VIRTUAL ELEMENT METHOD HOW TO

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Abstract. We give the basic ingredients for the implementation of the Virtual Element Method for self-adjoint elliptic problems. This notes are based on [2].

1. Model problem

As model problem let's take:

$$-\nabla\cdot(\boldsymbol{\kappa}\nabla u)+\gamma u = f(\boldsymbol{x}) \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega\subset\mathbb{R}^2$ is a polygonal domain. In weak form: find $u\in H^1_0(\Omega)$ such that

$$a(u,v) := (\boldsymbol{\kappa}\nabla u,\nabla v)+(\gamma u,v) = (f,v) \qquad \forall v\in H^1_0(\Omega), \tag{1.2}$$

with $(\cdot,\cdot)$ denoting the L^2 inner product on Ω.

2. VEM discretisation

Denote by $\{\mathcal{T}_h\}$ a finite decomposition (mesh) of the domain Ω into non-overlapping polygonal elements with maximum size h. The elements are referred to as $E\in\mathcal{T}_h$. We assume a constant polynomial degree p is used in the definition of the method.

We define the VEM trial space $V_h\subset H^1_0(\Omega)$ and discrete bilinear form A_h, and right-hand side linear functional $(f,\cdot)_h$ from an element-wise construction as follows.

The global VE space is continuous. Hence elemental boundary degrees of freedom (DoF) are always in common to the adjacent elements. I am *not* going to detail here the actual definition of the local space $V_h^E = V_h|_E$. From the practical point of view, all you need to know is that

- $\mathcal{P}_p(E)\subset V_h^E$, with $\mathcal{P}_p(E)$ the space of polynomials of **total degree** up to p;
- $V_h^E|_s = \mathcal{P}_p(s)$ for all edges $s\in\partial E$;

and it is identified by the following set of DoF.

Definition 1. *The degrees of freedom for the local VE space V_h^E are*

- *the value of v_h at each vertex of E;*
- *for $p>1$, the moments of v_h of up to order $p-2$ on each mesh interface (edge) $s\subset\partial E$*
$$\frac{1}{|s|}\int_s v_h m_\alpha \,\mathrm{d}s \quad \forall m_\alpha\in\mathcal{M}_{p-2}(s);$$
- *for $p>1$, the moments of v_h of up to order $p-2$ inside the element E*
$$\frac{1}{|E|}\int_E v_h m_\alpha \,\mathrm{d}\boldsymbol{x} \quad \forall m_\alpha\in\mathcal{M}_{p-2}(E).$$

We denote by N_E the total number of DoF.

Here $\mathcal{M}_\ell(E)$ denotes a scaled **basis** for $\mathcal{P}_\ell(E)$, $\ell\le p$. This could be the following. Denote by $\mathcal{M}^*_\ell(E)$, $\ell\in\mathbb{N}$, the set of *scaled monomials*

$$\mathcal{M}^*_\ell(E) := \left\{\left(\frac{\boldsymbol{x}-\boldsymbol{x}_E}{h_E}\right)^{\boldsymbol{s}}, |\boldsymbol{s}|=\ell\right\},$$

where $\boldsymbol{s}$ is a multi-index with $|\boldsymbol{s}| := s_1+s_2$ and $\boldsymbol{x}^{\boldsymbol{s}} := x_1^{s_1}x_2^{s_2}$. Further, we define $\mathcal{M}_p(E) := \bigcup_{l\le p}\mathcal{M}^*_l(E) =: \{m_\alpha\}_{\alpha=1}^{N_p}$, a basis of $\mathcal{P}_p(E)$, where $N_p := \dim(\mathcal{P}_p(\mathbb{R}^2))$.

Remark 1. *We can also take* any *basis for the space of polynomials we like. The important thing is that* (2.2) *below needs to be correctly scaled. With the scaled monomials above, the corresponding DoF scale like one. For simplicity, I am also assuming below that the basis is hierarchical with respect to p.* **The truncated (!) and correctly scaled Legendre basis defined on the bounding box would do.**

Remark 2. *Note that the values at $p-1$ distinct point (for instance, the Gauss-Lobatto quadrature points) could be used on the edges. The reason why we like to use moments instead is to imply a recursive definition across space dimensions useful in 3D. Indeed the 3D virtual space is obtained starting from the 2D virtual space on each face.*

