M390C NOTES: ALGEBRAIC GEOMETRY

ARUN DEBRAY
MAY 10, 2016

Note: this example was adapted from notes I took in David Ben-Zvi's algebraic geometry course, Spring 2016.

CONTENTS

1. THE COURSE AWAKENS: 1/19/16

"There was a mistranslation in Grothendieck's quote, 'the rising sea:' he was actually talking about raising an X-wing fighter out of a swamp using the Force."

There are a lot of things that go under the scheme of algebraic geometry, but in this class we're going to use the slogan "algebra = geometry;" we'll try to understand algebraic objects in terms of geometry and vice versa.

There are two main bridges between algebra and geometry: to a geometric object we can associate algebra via functions, and the reverse construction might be less familiar, the notion of a spectrum. This is very similar to the notion of the spectrum of an operator.

We will follow the textbook of Ravi Vakil, *The Rising Sea*. There's also a course website.[1] The prerequisites will include some commutative algebra, but not too much category theory; some people in the class might be bored. Though we're not going to assume much about algebraic sets, basic algebraic geometry, etc., it will be helpful to have seen it.

Let's start. Suppose X is a space; then, there's generally a notion of $\mathbb{C}$-valued functions on it, and this space might be $F(X)$. For example, if X is a smooth manifold, we have $C^\infty(X)$, and if X is a complex manifold, we have the holomorphic functions $\mathrm{Hol}(X)$.[2] Another category of good examples is *algebraic sets*, $X \subset \mathbb{C}^n$ that is given by the common zero set of a bunch of polynomials: $X = \{f_1(x) = \cdots = f_k(x) = 0\}$ for some $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$. These have a natural notion of function, *polynomial functions*, which are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ restricted to X, If $I(X)$ is the functions vanishing on X, then these functions are given by $\mathbb{C}[x_1, \ldots, x_n]/I$.

The point is, on all of our spaces, the functions have a natural ring structure.[3] In fact, there's more: the constant functions are a map $\mathbb{C} \to F(X)$, and since $\mathbb{C}$ is a field, this map is injective. This means $F(X)$ is a $\mathbb{C}$-algebra, i.e. it is a $\mathbb{C}$-vector space with a commutative, $\mathbb{C}$-linear multiplication.

One of the things Grothendieck emphasized is that one should never look at a space (or an anything) on its own, but consider it along with maps between spaces. For example, given a map $\pi : X \to Y$ of spaces, we always have a *pullback* homomorphism $\pi^* : F(Y) \to F(X)$: if $f : Y \to \mathbb{C}$, then its pullback is $\pi^* y(x) = y(\pi(x))$. This tells us that we have a *functor* from spaces to commutative rings.

[1] `https://www.ma.utexas.edu/users/benzvi/teaching/alggeom/syllabus.html`.

[2] The best examples here are Riemann surfaces; when the professor imagines a "typical" or example algebraic variety, he sees a Riemann surface.

[3] In this class, all rings will be commutative and have a 1. Ring homomorphisms will send 1 to 1.

Categories and Functors. This is all done in Vakil's book, but in case you haven't encountered any categories in the streets, let's revisit them.

Definition 1.1. A *category* C consists of a set[4] of *objects* $\operatorname{Ob}\mathsf{C}$; if $X \in \operatorname{Ob}\mathsf{C}$, we just say $X \in \mathsf{C}$. We also have for every $X, Y \in \mathsf{C}$ the set $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ of *morphisms*. For every $X, Y, Z \in \mathsf{C}$, there's a *composition map* $\operatorname{Hom}_{\mathsf{C}}(X, Y) \times \operatorname{Hom}_{\mathsf{C}}(Y, Z) \to \operatorname{Hom}_{\mathsf{C}}(Y, Z)$ and a unit $1_X \in \operatorname{Hom}_{\mathsf{C}}(X, X) = \operatorname{End}_{\mathsf{C}}(X)$ satisfying a bunch of axioms that make this behave like associative function composition.

To be precise, we want categories to behave like monoids, for which the product is associative and unital. In fact, a category with one object is a monoid. Thus, we want morphisms of categories to act like morphisms of monoids: they should send composition to composition.

Definition 1.2. A *functor* $F : \mathsf{C} \to \mathsf{D}$ is a function $F : \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{D}$ with an induced map on the morphisms:

- If the map acts as $\operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{D}}(F(X), F(Y))$, F is called a *covariant* functor.
- If it sends $\operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{D}}(F(Y), F(X))$, then F is *contravariant*.

When we say "functor," we always mean a covariant functor, and here's the reason. Recall that for any monoid A there's the *opposite monoid* A^{op} which has the same set, but reversed multiplication: $f \cdot_{\mathrm{op}} g = g \cdot f$. Similarly, given a category C, there's an *opposite category* C^{op} with the same objects, but $\operatorname{Hom}_{\mathsf{C}^{\mathsf{op}}}(X, Y) = \operatorname{Hom}_{\mathsf{C}}(Y, X)$. Then, a contravariant functor $\mathsf{C} \to \mathsf{D}$ is really a covariant functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$. Hence, in this class, we'll just refer to functors, with opposite categories where needed.

Exercise 1.3. Show that a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ induces a functor $\mathsf{C} \to \mathsf{D}^{\mathsf{op}}$.

When presented a category, you should always ask what the morphisms are; on the other hand, if someone tells you "the category of smooth manifolds," they probably mean that the morphisms are smooth functions.

Now, we see that pullback is a functor $F : \mathsf{Spaces} \to \mathsf{Ring}^{\mathsf{op}}$. One of the major goals of this class is to define a category of spaces on which this functor is an equivalence. This might not make sense, *yet*. This is the seed of "algebra = geometry."

Definition 1.4. Let $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ is a collection of maps: for every $X \in \mathsf{C}$, there's a map $\eta_X : F(X) \to G(X)$ satisfying a consistency condition: for every $f : X \to Y$ in C, there's a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

That is, a natural transformation relates the objects and the morphisms, and reflects the structure of the category.

Definition 1.5. A natural transformation η is a *natural isomorphism* if for every $X \in \mathsf{C}$, the induced $\eta_X \in \operatorname{Hom}_{\mathsf{D}}(F(X), G(X))$ is an isomorphism.

This is equivalent to having a natural inverse to η.

So one might ask, what is the notion for which two categories are "the same?" One might naïvely suggest two functors whose composition is the identity functor, but this is bad. The set of objects isn't very useful: it doesn't capture the structure of the category. In general, asking for equality of objects is worse than asking for isomorphism of objects. Here's the right notion of sameness.

Definition 1.6. Let C and D be categories. Then, a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there's a functor $G : \mathsf{D} \to \mathsf{C}$ such that there are natural isomorphisms $FG \to \mathrm{id}_{\mathsf{D}}$ and $GF \to \mathrm{id}_{\mathsf{C}}$.

This is a very useful notion, and as such it will be useful to see an equivalence that is not an isomorphism.

[4] This is wrong. But if you already know that, you know that worrying about set-theoretic difficulties is a major distraction here, and not necessary for what we're doing, so we're not going to worry about it.

Exercise 1.7. Let k be a field, and let $\mathsf{D} = \mathsf{fdVect}_k$, the category of finite-dimensional vector spaces and linear maps, and let C be the category whose objects are $\mathbb{Z}_{\geq 0}$, the natural numbers, with an object denoted $\langle n \rangle$, and with $\mathrm{Hom}(\langle n \rangle, \langle m \rangle) = \mathrm{Mat}_{m \times n}$. This is a category with composition given by matrix multiplication.

Let $F : \mathsf{C} \to \mathsf{D}$ send $\langle n \rangle \mapsto k^n$, and with the standard realization of matrices as linear maps. Show that F is an equivalence of categories.

This category C has only some vector spaces, but for those spaces, it has all of the morphisms.

Definition 1.8. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor.

- F is *faithful* if all of the maps $\mathrm{Hom}_{\mathsf{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ are injective.
- F is *fully faithful* if all of these maps are isomorphsism.
- F is *essentially surjective* if every $X \in \mathsf{D}$ is isomorphic to $F(Z)$ for some $Z \in \mathsf{C}$.

The following theorem will also be a useful tool.

Theorem 1.9. *A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff it is fully faithful and essentially surjective.*

So, to restate, we want a category of spaces that is the opposite category to the category of rings; this is what Grothendieck had in mind. In fact, let's peek a few weeks ahead and make a curious definition:

Definition 1.10. The *category of affine schemes* is $\mathsf{Rings}^{\mathsf{op}}$.

Of course, we'll make these into actual geometric objects, but categorically, this is all that we need.

Recall that if $f : M \to N$ is a set-theoretic map of manifolds, then f is smooth iff its pullback sends C^∞ functions on N to C^∞ functions on M. The first step in this direction is the following theorem, sometimes called *Gelfand duality*.

Theorem 1.11 (Gelfand-Naimark). *The functor $X \mapsto C^0(X)$ (the ring of continuous functions) defines an equivalence between the category of compact Hausdorff spaces and the (opposite) category of commutative C^*-algebras.*

This is an algebro-geometric result: it identifies a category of spaces with the opposite category of a category of algebraic objects.

