AN INTRODUCTION TO SPECTRAL SEQUENCES

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1. Monday, August 10: introduction and the Serre spectral sequence

2. Tuesday, August 11: group cohomology and the homotopy fixed-points spectral sequence

3. Wednesday, August 12: the Atiyah-Hirzebruch spectral sequence

TODO: this is taken directly from my notes when I gave this lecture three years ago, and will need to be modified. I'll indicate what needs to be changed where it needs to be changed.

Today, I'm going to talk about the Atiyah-Hirzebruch spectral sequence. This spectral sequence computes generalized (co)homology of a space or spectrum, with input data its ordinary homology.

Let D be a spectrum and X be a CW complex. The *homological Atiyah-Hirzebruch spectral sequence* is a spectral sequence with signature

$$E^2_{p,q} \coloneqq H_p(X; D_q(\mathrm{pt})) \Longrightarrow D_{p+q}(X). \tag{3.1}$$

The *cohomological Atiyah-Hirzebruch spectral sequence* is a spectral sequence with signature

$$E_2^{p,q} \coloneqq H^p(X; D^q(\mathrm{pt})) \Longrightarrow D^{p+q}(X). \tag{3.2}$$

(Briefly recall what this means. Briefly discuss convergence, and where it can go wrong.)

If D is a ring spectrum, (3.2) has a multiplicative structure.

TODO: this would go in day 1, if we want to include it at all. Feel free to port it over there; if not, comment it out.

Convergence. Sometimes you're reading a book and it feels like it goes on forever. It's nice when spectral sequences don't do that. As an example, we'll look at a *first-quadrant spectral sequence*, one where $E_2^{p,q} = 0$ when $p < 0$ or $q < 0$. In this setup, if you pick any (p, q), then after finitely many pages, the differentials are so long that they leave the first quadrant, so you get a sequence $0 \to E^r_{p,q} \to 0$, and therefore when you take homology, nothing changes. Thus it makes sense to say what the end of the spectral sequence is.

Definition 3.3. Whenever it makes sense, we'll define the E_∞*-page* of the spectral sequence to be $E_\infty^{p,q} = E^r_{p,q}$ for $r \gg 0$. One says $E_r^{p,q}$ *converges* or *abuts* to $E_\infty^{p,q}$.

Typically this is something interesting we want to calculate.

Definition 3.4. Let $A_\bullet$ be a graded abelian group together with an exhaustive filtration $\{F_p\}$.

- The *associated graded* of the filtration $\{F_i\}$ is

$$(grA)_{p,q} \coloneqq F_pA_{p+q}/F_{p-1}A_{p+q}.$$

- A spectral sequence $E_r^{p,q}$ *converges (weakly)* to $A_\bullet$, written
$$E_r^{p,q} \Longrightarrow A_\bullet,$$
if it has an E_∞ page and the E_∞ page is the associated graded of $A_\bullet$.

Remark 3.5. There is a notion of *conditional convergence*, due to Boardman, which essentially means "not always weakly convergent, but converges under hypotheses often met in practice." Unfortunately, defining this precisely would be a huge digression.

TODO: these are some examples of spectra. Probably I won't delve into this level of detail, just introduce KO, KU, ko, ku, and MSO and say what their coefficient rings are, as well a little bit about what they mean. Some of this may have already been covered in Tuesday's lecture.

Example 3.6 (K-theory)**.** Let X be a compact Hausdorff space. Then, the set of isomorphism classes of complex vector bundles on X is a semiring, so we can take its group completion and obtain a ring $K^0(X)$.

The following theorem is foundational and beautiful.

Theorem 3.7 (Bott periodicity)**.** $K^0(\Sigma^2 X) \cong K^0(X)$.

This allows us to promote K^* into a 2-*periodic* generalized cohomology theory K^*, called *complex K-theory*, by setting $K^{2n}(X) = K^0(X)$ and $K^{2n+1}(X) = K^0(\Sigma X)$.[1]

Like cohomology, K-theory is *multiplicative*, i.e. it spits out $\mathbb{Z}$-graded rings. However, $K^i(X)$ is often nonzero for negative i.

Exercise 3.8. For example, show that as graded abelian groups, $K^*(\mathrm{pt}) = \mathbb{Z}[t, t^{-1}]$, where $|t| = 2$.

K-theory admits a few variants.

