PROBLEM STATEMENT

Spin sectors

Quantum particles have a property called *spin*, which is an intrinsic angular momentum. The spin of a particle is restricted to be a multiple of $\hbar/2$. In units where $\hbar = 1$, the spin is either an integer or an odd multiple of one half. Electrons have spin $s = 1/2$.

The total spin S of a collection of electrons is determined by the angular momentum summation rule $\frac{1}{2} \otimes S = (S - \frac{1}{2}) \oplus (S + \frac{1}{2})$. The possible spin sectors for two, three, and four electrons are shown below.

$$\begin{aligned}
\tfrac{1}{2} \otimes \tfrac{1}{2} &= 0 \oplus 1 \\
\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} &= \tfrac{1}{2} \oplus \tfrac{1}{2} \oplus \tfrac{3}{2} \\
\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} &= 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \\
&\vdots \\
\underbrace{\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \cdots \otimes \tfrac{1}{2}}_{2N \text{ times}} &= \prod_{2S=0}^{N} \underbrace{S \oplus S \oplus \cdots \oplus S}_{C_S^{(N)} \text{ times}}
\end{aligned}$$

The final line shows the general result for N electrons. The coefficient $C_S^{(N)}$ denotes the number of states in a given spin sector. One can show that the total number of states in the singlet $(S = 0)$ sector is

$$C_0^{(N)} = \frac{1}{N/2+1}\binom{N}{N/2} = \frac{N!}{(N/2)!(N/2+1)!}$$

and that the total number of states—counting the $(2S + 1)$-fold degeracy—is

$$\sum_{2S=0}^{N} (2S+1) C_S^{(N)} = 2^N.$$

The 2^N result is just the number of ways to orient N electrons either spin-up or spin-down.

Read over the file `moments.cpp`. It provides the skeleton of a program that computes the coefficients $C_S^{(N)}$ and displays them in a table. Determine how elements of the array `current` (indexed by the integer 2S) should be incremented in terms of the values in `last`. Accumulate the total number of states into the variable `num_states`. The output of your program should be identical to the following.

```
$ make moments
g++ -o moments moments.cpp -O2 -ansi -pedantic -Wall -lm
$ ./moments
         0    0.5      1    1.5      2    2.5      3    num
    +-------------------------------------------------+-------
   1|           1                                     |      2
   2|     1             1                             |      4
   3|           2             1                       |      8
   4|     2             3             1               |     16
   5|           5             4             1         |     32
   6|     5             9             5             1 |     64
   7|          14            14             6         |    128
   8|    14            28            20             7 |    256
   9|          42            48            27         |    512
  10|    42            90            75            35 |   1024
  11|         132           165           110         |   2048
  12|   132           297           275           154 |   4096
  13|         429           572           429         |   8192
  14|   429          1001          1001           637 |  16384
  15|        1430          2002          1638         |  32768
  16|  1430          3432          3640          2548 |  65536
    +-------------------------------------------------+-------
```

Write a function `verify_singlet` that computes $C_0^{(N)}$ *explicitly*. Check its result against `last[0]` for each even value of `n` from `2` to `Nmax`.