However, we need to think harder than Gelfand duality in terms of compact, complex manifolds or in terms of algebraic spaces: for example, for $X = \mathbb{CP}^1$, $\mathrm{Hol}(X) = \mathbb{C}$: the only holomorphic functions are constant. The issue is that there are no partitions of unity in the holomorphic or algebraic world. This means we'll need to keep track of local data too, which will lead into the next few lectures' discussions on *sheaf theory*.

Returning to the example of algebraic sets, suppose X and Y are algebraic sets. What is the set of their morphisms? We decided the ring of functions was the polynomial functions $Y \to \mathbb{C}$, so we want maps $X \to Y$ to be those whose pullbacks send polynomial functions to polynomial functions. To be precise, the *ideal of* X is $I(X) = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f|_X = 0\}$, defining a map I from algebraic subsets of $\mathbb{C}^n$) to ideals in $\mathbb{C}[x_1, \dots, x_n]$. There's also a reverse map V,[5] sending an ideal I to $V(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in I\}$. From classical commutative algebra, it's a fact that this is finitely generated, so it's the vanishing locus of a finite number of polynomials, and therefore in fact an algebraic set.

The dictionary between algebraic sets and ideals of $\mathbb{C}[x_1, \dots, x_n]$ is one of many versions of the Nullstellensatz (more or less German for the "zero locus theorem"): if J is an ideal, $I(V(J)) = \sqrt{J}$, its radical.

Definition 1.12. Let R be a ring and $J \subset R$ be an ideal. Then, the *radical* of J is $\sqrt{J} = \{r \in R \mid r^n \in J \text{ for some } n > 0\}$. One says that J is *radical* if $J = \sqrt{J}$.

What this says is that J is radical iff R/J has no nonzero nilpotents.[6] Why are these kinds of ideals relevant? If $X \subset \mathbb{C}^n$ and f vanishes on X, then so does f^n for all n. That is, radicals encode the geometric property of vanishing, which is why $I(X)$ is a radical ideal.

This is an outline of what classical algebraic geometry studies: it starts by defining algebraic subsets, and establishing a bijection between algebraic subsets of $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[x_1, \dots, x_n]$. This isn't yet an equivalence of categories. Radical ideals correspond to finitely generated $\mathbb{C}$-algebras with no (nonzero) nilpotents: an ideal I corresponds to the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \dots, x_n]/I$.

[5] V stands for "vanishing," "variety," or maybe "vendetta."

[6] Recall that if R is a ring, an $r \in R$ is *nilpotent* if $r^n = 0$ for some n.

This is all what the course is *not* about; we're going to replace the category of finitely generated, nilpotent-free $\mathbb{C}$-algebras with the category of *all* rings, but we want to keep some of the same intuition. This involves generalizing in a few directions at once, but we'll try to write down a dictionary; the defining principle is to identify spaces X with rings $R = F(X)$, their ring of functions.

A point $x \in X$ is a map $i_x : x \to X$, so we get a pullback $i_x^* : F(X) \to \mathbb{C}$ given by evaluation at x. Let $\mathfrak{m}_x = \ker(i_x^*)$; since $\mathbb{C}$ is a field, this is a maximal ideal.[7] If k is a field and R is a k-algebra, then R/I is also a k-algebra, so in particular if I is maximal, then $k \hookrightarrow R/I$ is a map of fields, and therefore a field extension. Thus, if k is algebraically closed (e.g. we're studying $\mathbb{C}$) and R is a finitely generated k-algebra, then maximal ideals of R are in bijection with homomorphisms $R \to k$.

Thus, given a ring R, we'll associate a set $\mathrm{MSpec}(R)$, the set of maximal ideals of R, such that R should be its ring of functions. To do this, we'll say that an $r \in R$ is a "function" on $\mathrm{MSpec}(R)$ by acting on an $\mathfrak{m}_x \subset R$ as $r \bmod \mathfrak{m}_x$. This is a "number," since it's in a field, but the notion may be different at every point in $\mathrm{MSpec}(R)$! For example, if $R = \mathbb{Z}$, then $\mathrm{MSpec}(\mathbb{Z})$ is the set of primes, and $n \in \mathbb{Z}$ is a function which at 2 is $n \bmod 2$, at 3 is $n \bmod 3$, and so on.

A perhaps nicer example is when $R = \mathbb{R}[x]$, which has maximal ideals $(x - t)$ for all $t \in \mathbb{R}$. Here, evaluation sends $f(x) \mapsto f(x) \bmod (x - t) = f(t)$. That is, this is really evaluation, and here the quotient field is $\mathbb{R}$. So these look like good old real-valued functions, but these aren't all the maximal ideals: $(x^2 + 1)$ is also a maximal ideal, and $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$. Then, we do get a kind of evaluation again, but we have to identify points and their complex conjugates.

So we'll have to find a good notion of geometry which generalizes from $\mathbb{C}$-algebras to k-algebras for any field k, to any commutative rings. We'll also have to think about nilpotents: we threw them away by thinking about zero sets, but they play a huge role in ring theory.

2. Attack of the Cones: 1/21/16

> *"To this end, we're going to give a crash course in category theory over the next few lectures; the door is over there."*

Remember that our general agenda is to match algebra and geometry; one way to express this idea is to take the category of rings and identify it with some category of geometric objects. However, we're going to reverse the arrows, and we'll get the category of affine schemes. These are some geometric spaces, with a contravariant functor from affine schemes to rings given by taking the ring of functions and a functor in the opposite direction called Spec.

One potential issue is that spaces may not have enough functions, e.g. $\mathbb{CP}^1$ as a complex manifold only has constant functions; as such, we'll enlarge our category to a whole category of schemes, which will also have an algebraic interpretation. Another weird aspect is that functions may take values in varying fields.

Schemes generalize geometry in three different directions: gluing spaces together to ensure we have enough functions is topology, like making manifolds; functions having varying codomains is useful for arithmetic and number theory; and allowing for rings with nilpotents feels a little like analysis.

Last time, we defined $\mathrm{MSpec}(R)$ for a ring R, the set of maximal ideals. It turns out that topology is not sufficient to understand these spaces; for example, the class of *local rings* are those with only one maximal ideal. There are many such rings, e.g. $\mathbb{C}[x]/(x^n)$, whose maximal ideal is (x). In short, MSpec doesn't see nilpotents.

To any ring R, one can attach the category Mod_R, whose objects are R-modules and morphisms are R-linear maps (those commuting with the action of R). This category is one of the more important things one studies in algebra, and we also want to express them in terms of geometric objects that are related somehow to $\mathrm{Spec}\, R$. This should also help us understand the algebraic properties of R-modules too.

Crash Course in Categories. There's a lot of categorical notions in algebraic geometry; it does strike one as a painful way to start a course, but hopefully we can get it out of our systems and move on to geometry knowing what we need. This corresponds to chapters 1 and 2 in the book.

We've seen several examples of categories: sets, groups, rings, etc. The next example is a useful class of categories.

Definition 2.1. A *poset* is a set S and a relation $\leq$ on S that is

[7] Recall that an ideal $I \subset R$ is *maximal* iff R/I is a field. This is about the level of commutative algebra that we'll be assuming.

- *reflexive*, so $x \leq x$ for all $x \in \mathsf{S}$;
- *transitive*, so if $x \leq y$ and $y \leq z$, then $x \leq z$; and
- *antisymmetric*, so if $x \leq y$ and $y \leq x$, then $x = y$.

S has the structure of a category: the objects are the elements of S, and $\mathrm{Hom}(x, y)$ is $\{\mathrm{pt}\}$ if $x \leq y$ and is empty otherwise.

Transitivity means that we have composition, and reflexivity gives us identity maps.

This is an unusual example compared to things like "the category of all (somethings)," but is quite useful: a functor from the poset $\bullet \to \bullet$ to another category C is a choice of $A, B \in \mathsf{C}$ and a map $A \to B$; a functor from the poset $\mathbb{N}$ is the same as an infinite sequence in C, and a commutative diagram is the same as a functor out of the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

into C.

Example 2.2. A particularly important example of this: if X is a topological space, then its open subsets form a poset under inclusion. Hence, they form a category, called $\mathsf{Top}(X)$. This category is important for sheaf theory, which we will say more about later. For example, if A is an abelian group and $U \subset X$ is open, then let $\mathscr{O}_A(U)$ denote the abelian group of A-valued functions on U (for example, A might be $\mathbb{C}$, so $\mathscr{O}_A(U) = C^\infty(U)$). If $V \subset U$, then restriction of functions defines a map $\mathrm{res}_U^V : \mathscr{O}_A(U) \to \mathscr{O}_A(V)$. Since restriction obeys composition, then we've defined a functor $\mathscr{O}_A : \mathsf{Top}(X)^{\mathsf{op}} \to \mathsf{Ab}$ (or perhaps to $\mathbb{C}$-algebras, or another category); this is a *presheaf of abelian groups* (or $\mathbb{C}$-algebras, etc.).

