Notes on Mathematical Analysis

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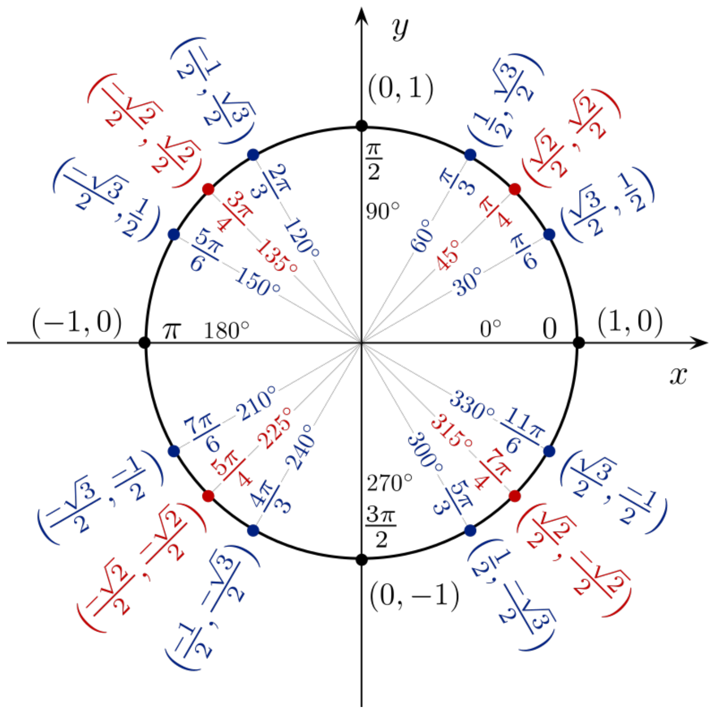
Trigonometric Identities

The Unit Circle



Periodic formulas

If n is an integer.

1. $\sin(\theta + 2\pi n) = \sin\theta$
2. $\cos(\theta + 2\pi n) = \cos\theta$
3. $\csc(\theta + 2\pi n) = \csc\theta$
4. $\sec(\theta + 2\pi n) = \sec\theta$
5. $\tan(\theta + \pi n) = \tan\theta$
6. $\cot(\theta + \pi n) = \cot\theta$

Co-function formulas

1. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
2. $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$
3. $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

Trigonometric functions: domain, range, and periodicity

Function	Period	Domain	Range
$\sin\theta$	2π	$(-\infty,\infty)$	$[-1,1]$
$\cos\theta$	2π	$(-\infty,\infty)$	$[-1,1]$
$sec\ \theta$	2π	$\bigcup_{k\in Z}\left(\frac{(2k+1)\pi}{2},\frac{(2k+3)\pi}{2}\right)$	$(-\infty,-1]\cup$ $[1,\infty)$
$csc\ \theta$	2π	$\bigcup_{k\in Z}\left(k\pi,(k+1)\pi\right)$	$(-\infty,-1]\cup$ $[1,\infty)$
$tan\ \theta$	π	$\bigcup_{k\in Z}\left(\frac{(2k+1)\pi}{2},\frac{(2k+3)\pi}{2}\right)$	$(-\infty,\infty)$
$\cot\theta$	π	$\bigcup_{k\in Z}\left(k\pi,(k+1)\pi\right)$	$(-\infty,\infty)$

Definitions

- $\sin(\theta)=\frac{op}{hyp}$
- $\cos(\theta)=\frac{ad}{hyp}$
- $\tan(\theta)=\frac{op}{ad}$
- Fundamental identities
  - Reciprocal identities
    - $\sin\theta=\frac{1}{\csc\theta}$
    - $\cos\theta=\frac{1}{\sec\theta}$
    - $\tan\theta=\frac{1}{\cot\theta}$
  - Pythagorean identities
    - $\sin^2\theta+\cos^2\theta=1$
    - $1+\tan^2\theta=\sec^2\theta$
    - $1+\cot^2\theta=\csc^2\theta$
  - Ratio identities
    - $\tan\theta=\frac{\sec\theta}{\csc\theta}$
    - $\tan\theta=\frac{\sin\theta}{\cos\theta}$
    - $\cot\theta=\frac{\cos\theta}{\sin\theta}$

