18.906: Problem Set V

Due Wednesday, April 26, 2017, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

Extra credit for finding mistakes and telling me about them early!

22. In this problem, BG will denote a classifying space for the topological group G. One construction is via Graeme Segal's simpicial space, but when you regard it as classifying the set of isomorphism classes of numerable principal G-bundles it is only well defined up to homotopy.

(a) Let G be a Lie group and H a closed subgroup. The orbit projection $G \to G/H$ is a submersion, therefore a fiber bundle; so the translation action of H on G is principal. $H \to G \to G/H$ is a fibration sequence. We can classify it by a map $G/H \to BH$. Show that $G \to G/H \to BH$ is a fibration sequence – in fact, that there is a space, homotopy equivalent to G/H, that is the total space of a fiber bundle over BH with fiber G.

(b) The homomorphism $H \to G$ induces a map $BH \to BG$. Show that $G/H \to BH \to BG$ is a fibration sequence (again locally trivial).

(c) Let N be a closed normal subgroup of the Lie group G. Show that there is a fibration sequence $BN \to BG \to B(G/N)$ (again locally trivial).

(d) Let N be a central subgroup of the Lie group G. Show that there is a fibration sequence $BG \to B(G/N) \to B^2N$.

(e) Let π be a Lie group and suppose it acts on the Lie group G by group automorphisms. This action defines the *semi-direct product* group $G\tilde{\times}\pi$. This is the set $G \times \pi$ with group structure given by $(g, x) \cdot (h, y) = (g \cdot (xh), xy)$. Show that $E\pi \times_\pi BG$ serves as a model for $B(G\tilde{\times}\pi)$.

23. Let B and B' be path connected spaces, and suppose given fibrations $p : E \to B$ and $p' : E' \to B'$. Let $* \in B$ and $* \in B'$ and write F and F' for the fibers over these points. Suppose given a map of fibrations

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

sending $*$ to $*$. It induces a map $F' \to F$. Suppose that two of these three horizontal maps are weak equivalences, and show that the third is also.

24. Give an example of a filtered complex F_*C such that $\cap_s F_s C = 0$ but $\cap_s F_s H_*(C) \neq 0$.

25. Let F_* be a filtration of a chain complex C, and define a new filtration on C by declaring $\tilde{F}_p C_n = F_{p-n} C_n$. Check that this does give a filtered complex, and provide a relationship between the spectral sequences associated to these two filtered complexes.

26. Let p be a prime number and $\mathbb{Z}/p^\infty$ the Prüfer group given by the union or direct limit

$$\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \cdots \hookrightarrow \mathbb{Z}/p^\infty$$

Let $C \to C'$ be a map of chain complexes of free abelian groups. The short exact sequences $0 \to \mathbb{Z}/p^{n-1} \to \mathbb{Z}/p^n \to \mathbb{Z}/p \to 0$ show that if $H_*(C \otimes \mathbb{Z}/p) \to H_*(C' \otimes \mathbb{Z}/p)$ is an isomorphism, then so is $H_*(C \otimes \mathbb{Z}/p^n) \to H_*(C' \otimes \mathbb{Z}/p^n)$ for all $n \geq 1$. Then the fact that tensor product commutes with direct limits shows that $H_*(C \otimes \mathbb{Z}/p^\infty) \to H_*(C' \otimes \mathbb{Z}/p^\infty)$ is also an isomorphism.

This problem asks you to provide an "absolute" form of this result; namely, what additional structure on $H_*(C \otimes \mathbb{Z}/p)$ does one need to compute $H_*(C \otimes \mathbb{Z}/p^\infty)$?

Define a filtration on an abelian group A by declaring $F_{-1}A = 0$ and for $s \geq 0$

$$F_s A = \ker\left(p^{s+1} : A \to A\right).$$

This filtration is exhaustive whenever A is a torsion abelian group; for example, when $A = C_n \otimes \mathbb{Z}/p^\infty$.

(a) Compute (E^0, d^0) and (E^1, d^1) in the associated spectral sequence. Explain how d^1 is related to the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

(b) Make a sketch of this spectral sequence in case the chain complex has $\mathbb{Z}$ in dimensions 0 and 1 and zero elsewhere, and $d : C_1 \to C_0$ is multiplication by p^n.

(c) Multiplication by p defines an operator on C that sends $F_s C$ to $F_{s-1}C$. Describe its effect on the spectral sequence.

(d) Use this operator to "solve the extension problems," and describe $H_*(C \otimes \mathbb{Z}/p^\infty)$ (say under the assumption that $H_n(C)$ is finitely generated for all n) in terms of this spectral sequence.

(e) Explain what this information says about $H_*(C)$ itself (under the same assumption).

This is the "Bockstein spectral sequence." It's usually thought of as arising from the singly-graded exact couple

$$\begin{tikzcd}
H_*(C) \arrow[rr, "p"] & & H_*(C) \arrow[dl] \\
& H_*(C/p) \arrow[ul, dotted] &
\end{tikzcd}$$

(where the dotted arrow is the boundary map), or from the p-adic filtration on $\mathbb{Z}$. Neither of these perspectives makes the convergence properties transparent, but ...

(f) Explain how the spectral sequence associated with the exact couple drawn above is related to the Bockstein spectral sequence as presented in the problem.