To be precise, a *presheaf* on X is a functor out of $\mathsf{Top}(X)^{\mathsf{op}}$. This is a way of organizing functions in a way that captures restriction; it will be very useful throughout this class.

Returning to category theory, one of its greatest uses is to capture structure through universal properties, rather than using explicit details of a given category. We'll give a few universal properties here.

Definition 2.3. Let C be a category.

- A *final* (or *terminal*) object in C is a $* \in \mathsf{C}$ such that for all $X \in \mathsf{C}$, there's a unique map $X \to *$.
- An *initial* object is a $* \in \mathsf{C}$ such that for all $X \in \mathsf{C}$, there's a unique map $* \to X$.

This is not the last time we'll have dual constructions produced by reversing the arrows.

Example 2.4. If C is a poset, then a terminal object is exactly a maximum element, and an initial object is a minimum element. Thus, in particular, they do not necessarily exist.

Nonetheless, if a final (or initial) object exists, it's necessarily unique.

Proposition 2.5. *Let $*$ and $*'$ be terminal objects in C; then, there's a unique isomorphism $*$ to $*'$.*

Proof. There's a unique map $* \to *$, which therefore must be the identity, and there are unique maps $* \to *'$ and $*' \to *$, so composing these, we must get the identity, so such an isomorphism exists, and it must be unique, since there's only one map $* \to *'$. ☐

By reversing the arrows, the same thing is true for initial objects. Thus, if such an object exists, it's unique, so one often hears "the" initial or final object. These will be useful for constructing other universal properties.

Example 2.6.

(1) In the category of sets, or in the category of topological spaces, the final object is a single point: everything maps to the point. The initial object is the empty set, since there's a unique (empty) map to any set or space.
(2) In Ab or Vect_k (abelian groups and vector spaces, respectively), 0 is both initial and terminal: the unique map is the zero map. An object that is initial and final is called a *zero object*; as in the case of sets, it may not exist.

(3) In the category of rings, 0 is terminal, but not initial (since a map out of 0 must send $0 = 1$ to 0 and 1). $\mathbb{Z}$ is initial, with the unique map determined by $1 \mapsto 1$.[8]
(4) Even though we don't really understand what an affine scheme is yet, we know that $\operatorname{Spec} \mathbb{Z}$ has to be a terminal object, and $\operatorname{Spec} 0$ has to be the initial object. Since we want this to be geometric, then $\operatorname{Spec} \mathbb{Z}$ will play the role of a point. It might not look like a point, but categorically it behaves like one.
(5) The category of fields is also interesting: setting $1 = 0$ isn't allowed, so there are neither initial nor terminal objects! If we specialize to fields of a given characteristic, then we get a unique map out of $\mathbb{Q}$ or $\mathbb{F}_p$, so the category of fields of a given characteristic is initial.
(6) The poset $\mathsf{Top}(X)$ has $\varnothing$ initial and X terminal: it has top and bottom objects.

The fact that initial and terminal objects are unique means that if you characterize an object in terms of initial or terminal objects, then you know they're unique as soon as they exist.

Definition 2.7. If R is a ring, we have the category Alg_R of R-algebras (rings T with the extra structure of a map $R \to T$; morphisms must commute with this map). This is an example of something more general, called an *undercategory*: if C is a category and $X \in \mathsf{C}$, then the undercategory $X \downarrow \mathsf{C}$ is the category whose objects are data of $Y \in \mathsf{C}$ with C-morphisms $a_Y : X \to Y$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] & & Y_2 \\ & X. \arrow[ul, "a_{Y_1}"] \arrow[ur, "a_{Y_2}"'] & \end{tikzcd}$$

In the same way, the *overcategory* $X \uparrow \mathsf{C}$ is the same idea, but with maps to X rather than from X (e.g. spaces over a given space X).

Thus, it's possible to concisely define $\mathsf{Alg}_R = R \downarrow \mathsf{Ring}$. We will see other examples of this.

Example 2.8 (Localization)**.** Let R be a ring and $S \subset R$ be a multiplicative subset. Then, the *localization at S* is $S^{-1}R = \{r/s \mid r \in R, s \in S : r/s = r/s' \text{ when } s''(rs' - r's) = 0 \text{ for some } s'' \in S\}$. This is a construction we'll use a lot, so it will be useful to have a canonical characterization of them.

Now, let C be the category of R-algebras T with maps ($\varphi_T : R \to T$ such that (and this is a property, not structure) $\varphi_T(s)$ is invertible in T for all $s \in S$.

Exercise 2.9. Show that $S^{-1}R$ is the initial object in C.

The naïve idea that localization is "fractions in S" is true if R is an integral domain, but if we have zero divisors, the R-algebra structure map $R \to S^{-1}R$ need not be injective. But the point is that if T is an R-algebra where the elements of S become invertible, the map φ_T factors through $S^{-1}R$; this means that $S^{-1}R$ is the element of C that's "closest to R." However, you still have to concretely build it to show that it exists; however, we know already that it's determined up to unique isomorphism, so we say "the" localization.

Another very fundamental language for making constructions is that of limits and colimits. It may seem a little strange, but it's quite important.

Definition 2.10. Let I be a *small category* (so its objects form a set); in the context of limits, we will refer to it as an *index category*. Then, a functor $A : I \to \mathsf{C}$ is called a *I-shaped* (or *I-indexed*) *diagram in* C.

That is, if $m : i \to j$ is a morphism in I, then this diagram contains an arrow $A(m) : A_i \to A_j$.

Definition 2.11. Let A be an I-shaped diagram in C. Then, a *cone* on A is the data of an object $B \in \mathsf{C}$ and maps $A_i \to B$ for every $i \in I$ commuting with the morphisms in I. The cones on A form a category Cones_A, where the morphisms are maps $B \to B'$ commuting with all the maps in the cone.

We can also take the category of "co-cones," which are data of maps from B *into* the diagram. This is not quite the opposite category (since we want maps $B \to B'$ commuting with the maps into the diagram).[9]

[8]That rings and ring homomorphisms are unital is important for this to be true.

[9]Some people switch the definitions of cones and co-cones, but since we're not going to use these words very much, it doesn't matter all that much.

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] \arrow[drr, bend left] & A_3 \arrow[dr] & \\
A_2 \arrow[ur] \arrow[rr] & & B
\end{tikzcd}$$

FIGURE 1. A cone on a diagram A.

Definition 2.12.

- The *colimit* $\varinjlim_I A$ is the initial object in the category of cones of A.
- The *limit* $\varprojlim_I A$ is the terminal object in the category of co-cones of A.

As before, colimits and limits may or may not exist, but if they do, they're unique up to unique isomorphism.

Colimits act like a quotient, and it's easier to map out of them. Correspondingly, limits behave like a subobject, and it's easier to map into them.

Example 2.13 (Products and Coproducts)**.** Let $I = \bullet\ \bullet$ be a two-element discrete set (no non-identity arrows). Thus, an I-shaped diagram is just a choice of two spaces A_1 and A_2, so a colimit C_1 is the data of a unique map φ_B for each $B \in \mathsf{C}$ fitting into the following diagram.

$$\begin{tikzcd}
A_1 \arrow[dr] \arrow[drr, bend left] & & \\
& C_1 \arrow[r, "\varphi_B"] & B \\
A_2 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

This is called the *coproduct* of A_1 and A_2, denoted $A_1 \sqcup A_2$ or $A_1 \amalg A_2$.

Similarly, the limit of A is called the *product* of A_1 and A_2, is denoted $A_1 \times A_2$, and fits into the diagram

$$\begin{tikzcd}
& & A_1 \\
B \arrow[urr] \arrow[r] \arrow[drr] & A_1 \times A_2 \arrow[ur] \arrow[dr] & \\
& & A_2
\end{tikzcd}$$

In the same way, if I is a larger discrete set, we get coproducts and products of objects in C indexed by I, denoted $\coprod_I A_i$ and $\prod_I A_i$, respectively.

In the category of sets, the product is Cartesian product, and the coproduct is disjoint union. The same is true in topological spaces.

In the category of groups, the product is once again Cartesian product, but the coproduct is the free product (mapping out of it is the same as mapping out of the individual components, which is not true of the direct product). As underlying sets, this is distinct from the coproduct of sets.

In linear categories, e.g. Ab, Mod_R, or Vect_k, $V \oplus W$ is the product and coproduct, and the same is true over all finite I. However, this is *not* true when I is infinite: the coproduct is the direct sum, which takes finite sums of elements, and the product is the Cartesian product, which takes arbitrary sums of elements. It's worth working out why this is, and how it works.

Many of these categories are "sets with structure," e.g. groups, vector spaces, topological spaces, and so on. In these cases, there is a *forgetful functor* which forgets this structure: indeed, a group homomorphisms (continuous map, linear map) is a map of sets too.[10]

There's a useful principle here: *forgetful functors preserve limits*: if F is a forgetful functor, then there is a canonical isomorphism $F(\varprojlim A) \cong \varprojlim F(A)$. This is something that can be defined more rigorously and proven. But one important corollary is that if you know what the limit looks like for sets, it's the same in groups, rings, vector spaces, topological spaces, and so on. However, this is very false for coproducts, e.g. the coproduct on groups is not the same as the one on sets.