The bilinear form A_h is written as the sum of elemental contributions:

$$A_h(u_h, v_h) = \sum_{E\in\mathcal{T}_h} a_h^E(u_h, v_h),$$

with a_h^E so defined:

$$a_h^E(u_h, v_h) := (\boldsymbol{\kappa}\Pi^0_{p-1}\nabla u_h, \Pi^0_{p-1}\nabla v_h)_E + (\gamma\Pi^0_p u_h, \Pi^0_p v_h) + S^E((\mathrm{I}-\Pi^0_p)u_h, (\mathrm{I}-\Pi^0_p)v_h), \qquad (2.1)$$

where the operator $\Pi^0_\ell : L^2(E) \to \mathcal{P}_\ell(E)$ for $\ell \le p$ denotes the $L^2(E)$-orthogonal projection onto the polynomial space $\mathcal{P}_\ell(E)$. For vector functions, e.g. ∇v_h, the projection is taken componentwise. Here, the *VEM stabilisation* term S^E is given by:

$$S^E(v_h, w_h) := (\overline{\boldsymbol{\kappa}}_E h_E^{d-2} + \overline{\gamma}_E h_E^d) \sum_{r=1}^{N_E} \mathrm{dof}_r(v_h)\,\mathrm{dof}_r(w_h) \qquad (2.2)$$

for $v_h, w_h \in V_h^E/\mathcal{P}_p(E)$, with $\overline{\boldsymbol{\kappa}}_E$ and $\overline{\gamma}_E$ any constant approximations of $\boldsymbol{\kappa}$ and μ over E, respectively. These could be, for instance, the value at the barycentre or the mean of the vertex values.

Similarly, the right-hand side is defined for all $v_h \in V_h$ by

$$(f, v_h)_h := \sum_{E\in\mathcal{T}_h} (f, \Pi^0_p v_h)_E.$$

Now the VEM reads: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h)_h \quad \forall v_h \in V_h. \qquad (2.3)$$

3. Implementation

From the above we see that he following ingredients are needed:

- The basis for $\mathcal{P}_p(E)$ on each element $E \in \mathcal{T}_h$.
- Both a local and global way to count vertices, edges, and internal entities in view of distributing the DoF.
- A way to compute

  $$\Pi^0_{p-1}\nabla v_h \quad \text{and} \quad \Pi^0_p v_h, \qquad (3.1)$$

  just by accessing the DoF of v_h and *the monomial basis.*

Assuming the above are achievable, the set up of the linear system is *identical* to that of standard FEM for the first two terms in (2.1) and for the right-hand side as all these terms are in the form of *integrals of coefficients times polynomials.* The stabilising term (2.2) is different but straightforward to compute (see below).

It remains to write the explicit (magic) formulas for the **local** polynomial projections. We shall use the following notation:

- Greek indices α, β are used to count within the polynomial basis;
- Latin indices i, j are used to count the DoF.

For the projection problems (3.1) we need (on each element, obviously) the $N_E \times N_p$ matrix D and $N_\ell \times N_\ell$, $\ell = p-1, p$, matrix H_ℓ whose entries are given by:

$$(\mathsf{D})_{i\alpha} = \mathrm{dof}_i(m_\alpha) \qquad i = 1, \dots, N_E, \quad \alpha = 1, \dots, N_p,$$
$$(\mathsf{H}_\ell)_{\alpha\beta} = (m_\alpha, m_\beta)_E \qquad \alpha, \beta = 1, \dots, N_\ell.$$

Remark 3. *In the special case $p = 1$ only $\mathsf{H}_{p-1} = \mathsf{H}_0$ is needed. Note that this is just the scalar $\mathsf{H}_0 = |E|$, the area of E.*

Let $\{\phi_i\}_{i=1}^{N_E}$ be the Lagrangian basis of V_h^E with respect to the DoF of Definition 1. The L^2 projection of each ϕ_i is found as the polynomial $\Pi_p^0 \phi_i \in \mathcal{P}_p(E)$ solution of

$$\left(m_\alpha, \Pi_p^0 \phi_i\right)_E = (m_\alpha, \phi_i)_E \qquad \forall \alpha = 1, \dots, N_k, \tag{3.2}$$

and the projection of the gradient $\Pi_{p-1}^0 \nabla \phi_i$ is defined component-wise for $l = 1, 2$ by

$$\begin{aligned}(\Pi_{p-1}^0 \frac{\partial \phi_i}{\partial x_l}, m_\alpha)_E &= (\frac{\partial \phi_i}{\partial x_l}, m_\alpha)_E \qquad \forall m_\alpha \in \mathcal{M}_{p-1}(E) \\ &= \int_{\partial E} m_\alpha n_l \phi_i \, \mathrm{d}s - (\phi_i, \frac{\partial m_\alpha}{\partial x_l})_E,\end{aligned} \tag{3.3}$$

where n_l is the l-th component of the outwards normal $\boldsymbol{n}$.