Law of sines

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Law of cosines

$$c^2 = a^2 + b^2 - 2abcosC$$

Law of tangents

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}\left(A-B\right)\right]}{\tan\left[\frac{1}{2}\left(A+B\right)\right]}$$

Taylor series

$$\begin{array}{cccc}
\sin x & = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} & = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots & \text{for all } x \\
\cos x & = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} & = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots & \text{for all } x \\
\tan x & = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} & = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots & \text{for } |x| < \frac{\pi}{2} \\
\sec x & = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} & = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots & \text{for } |x| < \frac{\pi}{2} \\
\arcsin x & = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} & = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots & \text{for } |x| \le 1 \\
\arccos x & = \frac{\pi}{2} - \arcsin x & & \\
 & = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} & = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots & \text{for } |x| \le 1 \\
\arctan x & = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} & = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots & \text{for } |x| \le 1,\ x \ne \pm i
\end{array}$$

Sum and difference identities

Circular functions

- $\sin\left(\alpha \pm \beta\right) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$
- $\cos\left(\alpha \pm \beta\right) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$
- $\tan\left(\alpha \pm \beta\right) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$

Inverse functions

- $\arcsin x \pm \arcsin y = \arcsin\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$
- $\arccos x \pm \arccos y = \arccos\left(xy \mp \sqrt{\left(1-x^2\right)\left(1-y^2\right)}\right)$
- $\arctan x \pm \arctan y = \arctan\left(\frac{x \pm y}{1 \mp xy}\right)$

Multiple-angle formulas

Half-angle formula

- $$\sin\frac{\theta}{2} = sgn\left(2\pi - \theta + 4\pi\lfloor\frac{\theta}{4\pi}\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$$
  where $sgnx = \pm 1$ according to whether x is positive or negative.
- $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$
- $\cos\frac{\theta}{2} = sgn\left(\pi + \theta + 4\pi\lfloor\frac{\pi-\theta}{4\pi}\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$
- $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$
- $$\begin{aligned}\tan\frac{\theta}{2} &= \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{1-\cos\theta}{\sin\theta} = \frac{-1\pm\sqrt{1+\tan^2\theta}}{\tan\theta} = \frac{\tan\theta}{1+\sec\theta}\end{aligned}$$
- $\cot\frac{\theta}{2} = \csc\theta + \cot\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$

Double-angle formula

- $\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$
- $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$
- $\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$
- $\sec(2\theta) = \frac{\sec^2\theta}{2-\sec^2\theta}$
- $\csc(2\theta) = \frac{\sec\theta\csc\theta}{2}$

Triple-angle formula

- $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta = 4\sin\theta\sin\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)$
- $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta = 4\cos\theta\cos\left(\frac{\pi}{3} - \theta\right)\cos\left(\frac{\pi}{3} + \theta\right)$
- $\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \tan\theta\tan\left(\frac{\pi}{3} - \theta\right)\tan\left(\frac{\pi}{3} + \theta\right)$
- $\cot(3\theta) = \frac{3\cot\theta - \cot^3\theta}{1-3\cot^2\theta}$
- $\sec(3\theta) = \frac{\sec^3\theta}{4-3\sec^2\theta}$
- $\csc(3\theta) = \frac{\csc^3\theta}{3\csc^2\theta - 4}$

Power-reduction formula

If n is odd

$\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos((n-2k)\theta)$

$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\left(\frac{n-1}{2}-k\right)}\binom{n}{k}\sin((n-2k)\theta)$

If n is even

$\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos\left((n-2k)\,\theta\right)$

$\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\left(\frac{n}{2}-k\right)}\binom{n}{k}\cos\left((n-2k)\,\theta\right)$

Product-to-sum and sum-to-product identities

Product-to-sum

$$\begin{aligned}\prod_{k=1}^{n}\cos\theta_k &= \frac{1}{2^n}\sum_{e\in S}\cos\left(e_1\theta_1+\cdots+e_n\theta_n\right)\\ &\text{where } S=\{1,-1\}^n\end{aligned}$$

- $2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$
- $2\sin\theta\sin\varphi = \cos(\theta-\varphi)-\cos(\theta+\varphi)$
- $2\sin\theta\cos\varphi = \sin(\theta+\varphi)+\sin(\theta-\varphi)$
- $2\cos\theta\sin\varphi = \sin(\theta+\varphi)-\sin(\theta-\varphi)$
- $\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{\cos(\theta-\varphi)+\cos(\theta+\varphi)}$