[10] If this seems vague, that's all right; it's possible to define and find forgetful functors more formally.

This becomes a little cooler once we see limits that aren't just products.

Example 2.14. Consider the diagram of rings

$$\cdots \longrightarrow \mathbb{Z}/p^n \longrightarrow \cdots \longrightarrow \mathbb{Z}/p^3 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p,$$

where each map is given by modding out by p. One can show that the limit exists, and it'll be the same as the limit of the underlying sets, a sequence of compatible elements; this limit is called the *p-adic integers*, denoted $\mathbb{Z}_p$. More generally, the same thing works for $\varprojlim R/I^n$ for an ideal $I \subset R$, and defines the *I-adic completion* $\widehat{R}_I$, which we'll revisit, since it has useful geometric meaning.

3. Locally Ringed Spaces: 2/23/16

"I didn't give it a good notation because I didn't like it."

Recall that last time, we defined sheafification, which can be thought of projecting presheaves onto sheaves in a particularly nice way. This allows us to forget the difference between sheaves and presheaves, so to speak; we'll use this to understand colimits of sheaves.

Example 3.1. First, a quick digression, since we got confused last time, on the espace étalé of a skyscraper sheaf. Directly from the sheaf axioms, one can show that if $\mathscr{F}$ is a C-valued sheaf, then $\mathscr{F}(\varnothing)$ is the terminal object (a point for Set, 0 for Ab, and so on). This follows from abstract nonsense: the empty product $\prod_\varnothing S$ is necessarily the terminal object (there's more to think through here). This is what motivates the definition of the skyscraper sheaf $i_*S = i_{x,*}S$ in Example **??**. For simplicity, assume $x \in X$ is a closed point.

Now, let's construct its espace étalé $\pi : Y_{i_*S} \to X$; for any $y \in X$, $\pi^{-1}(y)$ is the stalk of (i_*S) at y, which is S if $y = x$ or the terminal object $*$ otherwise. Thus, Y_{i_*S} is as a set a copy of X, but with S over the basepoint x instead of a single point; then, we glue each of these points of S to the rest of Y_{i_*S} as if they were all x. The result is X with multiple basepoints, so to speak, and is not at all Hausdorff. However, as topological spaces, we have a pullback diagram

$$\begin{tikzcd} (U \setminus \{x\}) \times S \arrow[r] \arrow[d] & U \times S \arrow[d] \\ U \setminus \{x\} \arrow[r] & Y_{i_*S}. \end{tikzcd}$$

We can also use the espace étalé to define sheafification: the sheafification $\mathscr{F}^{\mathrm{sh}}$ is just the sheaf of sections of $Y_{\mathscr{F}}$.

Kernels and Cokernels. Before discussing limits and colimits more generally, let's focus on kernels and cokernels. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on a space X, and for every open $U \subset X$, define $(\ker\varphi)(U) = \ker(\varphi|_U)$. It's easy to check that this is a presheaf, and a little more work to check that it's a sheaf, too. And this is actually the kernel in Ab_X, in that it satisfies the universal property: it fits into the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d] & \mathscr{F} \arrow[d, "\varphi"] \\ 0 \arrow[r] & \mathscr{G}, \end{tikzcd}$$

and any other sheaf $\mathscr{H}$ that fits into the same place in the above diagram has a unique map to $\ker\varphi$.

Likewise, a morphism in Ab_X is injective (meaning a monomorphism) exactly when $\varphi|_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open $U \subset X$.

Cokernels are a little more interesting. The sheaf assigning $U \mapsto \operatorname{coker}(\varphi|_U)$ is a presheaf, and is the cokernel in the category of presheaves, but it is *not* the cokernel in the category of sheaves; it fails to satisfy the universal property. This is where some of the interesting nuances of sheaf theory pop up.

Example 3.2. We'll let $X = \mathbb{C}$, and let $\mathscr{O}$ be the sheaf of holomorphic functions and $\mathscr{O}^*$ be the sheaf of "invertible," i.e. nonvanishing, holomorphic functions (an abelian group under multiplication). The exponential map $f(z) \mapsto e^{f(z)}$ sends holomorphic functions to nonvanishing holomorphic functions, and commutes with restriction, so it's a morphism $\exp : \mathscr{O} \to \mathscr{O}^*$ in $\mathsf{Ab}_{\mathbb{C}}$.

If a function maps to 1 in $\mathscr{O}^*$, then it must be an integer multiple of $2\pi i$, so it must be locally constant, Thus, it's constant on each connected component of the given open set. Thus, $\ker(\exp) = 2\pi i\underline{\mathbb{Z}}$: the constant sheaf, not the constant presheaf. This agrees with what we just learned about kernels.

Then, $\mathrm{Im}(\exp)(U)$ is the $f^* \in \mathscr{O}^*(U)$ such that $f = e^{2\pi i g}$ for some $g \in \mathscr{O}(U)$. That is, $\log f$ must have a well-defined branch on U. In particular, if $U = \mathbb{C}^*$ and $f = z$, then $f \notin \mathrm{Im}(\exp(U)$. This is a problem: $\mathbb{C}^*$ can be covered by simply connected open sets on which the logarithm exists, but the gluing axiom fails.

However, since $\exp : \mathscr{O} \to \mathscr{O}^*$ is surjective on simply connected open sets, then it's surjective on the level of stalks, even though it's not surjective as a map of sheaves. In other words, we want the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{j} \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \longrightarrow 0$$

to be a short exact sequence of sheaves, but if we naïvely define the cokernel like the kernel, it isn't. This means that *to define the sheaf cokernel, we sheafify the presheaf cokernel*. In this case, the sheafification of the presheaf cokernel $\mathrm{coker}(j)$ stitches together the stalks, but on stalks exp is surjective, so since a sheaf is completely determined by stalks, this is just $\mathscr{O}^*$ again, which jives with the idea of surjectivity. In the same way, we get that $\mathrm{coker}(\exp) = 0$, as one would expect.

In other words, a surjective map of sheaves (categorically, an epimorphism), is surjectivity on stalks, but *not* surjectivity on all open subsets. Injectivity is equivalent to injectivity on stalks and on open subsets, though.

Since sheafification preserves colimits, this can be generalized: the colimit of a diagram of sheaves is the sheafification of the presheaf colimit (which is just the colimit on every open set).

Example 3.3. This next example is in some sense the same example. Let X be a smooth manifold, $\mathscr{F}$ be the sheaf of smooth maps to S^1, C^∞ be the smooth maps to $\mathbb{R}$ (so just the smooth functions), and $\underline{\mathbb{Z}}$ is the constant sheaf (which is also smooth maps to $\mathbb{Z}$, since $\mathbb{Z}$ is discrete); each of these is a sheaf of abelian groups.

We'd like to understand that $S^1 = \mathbb{R}/\mathbb{Z}$. This comes from the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow C^\infty \longrightarrow \mathscr{F} \longrightarrow 0,$$

which is short exact. The injectivity of $\underline{\mathbb{Z}} \hookrightarrow C^\infty$ comes from the fact that every map to $\mathbb{Z}$ can be lifted to a smooth map to $\mathbb{R}$, and surjectivity comes from the fact that germs of functions can be lifted on a small neighborhood, so it's surjective on stalks. However, there are open subsets where functions can't be lifted: if $X = S^1$, then the identity map $S^1 \to S^1$ can't be lifted to a map to $\mathbb{R}$. Thus, this is surjective, even though it's not so on the level of open sets.

Example 3.4. Our next example will be the de Rham complex. Let X be a smooth manifold. Let $\underline{\mathbb{R}}$ denote the constant sheaf on $\mathbb{R}$ (locally constant functions) and Ω^1 denote the sheaf of one-forms on X. The exterior derivative gives us an exact sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow C^\infty \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \xrightarrow{\mathrm{d}} \dots$$

However, this is not in general short exact; if Ω^1_{cl} denotes the space of closed one-forms, then the Poincaré lemma just states that the following sequence is short exact.

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow C^\infty \xrightarrow{\mathrm{d}} \Omega^1_{\mathrm{cl}} \longrightarrow 0$$

In other words, even considering something very simple about short exact sequences of sheaves gives us cohomology. This can be used to define sheaf cohomology, though we won't return to that anytime soon.

In fact, Example 3.2 is a special case of this, since $\mathrm{d}z/z \in \Omega^1(\mathbb{C}^*)$ is a closed form that's not exact.

Ringed Space. Anyways, we were going to talk about schemes, right? These are not just topological spaces, but ringed spaces: topological spaces with a notion of a ring of functions.

Definition 3.5. A *ringed space* is the data $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X.

The motivating examples are a topological space with continuous functions to $\mathbb{R}$ (since these form a ring), or a smooth manifold with the sheaf C^∞, or an analytic manifold with C^ω (analytic functions). Thus, there are definitely different notions of "function" on a manifold, but the ringed space structure means knowing what kinds of functions (geometric structure) is.