- If you use real vector bundles instead of complex vector bundles, everything still works, but Bott periodicity is 8-fold periodic. Thus we obtain a periodic, multiplicative cohomology theory called *real K-theory*, denoted $KO^*(X)$. Its value on a point is encoded in the *Bott song*.
- Sometimes it will be simpler to consider a smaller variant where we only keep the negative-degree elements. This is called *connective K-theory*, and is denoted ku^* (for complex K-theory) or ko^* (for real K-theory). These are also multiplicative.

Example 3.9 (Bordism)**.** Let X be a space and define $\Omega_n^{\mathrm{O}}(X)$ to be the set of equivalence classes of maps of n-manifolds $M \to X$, where $[f_0\colon M \to X] \sim [f_1\colon N \to X]$ if there's a cobordism $Y\colon M \to N$ and a map $F\colon Y \to X$ extending f_0 and f_1. This is an abelian group under disjoint union, and the collection $\{\Omega_n^{\mathrm{O}}\}$ defines a generalized homology theory called *unoriented bordism*.[2]

The following theorem was the beginning of differential topology.

Theorem 3.10 (Thom)**.** *As graded abelian groups, $\Omega_n^{\mathrm{O}}(\mathrm{pt}) \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, \dots] = \mathbb{F}_2[x_i \mid i \neq 2^j - 1]$. Moreover, Ω_*^{O} is a direct sum of (suspended) ordinary cohomology theories.*

There's a lot of variations, based on whatever flavors of manifolds you consider. Using oriented manifolds produces *oriented bordism* Ω_*^{SO}, spin manifolds produce *spin bordism* Ω_*^{Spin}, and so forth. These are not direct sums of ordinary cohomology theories in general.

Example 3.11. We'll use the Atiyah-Hirzebruch spectral sequence to compute $K^*(\mathbb{CP}^n)$. Recall that
$$H^p(\mathbb{CP}^k; A) = \begin{cases} A, & p \text{ even} \\ 0, & \text{odd}. \end{cases}$$
Hence
$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & p, q \text{ even}, 0 \leq p \leq 2k \\ 0, & \text{otherwise}. \end{cases}$$

[1] Extending from compact Hausdorff spaces to all of Top is possible, but then one loses the vector-bundle-theoretic description.

[2] The corresponding cohomology theory is called *cobordism*.

Thus all the differentials are zero! So $E_2^{p,q} \cong E_\infty^{p,q}$. Hence the E_∞ page has no torsion, and therefore $K^*(\mathbb{CP}^n)$ is isomorphic to its associated graded.

$$K^i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{n+1}, & i \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

TODO: more examples.

TODO: define k-invariants, introduce Steenrod squares, and state what the first differential in the AHSS is. Use this in examples to compute ku^* and ko^* of simple things.

4. Thursday, August 13: comparison tools and other tricks

5. Friday, August 14: the Adams spectral sequence over $\mathcal{A}(1)$

The Adams spectral sequence is an important tool in homotopy theory. It converges to the stable homotopy groups of the spheres (well, the most commonly studied variation), and so by conservation of effort, running the spectral sequence must be very hard. Indeed,

- determining the E_2-page is complicated, but is purely algebraic so can be aided with computer calculations, and
- determining the differentials is art and/or dark magic!

Though we understand a great deal of the structure of the Adams spectral sequence, including vanishing slopes and infinite families of elements that survive to the E_∞-page, Mahowald's uncertainty principle (not a real theorem, at least not yet) states that any single method to determine Adams differentials must leave infinitely many unresolved. Again, this is not a theorem, just an observation made after years of experience.

All this posturing aside, what does the Adams spectral sequence look like? Well, fix a prime p and two spectra X and Y. Then the Adams spectral sequence has signature

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \Longrightarrow [Y,X]_p^\wedge. \tag{5.1}$$

That is, it begins with Ext over the mod p Steenrod algebra $\mathcal{A}_p$, and converges to the p-completion of the abelian group of homotopy classes of maps $Y \to X$. TODO: convergence. Often, one takes $Y = \mathbb{S}$, so that this calculates the p-completed homotopy groups of X. Most research has been done for the case $X = \mathbb{S}$; in this case $p = 2$ is the hardest and most-studied; we understand the spectral sequence entirely up to about degree $t - s = 63$, and blearily up to about degree 90. (TODO: many names go here.)