Below we shall represent the above projections as follows:

- the $N_p \times N_E$ matrix $\mathbf{\Pi}_p$ represents the L^2 projection operator with respect to the Lagrangian basis $\{\phi_i\}$ and polynomial basis $\{m_\alpha\}$; similarly,
- the $N_{p-1} \times N_E$ matrix $\mathbf{\Pi}_{p-1}^{x_l}$ represents the partial derivative projection operator.

3.1. **The case $p = 1$.** In this case (see the general case for a few details) it happens that the L^2 projection matrix is simply given by the pseudo (or left) inverse of D:

$$\mathbf{\Pi}_1 = (\mathsf{D}^T \mathsf{D})^{-1} \mathsf{D}^T.$$

The DoF of the projection can be obtained by changing basis from the polynomials to the VEM basis, so this is given by

$$\mathbf{\Pi}_1^\phi := \mathsf{D} \mathbf{\Pi}_1 = \mathsf{D}(\mathsf{D}^T \mathsf{D})^{-1} \mathsf{D}^T.$$

As for the H^1 projection, with $p = 1$ we have just the constant function in $\mathcal{M}_0(E)$ given by[1] m_1, so the second term in the right-hand side of (3.3) is zero and

$$(\Pi_0^0 \frac{\partial \phi_i}{\partial x_l}, m_1)_E = \int_{\partial E} m_1 n_l \phi_i \, \mathrm{d}s = m_1(\boldsymbol{x}_E) \frac{\left|s_i^-\right| (\boldsymbol{n}(s_i^-))_l + \left|s_i^+\right| (\boldsymbol{n}(s_i^+))_l}{2} =: (R^{x_l})_i,$$

with $s_i^\pm$ denoting the two edges on which $\phi \neq 0$ and $\boldsymbol{n}(s_i^\pm)$ the normal to $s_i^\pm$. The so computed R^{x_l} is a $1 \times N_E$ row vector. With this, the partial derivative projection operator is represented by the $1 \times N_E$ row vector

$$\mathbf{\Pi}_0^{x_l} = R^{x_l} / \mathsf{H}_0.$$

We are now ready to write all local contributions to the virtual element problem. Let K be the 2×2 matrix with entries

$$(K)_{ln} = (\boldsymbol{\kappa}_{ln} m_1, m_1)_E \qquad l, n = 1, 2.$$

[1] This is equal to 1 according to Definition 1 but I won't use this fact.

Then, the ij-th entry of the local stiffness matrix is given by

$$\begin{aligned}(\boldsymbol{\kappa}\Pi_0^0\nabla\phi_i, \Pi_0^0\nabla\phi_j)_E &= \sum_{l,n=1}^{2} \Big(\boldsymbol{\kappa}_{ln}\Pi_0^0\frac{\partial\phi_i}{\partial x_l}, \Pi_0^0\frac{\partial\phi_j}{\partial x_n}\Big)_E \\ &= \sum_{l,n=1}^{2} \big(\boldsymbol{\Pi}_0^{x_l}\big)_i(\boldsymbol{\kappa}_{ln}\, m_\alpha,\, m_\beta)_E\big(\boldsymbol{\Pi}_0^{x_n}\big)_j \\ &= \big[\boldsymbol{\Pi}_0^{x_1} \quad \boldsymbol{\Pi}_0^{x_2}\big]_i K \begin{bmatrix} \boldsymbol{\Pi}_0^{x_1} \\ \boldsymbol{\Pi}_0^{x_2} \end{bmatrix}_j.\end{aligned}$$