Sum-to-product

- $\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$
- $\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$
- $\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$

Hyperbolic functions

Exponential definition

- ◦ ∗ † $\sinh x = \frac{e^x-e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} = \frac{1-e^{-2x}}{2e^{-x}}$
  - † $\cosh x = \frac{e^x+e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} = \frac{1+e^{-2x}}{2e^{-x}}$
  - † $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$
  - † $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$
  - † $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}} = \frac{2e^x}{e^{2x}+1}$
  - † $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x-e^{-x}} = \frac{2e^x}{e^{2x}-1}$

Trigonometric definition

- ◦ ∗ † $\sinh x = -i \sin(ix)$
  † $\cosh x = \cos(ix)$
  † $\tanh x = -i \tan(ix)$
  † $\coth x = i \cot(ix)$
  † $\operatorname{sech} x = \sec(ix)$
  † $\operatorname{csch} x = i \csc(ix)$

Taylor series

$$
\begin{array}{cccc}
\sinh x & = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} & = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots & \text{for all } x \\
\cosh x & = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} & = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots & \text{for all } x \\
\tanh x & = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n (4^n - 1)}{(2n)!} x^{2n-1} & = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots & \text{for } |x| < \frac{\pi}{2} \\
arsinhx & = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} & & \text{for } |x| \le 1 \\
artanhx & = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} & & \text{for } |x| \le 1, \ x \neq \pm 1
\end{array}
$$

Useful relations

$$
\begin{aligned}
\sinh(-x) &= -\sinh x \\
\cosh(-x) &= \cosh x
\end{aligned}
$$

$$
\begin{aligned}
\tanh(-x) &= -\tanh x \\
\coth(-x) &= -\coth x \\
\operatorname{sech}(-x) &= \operatorname{sech} x \\
\operatorname{csch}(-x) &= -\operatorname{csch} x
\end{aligned}
$$

$$
\begin{aligned}
arsechx &= arcosh\left(\frac{1}{x}\right) \\
arcschx &= arsinh\left(\frac{1}{x}\right) \\
arcothx &= artanh\left(\frac{1}{x}\right)
\end{aligned}
$$

Logarithmic Properties

Trivial identities

- $\log_b(1) = 0 \because \ b^0 = 1$
- $\log_b(b) = 1^1 = b$

Cancelling exponentials

- $b^c = x \Leftrightarrow \log_b(x) = c$
  - $b^{\log_b(x)} = x_b(\log_b(x)) = x$
  - $\log_b(b^x) = x \because \log_b(antilog_b(x)) = x$

Using Simpler operations

1. $\log_b(xy) = \log_b(x) + \log_b(y)^c\, b^d = b^{c+d}$
2. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \because \frac{b^c}{b^d} = b^{c-d}$
3. $\log_b\left(x^d\right) = d\log_b(x) \because (b^c)^d = b^{c\times d}$
4. $\log_b(\sqrt[y]{x}) = \frac{\log_b(x)}{y} \because \sqrt[y]{x} = x^{\frac{1}{y}}$
5. $x^{\log_b(y)} = y^{\log_b(x)\log_b(y)} = b^{\log_b(x)\log_b(y)} = \left(b^{\log_b(y)}\right) = y^{\log_b(x)}$
6. $c\log_b(x) + d\log_b(y) = \log_b\left(x^c \times y^d\right)$

Changing the base

- $\log_b a = \frac{\log_{10} a}{\log_{10} b}$
  - $\log_b a = \frac{1}{\log_a b}$
  - $\log_{b^n} a = \frac{\log_b a}{n}$
  - $b^{\log_a d} = d^{\log_a b}$
  - $-\log_b a = \log_b\left(\frac{1}{a}\right) = \log_{\frac{1}{b}} a$

Limits

- $\lim_{x\to 0^+} \log_a(x) = -\infty; a > 1$
- $\lim_{x\to 0^+} \log_a(x) = \infty; 0 < a < 1$
- $\lim_{x\to \infty} \log_a(x) = \infty; a > 1$
- $\lim_{x\to \infty} \log_a(x) = -\infty; 0 < a < 1$
- $\lim_{x\to 0^+} x^b\log_a(x) = 0; b > 0$
- $\lim_{x\to \infty} \frac{\log_a(x)}{x^b} = 0; b > 0$

Theorems on Limits

Definition:

The (ε , δ)-definition of limit "epsilon–delta definition of limit" is a formalization of the notion of limit.