We'd also like to know how to evaluate functions on a ringed space. For an arbitrary $x \in U$ and $f \in \mathscr{O}_X(U)$, it's not clear how to define $f(x)$; we have stalks, but then what? In each of our examples (continuous functions, smooth functions, analytic functions, holomorphic functions, etc.), the stalks $\mathscr{O}_{X,x}$ aren't just rings, but local rings,[11] with the maximal ideal $\mathfrak{m}_x$ of functions which vanish at x. $\mathfrak{m}_x$ is unique, because if $f \in \mathscr{O}_{X,x} \setminus \mathfrak{m}_x$, then $f(x) \neq 0$, so it's nonzero on a neighborhood of x, and therefore invertible in that subset! Thus, $f \in \mathscr{O}_{X,x}^\times$, so $\mathfrak{m}_x$ must be unique.

The point is, evaluating at x is exactly quotienting by $\mathfrak{m}_x$, producing an element of $\mathbb{R}$. The sheaves we care about have local rings for stalks, which is what makes this evaluation work. We'll turn this into a definition of something much more useful than a ringed space.

Definition 3.6. A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ such that for every $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring.

Thus, all of our basic examples are locally ringed spaces, and in general, given an $f \in \mathscr{O}_X(U)$, we can define $V(f) = \{x \in U : f(x) = 0\}$, and this will end up being a closed set.

Schemes are particular examples of locally ringed spaces. We'll have to define how to produce a sheaf of functions, which we'll probably do next time, but we're almost there. One major takeaway is that schemes behave somewhat like these examples we already have.

We also need to define morphisms. An isomorphism is evident: $(X, \mathscr{O}_X) \cong (Y, \mathscr{O}_Y)$ is the data of a homeomorphism $f : X \to Y$ that identifies the sheaves, i.e. for all open $U \subset Y$, there's an isomorphism $f_* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$.

It's less obvious how to define morphisms in general; clearly, we need a continuous $f : X \to Y$, and we want to compare $\mathscr{O}_X$ and $\mathscr{O}_Y$. Functions pull back (because the preimage of an open set is open); in the examples we had before, we checked that the pullbacks of continuous (smooth, etc.) functions were continuous (smooth, etc.). More generally, given an open $U \subset Y$, we have the two rings $\mathscr{O}_Y(U)$ and $\mathscr{O}_X(f^{-1}(U))$, and we want the pullback of functions $f_* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ to be a ring homomorphism. This is exactly how we defined the pushforward of a sheaf.

Definition 3.7. A *morphism of ringed spaces* is a pair $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ in which
- $f : X \to Y$ is continuous, and
- $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism in Ring_Y.

That is: for every open subset, we can pull functions back into that open subset. But we can say that more concisely with the sheaf theory we have developed.

It's worth remembering that nilpotents on affine schemes give us functions that aren't determined by their values (well, we do have to set up the structure of a locally ringed space first, but we'll get there), so a function isn't quite a bunch of values at points; it's something that we care to pull back.

This is cool, but we care about ringed spaces. What about these maximal ideals? They tell us what it means for a function to vanish. Back in the world of smooth functions, if $\varphi(y) = 0$ and $x \in f^{-1}(y)$, then $(f^*\varphi)(x) = \varphi(f(x))$ had better be 0 too. This is not preserved by morphisms of ringed spaces (since evaluation isn't defined for germs of functions on ringed spaces), so we need an additional axiom.

If $(f, f^\sharp)$ is a morphism of ringed spaces, passing to colimits induces a map $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$, whenever $f(x) = y$ (this is generally true for a map of sheaves, thanks to the property of colimits). Then, we want this map to send $\mathfrak{m}_y \to \mathfrak{m}_x$.

Definition 3.8. A morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces if for every $x \in X$, $y \in Y$ such that $f(x) = y$, the induced map $f_x^\sharp : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ maps $\mathfrak{m}_y$ into $\mathfrak{m}_x$.

This is actually all the data that we'll need to define schemes: schemes are a full subcategory of locally ringed spaces; specifically, they are the ones that are locally isomorphic to $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ (as soon as we define the locally ringed space structure on $\operatorname{Spec} R$), i.e. there are actual isomorphisms on an open cover.

[11] Recall that a *local ring* is a ring with a unique maximal ideal.

Does this look weird? It's actually not unfamiliar: a smooth manifold is a locally ringed space that's locally isomorphic to $(\mathbb{R}^n, C^\infty)$. This encodes a lot of information; in particular, a continuous map of manifolds is smooth iff it pulls smooth functions back to smooth functions. In the same way, a topological manifold is a locally ringed space locally isomorphic to $(\mathbb{R}^n, C)$ (the sheaf of continuous functions). All the structure of an atlas is encapsulated in this notion of locally ringed spaces.

This notion is extremely general. For example, we can define a complex analytic manifold to be a locally ringed space locally isomorphic to $(U \subset \mathbb{C}^n, \mathrm{Hol})$ (since small discs in $\mathbb{C}^n$ aren't necessarily biholomorphic to all of $\mathbb{C}^n$). In all of the cases we've seen, though, $\mathscr{O}_X(U)$ is always a subset of set maps $U \to \mathbb{R}$ (or $\mathbb{C}$), and in particular functions are determined by their values. This is something that will not be true for schemes.

Next time, we will define $\operatorname{Spec} R$, as a scheme.

4. Differentials: 3/31/16

"This is a proof by intimidation."

We're going to be doing some calculus in the algebraic geometry setting over the next few weeks, starting with differentials and the language to say all the things we need to say. In order to talk about differentials, we need to spend some time talking about duals. To do this, we need to make one caveat.

Let R be a ring and M be an R-module, so we have an associated quasicoherent sheaf $\mathscr{M} = \Delta(M)$, which is an $\mathscr{O}_X$-module. We can form the dual module $M^\vee = \mathrm{Hom}_R(M, R)$, and therefore this suggests taking a dual sheaf $\mathscr{M}^\vee = \mathit{Hom}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$. This is the correct notion if $\mathscr{M}$ is locally free (akin to vector bundles), but not in general.

Suppose $\mathscr{M}$ is the skyscraper sheaf at 0 on $\mathbb{A}^n$, which is the localization of the $k[x,y]$-module $M = k$, where the action of a $p \in k[x,y]$ is multiplying by $p(0)$. Thus, M is torsion, so $M^\vee = \mathrm{Hom}_{k[x,y]}(k, k[x,y]) = 0$: there are no maps from a torsion module to a free one.

There's always a map from M to its double-dual $M \to M^{\vee\vee}$ given by sending $m \in M$ to the map $(\varphi \mapsto \varphi(m))$, which is a map $\mathrm{Hom}_R(M,R) \to R$, i.e. an element of $M^{\vee\vee}$. This is a natural map, but it doesn't have to be an isomorphism: for the module that induces the skyscraper sheaf, it's zero.

Definition 4.1. An R-module M is *reflexive* if the natural map $M \to M^{\vee\vee}$ is an isomorphism.

The modules corresponding to free and locally free sheaves are reflexive. This is the sense in which the dual is actually dual; in general, it doesn't behave like you might expect.

Example 4.2. Let's consider a less silly example than the skyscraper sheaf: $\mathfrak{m}_0 = (x,y) \subset k[x,y]$ is an ideal, and therefore a $k[x,y]$-module. Then, $\mathfrak{m}_0^\vee = \mathrm{Hom}_{k[x,y]}(\mathfrak{m}_0, k[x,y])$. If $\varphi : \mathfrak{m}_0 \to k[x,y]$ is a homomorphism, then $\alpha = \varphi(x)$ and $\beta = \varphi(y)$ determine the homomorphism. However, they're not linearly independent: we need $y\alpha = x\beta$, since they're both $\varphi(x,y)$. Therefore, $\alpha = fx$ and $\beta = fy$ for some $f \in k[x,y]$, and more generally $\varphi(r) = fr$: $f = \alpha/x = \beta/y \in k[x,y]$. Thus, $\mathrm{Hom}_{k[x,y]}(\mathfrak{m}_0, k[x,y]) = k[x,y]$. And certainly $(k[x,y])^\vee = k[x,y]$, since it's free over itself, and the canonical map $\mathfrak{m}_0 \to \mathfrak{m}_0^{\vee\vee} = k[x,y]$ is the inclusion.

Thus, torsion-free does not imply reflexive! Duals are weird: they forget information.

The point of this is that to construct the ring of total sections, we took the dual, so we have to be careful. Let M be an R-module, so that $\mathrm{Sym}_R M$ is an R-algebra, given by $R \oplus M \oplus \mathrm{Sym}^2 M \oplus \mathrm{Sym}^3 M \oplus \cdots$.

Definition 4.3. If R is a ring, an R-algebra M is an *augmented R-algebra* if there's a *augmentation map* $\varepsilon : M \to R$ that's a section for the R-algebra map $\varphi : R \to M$, i.e. $\varepsilon \circ \varphi = \mathrm{id}$. $\ker(\varepsilon)$ is known as the *augmentation ideal*.