Collecting together the vectors $\boldsymbol{\Pi}_0^{x_l}$ into the $2 \times N_E$ gradient projection matrix $\widetilde{\boldsymbol{\Pi}}_0$ we find that the first term in (2.1) can be assembled as

$$\widetilde{\boldsymbol{\Pi}}_0^T \mathsf{K} \widetilde{\boldsymbol{\Pi}}_0.$$

For the second term in (2.1), defining the $N_1 \times N_1$, that is 3×3, matrix M as

$$(M)_{\alpha\beta} = (\gamma\, m_\alpha, m_\beta)_E \qquad \alpha, \beta = 1, \dots, N_1,$$

we easily find that:

$$(\mu\Pi_1^0\phi_i, \Pi_1^0\phi_j)_E = \sum_{\alpha,\beta=1}^{N_{d,k}} \big(\boldsymbol{\Pi}_1\big)_{\alpha i}(\gamma\, m_\alpha, m_\beta)_E\big(\boldsymbol{\Pi}_1\big)_{\beta j}.$$

Hence, the reaction term can be assembled as

$$\boldsymbol{\Pi}_1{}^T \mathsf{M} \boldsymbol{\Pi}_1.$$

Finally, according to 2.2, the stabilising term is given by

$$S^E((\mathrm{I} - \Pi_1^0)\phi_i, (\mathrm{I} - \Pi_1^0)\phi_j) = \big(\overline{\boldsymbol{\kappa}}_E h_E^{d-2} + \overline{\gamma}_E h_E^d\big)\Big(\big(\mathsf{I} - \boldsymbol{\Pi}_1^\phi\big)^T\big(\mathsf{I} - \boldsymbol{\Pi}_1^\phi\big)\Big)_{ij}$$

since $\mathrm{dof}_r((\mathrm{I} - \Pi_1^0)\phi_i) = \big(\mathsf{I} - \boldsymbol{\Pi}_1^\phi\big)_{ir}$.

Similarly, the right-hand side vector is given by:

$$(f_h, \Pi_p^0\phi_j)_E = \sum_{\alpha=1}^{N_k} (f, m_\alpha)_E\big(\boldsymbol{\Pi}_1^\phi\big)_{\alpha i}.$$

This is all we need to implement the method in the case $p = 1$.

Remark 4. *Note that in this special case* **no quadrature is needed by the VEM!** *Indeed all we need is: 1) the evaluation of the polynomial basis on the vertices to compute matrix* D*; 2) the evaluation of terms* K, M*, and* $(f, m_\alpha)_E$*. These can be evaluated by replacing the problem's data with some constant approximation and then by using again the values of the polynomial basis on the vertices to compute the remaining integrals exactly.*

3.2. **The general case.** Recall that $\{\phi_i\}_{i=1}^{N_E}$ is the Lagrangian basis of V_h^E with respect to the DoF. To ease writing only, I will assume that **in the local counting of the DoF, the internal DoF are counted first**. In other words, for the first $\alpha = 1, \dots, N_{p-2}$ functions ϕ_α we have $(\phi_\alpha, m_\alpha)_E = |E|$.

With this assumption, from the above matrices we form matrix C as[2]

$$\mathsf{C}_{\alpha i} = \begin{cases} |E|\delta_{\alpha i} & \text{if } \alpha \le N_{p-2}, \\ \big(\mathsf{H}_p(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T\big)_{\alpha i} & \text{if } N_{p-2} < \alpha \le N_p, \end{cases}$$

for $\alpha = 1, \dots, N_p$ and $i = 1, \dots, N_E$.

[2]Here $\delta_{\alpha i}$ is the Kronecker delta, so $\delta_{\alpha i} = 1$ if $\alpha = i$ and $\delta_{\alpha i} = 0$ otherwise.

Matrix C corresponds to[3] the right-hand side of the L^2 projection problem (3.2). The L^2 projection problem then reads $\mathsf{H}_p\mathbf{\Pi}_p = \mathsf{C}$, hence

$$\mathbf{\Pi}_p = \mathsf{H}_p^{-1}\mathsf{C},$$

and, as before, we can write this in terms of the virtual DoF as

$$\mathbf{\Pi}_p^{\phi} = \mathsf{D}\mathbf{\Pi}_p = \mathsf{D}\mathsf{H}_p^{-1}\mathsf{C}.$$

As for the partial derivatives projection problem, this time the second term in the right-hand side of (3.3) is not zero. Hence we re-define the $N_{p-1} \times N_E$ matrix R^{x_l} as

$$(R^{x_l})_{\alpha i} = \sum_{s\subset\partial E} n_l \int_s m_\alpha \phi_i \,\mathrm{d}s - (\phi_i, \frac{\partial m_\alpha}{\partial x_l})_E \qquad \alpha = 1,\dots,N_{p-1}, \quad i = 1,\dots,N_E.$$