Let f be a real-valued function defined on a subset D of the real number. Let c be a limit point of D and let L be a real number. We say that $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exist a δ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Thus, when we say $f(x)$ is close to L we mean $|f(x) - L|$ is small. When we say that x and a are close, we mean that $|x - a|$ is small

Properties:

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

- $\lim_{x\to a} c = c$
- $\lim_{x\to a} x = a$
- $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
- $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} [f(x) g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$

Note: the root function is a special case of exponential function. The same rules apply

One-sided limits

- Left-hand limit: $\lim_{x\to a^-} f(x) = f(a)$. The definition of a limit still holds but bounded to $x < a$.
- Right-hand limit: $\lim_{x\to a^+} f(x) = f(a)$. The definition of a limit still holds but bounded to $x > a$.

Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x) = f(a) \to \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$
$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \to \lim_{x\to a} f(x)$ Does Not Exist

Theorems

Theorem1.0: if $f(x)$ and $g(x)$ are continuous at a and c is a constant, then the following functions are also continuous at a :

- $f(a) \pm g(a)$
- $cf(a)$ or $cg(a)$
- $f(a) \times g(a)$
- $\frac{f(a)}{g(a)} \mid g(a) \neq 0$

Theorem (1.1)
a. any *polynomial* is *continuous everywhere*; that is, it is continuous on $R = (-\infty, \infty)$.
b. any *rational function* is continuous wherever it is defined; that is, it is *continuous on its domain.*

Theorem (1.2): if n is a positive integer, then $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty]$. If n is a positive odd integer, then f is continuous on $(-\infty, \infty)$.

Theorem (1.3): if f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b) = f(\lim_{x \to a} g(x))$.

Theorem (1.4): if g is continuous at a and f is continuous at $g(a)$, then $(f)(x) = f(g(x)$ is continuous at a .

Limits at Infinity

Definition: Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then $\lim_{x \to a} f(x) = \infty$ means that for every positive number M there is a corresponding number $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

Definition: Let f be a function defined on some open interval such contains the number a, except possibly at a itself. Then $\lim_{x \to a} f(x) = -\infty$ means that for every positive number N there is a corresponding number $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - a| < \delta$.

Continuous and Discontinuous functions

Definition (a): a function $f(x)$ is said to be **continuous** *if and only if* the following are satisfied:

- $f(x)$ exists
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(x)$.

Otherwise, it is said to be *discontinuous.*

Definition (b): a function f is continuous from the **right** at a number a if $\lim_{x \to a^+} f(x) = f(a)$. And f is continuous from the **left** at a if $\lim_{x \to a^-} f(x) = f(a)$.
Definition (c): a function f is continuous on an **interval** if it is continuous at every number in the interval.

Differential Calculus

Differentiability and Continuity

Theorem (continuity): A function f is continuous at $x = a$ provided that:

1) f has a limit as $x \to a$
2) f is defined at $x = a$, and
3) $\lim_{x \to a} f(x) = f(a)$

Theorem (differentiability): a function f , defined on an open set U , is said to be differentiable at $a \in U$ if any of the following equivalent conditions is satisfied:

1) The derivative $f'(a)$ exists
2) There exists a real number L such that $\lim_{(h \to 0)} \frac{(f(a+h) - f(a) - Lh)}{h} = 0$. The number L , when it exists, is equal to $f'(a)$
3) There exists a function g , such that $f(a+h) = f(a) + f^{'}(a)\, h + g(h)$ and $\lim_{h \to 0} \frac{g(h)}{h} = 0$.

Differentiation Rules

1. Differentiation is linear: for any functions f and g and any real numbers a and b , the derivative of the $h(x) = af(x) + bg(x)$ with respect to x is: $h^{'}(x) = af^{'}(x) + bg'(x)$.

   Special cases include:

- Constant rule: $af(x) \frac{d}{dx} = af'(x)$
- Sum & Subtraction rule: $(f(x) \pm g(x)) \frac{d}{dx} = f^{'}(x) \pm g^{'}(x)$

1. Product rule: for any functions f and g , the derivative of the function $h(x) = f(x)\, g(x)$ with respect to x is: $f^{'}(x)\, g(x) + g^{'}(x)\, f(x)$.