For example, quotienting out by all terms of positive degree in $\mathrm{Sym}_R M$ defines an augmentation map $\varepsilon : \mathrm{Sym}_R M \to R$, so $\mathrm{Sym}_R M$ is an augmented R-algebra, and the augmentation ideal is (generated by) the terms of positive degree.

Geometrically, taking Spec turns everything around: if $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} \mathrm{Sym}_R M$, then the data that $\mathrm{Sym}_R M$ is an R-algebra gives us a map $Y \to X$, and $\varepsilon^* : X \to Y$ is a section for this map. $\mathrm{Sym}_R M$ is in fact a graded R-algebra, and therefore there's a $\mathbb{G}_m$-action on Y.

For example, let $R = k[x]$ and $M = k[x]/(x)$, which can be thought of as k plus an action of x. Let y be a generator of M, so that $\mathrm{Sym}_R M = k[x] \oplus k \cdot y \oplus k \cdot y^2 \oplus k \cdot y^3 \oplus \cdots \cong k[x,y]/(xy)$.

If we take Spec, this is just the union of the x- and y-axes in $\mathbb{A}^2_k$, projecting down onto $\operatorname{Spec} R = \mathbb{A}^1_k$. This isn't quite a vector bundle: the fiber over every point is a k-vector space, but at 0 it jumps (really, it's a skyscraper over 0), and this is bad.

We can recover M from the algebraic data of $\operatorname{Sym}_R M$ as the degree-1 elements, and there is also a way to do this geometrically. Explicitly, to get the terms of degree-1, take the augmentation ideal and remove all terms of degree at least 2: $M \cong I/I^2$ as R-modules.

Alternatively, one could take the ring $\operatorname{Sym}_R M/I^2$, which is the split square-zero extension $R \oplus M$, in the way that we talked about last time (so M multiplies to 0). The augmentation survives as the map $\varepsilon : \operatorname{Sym}_R M/I^2 \to \operatorname{Sym}_M I \cong R$.

This is equivalent data to what we talked about last time, but is more geometric. The maps of rings induce maps of schemes $\operatorname{Spec} R \to \operatorname{Spec}(R \oplus M) \to \operatorname{Spec} \operatorname{Sym}_R M$. Here, $\operatorname{Spec}(R \oplus M)$ is the first-order neighborhood of the "zero section" in the "total space" of M, which is $\operatorname{Spec} \operatorname{Sym}_R M$. This is because I cuts out the zero section, but we're modding out by I^2, which gives us the first-order neighborhood, as with the dual numbers. In fact, if M is free of dimension n, $\operatorname{Spec}(R \oplus M) = \operatorname{Spec} R \times (\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)))^{\oplus n}$; if M is locally free, then this is true locally.[12]

Definition 4.4. With R, M, and I as in the preceding discussion, let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} \operatorname{Sym}_R M$. Then, the *conormal sheaf* to $X \hookrightarrow Y$ is the sheaf associated to the R-module I/I^2.

This seems like it should be the normal sheaf (analogous to the normal bundle), but if you look carefully, this is really linear functionals, so it is more like a dual space.

For example, suppose $R = k$, so $X = \operatorname{Spec} k = \operatorname{Spec}(\operatorname{Sym}_R M/\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal of the symmetric algebra. In this case, $I/I^2 = \mathfrak{m}/\mathfrak{m}^2$, which is the cotangent space.

We'll talk about cotangents in order to make conormals make more sense, and hence talk about derivations.

Recall that if A is a B-algebra and M is an A-module, we make $A \oplus M$ into a B-algebra as a square-zero extension, like last time. Then, the space of derivations is $\operatorname{Der}_B(A, M) = \operatorname{Hom}_B(A, A \oplus M)$. These are the B-linear functions $\partial : A \to M$ such that $\partial(fg) = f\partial g + g\partial f$. These feel like differential operators; for example, $\frac{\partial}{\partial x} \in \operatorname{Der}_k(k[x], k[x])$.

If you've been sufficiently Grothendiecized by this class, you should expect some sort of "universal" of "free" derivation given the B-algebra structure $\varphi : B \to A$. This will be an A-module $\Omega_{A/B}$ along with a map $\mathrm{d} : A \to \Omega_{A/B}$.

Definition 4.5. Define the A-module $\Omega_{A/B}$, the module of *(Kähler) differentials of A over B*, to be the A-module spanned by elements $\mathrm{d}a$ for all $a \in A$ subject to the following relations for all $a, a' \in A$:

- $\mathrm{d}a + \mathrm{d}a' = \mathrm{d}(a + a')$,
- $\mathrm{d}(aa') = a\mathrm{d}a' + a'\mathrm{d}a$, and
- $\mathrm{d}(\varphi(b)) = 0$.

Then, define $\mathrm{d} : A \to M$, the *de Rham differential*, to send $a \mapsto \mathrm{d}a$.

The first relation forces d to be A-linear, and the second is the Leibniz rule. The last rule makes this compatible with the structure of B.

This feels nostalgically like the construction of the tensor product, and so we should expect a universal property.

Proposition 4.6. *Let M be an A-module and $\partial : A \to M$ be a derivation. Then, there is a unique derivation $\widetilde{\partial} : \Omega_{A/B} \to M$ such that the following diagram commutes.*

$$\begin{tikzcd} & \Omega_{A/B} \arrow[d, dashed, "\widetilde{\partial}"', "\exists!"] \\ A \arrow[ur, "\mathrm{d}"] \arrow[r, "\partial"'] & M \end{tikzcd}$$

[12] More generally, one can consider things such as $\operatorname{Spec}(\operatorname{Sym}_R M/I^n)$, which is something people do, though in this context it's not so useful. The point is that we're going to eventually think of I/I^2 as the conormal bundle. We'll return to things like this.

The construction is to let $\widetilde{\partial}(\mathrm{d}a) = \partial a$ and extend A-linearly. As a consequence, $\mathrm{Hom}_A(\Omega_{A/B}, M) = \mathrm{Der}_B(A, M)$ as B-modules.

For example, $T_{A/B} = \Omega_{A/B}^\vee = \mathrm{Hom}_A(\Omega_{A/B}, A)$ can be thought of as vector fields, because it's identified with $\mathrm{Der}_B(A, A)$. Unlike in differential geometry, we're already doing everything relatively, and so these are "relative vector fields," compatible with the map $\operatorname{Spec} B \to \operatorname{Spec} A$. If you want to understand absolute vector fields, you can take $B = \mathbb{Z}$, since $\mathbb{Z}$-linearity is just additivity, which doesn't tell us anything. But the flexibility of taking something relative (which might be a point) is still very useful.

An example of $T_{A/B}$ is $A = k[x, y]$ as a $B = k[x]$-module; then, $\mathrm{Der}_{k[x]}(k[x, y], k[x, y]) \cong k[x, y]$, where $1 \mapsto \frac{\partial}{\partial y}$.

Here's a neat definition, though we haven't earned it.

Definition 4.7. A map $\operatorname{Spec} B \to \operatorname{Spec} A$ is *smooth* if the module of differentials $\Omega_{A/B}$ induced from this map is locally free.

There are many questions here: what does it mean for a module to be locally free? It's the same notion turned around, so it's free after sufficiently strong localizations. Over affine schemes, this is the same as free, but this is very far from true in general. Another nice consequence is that smoothness of any scheme is smoothness of the induced map to $\operatorname{Spec} \mathbb{Z}$. This probably isn't very enlightening; the next time we return to smoothness, we'll have the context to appreciate it more.

A vector bundle over a contractible manifold is trivial, which isn't very hard to show. Is the same true in algebraic geometry? The best example of something "contractible" is affine space, right?

Theorem 4.8 (Serre's conjecture/Quillen-Suslin theorem)**.** *If k is an algebraically closed field, all vector bundles on $\mathbb{A}_k^n$ are trivial.*

This is a scary, hard theorem: look at those big names! More seriously, the proof of this theorem was one of the first major breakthroughs demonstrating the power of algebraic K-theory. We definitely haven't earned this theorem.

Anyways, the point is, "smooth things should have tangent bundles." This is a philosophy, but we can just define it.

Definition 4.9. Let A be a k-algebra. Then, the *tangent bundle* of $X = \operatorname{Spec} A$ is $TX = \operatorname{Spec}(\operatorname{Sym}_k \Omega_{A/k})$, and the *projectivized tangent bundle* is $\mathbb{P}(TX) = \operatorname{Proj}(\operatorname{Sym}_k \Omega_{A/k})$.

TX locally looks like $X \times \mathbb{A}_k^n$, and $\mathbb{P}(TX)$ locally looks like $X \times \mathbb{P}_k^{n-1}$.

The point is that this will be a vector bundle iff X is smooth. We'll have to unwrap this later. But it advertises another good fact about algebraic geometry: from the beginning, we care about singularities, because rings have singularities. To understand smoothness in a geometric sense, we need calculus, which is why we're talking about differentials.