Remark 5.2. There are variants of the Adams spectral sequence where you begin with a spectrum E and feed it the E-cohomology of X and Y, and work over the algebra of stable E-cohomology operations. Our case corresponded to $E = H\mathbb{F}_p$. The case $E = BP$ is also common, in which case this is called the *Adams-Novikov spectral sequence*; one also sees $E = ko$. The E_2-page is not as nice as (5.1) in general, though.

Amusingly, you can take $E = H\mathbb{Q}$, and as soon as you know that the algebra of stable $\mathbb{Q}$-cohomology operations is just $\mathbb{Q}$ in degree 0, the $H\mathbb{Q}$-based Adams spectral sequence collapses, recovering a fact you may already know: rational stable homotopy is rational stable homology.

Remember the mod 2 Steenrod algebra from Wednesday? It's neither finitely generated nor commutative, which is to say it's a massive headache. So in our class today, focused as we are on getting our hands dirty, we will work with a special case of the Adams spectral sequence which is simple enough to actually teach in a day, yet complicated enough to both feature the structures and ideas present in a more general Adams spectral sequence calculation, and also something that's actually interesting and useful to calculate.

(Change-of-rings theorem here TODO)

The most common reason to invoke this is Stong's calculation that

$$H^*(ko;\mathbb{F}_2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2, \tag{5.3}$$

where $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$. So if you want to know the 2-primary ko-theory of something, the change-of-rings theorem and Künneth formula together imply the Adams spectral sequence for $ko \wedge X$ has signature

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(1)} H^*(X;\mathbb{F}_2), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*(X,\mathbb{F}_2), \mathbb{F}_2). \tag{5.4}$$

This is good, because $\mathcal{A}(1)$ is *much* smaller than $\mathcal{A}$: it's eight-dimensional over $\mathbb{F}_2$. In practice, running the Adams spectral sequence is often (if not always) completely tractable in these cases: determining Ext, resolving differentials, and resolving extensions. Our goal today is to teach you how to do this — not to develop everything by hand, but to build on what's already written down to use this as a tool to solve whatever probelms you might have in practice.

Remark 5.5. For the bordism-minded in the audience, at the prime 2, $MSpin$ splits as a sum of various connective covers of ko and $H\mathbb{F}_2$s; each of these pieces can be tackled with this change-of-rings trick. Shearing arguments generalize this to related kinds of bordism, e.g. spin^c, pin$^+$, or pin$^-$ bordism, and so these kinds of bordism of bounded-below spectra can be fed relatively routinely to this method of calculation. It is ultimately for this reason that you sometimes see Adams spectral sequence calculations in physics!

There are several references for this stuff, and I recommend Beaudry and Campbell's paper "A guide for computing stable homotopy groups," which develops the theory from relatively few assumptions, and includes several example calculations.

5.1. Drawing $\mathcal{A}(1)$-modules.

5.2. Determining Ext. Let M be an $\mathcal{A}(1)$-module that we care about. There are many ways to compute $\mathrm{Ext}(M)$.

1. Use someone else's preexisting computation (yes, really!).
2. Write down a minimal resolution of M.
3. Put M into a short exact sequence of $\mathcal{A}(1)$-modules and use the induced long exact sequence in Ext.
4. Notice that M looks a lot like another $\mathcal{A}(1)$-module whose Ext you already know.
5. Clever applications of the change-of-rings formula.

But before that, I'd like to discuss a little more structure that $\mathrm{Ext}(M)$ has: it is a module over the algebra $H^{*,*}(\mathcal{A}(1)) \coloneqq \mathrm{Ext}(\mathbb{F}_2)$.

By definition, Ext is the derived functor of Hom, but elements of Ext can be concretely represented by extensions of $\mathcal{A}(1)$-modules. For $s > 0$, as a set, $\mathrm{Ext}^{s,t}(M, N)$ is the equivalence classes of long exact sequences

$$0 \longrightarrow \Sigma^t N \longrightarrow P_s \longrightarrow P_{s-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow M \longrightarrow 0, \tag{5.6}$$

where two such sequences are equivalent if there are maps between all these modules such that the maps on M and N are the identity, and that the relevant big ol' diagram commutes. This has an abelian group structure using something called *Baer sum*, but since $2 = 0$ it doesn't come up in practice much.

