Physics 540: Class 17

Wednesday, June 23, 2021

The shape of a string suspended from two fixed points is called a *catenary* and has the well-known form

$$y(x) = a[\cosh((x - x_0)/a) - 1] + y_0.$$

The parameters x_0 and y_0 are translations, and $a = \rho g$ is a product of the line tension of the string and the local gravitational acceleration. We're going to use measurements to fit these three unknowns (of which only a is physically meaningful).

To start, compile and run the `catenary` program:

```
$ make catenary
g++ -c catenary.cpp -O2 -ansi -pedantic -Wall
g++ -c catenary_openGL.cpp -O2 -ansi -pedantic -Wall
g++ -o catenary catenary.o catenary_openGL.o  -framework OpenGL -framework GLUT -lobjc
$ ./catenary
```

Maximize the window so that it covers most of your screen. Then, hang a piece of string from the two sides of your monitor. Using your mouse, carefully trace the path of the string using 20 to 50 clicks. Press 'n' to start a new set of measurements. Repeat this at least ten times. Press 'q' to quit the program. You should find that your measurements are stored in a file named `string.dat`:

```
$ more string.dat
$ gnuplot
> plot "string.dat"
> plot "string.dat" index 0
> plot "string.dat" index 1
> plot "string.dat" index 2
> quit
```

The non-negative quantity

$$\chi^2 = \sum_{i=1}^{N} (y - y_i)^2 = \sum_{i=1}^{N} (\Delta y_i)^2 = \sum_{i=1}^{N} \left[a \cosh\left(\frac{x_i - x_0}{a} \right) - a + y_0 - y_i \right]^2$$

measures the deviation of the measured data from the predicted curve for a particular set of parameters (x_0, y_0, a). The best possible fit corresponds to the smallest value of χ^2. The derivatives of χ^2 along each of the parameter directions are summarized below.

$$\frac{\partial \chi^2}{\partial x_0} = -\sum_{i=1}^{N} \Delta y_i \sinh\left(\frac{x_i - x_0}{a} \right)$$

$$\frac{\partial \chi^2}{\partial y_0} = \sum_{i=1}^{N} \Delta y_i$$

$$\frac{\partial \chi^2}{\partial a} = \sum_{i=1}^{N} \Delta y_i \left[\cosh\left(\frac{x_i - x_0}{a} \right) - 1 - \left(\frac{x_i - x_0}{a} \right) \sinh\left(\frac{x_i - x_0}{a} \right) \right]$$

We can think of the three parameters as constituting a vector $\mathbf{v} = (v_1, v_2, v_3) \equiv (x_0, y_0, a)$ whose local downhill direction is given by the gradient $-\partial \chi^2 / \partial v_i$.

The program `string_fit` is designed to read in your measurements from `string.dat` and fit a catenary curve to it. A bootstrapping technique is used in which the fit is performed repeatedly on a random sample of the data. The mean and variance of the parameters then provide a best fit value and a statistical uncertainty.

In the file `string_fit.cpp`, write routines `steepest_descent` and `statistics` that (i) compute the best-fit value of $\mathbf{v}$ using steepest descent and (ii) compute the mean and variance of a list of bootstrap samples. A completed program will give you results something like the following and a nice graph.

```
$ ./string_fit
Processing...
iteration 1
iteration 2
iteration 3
.
.
iteration 95
iteration 96
iteration 97
iteration 98
iteration 99
iteration 100
  x0 = -0.01583404978 +/- 0.00045
  y0 = -0.3950524019 +/- 0.0032
   a = 1.044980718 +/- 0.003
$ gnuplot -persist view.gp
```