that any quotient of a  $k$ -space is a  $k$ -space, and then use the dual of 1 (a).)

(c) A category “has finite products” if any functor from a finite discrete (only morphisms are identity morphisms) category has a limit. This limit is the “product” of the objects involved. In this part  $\mathbf{C}$  will be a category with finite products.

(i) What is the limit of a functor from the empty category? Write  $*$  for it, and show that the projection map  $X \times * \rightarrow X$  is an isomorphism for any  $X$ .

(ii) For each pair of objects  $X$  and  $Y$  in  $\mathbf{C}$ , pick a product object  $X \times Y$  along with the projection maps  $X \leftarrow X \times Y \rightarrow Y$  establishing it as the product of  $X$  and  $Y$ . Show that this choice extends (by defining values on morphisms) to a functor  $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$

in such a way that the product projections are natural transformations; and that such an extension is unique.

A category with finite products is *Cartesian closed* provided that for every object  $X$  the functor  $X \times - : \mathbf{C} \rightarrow \mathbf{C}$  has a right adjoint (which is written  $Z \mapsto Z^X$ ). The adjunctions

$$\alpha_Y : Y \rightarrow (X \times Y)^X, \quad \beta_Z : X \times Z^X \rightarrow Z,$$

are the “slice inclusion” and “evaluation” maps.

(iii) Pick such a right adjoint for each  $X$ . Show that they assemble into a functor

$$\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}, \quad (X, Z) \mapsto Z^X.$$

(iv) Verify the exponential laws: construct natural isomorphisms

$$Z^{X \times Y} \cong (Z^X)^Y, \quad (Y \times Z)^X \cong Y^X \times Z^X.$$

The first of these shows that the adjunction bijection  $\mathbf{C}(X \times Y, Z) \cong \mathbf{C}(Y, Z^X)$  “enriches” to an isomorphism in  $\mathbf{C}$ . The second says that the product in  $\mathbf{C}$  is actually an “enriched” product.

(v) Construct a “composition” natural transformation

$$Y^X \times Z^Y \rightarrow Z^X$$

using the evaluation maps, and show that it is associative and unital.

**(extra credit)** It appears that there must be a (non-Hausdorff) compact space that does not admit a surjection from a compact Hausdorff space. Can you find such an example?

**3.** Show that the smash product is associative as a functor  $\mathbf{C}_* \times \mathbf{C}_* \rightarrow \mathbf{C}_*$  when  $\mathbf{C} = k\mathbf{Top}$  ( $k$ -spaces) and when  $\mathbf{C} = \mathbf{CG}$  (weakly Hausdorff  $k$ -spaces). (This is not true with  $\mathbf{C} = \mathbf{Top}$ .) It’s commutative and unital also, with unit given by  $*_+$ .

**4.** Let  $X$  be an object of the category  $\mathbf{C}$ . The *over-category* or *slice category* over  $X$ ,  $\mathbf{C}/X$ , has as objects morphisms  $p : Y \rightarrow X$  in  $\mathbf{C}$ , and given also  $q : Z \rightarrow X$ , the set of morphisms from  $p$  to  $q$  consists of the set of morphisms  $f \in \mathbf{C}(Y, Z)$  such that  $qf = p$ .

**(a)** Suppose that  $\mathbf{C}$  has finite products. Construct a left adjoint to the functor  $X \times - : \mathbf{C} \rightarrow \mathbf{C}/X$  that sends  $Y$  to  $\text{pr}_1 : X \times Y \rightarrow X$ .

**(b)** So  $X \times - : \mathbf{C} \rightarrow \mathbf{C}/X$  preserves limits, while the composite  $\mathbf{C} \rightarrow \mathbf{C}/X \rightarrow \mathbf{C}$  probably does not. The only question is: What is the limit of a diagram in  $\mathbf{C}/X$ ? Answer this if you want (and you may need to assume that  $\mathbf{C}$  has all finite limits, not just finite products), but at least show that if  $Y : J \rightarrow \mathbf{C}$  then

$$\lim_{j \in J} {}^{\mathbf{C}/X} (X \times Y_j) \cong X \times \lim_{j \in J} {}^{\mathbf{C}} Y_j.$$

Discuss the special case in which  $J$  has only two objects and only identity morphisms.