Sets & Relations

Dave Arthur D. Amiana *

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1 Sets

Sets can be defined as a collection of well-defined objects; it does not contain any duplicates of an entity that is already a member of the set. The term "well-defined" means that for a given value it is possible to determine whether or not it is a member of a set. Sets may be defined by using a predicate to constrain set membership (intentional), or by listing all the members of the set (extensional).

We can define the set of even numbers from 1, 10 in the following manner:

- Intentional:$S = \{n \in N : n \leq 10 \wedge n \; mod \; 2 = 0\}$
- Extensional: $S = \{2, 4, 6, 8, 10\}$

The intentional and extensional definitions of a set may be known in some context as Set-builder and Roster notations, respectively.

We say that a belongs to set A if $a \in A$. When x does not belong to set A, we say that $x \notin A$, or x is not an *element* of set A. Whenever set A and set B has some a common memember, we say that $A \cap B$, or A *intersects* B.Whenever a new set, say, C is defined as a collection of entities that belongs to set A or set B, we say that $C := A \cup B$ or C is the *union* of A and B. Finally, we can produce new sets by using our primitive operators.

Set-theoretic Operations

Since set theory can be more appropriately viewed as a language where we intend to define a mathematical construct, we shall first introduce the alphabet of sets.

- $Operators := \{\times, \; \neg, \; \vee, \; \wedge, \; \leftrightarrow, \; \rightarrow, \; \oplus\}$
- $Relations := \{=, \; >, \; <, \; \leq, \; \geq, \; \neq, \; \cup, \; \cap\}$

*FEU Institute of Technology, Computer Science

Given our familiarity with the primitive operators between sets or within sets themselves, we can now further expand our list of definitions by means of these primitive operators $\cup, \cap, \in, \notin$.

Symbols	Interpretation
$x \in A$	x is an element of set A
$A \subseteq B$	A is a proper subset of B
$B \supseteq A$	A is a superset of B
$\emptyset$	Empty set
$A \cup B$	The union of set A and B
$A \cap B$	The intersection of A and B
$P(A)$	Power set of set A
A^C	Complement of set A
$A \Delta B$	The symmetric difference of set A and B
$A \setminus B$	Difference of set A and set B
$\lvert A \rvert$	The cardinality of set A

Definition 1.1 (Proper Subset) *If A and B are sets, then:*

- *A is a **subset** of B if an only if their intersection is equal to A: $A \subseteq B \Leftrightarrow A \cap B = A$.*
- *A is a **subset** of B if an only if their union is equal to B: $A \subseteq B \Leftrightarrow A \cup B = B$*

Remark: A proper subset of a set A is a subset of A that is not equal to A. In other words, if B is a proper subset of A, then all elements of B are in A but A contains at least one element that is not in B.

Definition 1.2 (Super set) *A set containing all elements of a smaller set. If B is a subset of A, then A is a superset of B, written AB. If A is a proper superset of B this is written AB.*

Definition 1.3 (Power set) *The power set of any set S is the set of all the subsets of S, including the empty set $\emptyset$ and S itself.*

Definition 1.4 (Complement of Set) *Let A be a set, then the complement of A is the set of elements that are not in A.*

$$A^c = \{x \in U \mid x \notin A\}.$$

Definition 1.5 (Symmetric Difference) *The **symmetric difference** between two sets S and T is written $S \Delta T$ may be defined as follows:*

- $S \Delta T := S \setminus T \cup T \setminus S$
- $S \Delta T = S \cup T \setminus S \cap T$

Definition 1.6 (Set Difference) *The (set) difference between two sets S and T is written $S \setminus T$, and means the set that consists of the elements of S which are not elements of T:*

$$x \in S \setminus T \iff x \in S \wedge x \notin T$$

It can also be defined as:

$$S \setminus T = x \in S : x \notin T$$

$$S \setminus T = x : x \in S \wedge x \notin T$$

Example. Let S and T be sets such that:

$$S := \{1, 2, 3\}$$

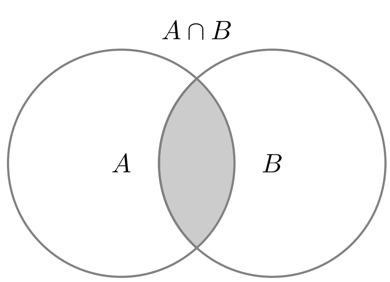
$$T := \{2, 4, 5, 6\}$$

Then, $S \setminus T := \{1, 3\}$

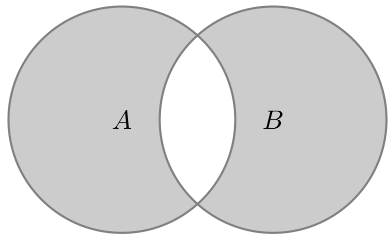
Definition 1.7 (Cardinality of Sets) *The cardinality of set A, denoted by $|A|$ is a measure of the number of elements in set A.*