We defined $\Omega_{A/B}$ with a lot of generators and a lot of relations; if we have generators and relations for A as a B-algebra, we can simplify this. In particular, we can always assume $A \cong B[x_i : i \in I]/(r_j : j \in J)$. In this case, $\Omega_{A/B}$ is much simpler:

$$\Omega_{A/B} \cong \left(\bigoplus_{i \in I} \mathrm{d}x_i \right) / (\mathrm{d}r_j : j \in J).$$

The de Rham differential of a relation is given by expanding A-linearly and using the Leibniz rule, e.g. $\mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x$. This construction of $\Omega_{A/B}$ makes it a little more apparent that $\Omega_{A/B}$ is a "linearization" of the structure of A. It also makes some nice properties apparent.

Corollary 4.10. *If A is a finitely generated (resp. finitely presented) B-algebra, then $\Omega_{A/B}$ is a finitely generated (resp. finitely presented) A-module.*

Now, suppose $\varphi : B \twoheadrightarrow A$ is surjective, so $A \cong B/I$ for an ideal $I \subset B$; geometrically, we'd have an inclusion of schemes. Then, $\Omega_{A/B} = 0$, because every $a \in A$ is $\varphi(b)$ for some $b \in B$, so $\mathrm{d}a = \mathrm{d}(\varphi(b)) = 0$.

Localizations (open subsets) also don't have any relative differentials: if $f/g \in S^{-1}B$, then

$$\partial\left(\frac{f}{g}\right) = \frac{g\partial f - f\partial g}{g^2} = 0,$$

because $f, g \in B$, so $\partial f = \partial g = 0$. Hence, $\Omega_{S^{-1}B/B} = 0$.

Example 4.11. Suppose $A = k[x,y]/(y^2 - x^3 + x)$, which corresponds to the elliptic curve $y^2 = x^3 - x$. Then, $\Omega_{A/k} = (A\,\mathrm{d}x \oplus A\,\mathrm{d}y)/(2y\,\mathrm{d}y = (3x^2+1)\,\mathrm{d}x)$.

Is this smooth? In other words, is $\Omega_{A/k}$ locally free? If $y \neq 0$, then $\mathrm{d}x$ generates, and if $3x^2 + 1 \neq 0$, then $\mathrm{d}y$ is a generator. If $y = 0$, then $x^3 - x = 0$, so $x = 0, \pm 1$, and therefore $3x^2 + 1 \neq 0$. Thus, these cover everything, so $\Omega_{A/k}$ is a line bundle (locally free of rank 1)![13] In particular, this curve is smooth.

Example 4.12. Our favorite singular curve (well, should be singular) is $A = k[x,y]/(xy)$: the singularity is at the origin. Then, $\Omega_{A/k} = (A\,\mathrm{d}x \oplus A\,\mathrm{d}y)/(x\,\mathrm{d}y + y\,\mathrm{d}x)$. Thus, on the y-axis, $\mathrm{d}y$ generates, and on the x-axis, $\mathrm{d}x$ generates, but at the origin, we need both of them. Thus, $\Omega_{A/k}$ isn't locallty free, so this curve isn't smooth. However, $\Omega_{A/k}|_0 \cong \mathfrak{m}_0/\mathfrak{m}_0^2$, where $\mathfrak{m}_0$ is the maximal ideal corresponding to the origin.

This fact about the fiber is more general: $\Omega_{A/k}$ is a nice way to put all the contangent spaces together.

Proposition 4.13. *Let A be a k-algebra and $x \in \operatorname{Spec} A$ correspond to the maximal ideal $\mathfrak{m}_x$. If $\mathfrak{m}_x$ has residue field k, then there's an isomorphism $\delta : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{A/k}|_x = \Omega_{A/k} \otimes_A k_x$.*

Geometrically, we're tensoring with the skyscraper sheaf to obtain the fiber.

We're not going to prove this today, for a lack of time. But one can use this to construct a quasicoherent sheaf $\widetilde{\Omega}_{A/k}$, defined by $\widetilde{\Omega}_{A/k}(D(f)) = \Omega_{A_f/k}$, so localizing as usual. We'll also consider more geometric ways of understanding this.

Another useful word is the analogue of a covering space: there's no way to differentiate along the fibers.

Definition 4.14. If A and B are k-algebras, where k is characteristic zero, then if $\Omega_{A/B} = 0$, then the map $B \to A$ is called *étale*.

There is a definition of étale in positive characteristic, but this isn't the correct definition.

5. The Affine Communication Lemma: 4/7/16

Today and the next few lectures, we're going to have a change of pace, developing some more fundamental properties of schemes before using them to talk more about quasicoherent sheaves, differentials, etc. Today, for example, we'll talk about affine covers, and what properties can be checked affine-locally. This will enable us to turn our story about differentials from one about rings and modules to one about schemes and quasicoherent sheaves.

Definition 5.1. An *affine open* of a scheme X is an open subset U that is isomorphic to $\operatorname{Spec} A$ for some ring A.

The intersection of affine opens may not be itself open. For example, let $X = \mathbb{A}^2 \amalg_{\mathbb{A}^2 \setminus 0} \mathbb{A}^2$, the *plane with two opens* (we've already done this with a line, and this isn't very different). The intersection of the two copies of $\mathbb{A}^2$, which are both affine opens, is $\mathbb{A}^2 \setminus 0$, which we know is not affine.

One can define what it means for a scheme to be separated, which is analogous to the Hausdorff condition on manifolds, and will guarantee that the intersection of affine opens is affine.

Proposition 5.2. *Let X be a scheme and $\operatorname{Spec} A, \operatorname{Spec} B \subset X$ be affine opens. Then, $\operatorname{Spec} A \cap \operatorname{Spec} B$ is a union of open subsets that are distinguished open subsets for both $\operatorname{Spec} A$ and $\operatorname{Spec} B$.*

Proof. Suppose $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$. Then, there's an $f \in A$ such that $p \in \operatorname{Spec} A_f \subset \operatorname{Spec} A \cap \operatorname{Spec} B$, and therefore a $g \in B$ such that $p \in \operatorname{Spec} B_g \subset \operatorname{Spec} A_f \subset \operatorname{Spec} A \cap \operatorname{Spec} B$.

The claim is that $\operatorname{Spec} B_g$ is a distinguished open in $\operatorname{Spec} A$. Restriction defines a map $B = \mathscr{O}_X(\operatorname{Spec} B) \to \mathscr{O}_X(\operatorname{Spec} A_f) = A_f$; let $\widetilde{g}$ be the image of g under this map. Then, the points of $\operatorname{Spec} A_f$ where g vanishes are the same as the same points where $\widetilde{g}$ vanishes (which is exactly what restriction does). Thus, $\operatorname{Spec} B_g = \operatorname{Spec}(A_f)_{\widetilde{g}} = \operatorname{Spec} A_{f\widetilde{g}}$. $\square$

[13] This is actually a trivial line bundle: one can write down a nowhere-vanishing differential.

One way to think of this is that a distinguished open of a distinguished open is distinguished in the original scheme.

We'll use this to prove an extremely useful lemma, the affine communication lemma. This is sometimes called the ACL (not to be confused with the Austin City Limits music festival nor the anterior cruciate ligament in your knee, though these are both great things too).

Lemma 5.3 (Affine communication lemma)**.** *Suppose P is a property*[14] *enjoyed by some affine opens of a scheme X, such that:*

(1) *The property is preserved by restriction: if* $\operatorname{Spec} A \in P$*, then for all* $f \in A$, $\operatorname{Spec} A_f \in P$.
(2) *The property is preserved by finite gluing: if* $A = (f_1, \ldots, f_n)$ *and* $\operatorname{Spec} A_{f_i} \in P$ *for all i, then* $\operatorname{Spec} A \in P$.
(3) *There is a cover of X by affine opens that satisfy P.*

Then, every affine open subset of X is in P.

Definition 5.4. If P is a property of affine opens that satisfies the hypotheses of Lemma 5.3, then we'll call P an *affine-local* property.

We'll generally use this to prove things about a scheme using only a particular cover, rather than having to check all affine opens. Another nice fact about affine-local properties is that any open subset $U \subset X$ (not just affine opens) inherits any affine-local property of X. Many of these properties will ultimately be geometric, and we'll give a long list of them.

Proof of Lemma 5.3. The proof is actually trivial: the perfect lemma is non-obvious, extremely useful, and easy to prove.

Suppose $\operatorname{Spec} A \subset X$ is open. There's a cover $\mathfrak{U}$ of X by affine opens $\operatorname{Spec} B_i \in P$ by hypothesis (3). By Proposition 5.2, $\operatorname{Spec} B_i \cap \operatorname{Spec} A$ can be covered by open subsets which are distinguished opens of both $\operatorname{Spec} A$ and X, and by hypothesis (1), each of these has property P. Thus, there's a cover of $\operatorname{Spec} A$ by distinguished opens $\operatorname{Spec} A_f \in P$. Since $\operatorname{Spec} A$ is quasicompact, then we may assume this cover is finite, so by hypothesis (2), $\operatorname{Spec} A \in P$. ☐

There's a lot of verifications that various properties are affine-local; we'll skip over some of these.