1. Power rule: $x^n \frac{d}{dx} = nx^{n-1}$
2. Chain Rule: for any function, the derivative of the function $h(x) = f(g(x))$ is: $f^{'}(g(x))\, g^{'}(x)$ or $\frac{d(f(g(x))}{dx} \frac{dg(x)}{dx}$

1. Quotient rule: if f and g are functions, then:

$$\left(\frac{f(x)}{g(x)}\right) \frac{d}{dx} = \frac{f^{'}(x)\, g(x) - g^{'}(x)\, f(x)}{(g(x))^2}$$

- Rules on exponentials: for any function g and a constant a .
  - $e^{g(x)}\frac{d}{dx} = g^{'}\left(x\right)e^{g(x)}$
  - $a^{g(x)}\frac{d}{dx} = g^{'}\left(x\right)a^{g(x)}ln\left(a\right)$
  - $\ln\left(g\left(x\right)\right)\frac{d}{dx} = \frac{g^{'}(x)}{g(x)}$
  - $\log_{a}g\left(x\right)\frac{d}{dx} = \frac{g'(x)}{g(x)ln(a)}$

Derivative of Transcendental functions

- $\sin\left(x\right)\frac{d}{dx} = \cos\left(x\right)$
- $\cos\left(x\right)\frac{d}{dx} = -sin\left(x\right)$
- $\tan\left(x\right)\frac{d}{dx} = \sec^{2}\left(x\right)$
- $\cot\left(x\right)\frac{d}{dx} = -\csc^{2}\left(x\right)$
- $\sec\left(x\right)\frac{d}{dx} = \tan\left(x\right)sec\left(x\right)$
- $\csc\left(x\right)\frac{d}{dx} = -\cot\left(x\right)csc\left(x\right)$

Higher-order derivatives

Higher-order derivatives are expressed as $\frac{d^{n}f(x)}{dx^{n}}$ or $f^{n}\left(x\right)$ in Langrangian notation for a function $f\left(x\right)$.

Illustrative example:

- $f\left(x\right) = cos\left(x^{3}\right)$; find $f^{2}\left(x\right)$

$$\begin{aligned} f^{1}\left(x\right) &= -3x^{2}\sin\left(x^{3}\right) \\ f^{2}\left(x\right) &= -6x\sin\left(x^{3}\right) - 9x^{4}\left(\sin\left(x^{3}\right)\right) \end{aligned}$$

Implicit differentiation

Illustrative example:

$$\frac{dy}{dx}\left(x^{2}+y^{2}\right) = 0$$

Solution: (solving for dy/dx)

$$\begin{aligned} \tfrac{dy}{dx}\left(x^{2}+y^{2}\right) &= 2x\ dx + 2y\ dy \\ \tfrac{dy}{dx} - 2y &= 2x \\ \tfrac{dy}{dx} &= \tfrac{2x}{-2y} \\ &= -\left(\tfrac{x}{y}\right) \end{aligned}$$

Integral Calculus

Basic Integration Rules:

- $F^{'}\left(x\right)=f\left(x\right)$
- $\int\left[f\left(x\right)\pm g\left(x\right)\right]dx=\int f\left(x\right)dx\pm\int g\left(x\right)dx$
- $\int kdx=kx+C$
- $\int kf\left(x\right)dx=k\int f\left(x\right)dx$
- $\int x^{n}dx=\frac{x^{n+1}}{n+1}+C$

Integrals of Transcendental Functions

- $\int\sin x\ dx=-\cos\left(x\right)+C$
- $\int\cos\left(x\right)\ dx=\sin\left(x\right)+C$
- $\int\sec^{2}\left(x\right)\ dx=\tan\left(x\right)+C$
- $\int\csc^{2}\left(x\right)\ dx=-\cot\left(x\right)+C$
- $\int\sec\left(x\right)\tan\left(x\right)dx=\sec\left(x\right)+C$
- $\int\csc\left(x\right)\cot\left(x\right)dx=-\csc\left(x\right)+C$

Integrals Yielding Natural Logarithmic Functions

- $\int\frac{1}{x}dx=\ln x+C$
- $\int\frac{f^{'}(x)}{f(x)}dx=ln|f\left(x\right)|+C$