Representation of Sets

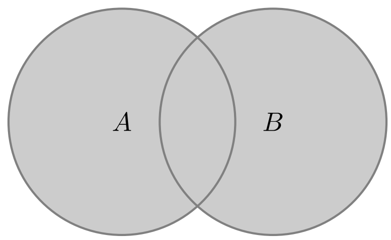
Primitive operations on sets can be represented using Venn Diagrams.



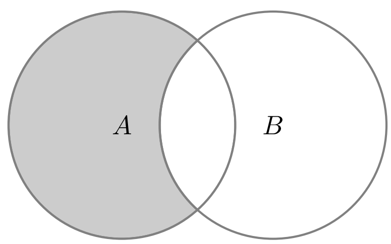
$\overline{A \cap B}$



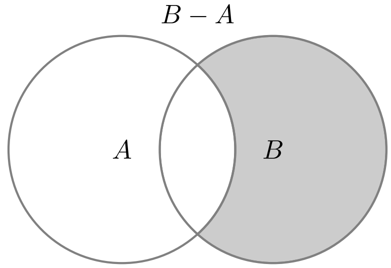
$A \cup B$



$A - B$



$B - A$



Properties of Sets

The algebra of sets defines the properties of sets, the set-theoretic operations of union, intersection, and complementation and the relations of set equality and set inclusion. It also provides systematic procedures for evaluating expressions, and performing calculations, involving these operations and relations.

Furthermore, algebra of sets is the set-theoretic analogue of the algebra of numbers. Just as arithmetic addition and multiplication are associative and commutative, so are set union and intersection; just as the arithmetic relation "less than or equal" is reflexive, anti-symmetric and transitive, so is the set relation of *subset*.

Associative laws:

$$A \cup (B \cup C) = (A \cup C) \cup B$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Commutative laws:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Distributive laws:

$$A \cap (B \cup B) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap B) = (A \cup B) \cap (A \cup C)$$

Identity law:

$$A \cup \emptyset = A; A \cap U = A$$

Domination law:

$$A \cup U = U; A \cap \emptyset = \emptyset$$

Complement laws:

$$A \cup A^C = U$$

$$(A')' = A$$

De Morgan's law:

$$(A \cup B)^C = A^C \cap B^C$$

$$(A \cap B)^C = A^C \cup B^C$$

Indempotent law:

$$A \cup A = A$$

$$A \cap A = A$$

Absorption Law:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Computer Representation of Sets

The representation of a set on a computer requires a change from the set-theoretic conception where the order is deemed irrelevant. In the case of computers, we have to assume definite order in the underlying universal set; that is to say that a set is always defined in a computer program with respect to an underlying universal type (set) and the elements in the the universal set are listed in an ordered manner.

For an instance, let $M = \{x \in N | 0 < x \leq 10\}$ be the universal set, then the representation of $\bar{M} := \{n \in M | n \ mod \ 2 = 0\}$ is given by a memory representation in bits of 10 strings as 0101010101. Where $0, 1$ are boolean representation of memberships in the set [2].

There is a one-to-one correspondence between the subsets of M and all possible n-bit strings. Further, the set theoretical operations of set can be carried out directly with the bit strings (provided that the sets involved are defined with respect to the same universal set).

This representation stems not only from an engineering perspective of computer memory, but to a more abstract piece of mathematics that made computer science its name as becoming a highly theoretical subject called Type theory. However, we will not talk about Types and its more rigorous properties over sets. For now, it is interesting to note that we can, indeed, construct sets from primitive data types. And in that way, our membership assumptions will not be taken for granted. For the interested reader, I personally recommend to read the Philosophical entries of this theory first which I link here [1].

References

[1] Thierry Coquand. “Type Theory”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2018. Metaphysics Research Lab, Stanford University, 2018.

[2] Gerard OâRegan. *Guide to discrete mathematics*. Springer, 2016.

[3] Anthony Ralston. *Mathematical Structures for Computer Science*. 1983.

