AN INTRODUCTION TO SPECTRAL SEQUENCES

ARUN DEBRAY AND RICHARD WONG

Contents

1. Monday, August 10: introduction and the Serre spectral sequence

2. Tuesday, August 11: group cohomology and the homotopy fixed-points spectral sequence

3. Wednesday, August 12: the Atiyah-Hirzebruch spectral sequence

TODO: this is taken directly from my notes when I gave this lecture three years ago, and will need to be modified. I'll indicate what needs to be changed where it needs to be changed.

Today, I'm going to talk about the Atiyah-Hirzebruch spectral sequence. This spectral sequence computes generalized (co)homology of a space or spectrum, with input data its ordinary homology.

Let D be a spectrum and X be a CW complex. The *homological Atiyah-Hirzebruch spectral sequence* is a spectral sequence with signature

$$E^2_{p,q} \coloneqq H_p(X; D_q(\mathrm{pt})) \Longrightarrow D_{p+q}(X). \tag{3.1}$$

The *cohomological Atiyah-Hirzebruch spectral sequence* is a spectral sequence with signature

$$E_2^{p,q} \coloneqq H^p(X; D^q(\mathrm{pt})) \Longrightarrow D^{p+q}(X). \tag{3.2}$$

(Briefly recall what this means. Briefly discuss convergence, and where it can go wrong.)

If D is a ring spectrum, (3.2) has a multiplicative structure.

TODO: this would go in day 1, if we want to include it at all. Feel free to port it over there; if not, comment it out.

Convergence. Sometimes you're reading a book and it feels like it goes on forever. It's nice when spectral sequences don't do that. As an example, we'll look at a *first-quadrant spectral sequence*, one where $E_2^{p,q} = 0$ when $p < 0$ or $q < 0$. In this setup, if you pick any (p, q), then after finitely many pages, the differentials are so long that they leave the first quadrant, so you get a sequence $0 \to E^r_{p,q} \to 0$, and therefore when you take homology, nothing changes. Thus it makes sense to say what the end of the spectral sequence is.

Definition 3.3. Whenever it makes sense, we'll define the E_∞*-page* of the spectral sequence to be $E_\infty^{p,q} = E^r_{p,q}$ for $r \gg 0$. One says $E_r^{p,q}$ *converges* or *abuts* to $E_\infty^{p,q}$.

Typically this is something interesting we want to calculate.

Definition 3.4. Let $A_\bullet$ be a graded abelian group together with an exhaustive filtration $\{F_p\}$.

- The *associated graded* of the filtration $\{F_i\}$ is

$$(grA)_{p,q} \coloneqq F_pA_{p+q}/F_{p-1}A_{p+q}.$$

- A spectral sequence $E_r^{p,q}$ *converges (weakly)* to $A_\bullet$, written
$$E_r^{p,q} \Longrightarrow A_\bullet,$$
if it has an E_∞ page and the E_∞ page is the associated graded of $A_\bullet$.

Remark 3.5. There is a notion of *conditional convergence*, due to Boardman, which essentially means "not always weakly convergent, but converges under hypotheses often met in practice." Unfortunately, defining this precisely would be a huge digression.

TODO: these are some examples of spectra. Probably I won't delve into this level of detail, just introduce KO, KU, ko, ku, and MSO and say what their coefficient rings are, as well a little bit about what they mean. Some of this may have already been covered in Tuesday's lecture.

Example 3.6 (K-theory)**.** Let X be a compact Hausdorff space. Then, the set of isomorphism classes of complex vector bundles on X is a semiring, so we can take its group completion and obtain a ring $K^0(X)$.

The following theorem is foundational and beautiful.

Theorem 3.7 (Bott periodicity)**.** $K^0(\Sigma^2 X) \cong K^0(X)$.

This allows us to promote K^* into a 2-*periodic* generalized cohomology theory K^*, called *complex K-theory*, by setting $K^{2n}(X) = K^0(X)$ and $K^{2n+1}(X) = K^0(\Sigma X)$.[1]

Like cohomology, K-theory is *multiplicative*, i.e. it spits out $\mathbb{Z}$-graded rings. However, $K^i(X)$ is often nonzero for negative i.

Exercise 3.8. For example, show that as graded abelian groups, $K^*(\mathrm{pt}) = \mathbb{Z}[t, t^{-1}]$, where $|t| = 2$.

K-theory admits a few variants.

- If you use real vector bundles instead of complex vector bundles, everything still works, but Bott periodicity is 8-fold periodic. Thus we obtain a periodic, multiplicative cohomology theory called *real K-theory*, denoted $KO^*(X)$. Its value on a point is encoded in the *Bott song*.
- Sometimes it will be simpler to consider a smaller variant where we only keep the negative-degree elements. This is called *connective K-theory*, and is denoted ku^* (for complex K-theory) or ko^* (for real K-theory). These are also multiplicative.

Example 3.9 (Bordism)**.** Let X be a space and define $\Omega_n^{\mathrm{O}}(X)$ to be the set of equivalence classes of maps of n-manifolds $M \to X$, where $[f_0 \colon M \to X] \sim [f_1 \colon N \to X]$ if there's a cobordism $Y \colon M \to N$ and a map $F \colon Y \to X$ extending f_0 and f_1. This is an abelian group under disjoint union, and the collection $\{\Omega_n^{\mathrm{O}}\}$ defines a generalized homology theory called *unoriented bordism*.[2]

The following theorem was the beginning of differential topology.

Theorem 3.10 (Thom)**.** *As graded abelian groups,* $\Omega_n^{\mathrm{O}}(\mathrm{pt}) \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, \dots] = \mathbb{F}_2[x_i \mid i \neq 2^j - 1]$. *Moreover,* Ω_*^{O} *is a direct sum of (suspended) ordinary cohomology theories.*

There's a lot of variations, based on whatever flavors of manifolds you consider. Using oriented manifolds produces *oriented bordism* Ω_*^{SO}, spin manifolds produce *spin bordism* Ω_*^{Spin}, and so forth. These are not direct sums of ordinary cohomology theories in general.

Example 3.11. We'll use the Atiyah-Hirzebruch spectral sequence to compute $K^*(\mathbb{CP}^n)$. Recall that
$$H^p(\mathbb{CP}^k; A) = \begin{cases} A, & p \text{ even} \\ 0, & \text{odd}. \end{cases}$$
Hence
$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & p, q \text{ even}, 0 \le p \le 2k \\ 0, & \text{otherwise}. \end{cases}$$

[1] Extending from compact Hausdorff spaces to all of Top is possible, but then one loses the vector-bundle-theoretic description.

[2] The corresponding cohomology theory is called *cobordism*.

Thus all the differentials are zero! So $E_2^{p,q} \cong E_\infty^{p,q}$. Hence the E_∞ page has no torsion, and therefore $K^*(\mathbb{CP}^n)$ is isomorphic to its associated graded.

$$K^i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{n+1}, & i \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

TODO: more examples.

TODO: define k-invariants, introduce Steenrod squares, and state what the first differential in the AHSS is. Use this in examples to compute ku^* and ko^* of simple things.

4. Thursday, August 13: comparison tools and other tricks

5. Friday, August 14: the Adams spectral sequence over $\mathcal{A}(1)$