What *does* come up is the *Yoneda product*. Given an extension $0 \to \Sigma^t L \to \cdots \to M \to 0$ of length s and an extension $0 \to \Sigma^{t'} M \to \cdots \to N \to 0$ of length s', you can compose them to an extension

$$\begin{tikzcd} 0 \arrow[r] & \Sigma^{t+t'} L \arrow[r] & \cdots \arrow[r] & \Sigma^t P'_{s'} \arrow[rr] \arrow[dr] & & \cdots P_1 \arrow[r] & \cdots \arrow[r] & N \\ & & & & \Sigma^t M & & & \end{tikzcd} \tag{5.7}$$

and this defines a product

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(L, M) \otimes \mathrm{Ext}^{s',t'}_{\mathcal{A}(1)}(M, N) \longrightarrow \mathrm{Ext}^{s+s',t+t'}_{\mathcal{A}(1)}(L, N). \tag{5.8}$$

called the Yoneda product. When $L = M = N = \mathbb{F}_2$, this makes $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$ into a bigraded algebra, which is denoted $H^{*,*}(\mathcal{A}(1))$, and in general this naturally makes $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$ into an $H^{*,*}(\mathcal{A}(1))$-module. This structure is very useful in Adams spectral sequence calculations (and this is true no matter whether you work over $\mathcal{A}(1)$ or $\mathcal{A}$ or any other subalgebra).

Remark 5.9 (s = 0). By general derived functor machinery, $\mathrm{Ext}^{0,t}(M, N) = \mathrm{Hom}(M, \Sigma^t N)$. The Yoneda product is TODO.

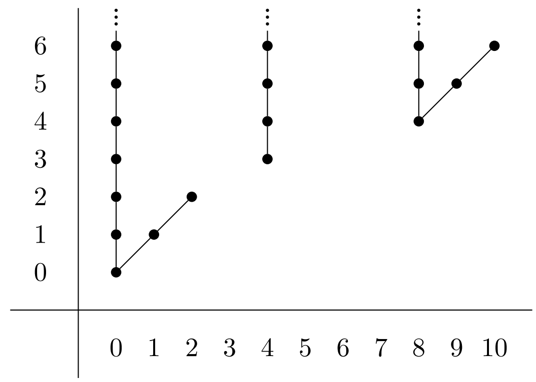
Let's dig into this algebra and module structure a bit. One can calculate that

$$H^{*,*}(\mathcal{A}(1)) \cong \mathbb{F}_2[h_0, h_1, v, w]/(h_0 h_1, h_1^3, h_1 v, v^2 = h_0^2 w) \tag{5.10}$$

with $|h_0| = (1, 1)$, $|h_1| = (1, 2)$, $|v| = (3, 7)$, and $|w| = (4, 12)$. This description, though complete, lacks a certain — you know, what does this *look* like?

As this is the E_2-page of a spectral sequence (converging to the *ko*-homology of a point), let's draw it like one! One important point is that the Adams spectral sequence grading is $(t-s, s)$, so, e.g. h_0 will be in bidegree $(0,1)$, h_1 will be in bidegree $(1,1)$, etc. Anyways, what you get is

(5.11)



TODO: label h_0, h_1, v, and w.

In general, given an $\mathcal{A}(1)$-module M, when drawing $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$, we will draw in the $H^{*,*}(\mathcal{A}(1))$-action as follows: h_0 by vertical lines, h_1 by diagonal lines. We won't draw the effects of the remaining generators, but they're occasionally helpful to keep in mind.

Remark 5.12. If you said, a single dot corresponds to a $\mathbb{F}_2$ summand, maybe an infinite tower corresponds to a $\mathbb{Z}$ somehow, then reading along the x-axis you get $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0 repeating. If this sounds familiar, this is not a coincidence. We'll get there in a sec.

Now, here are some examples of commonly arising $\mathcal{A}(1)$-modules and their Exts. These and others you can find fit into the "look it up" technique.

Example 5.13. Of course, $\mathrm{Ext}_{\mathcal{A}(1)}(\Sigma^t\mathcal{A}(1), \mathbb{F}_2) \cong \Sigma^t\mathbb{F}_2$: a single dot in the Ext chart.

Example 5.14. The $\mathcal{A}(1)$-module $\Sigma^{-2}\widetilde{H}^*(\mathbb{CP}^2)$, consisting of two points joined by a Sq^2, is often denoted $C\eta$...

Example 5.15. The joker J...

Example 5.16. The upside-down question mark (sometimes called the Spanish question mark) (TODO: symbol)...

Example 5.17. This thing called M_∞..., useful for studying $H^*(\mathbb{RP}^\infty)$ and its variants (stunted projective spaces).