The following proposition is Exercises 5.3.G and 5.3.H in Vakil's notes. Recall that a ring is reduced if it has no nilpotents other than 0.

Proposition 5.5. *Let A be a ring and $f_1, \ldots, f_n$ generate A.*

(1) *A is reduced iff A_{f_i} is reduced for all i.*
(2) *A is Noetherian iff A_{f_i} is Noetherian for all i.*
(3) *If B is another ring and $B \to A$ gives A the structure of a B-algebra, then A is finitely generated over B iff each A_{f_i} is finitely generated over B.*

This means that the analogoues of these properties for schemes are affine-local.

Proof of items (2) *and* (3)*.* Though we'll only prove one of these, most of the proofs of these things go the same way. There are a few exceptions, however.

First, suppose A is Noetherian, and let $I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq \cdots$ be an ascending chain of ideals in A_f. Let $\iota : A \hookrightarrow A_f$ be the canonical inclusion, and let $J_i = \iota^{-1}(I_i)$.[15] Then, $J_1 \subset J_2 \subset J_3 \subset \cdots$. Inside A_f, there must be some $x/f^N \in I_{n+1}\ I_n$, and therefore, by clearing denominators, $x \in J_{n+1} \setminus J_n$, so A isn't Noetherian.

In the other direction, we'll also prove the contrapositive. Suppose $J_1 \subsetneqq J_2 \subsetneqq J_3 \subsetneqq \cdots$ be an ascending chain of ideals in A, and for each i, let $I_{i,j}$ be the localization of J_j at f_i. Then, for all j, there's some i for which $I_{i,j} \subsetneqq I_{i,j+1}$. The idea is that since the f_i generate A, then $A \hookrightarrow \prod_{i=1}^n A_{f_i}$ sending $r_i \mapsto (r_i)_{i=1}^n$. If $r_j \in J_{j+1} \setminus J_j$, then $r_j \subset I_{i,j}$ for al i, but it can't be in all of the $I_{i,j+1}$. There's a little more to say here, but it's kind of annoying.

[14] Formally, a *property* is a subset of the set of affine opens of X; more generally, one could do this in a way independent of the scheme X by considering the set of all affine open embeddings $\operatorname{Spec} A \hookrightarrow X$ over all schemes X; a general property is a subset of this huge set. Examples will be Noetherianness, reducedness, etc.

[15] If A is an integral domain, we can think of this as $I_i \cap A \subset A \subset A_f$, but this isn't always true in general.

For (3) let $r_1, \dots, r_n$ be generators of A as a B-algebra. Then, A_f is generated by $\{r_1, \dots, r_n, 1/f\}$, so it's clearly finitely generated. Conversely, since the f_i generate A, then $1 = \sum r_i f_i$ for some $r_i \in A$. If A_{f_i} is generated by a finite set $\{s_{ij}/f_j^{k_j}\}$, with the $s_{ij} \in A$, then (again, there's something to check here) A is generated by $\{f_i, r_i, s_{ij}\}$, which is a finite set. ☐

Definition 5.6. A scheme X is *reduced* if for all open subsets $U \subset X$, $\mathscr{O}_X(U)$ is a reduced ring.

Since $\mathscr{O}_X(U) \hookrightarrow \prod_{x\in U} \mathscr{O}_{X,x}$ as rings, this is equivalent to $\mathscr{O}_{X,x}$ being reduced for all $x \in X$.

One could also define an affine scheme $\operatorname{Spec} A$ to be reduced if A is a reduced ring; then, our definition is equivalent to affine opens of X being reduced, and hence, by Lemma 5.3, there's a cover of X by reduced affine opens. That is, we've proven the following.

Corollary 5.7. *The following are equivalent for a scheme X.*

(1) *X is reduced.*
(2) *For every $x \in X$, $\mathscr{O}_{X,x}$ is reduced as a ring.*
(3) *All affine opens of X are reduced.*
(4) *There exists a cover of X by reduced affine opens.*

The naïve definition of reducedness would be for $\Gamma(\mathscr{O}_X)$ to be a reduced ring; however, this is *not* equivalent. One example is the "first-order neighborhood of $\mathbb{P}^1$" in the total space $\mathscr{O}(1)$. Using split square-zero extensions, $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}(-1)$ is a ring, so we can define the scheme $X = \operatorname{Spec}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}(-1))$. This has no nonconstant global functions, so $\Gamma(\mathscr{O}_X)$ is reduced; however, locally, the functions on the first-order neighborhood looks like $k[\varepsilon]/(\varepsilon^2)$, which is not reduced. This is why we used the definition that we did.

One moral is that, on non-affine schemes, global ring-theoretic properties may be badly behaved, so it's better to use local ones.

Definition 5.8. Let X be a scheme.

- X is *locally Noetherian* if there's a cover $\mathfrak{U}$ by affine opens $X = \bigcup_{i\in I} \operatorname{Spec} A_i$ such that each A_i is a Noetherian ring.
- X is *Noetherian* if it's quasicompact and locally Noetherian.

Hence, if X is Noetherian, we can choose the cover $\mathfrak{U}$ to be finite. Moreover, by Lemma 5.3, if X is locally Noetherian, *all* affine open subsets of X are Noetherian.

Noetherianness is a very nice property, which is good because lots of schemes you and I might care about are Noetherian. It's a strong "finite-dimensionality" condition. The following property is one nice example.

Proposition 5.9. *If X is a Noetherian scheme, it's also Noetherian as a topological space, and therefore all open subsets of X are quasicompact.*

Hence, for Noetherian schemes, the quasicompactness propagates, which is nice.

Definition 5.10. A scheme X is *quasiseparated* (abbreviated QS) if the intersection of any two quasicompact opens of X is itself quasicompact.

This means that the intersection of two affines may not be affine, but is a finite union of affines. This is not just nice, but something that's scary to not have. Affine schemes are clearly QS, but so are locally Noetherian schemes, because all quasicompact open subsets U of a locally Noetherian scheme X are Noetherian, so by Proposition 5.9, any open subset of U is also quasicompact.

The QS condition is not a restriction, because almost all schemes you will come across will be quasiseparated; instead, it's a signal that QS is used in a proof.

Example 5.11. There are schemes that are not quasiseparated, however; if $\mathbb{A}_k^\infty = \operatorname{Spec} k[x_1, x_2, \dots]$, then $\mathbb{A}_k^\infty \setminus 0$ isn't quasiseparated (not quasicompact).

In some literature, "scheme" means "quasicompact, quasiseparated scheme," and a scheme lacking these hypotheses is explicity noted to not satisfy them. There is a lot of interesting infinite-dimensional geometry (e.g. classifying vector bundles), but we're not going to worry about them in this class.

We'll use the acronym QCQS to denote "quasicompact quasiseparated;" a scheme X is QCQS iff there's a finite cover of X by affine opens, all of whose pairwise intersections are finite unions of affines. This is an extremely useful hypothesis in arguments where you want to glue a hypothesis that we know about affine opens of a QCQS scheme. It's not a particularly interesting geometric property, but is a "reasonableness" property, akin to paracompactness of manifolds, that is a technical ingredient in proofs.

Properties of morphisms. Let $\pi : X \to Y$ be a map of schemes. The preceding discussion allows us to define lots of different nice properties that π could satisfy. In particular, we'll define a bunch of properties of π that can be determined affine-locally on both X and Y (since here they correspond to homomorphisms of rings in the opposite direction).

To be precise, let $\operatorname{Spec} B \subset Y$ be an affine open and $\operatorname{Spec} A \subset \pi^{-1}(\operatorname{Spec} B)$, so $\operatorname{Spec} A$ is an affine open subset of X. Thus, A is a B-algebra.

Definition 5.12. π is *locally of finite type* if for every affine open $\operatorname{Spec} B \subset Y$ and every affine open $\operatorname{Spec} A \subset \pi^{-1}(\operatorname{Spec} B)$, A is finitely generated as a B-algebra.

If X and Y are Noetherian, this is equivalent to such A being finitely presented as a B-algebra; in general, the two may be different, and if A is finitely presented as a B-algebra, one says π is *locally of finite presentation*. We're not going to use this very much.

By the affine communication lemma, π is locally of finite type iff for all affine opens $\operatorname{Spec} B$ of Y, $\pi^{-1}(\operatorname{Spec} B)$ can be covered by affine opens $\operatorname{Spec} A_i$ such that A_i is a finitely generated B-algebra, which is easier to check.

Now, we can generalize a few properties of schemes to properties of morphisms.

Definition 5.13.

- π is *quasicompact* (resp. *quasiseparated*) if for all affine opens $\operatorname{Spec} B \subset Y$, $\pi^{-1}(\operatorname{Spec} B)$ is quasicompact (resp. quasiseparated).
- π is *affine* if for all affine opens $\operatorname{Spec} B \subset Y$, $\pi^{-1}(\operatorname{Spec} B)$ is affine.

We'll prove these satisfy the hypotheses of Lemma 5.3; this will make them extremely useful.