We consider the problem of computing R^{x_l} starting from the internal term. As $m_\alpha \in \mathcal{P}_{p-1}(E)$ we have

$$\frac{\partial m_\alpha}{\partial x_l} = \sum_{\beta=1}^{N_{p-2}} (d_l)_\alpha^\beta m_\beta,$$

for some coefficients $(d_l)_\alpha^\beta$ which **need to be computed!** Then[4],

$$(\phi_i, \frac{\partial m_\alpha}{\partial x_l})_E = \sum_{\beta=1}^{N_{p-2}} (d_l)_\alpha^\beta (\phi_i, m_\beta)_E = \begin{cases} (d_l)_\alpha^i |E| & \text{if } i \le N_{p-2}, \\ 0 & \text{otherwise.} \end{cases}$$

The boundary term just involves integrals of polynomials (of degree up to $2p-1$) as $\phi_i|_s \in \mathcal{P}_p(s)$ and hence it can be computed exactly.

Collecting together all Lagrangian basis functions in $\boldsymbol{\phi} = (\phi_1,\dots,\phi_{N_E})$ we can write the partial derivative projection problem collectively as: find $\mathbf{\Pi}_{p-1}^{x_l}$ such that

$$\Pi_{p-1}^0 \frac{\partial \boldsymbol{\phi}}{\partial x_l} = \sum_{\alpha=1}^{N_{k-1}} m_\alpha \big(\mathbf{\Pi}_{p-1}^{x_l}\big)_{\alpha,\cdot},$$

and, similarly to the $p=1$ case, the solution is given by

$$\mathbf{\Pi}_{p-1}^{x_l} = \mathsf{H}_{p-1}^{-1} R^{x_l} \qquad l = 1,2.$$

Now, for $l,n = 1,2$, let K_{ln} be the $N_{p-1} \times N_{p-1}$ matrix with entries

$$(K_{ln})_{\alpha\beta} = (\boldsymbol{\kappa}_{ln} m_\alpha, m_\beta)_E \qquad \alpha,\beta = 1,\dots,N_{p-1}.$$

Then, we may compute the local stiffness matrix as

$$\begin{aligned}
\Big(\boldsymbol{\kappa}\Pi_{p-1}^0\nabla\phi_i, \Pi_{p-1}^0\nabla\phi_j\Big)_E &= \sum_{l,n=1}^{2} \Big(\boldsymbol{\kappa}_{ln}\Pi_{p-1}^0 \frac{\partial\phi_i}{\partial x_l}, \Pi_{p-1}^0 \frac{\partial\phi_j}{\partial x_n}\Big)_E \\
&= \sum_{l,n=1}^{2} \sum_{\alpha,\beta=1}^{N_{k-1}} \big(\mathbf{\Pi}_{p-1}^{x_l}\big)_{\alpha i} (\boldsymbol{\kappa}_{|ln}\, m_\alpha,\, m_\beta)_E \big(\mathbf{\Pi}_{p-1}^{x_n}\big)_{\beta j} \\
&= \sum_{l,n=1}^{2} \big(\mathbf{\Pi}_{p-1}^{x_l}\big)_{\cdot,i}^T K_{ln} \big(\mathbf{\Pi}_{p-1}^{x_n}\big)_{\cdot,j}.
\end{aligned}$$

All remaining terms are dealt with exactly as in the $p=1$ case.

[3]The issue here is the computation of the right-hand side as it depends on the virtual function ϕ. For $\alpha = 1,\dots,N_{k-2}$ the right-hand side is nothing else than an internal moment (apart from the scaling by $|E|$), hence it results to either $|E|$ or 0. The computation of the higher-order moments is obtained through the so-called virtual enhancement, cf [1, 2]. The result is what you see in the second part of the definition of C.

[4]The notation used here assumes that the polynomial basis are constructed hyerarcically in the degree. This is not strictly necessary, but it certainly simplifies the exposition.

References

[1] Ahmad, B., Alsaedi, A., Brezzi, F., Marini, L. D., and Russo, A. Equivalent projectors for virtual element methods. *Computers & Mathematics with Applications 66*, 3 (Sept. 2013), 376–391.

[2] Cangiani, A., Manzini, G., and Sutton, O. J. Conforming and nonconforming virtual element methods for elliptic problems. arXiv:1507.03543, 2015.