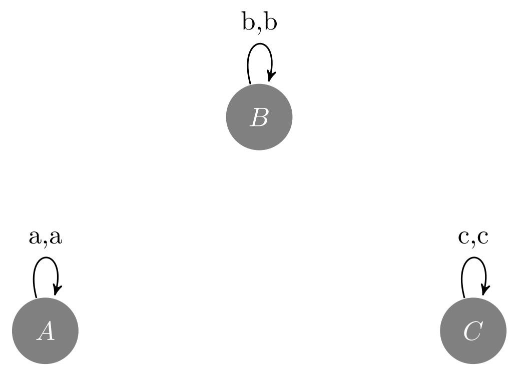


Figure 1: Reflexive relation

2 Relations

We can define relations as a subset of a cross-product between members of two sets A, B. For this section, we will be introduced to some properties of relations. As well as some conceptual illustrations for understanding relations.

Definition 2.1 (Domain and Range of a Relation) *Domain is the set of all the first elements in the ordered pair. Range is the set of all the second elements in the ordered pair of R.*

Example. Let $(a, b) \in P$ represent a relation of set A, B where $a \in A, b \in B$. Then, a is the domain of P, and b is the range of P.

Definition 2.2 (Inverse of a Relation) *The inverse of a relation is denoted as R^{-1} and is subset of $B \times A$.*

Example. Let $(a, b) \in R$ represent a relation between sets A, B, where $a \in A$ and $b \in B$. Then the inverse of R is, $(b, a) \in R^{-1}$

Definition 2.3 (Reflexive Relations)

$$\forall a \in R.[(a, a) \in R]$$

Definition 2.4 (Symmetric Relations)

$$\forall a, b \in R.[(a, b) \in R \wedge (b, a) \in R]$$

Definition 2.5 (Transitive Relations)

$$\forall a, b, c \in R.[(a, b) \in R \wedge (b, c) \in R] \rightarrow (a, c) \in R$$

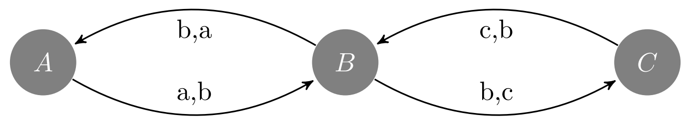


Figure 2: Symmetic Relation

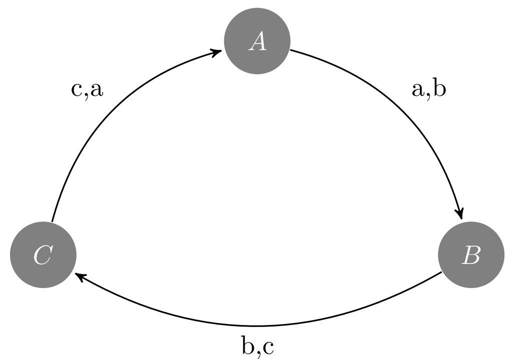


Figure 3: Transitive Relations

Definition 2.6 (Equivalence Relations) *Equivalence relations are those relations which are reflexive, symmetric, and transitive at the same time.*

Theorem. an equivalence relation on A gives rise to a partition of A where the equivalence classes are given by $Class(a) = \{x | x \in A \wedge (a, x) \in R\}$. Similarly, a partition gives rise to an equivalence relation R where $(a, b) \in R$ if and only if a and b are in the same partition.

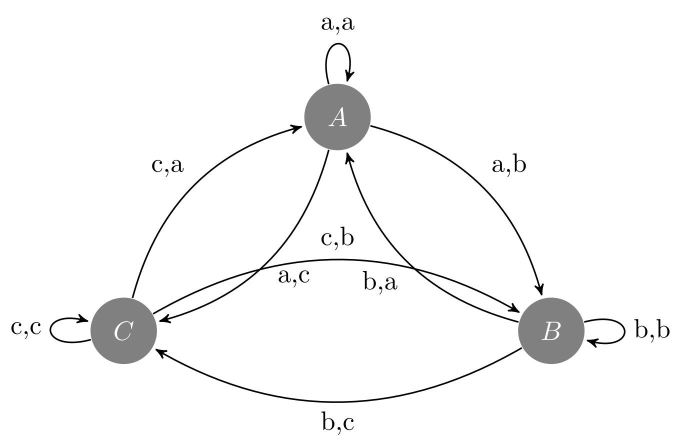


Figure 4: Equivalence Relation