Integrals of Exponential and Logarithmic Functions

- $\int e^{x}dx=e^{x}+C$
- $\int a^{x}dx=\frac{a^{x}}{\ln a}+C$
- $\int\log_{a}xdx=x\left(\log_{a}x-\log_{a}e\right)+C$

Techniques of Integration

The Substitution Method (Chain-rule in Integration)

Let $g\left(x\right)$ be differentiable and let I be the range of $g\left(x\right)$. Suppose that $f\left(x\right)$ is defined on I and $F\left(x\right)$ is an anti-derivative of f on I then:

- $\int f(g(x))[g'(x)]\ dx = F(g(x)) + C$

Letting $u = g(x),\quad du = g^{'}(x)\ dx,$

- $\int f(u)\,du = F(u) + C$

Integration by Parts (Product rule in Integration)

Integration by parts (or partial integration) is the analogue of the product rule of differentiation. Thus, the proof can be derived from the product rule of differentiation. It is important to note the hierarchy of functions – for assigning the variables u to dv – which are given as follows: (a) **L**ogarithms, (b) **I**nverse **T**rigonometric functions, (c) **A**lgebraic functions, (d) **T**rigonometric functions, (e) **E**xponential functions.

Let f(x) and g(x) be two differentiable functions, then

- $\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f^{'}(x)\,dx$

Letting $u = f(x)\,; du = f^{'}(x)\,dx; v = g(x)\,; dv = g^{'}(x)\,dx$

- $\int udv = uv - \int vdu$

Illustration:

$$\int (3t+5)\cos\left(\frac{t}{4}\right)dt$$

We let $u = 3t+5,\ du = 3dt; dv = \cos\left(\frac{t}{4}\right),\ v = 4\sin\left(\frac{t}{4}\right)$

By IBP rule, the integral is then:

$$\begin{aligned}\int (3t+5)\cos\left(\tfrac{t}{4}\right)dt\ &= 4(3t+5)\sin\left(\tfrac{t}{4}\right) - 12\int \sin\left(\tfrac{t}{4}\right)dt\\ &= 4(3t+5)\sin\left(\tfrac{t}{4}\right) + 48\cos\left(\tfrac{t}{4}\right) + C\end{aligned}$$

Trigonometric Substitution

Trigonometric substitution is the substitution of trigonometric functions for other expressions. One may use the trigonometric identities to simplify certain integrals containing radical expressions.

Expression	**Substitution**	**Simplification**	**Used identities**
$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$a\cos\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$a\sec\theta$	$1-tan^2\theta = \sec^2\theta$
$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$a\tan\theta$	$\sec^2\theta - 1 = \tan^2\theta.$

Partial Fraction Decomposition

Partial fraction decomposition is an operation that consists of expressing the fraction as a sum of a polynomial (possibly zero) and one or several fractions with a simpler denominator. The importance of the partial fraction decomposition lies to the fact that it provides algorithms for computations involving rational functions. Although, the proof of this method extends beyond to the context that is normally introduced in Calculus I and II, we will be going to highlight the theorem as well as the procedure in this section.

Theorem:
Let f and g be nonzero polynomials over a field K .

$$g = \prod_{i=1}^{k} p_i^{n} \quad {}_{i}$$

There are polynomials b and c_{ij} with deg $c_{ij} < \deg p_i$ such that

$$\frac{f}{g} = b + \sum_{i-1}^{k} \sum_{j=2}^{rj} \left(\frac{c_{ij}}{p_i^{j-1}} \right)^{'} + \sum \frac{c_{i1}}{p_i}$$

where $X^{'}$ denotes the derivative of X.

Procedure

Linear factors:	$\frac{P(x)}{(x-r_1)^m} = \frac{A}{(x-r_1)} + \frac{B}{(x-r_1)^2} + \ldots + \frac{Y}{(x-r_1)^{m-1}} + \frac{Z}{(x-r_1)^m}$
Quadratic factors	$\frac{P(x)}{(x^2-r_1)^m} = \frac{Ax+B}{(x^2-r_1)} + \frac{Cx+D}{(x^2-r_1)^2} + \ldots + \frac{Wx+X}{(x-r_1)^{m-1}} + \frac{Yx+Z}{(x-r_1)^m}$

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