Another way to compute Ext is using a minimal resolution. TODO: definition. Ideally add an example, but don't dwell on it during the lecture.

Another way to compute Ext is to use the fact that a short exact sequence of $\mathcal{A}(1)$-modules induces a long exact sequence in Ext. TODO: example computing Ext of upside-down question mark (or, since active learning, let's use the ordinary question mark?). This is the most flexible way to compute Ext in practice; specific other methods work better on specific modules, but the LES is quite general. (TODO: many more examples for the interested reader can be found...)

Note: I'll leave this in the notes but not discuss it in the lecture, for time reasons.

Example 5.18 (Use a preexisting minimal resolution)**.** Modules which look similar should have similar Exts. Here's a concrete implementation of this, for the augmentation ideal $I := \Sigma^{-1}\ker(\mathcal{A}(1) \to \mathbb{F}_2)$. This is the remainder of the first step of the minimal resolution $P^2 \to P^1 \to P^0 \to \mathbb{F}_1$, so it has a minimal resolution beginning $P^2 \to P^1 \to I$. This implies an $H^{*,*}(\mathcal{A}(1))$-equivariant identification

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(I, \mathbb{F}_2) \cong \mathrm{Ext}^{s+1,t+1}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2). \tag{5.19}$$

There'll be a few exercises using similar tricks, but this isn't as crucial of a method to know.

Again, this is appearing in the notes but will be skated over at best in the lecture.

Example 5.20. Using the change-of-rings formula to compute the Ext of $C\eta$ very quickly. Again, elegant but not as flexible.

5.3. **Running the spectral sequence: differentials.** In the Adams grading, a d_r differential moves one unit to the left and r units upward. Typically, Adams differentials are difficult, but we can take advantage of some structure.

(1) Differentials commute with the $H^{*,*}(\mathcal{A}(1))$-action. This allows you to compute one differential and then move it up an h_0-tower, for example. Another crucial use is to infer that if $h_i x = 0$ but $h_i y \neq 0$, then $d_r(x) \neq y$ for any r. For example, this method shows that all differentials in the Adams spectral sequence for *ko* (i.e., using $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2)$) vanish!
(2) There is a very useful theorem of Margolis which says that $\mathcal{A}$-module summands in the $\mathbb{F}_2$-cohomology of a spectrum X correspond precisely to summands of $H\mathbb{F}_2$ splitting off of X. The upshot is that in the E_2-page of the Adams spectral sequence for $ko \wedge Y$, anything coming from an $\mathcal{A}(1)$ summand in $H^*(Y;\mathbb{F}_2)$ neither admits nor receives nonzero differentials. These two methods are pretty powerful when used together.
(3) After this, one has to be crafty, and the standard tricks appear: use functoriality of the Adams spectral sequence to force differentials to vanish (or to not vanish); compute part of the answer using another spectral sequence, such as Atiyah-Hirzebruch, and use this to infer what differentials must vanish. There are ways to use Massey products and Toda brackets to infer differentials, but this is not something I understand; sorry about that.

5.4. **Extension questions.** So you've finished running your Adams spectral sequence calculations and arrived at the E_∞-page. How do you address extensions? Here are the ways.

(1) The $H^{*,*}(\mathcal{A}(1))$-action on the E_∞-page lifts to the ko_*-action on $ko_*(X)$! To be precise, h_0 lifts to multiplication by 2, h_1 lifts to the action by $\eta \in ko_1$, and w lifts to action by the periodicity element. This is the crucial first step to solving extension problems. For example, an infinite tower linked by h_0s lifts to the 2-adics (which came from $\mathbb{Z}$ if your spectrum, before 2-completion, was finite). If there are k dots linked by h_0s, you've found a $\mathbb{Z}/2^k$.

   The converse is not true: there can be "hidden extensions" not visible to the $H^{*,*}(\mathcal{A}(1))$-action.
(2) Margolis' theorem, the sequel: since your summands coming from $\mathcal{A}(1)$s split off of $ko \wedge X$, they cannot participate in hidden extensions.
(3) There are a few other tricks. One common one is that the relation $h_0 h_1 = 0$ lifts to $2\eta = 0$. That means that in $ko_*(X)$, if $x = 2y$, then $\eta x = 0$. Duh, you say, but this is useful for ruling out some hidden extensions. Mapping to/from other spectral sequences is also useful sometimes.

Remark 5.21 (Variants).